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Abstract
In a fundamental contribution, Prescott and Townsend (1984) [PT] have shown that the existence and efficiency properties of Walrasian equilibria extend to economies with moral hazard, when agents' trades are observable (exclusive contracts can be implemented). More recently, Bennardo and Chiappori (2003) [BC] have argued that Walrasian equilibria may (robustly) fail to exist when the class of moral hazard economies considered by Prescott and Townsend is generalized to allow for the presence of aggregate, in addition to idiosyncratic, uncertainty and for preferences which are nonseparable in consumption and effort. We re-examine here the existence and efficiency properties of Walrasian equilibria in the moral hazard economy considered by Bennardo and Chiappori. We show that Walrasian equilibria always exist in such economy and are incentive efficient, so the results of Prescott and Townsend continue to hold in the more general set-up considered by Bennardo and Chiappori.

Keywords
Moral Hazard, Aggregate Risk, Incentive Efficiency, Walrasian Markets
JEL Codes
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1 The Environment

Our notation is slightly adapted so as to be closer to the one in PT. There is a continuum of ex ante identical individuals with measure one and a single consumption good. Individuals are subject to both aggregate and idiosyncratic risk. Specifically, there are two aggregate states, $s = H, L$, and two idiosyncratic states, $\sigma = 1, 2$. The individual's endowment ω_s^σ depends on the realization of both s and σ. Without loss of generality ω_s^σ is assumed to be higher in aggregate state $s = H$ than in $s = L$ for each realization of the idiosyncratic state σ: $\omega_H^\sigma > \omega_L^\sigma$. Likewise, ω_s^σ is higher when the idiosyncratic state $\sigma = 1$ is realized, no matter what is the aggregate state $s : \omega_s^1 > \omega_s^2$. Idiosyncratic shocks are independently and identically distributed across all agents, and are independent of the aggregate shock. The probability π_s of aggregate state s is exogenous. On the other hand, the probability of idiosyncratic state σ depends on the level of effort of the individual. Effort can be high or low; the set of effort levels is then $E \equiv \{e_l, e_h\}$ and we assume that effort is undertaken by the individual prior to the realization of uncertainty (both aggregate and idiosyncratic). Let $\pi_{e\sigma}$ denote the probability of idiosyncratic state σ when the agent exerts effort $e \in E$; we assume that $0 < \pi_{e_l 1} < \pi_{e_h 1} < 1$. In words, the probability that the high-endowment idiosyncratic state is realized is higher when effort is high. The realization of the aggregate and idiosyncratic states is public, but the individual choice of effort is not.

Individuals have von Neumann-Morgernstern preferences described by the Bernoulli utility function $u : \Re_+ \times E \to \Re$. The utility of consumption c when effort e is undertaken, $u_e(c) \equiv u(c, e)$, is twice continuously differentiable, strictly increasing, and strictly concave with $\lim_{c\to 0} u_e'(c) = \infty$ and $\lim_{c\to\infty} u_e'(c) = 0$. Effort is costly, so $u_{e_l}(c) > u_{e_h}(c)$ for all $c \in C$.

Following PT, as well as BC, we assume that the set of possible consumption levels in any state is given by a finite subset of $\Re_+$, denoted by C. Let Z be then the set of possible state-contingent net trades of an individual. It is convenient to write $Z = Z_H \times Z_L$, where Z_s is the set of possible net trades in aggregate state s

$$Z_s = \left\{ (\, z_s^1, z_s^2) \in \Re^2 : \, z_s^\sigma + \omega_s^\sigma \in C, \ \sigma = 1, 2 \right\}.$$

Elements of Z are then denoted $z = (z_H, z_L)$.

2 The General Equilibrium Model

Commodities. The commodities traded in the model are *insurance contracts*. An insurance contract specifies an effort level $e \in E$ and a vector of state-contingent net trades $z \in Z$. This specification is allowed to be random. The commodity space L is so the set of measures on $E \times Z$. Since the set $E \times Z$ is finite, L is isomorphic to the Euclidean space of dimension n, where n is the cardinality of $E \times Z$:[1]

$$L = M(E \times Z) = \Re^n.$$

An insurance contract x is described as a probability measure on $E \times Z$; i.e., a vector $x = \{x(e,z)\}_{(e,z)\in E\times Z} \in P(E \times Z)$ where

$$P(E \times Z) \equiv \{x \in L^+ : \sum_{(e,z)\in E\times Z} x(e,z) = 1\}.$$

Here, $x(e,z)$ represents the probability that the contract specifies effort e and net trade z.

For a given contract x, let $x_e \in P(E)$ denote the marginal probability distribution with respect to e. This marginal distribution describes the probability that the contract specifies high and low effort. Also, let $x_{z/e} \in P(Z)$ denote the conditional distribution of z for a given effort e; this specifies a random vector of state-contingent net trades assigned when effort e is specified. Intuitively, the specification of the contract can be interpreted as follows. First, the lottery x_e prescribes an effort level. For any realization e of this lottery, a second lottery $x_{z/e}$ specifies the level of net trades in every state. Both lotteries are realized prior to the realization of the aggregate and idiosyncratic states. Since the effort undertaken by an individual is his private information, the specification of an effort level has to be understood only as a prescription, which to be effective must satisfy appropriate incentive compatibility constraints. It is also convenient to define the marginal probability distribution with respect to (e, z_s), which we denote by $x_s \in P(E \times Z_s)$; this determines the (random) effort level together with the (random) level of net trades in each idiosyncratic state when aggregate state s is realized.

The expected utility of an individual who exerts effort e and receives a net trade z is

$$v(e,z) \equiv \sum_{s=H,L} \pi_s \sum_{\sigma=1,2} \pi_{e\sigma} u_e(\omega_s^\sigma + z_s^\sigma).$$

[1] An equivalent (though slightly more involved) analysis can be carried out when C is an infinite set (e.g., $C = \mathcal{R}_+$), and $M(E \times Z)$ is then endowed with the weak-star topology (see for instance Jerez (2005)). The results presented here extend to that case.

The expected utility from a contract x is then given by the scalar product

$$v \cdot x = \sum_{(e,z)\in E\times Z} v\,(e,z)\, x(e,z). \tag{1}$$

Admissible Trades. Since trades are assumed to be observable, any restriction on trades can be imposed. Following PT, the set of contracts available for trade to any individual (in short, with some abuse of language, her consumption set) is the set of contracts that satisfy the incentive compatible (IC) constraints:

$$\bar{X} = \{x \in P(E \times Z) : \sum_{z\in Z} v\,(e,z)\, x_{z/e}(z) \geq \int_Z v\,(e',z)\, x_{z/e'}(z), \quad e \neq e';\ e,e' \in E\}.$$

The IC constraints require that, whenever contract x prescribes effort e, the implied conditional probability distribution $x_{z/e}$ is such that individuals prefer to conform to the effort prescription e rather than deviating to e'.

Prices. Prices are linear on the agents' consumption set, i.e. they are linear in probabilities. A price system is then an element of the linear space $L : p = \{p(e,z)\}_{(e,z)\in E\times Z} \in L$. The cost of a commodity bundle $x \in L$ is given by the scalar product

$$p \cdot x = \sum_{(e,z)\in E\times Z} p\,(e,z)\, x\,(e,z)\,. \tag{2}$$

Resource constraints. The economy is subject to two resource constraints, one for each aggregate state s. These constraints ensure that aggregate consumption in each state s does not exceed the aggregate endowment in that state. We will look at symmetric allocations, where all individuals trade the same contract x. By the Law of Large Numbers, when all individuals exert effort e, the fraction of individuals who end up in idiosyncratic state σ is $\pi_{e\sigma}$. Hence, the total (per capita) use of resources in state s when all individuals exert effort e and receive the net trade vector z_s is

$$r_s\,(e, z_s) \equiv \sum_{\sigma} \pi_{e\sigma} z_s^{\sigma}, \quad s = H, L.$$

Under contract $x \in X$, the total use of resources in state s is then given by the scalar product

$$r_s \cdot x_s = \sum_{(e,z_s)\in E\times Z_s} r_s\,(e, z_s)\, x_s(e, z_s), \tag{3}$$

where, as we have noted, x_s is the marginal probability distribution of x with respect to (e, z_s).

A contract (or symmetric allocation) x satisfies the resource constraints if the total net use of resources is non-positive in both aggregate states:

$$r_s \cdot x_s \leq 0, \quad s = H, L. \tag{4}$$

3 Incentive Efficient Allocations

A (symmetric) allocation x is feasible if $x \in \bar{X}$ and x satisfies the aggregate resource constraints (4). An allocation x is then *incentive efficient* if it maximizes the individual expected utility in the set of feasible allocations:

$$\max_{x \in \bar{X}} \quad v \cdot x \qquad \text{s.t} \quad r_s \cdot x_s \leq 0, \quad s = H, L. \tag{5}$$

The objective function in problem (5) is linear (and thus continuous) and the feasible set is a non-empty,[2] closed and bounded subset of the Euclidean space. Therefore, an optimal solution to the problem always exists. Also, it is easy to check that the feasible set is convex (in words, if contracts x and x' satisfy the incentive compatibility and resource constraints, so does any convex combination of these contracts).

4 Competitive equilibria

Following PT as well as BC, we introduce intermediation firms who supply contracts to consumers. Each firm is characterized by a technology:

$$Y = \{y \in L : \quad r_s \cdot y_s \leq 0, \quad s = H, L\}, \tag{6}$$

where y_s denotes the projection of the measure y on the set $E \times Z_s$. The specification in (6) says that a firm can offer any set of contracts, given by probability distributions over effort levels and net trades in every state, subject to the only constraint that the total net payments required by the contracts offered are self-financing. The Law of Large Numbers is applied to the set of contracts offered by a firm, allowing to write the self-financing constraint in expected terms in each aggregate state. Positive (resp. negative) components of y constitute commitments for the firm to pay (rights to receive) resources in a given state s, σ, given e.

Since Y displays constant returns to scale, profits will be zero in equilibrium and there is no loss of generality in assuming that there is a single firm in the market.

Definition A competitive equilibrium is a triple $(x^*, y^*; p^*) \in L^3$ such that: (i) x^* maximizes $v \cdot x$ over the set $\{x \in \bar{X} : \ p^* \cdot x \leq 0\}$; (ii) y^* maximizes $p^* \cdot y$ over the set Y; and (iii) markets clear, or $x^* = y^*$.

[2] The allocation where individuals exert low effort with probability one and consume their expected endowment in each aggregate state s is clearly always feasible.

Condition (i) requires that contract x^* gives every individual the highest utility among all budget feasible contracts lying in his consumption set. Condition (ii) says that y^* is a solution of the firm's problem, consisting in the choice of a vector y lying in the set Y that maximizes profits. The market clearing condition (iii) says that aggregate demand by consumers for insurance contracts equals aggregate supply by the firm.

5 Efficiency and Existence of Equilibrium

In this section, we show that a competitive equilibria always exists (in contrast with Proposition 5 of BC).

Theorem 1 In the economy under consideration, a competitive equilibrium always exists. In particular, any (symmetric) incentive efficient allocation can be supported as a competitive equilibrium.

Proof: Let $p_s(e, z_s)$ denote the projection of $p(e, z)$ on the set $E \times Z_s$, so that $p(e, z) = p_H(e, z_H) + p_L(e, z_L)$. It is immediate to verify (see also Lemma 3 in BC) that the constant returns to scale nature of the firms' technology Y implies that equilibrium prices satisfy the following property:

$$p_s(e, z_s) = \beta_s r_s(e, z_s), \tag{7}$$

for some $\beta_s \geq 0$, for each $s = H, L$; i.e., in each state s the price of (e, z_s) must either be actuarially fair, and be proportional to the expected use of resources at z_s when agents exert effort e, or be zero.

To make the comparison with BC easier, the rest of proof relies on a constructive argument: for any possible solution x^E of the planner's problem (5), we will find prices satisfying (7) which support x^E as a competitive equilibrium.

Since preferences are monotone, at x^E the resource constraint must bind in at least one state s. It is easy to see why: suppose both constraints were slack. Let x^l be a contract specifying low effort and an arbitrarily high level of net trades with probability one (regardless of the realization of s and σ). Contract x^l is incentive compatible ($x^l \in \bar{X}$) and strictly preferred to x^E by the individual, and hence so is any convex combination of x^E and x^l : $x^\alpha = (1-\alpha)x^E + \alpha x^l$ with $\alpha \in (0, 1]$. Also, if α is sufficiently small, x^α satisfies the resource constraints (4). But then x^E cannot be a solution to (5).

Consider then the case where at x^E the resource constraint does not bind in one aggregate state, in particular in $s = H$. This is the case analyzed in Proposition 5 of BC.[3] We will show that the following prices, given by (7) with $\beta_H = 0$ and $\beta_L = 1$, support x^E as a competitive equilibrium:

$$p^*{}_H(e, z_H) = 0 \text{ for all } e \in E, z_H \in Z_H, \tag{8}$$

$$p^*{}_L(e, z_L) = r_L(e, z_L) \text{ for all } e \in E, z_L \in Z_L. \tag{9}$$

The intuition is simple. The price associated to state s is the shadow cost of resources in s at an incentive-efficient allocation. Because the resource constraint in state $s = H$ is slack, the shadow cost is zero in this state for any pair $(e,\, z_H)$. On the other hand, the resource constraint in state $s = L$ is binding, so in this state the shadow cost is positive and proportional (in fact can be set equal) to the expected use of resources associated to any pair $(e,\, z_L)$. In sum, prices are zero in state $s = H$ and actuarially fair in state $s = L$.

When consumers face the price system in (8)-(9), the consumer's problem becomes:

$$\max_{x \in \bar{X}} \quad v \cdot x \quad \text{s.t} \quad p^* \cdot x = r_L \cdot x_L \leq 0. \tag{10}$$

We show next that x^E is a solution to this problem (so that condition (i) of the definition of a competitive equilibrium is satisfied). Since the resource constraint in state $s = H$ does not bind in the planner's problem (5), x^E is a local maximum of the same problem when this constraint is omitted. And, because the objective function is linear and the feasible set is convex in (10), a local maximum is also a global maximum by the local-global theorem (Intriligator (1971), p. 75).

Substituting the values in (8)-(9) for p^* in the expression of the firm's profits gives

$$p^* \cdot y = r_L \cdot y_L,$$

so (6) implies $p^* \cdot y \leq 0$ for all $y \in Y$. The market clearing condition (iii) requires then $y^* = x^E$. Since the resource constraint for $s = L$ binds in the planner's problem,

$$p^* \cdot y^* = r_L \cdot x^E_L = 0.$$

[3] BC derive some sufficient conditions for x^E to have this property and show that there is an open set of economies which satisfy them. Intuitively, if consumption and leisure are complements and the marginal utility of consumption decreases fast enough with effort, there is a limit to the level of consumption such that agents are still willing to provide high effort. Hence, when the aggregate endowment in $s = H$ is high enough, part of the aggregate endowment will not be consumed in that state.

Thus y^* is an optimal production plan for the firm. The firm's optimization condition (ii) then also holds, which completes the proof of the claim that a competitive equilibrium exists in the case under consideration, supporting the efficient allocation x^E.

It remains so to examine the case where both resource constraints bind at x^E. The allocation can now be decentralized with the following prices:

$$p^*{}_H(e, z_H) = \beta_L^E r_L(e, z_L) \text{ for all } e \in E, z_H \in Z_H \tag{11}$$

$$p^*{}_L(e, z_L) = \beta_H^E r_L(e, z_L) \text{ for all } e \in E, z_L \in Z_L, \tag{12}$$

where β_L^E and β_H^E are the shadow prices of the resource constraints at the solution of the planner's problem (5). One can again show that at such prices x^E solves the consumer's problem. Since x^E, β_L^E and β_H^E solve the first order conditions of the planner's problem, x^E also solves the first order conditions of the consumer's problem at the prices in (11)-(12) when the Lagrange multiplier of the budget constraint equals one. The reason is that, at the above values, the Lagrangean functions of the two problems have the same form. The rest of the argument is identical to the one of the previous case. ■

Remark 1 *In equilibrium aggregate consumption is lower than the aggregate endowment in state $s = H$ (i.e. there are resources not utilized in the high-endowment state). However, there is no incentive compatible and budget feasible contract that provides the consumer a higher utility than at x^E by allowing her to consume additional resources when $s = H$ is realized. This claim is in contrast with the one in the proof of Lemma 4 in BC. The authors argue that, if the price associated to net trades in state H were zero regardless of the effort level, the consumer could do better by buying a different contract x' where x'_H specifies low effort and a very high level of net trades (and hence of consumption) with probability one, whatever the idiosyncratic outcome. Since x' is clearly not feasible, BC concluded that $p^*{}_H(e, z_H)$ could not be zero at an equilibrium; the nonexistence result in Proposition 5 then relies on such claim. But this misses an important point. Namely, that effort is chosen before the realization of the aggregate state, so if x'_H specifies low effort with probability one so must x'_L. While a contract specifying low effort with probability one can provide a very high level of consumption if the high-endowment state $s = H$ is realized, consumption in the low-endowment state $s = L$ may have to be rather low. The consumer in fact needs to pay a positive price for the consumption goods received in state L, and the price can be quite high - and the value of the endowment quite low - in $s = L$ when the consumer exerts low effort.*

Formally, if (as claimed by BC) x' is feasible for the consumer it must induce agents to exert low effort (incentive compatibility has to hold) and the budget constraint must be satisfied:

$$p^* \cdot x' = \ p_H^* \cdot x'_H + \ p_L^* \cdot x'_L = r_L \cdot x'_L \leq 0.$$

Since x' is strictly preferred to x^E by the consumer, so is any convex combination of x^E and x' : $x^\beta = (1-\beta)x^E + \beta x'$ with $\beta \in (0,1]$. For any β, x^β satisfies the incentive compatibility constraints and the resource constraints in state L. Also, since at x^E the resource constraint in $s = H$ is slack, if β is sufficiently small, the same is true at x^β. But this would contradict the fact that x^E is a solution to (5).

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