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Departamento de Estadística y Econometría
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34)91 624-9849

CONSISTENT SPECIFICATION TESTING OF STATIONARY PROCESSES WITH LONG-RANGE DEPENDENCE: ASYMPTOTIC AND BOOTSTRAP TESTS.

Miguel Angel Delgado y Javier Hidalgo*

Abstract

The paper proposes goodness-of-fit tests for the class of covariance stationary FARIMA processes, which are consistent in the direction of a general covariance stationary linear process. The tests are based on functionals of marked empirical processes, whose marks are the integrated relative error of the empirical spectral density (periodogram) of the data to the estimated spectral density function under the specified FARIMA process. Two examples of such functionals are the T_p - Barlett and the Cràmer-Von Mises standardized ω - statistic. Moreover, the tests are able to detect contiguous alternatives which converge to the null at the parametric rate $n^{-1/2}$. Because distribution free tests are difficult to implement, we propose a bootstrap test showing its consistency and studying its small sample performance by means of a Monte Carlo experiment.

Keywords:
Goodness-of-fit; FARIMA processes; Marked empirical processes; Contiguous alternatives; Bootstrap tests.

*Delgado, Departamento de Estadística y Econometría, Universidad Carlos III de Madrid. C/ Madrid, 126 28903 Madrid. Spain. Ph: 34-1-624.98.04; Fax: 34-1-624.98.49, e-mail: delgado@est-econ.uc3m.es; Hidalgo, London School of Economics, Department of Economics, Houghton Street, London WC2A 2AE, U.K. This last author thanks the hospitality of the Department of Statistics and Econometrics at the Universidad Carlos III of Madrid while this paper was written. Research funded by the Spanish Dirección General de Enseñanza Superior, reference number: PB95-0292.

1. PRELIMINARIES AND STATEMENT OF THE PROBLEM

This paper proposes goodness-of-fit tests of a covariance stationary linear process $\{x_t\}$ which is observed at times $t = 1, 2..., n$. More precisely, we consider a process $\{x_t\}$ having mean that it is (without loss of generality) zero, and with absolute continuous spectral distribution, so that it has a spectral density function, denoted $f(\lambda)$, defined from

$$\gamma(j) = E(x_j x_0) = \int_{-\pi}^{\pi} f(\lambda) e^{ij\lambda} d\lambda \quad j = 0, 1, 2, ..., \tag{1}$$

so that the second moment properties of the process are equivalently specified either in the time domain, by the autocovariance function $\gamma(j)$, and $j = 0, 1, ...,$ or in the frequency domain, by its spectral density $f(\lambda)$ with $\lambda \in [0, \pi]$.

It is well known that if the spectral density $f(\lambda)$ satisfies the condition

$$\int_{-\pi}^{\pi} \log(f(\lambda)) d\lambda > -\infty,$$

then, it guarantees that the process x_t admits a backwards expansion, that is,

$$x_t = \sum_{j=0}^{\infty} \alpha(j) \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} |\alpha(j)|^2 < \infty, \tag{2}$$

where the innovations ε_t form a zero mean white noise process with variance σ_ε^2. Within a parametric framework, it is assumed that the coefficients $\alpha(j)$, in (2), depend on a finite set of parameters, say β, so that $\alpha(j) = \alpha(j, \beta)$ for all $j = 0, 1, 2, ...$ In this case, the spectral density function of the model (2) is

$$f(\lambda) = f(\lambda; \beta, \sigma_\varepsilon^2) = \frac{\sigma_\varepsilon^2}{2\pi} |A(\lambda; \beta)|^2,$$

where $A(\lambda; \beta) = \sum_{j=0}^{\infty} \alpha(j; \beta) e^{ij\lambda}$ is known as the spectral transfer function. It is noteworthy that, in the time domain, the counterpart of the spectral transfer function is

$$P(z; \beta) = \sum_{j=0}^{\infty} \alpha(j; \beta) z^j,$$

so that, it is readily observed the equivalence, in its formulation, of the process x_t given in the time domain, that is (2), and by the spectral density function, that is $f(\lambda; \beta, \sigma_\varepsilon^2)$.

Following the influential work by Box and Jenkins (1976), a classical finite parameterization of the model given in (2) is the stationary and invertible autoregressive moving average $ARMA(p, q)$ process defined as

$$\Phi(L, \phi) x_t = \Xi(L, \psi) \varepsilon_t,$$

where here $\beta = (\phi', \psi')'$, and $\Phi(\cdot)$ and $\Xi(\cdot)$ are the Autoregressive and Moving Average polynomials of order p and q, respectively. A generalization of the above model is the so-called stationary Fractional Integrated Autoregressive Moving Average ($FARIMA(p,d,q)$) process defined as

$$\Phi(L,\phi)(1-L)^d x_t = \Xi(L,\psi)\varepsilon_t, \tag{3}$$

where in (3), $\beta = (\phi', \psi', d)'$. The process given in (3), apparently originated in Adenstedt (1974), and introduced by Granger and Joyeux (1980) and Hosking (1981), has recently attracted a lot of attention in empirical research, see for instance Diebold and Rudebusch (1989), Porter-Hudak (1990), Sowell (1992) or Ray (1993) among others. When $d > 0$, we say that the process given in (3) exhibits long-range dependence, for $d = 0$, the process corresponds to the so-called weakly or short-range dependence, while for $d < 0$, we have an example of a process exhibiting the so-called negative or anti-persistent dependence.

Alternatively, the $FARIMA(p,d,q)$ model can be characterized as that process whose spectral density function is given by

$$f(\lambda,\theta) = \frac{\sigma_\varepsilon^2}{2\pi}\frac{1}{|1-e^{i\lambda}|^{2d}}\frac{|\Xi(e^{i\lambda},\psi)|^2}{|\Phi(e^{i\lambda},\phi)|^2}, \quad \lambda \in [0,\pi], \tag{4}$$

where $\theta = (\beta', \sigma_\varepsilon^2)'$, and using the relationship given in (1), with corresponding autocovariance function

$$\gamma(j,\theta) = \int_{-\pi}^{\pi} f(\lambda,\theta)e^{ij\lambda}d\lambda, \quad j = 0,1,2,....$$

Whereas under a correct specification of the stationary and invertible $FARIMA(p,d,q)$ process, with $p$and q finite, it is known that, say, the Whittle estimator of the parameters θ_0, $\widehat{\theta}_n$, is $n^{1/2}-$consistent and asymptotically normal, one possible criticism is that its statistical properties are very sensitive to the given specification. In particular, when the model is misspecified, estimators of d and, thus, those for the remaining parameters will be inconsistent. This would lead to incorrect statistical inferences about the process and, for instance, to inadecute predictions of the time series data.

Among alternatives processes to (3), but with a representation given by (2), we can include those processes where the order of the polynomials $\Phi(\cdot)$ and/or $\Xi(\cdot)$ are left unknown. Another set of alternatives is Bloomfield's (1973) exponential process, see also Robinson (1994) for its extension to processes which may exhibit long-range dependence, and which have recently attracted some attention and interest, see for instance Lobato and Robinson (1997), and also in empirical work, see for instance Gil-Alaña and Robinson (1996). Finally, we can include, in the set of alternative models, the fractional Gaussian noise process introduced by Mandelbrot and Van Ness (1968). It is noteworthy that the latter two models do not share the $FARIMA\,(p,d,q)$, with finite p and q, structure of the former, although they are characterized by a finite number of parameters.

In view of the many possible types of alternative processes and the importance and relevance of the $FARIMA\,(p,d,q)$ process in empirical work, it seems relevant to develop and study specification tests for the adequacy of the model given in (3) and, at the same time, consistent in the direction of the nonparametric alternatives given by the general covariance stationary linear process defined in (2). Observe that in our setup, to allow the alternative to be of a nonparametric form will be of practical relevance because, as was mentioned above, the true underlying structure of the time series may not even belong to the class of $FARIMA\,(p,d,q)$ processes. That is, the null may not be even nested in the set of alternative models. Finally, observe that the process given in (3) is nothing but a particular covariance stationary linear process defined in (2).

Specifically, in this paper we are interested in testing the null hypothesis that x_t belongs to the class of stationary invertible $FARIMA\,(p,d,q)$ processes, while the alternative is that the model is a covariance stationary linear process which is the negation of the null, that is,

$$H_0 : \forall\lambda \in [0,\pi] \text{ and for some } \theta_0 \in \Theta, \qquad f(\lambda) = f(\lambda;\theta_0) \tag{5}$$

against

$$H_1 : \exists C(\lambda) \subset [0,\pi] \text{ such that for all } \theta \in \Theta, \qquad f(\lambda) \neq f(\lambda;\theta), \tag{6}$$

where $C(\lambda)$, which may depend on θ, has Lebesgue measure greater than zero and $\Theta \subset \mathbb{R}^{p+q} \times [0,1/2) \times \mathbb{R}^{+}$ is the parameter space to be defined in Assumption A.1 below. That is, the null hypothesis H_0 is that the process x_t admits the representation (3), while under the alternative H_1, the process x_t belongs to the complementary set of models within the class of models defined by (2).

The outline of the paper is as follows. In the next section, we introduce and discuss the test, studying its asymptotic properties and showing that it has nontrivial power in the direction of contiguous alternatives converging to the null at the parametric rate $n^{-1/2}$. It is shown that, because the covariance structure of the limiting Gaussian process depends on $f(\lambda;\theta)$ and its derivative with respect to θ, the asymptotic null distribution is difficult to tabulate for the purpose of statistical inferences. Because of that, in Section 3, we propose a bootstrap approach to estimate the critical values of the test, showing its consistency. In Section 4, we perform a Monte-Carlo simulation study to shed some light about the finite sample performance of the bootstrap test. In Section 5, we provide the proofs of the results, which apply some technical lemmas given in Section 6.

2. THE TEST AND ITS STATISTICAL PROPERTIES

When the parameters θ_0 are known in (3), tests for the hypothesis testing given in (5) and (6), based on functionals of the integrated periodogram of the process and named by Anderson (1993) as the empirical spectral distribution function, are by no means new. They have a long tradition which go back to the work by Grenander and Rosenblatt (1957), who constructed Kolgomorov-Smirnov tests when $d = 0$, or Ibragimov (1963) under the assumption of a squared integrable spectral density function under Gaussianity, so effectively allowing for long-range dependence, although $d < 1/4$. Recently, Klüppelberg and Mikosch (1996) considered the innovations ε_t in (3) to have moments, possibly, smaller than 2, that is, the variance of ε_t may not exist. For a review, see for instance Anderson's (1993) paper. The tests employ the similarities between the empirical distribution for sequences of independent and identically distributed (iid) random variables and those features of the periodogram of a time series sequence.

Before we discuss the main ideas of tests for H_0 vs. H_1, let us introduce the following assumptions.

A.1: $\theta_0 = \left(\phi_0', \psi_0', d_0, \sigma_{0\varepsilon}^2\right)'$ is an interior point of the parameter space $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3 \subset \mathbb{R}^{p+q} \times [0, 1/2) \times \mathbb{R}^+$, which is assumed to be compact and such that for all $\theta = \left(\phi', \psi', d, \sigma_\varepsilon^2\right)' \in \Theta$, the Autoregressive and Moving Average polynomials $\Phi(\cdot)$ and $\Xi(\cdot)$ are of order p and q, respectively and they do not have roots in or on the unit circle.

A.2: In (3), the innovation sequence $\{\varepsilon_t\}$ is a stochastic process with finite eight moments, where $E(\varepsilon_t \mid \mathcal{F}_{t-1}) = 0$, $E\left(\varepsilon_t^2 \mid \mathcal{F}_{t-1}\right) = E\left(\varepsilon_t^2\right) = \sigma_\varepsilon^2$ a.s., $E\left(\varepsilon_t^\ell | \mathcal{F}_{t-1}\right) = \mu_\ell < \infty$ *a.s.*, $\ell = 3, ..., 8$, where $\mathcal{F}_t$ is the σ-algebra of events generated by $\varepsilon_s, s \leq t$, and with joint fourth cumulant of $\varepsilon_{t_1}, \varepsilon_{t_2}, \varepsilon_{t_3}$ and ε_{t_4} satisfying

$$cum\left(\varepsilon_{t_1}, \varepsilon_{t_2}, \varepsilon_{t_3}, \varepsilon_{t_4}\right) = \begin{cases} \kappa_4 & t_1 = t_2 = t_3 = t_4 \\ 0 & otherwise. \end{cases}$$

Some discussion about A.1 and A.2 is in place. First, it is noteworthy to mention that the parameterization given for the $FARIMA(p, d, q)$ implies that

$$\int_{-\pi}^{\pi} \log \left|A(\lambda; \beta)\right|^2 d\lambda = 0 \quad \text{for all } \beta \in \Theta_1 \times \Theta_2, \tag{7}$$

and therefore

$$\sigma_{0\varepsilon}^2 = 2\pi \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(f(\lambda; \theta_0)\right) d\lambda\right),$$

that is, the one step prediction error which is independent of β_0, so that

$$\frac{\partial}{\partial\beta}\int_{-\pi}^{\pi}\log\left(f\left(\lambda;\theta_0\right)\right)d\lambda=0. \tag{8}$$

Moreover, A.1 and A.2 imply the conditions in Fox and Taqqu (1986) or Giraitis and Surgailis (1990), needed for the $n^{1/2}$-consistency and asymptotic normality of the Whittle estimator of the parameters θ_0 under H_0.

Now, we discuss the main ideas of the test. Consider the periodogram of x_t

$$I_n\left(\lambda\right)=\frac{1}{2\pi n}\left|\sum_{t=1}^{n}x_te^{-it\lambda}\right|^2$$

and the spectral transfer function under H_0, that is

$$A\left(\lambda;\beta\right)=\frac{1}{\left|1-e^{i\lambda}\right|^d}\frac{\left|\Xi\left(e^{i\lambda};\psi\right)\right|}{\left|\Phi\left(e^{i\lambda};\phi\right)\right|},$$

which, except the constant $(2\pi)^{-1/2}\sigma_\varepsilon$, is nothing but the singular value decomposition of the spectral density $f\left(\lambda;\theta\right)$ which is assumed to be greater than zero $\forall\lambda\in[0,\pi]$.

Because we can expect that $E\left((2\pi)I_n\left(\omega\right)\right)\simeq\sigma_{0\varepsilon}^2\left|A\left(\omega;\beta_0\right)\right|^2$, under the null hypothesis H_0, then a test for $(5)-(6)$ can be based on whether or not

$$\frac{1}{2\pi}\int_0^\lambda\left(\frac{(2\pi)I_n\left(\omega\right)}{\sigma_{0\varepsilon}^2\left|A\left(\omega;\beta_0\right)\right|^2}-1\right)d\omega, \tag{9}$$

is statistically different than 0 for all $\lambda\in[0,\pi]$. (9) can be interpreted as the integrated relative error of the empirical spectral density (periodogram) of x_t compared to the true spectral density function. Observe that we could have defined the integrated periodogram from $-\pi$ instead of from 0, but because the periodicity, with period 2π, of $I_n\left(\omega\right)$ and $\left|A\left(\omega;\beta_0\right)\right|$, it suffices to consider only those frequencies λ in the interval $[0,\pi]$. On computational grounds, and to be able to use the Fast Fourier transform, following Barlett (1954), we can use the discrete approximation, or Riemann approximation of integrals by sums, of the above statistic, defined by

$$\overline{S}_n\left(\lambda,\theta_0\right)=\frac{1}{n}\sum_{j=1}^{[n/2]}\mathcal{I}\left(\lambda_j\leq\lambda\right)\left(\frac{I_{nj}}{f_j\left(\theta_0\right)}-1\right)\text{ where }\lambda\in[0,\pi],$$

and where $\mathcal{I}(B)$ denotes the indicator function of the event B, $\lambda_j=2\pi j/n$, $j=1,...,[n/2]$ with $f_j\left(\theta_0\right)\equiv f\left(\lambda_j,\theta_0\right)=(2\pi)^{-1}\sigma_{0\varepsilon}^2\left|A\left(\lambda_j;\beta_0\right)\right|^2$ and $I_{nj}=I_n\left(\lambda_j\right)$. Observe that $\overline{S}_n\left(\lambda,\theta_0\right)$ is a marked empirical process in $\mathbb{D}[0,\pi]$ where the marks are $\left(f_j^{-1}\left(\theta_0\right)I_{nj}-1\right)$. Moreover, because it could be convenient to rewrite $\overline{S}_n(\lambda, \theta_0)$ in such a way that the process belongs to the space $\mathbb{D}[0,1]$, we redefine $\overline{S}_n(\lambda, \theta_0)$ by

$$\widetilde{S}_n(\vartheta, \theta_0) = \frac{1}{n}\sum_{j=1}^{[n/2]} \mathcal{I}(\lambda_j \le \pi\vartheta)\left(\frac{I_{nj}}{f_j(\theta_0)} - 1\right), \text{ where } \vartheta \in [0,1]. \tag{10}$$

Remark 1 *Instead of the process* $\widetilde{S}_n(\vartheta, \theta_0)$*, we could have employed*

$$\frac{1}{n}\sum_{j=1}^{[n/2]} \mathcal{I}(\lambda_j \le \pi\vartheta)\left[I_{nj} - f_j(\theta_0)\right], \tag{11}$$

which would be the statistic obtained when analyzing whether or not

$$F(\lambda) - F(\lambda, \theta_0) = \int_0^\lambda (f(\omega) - f(\omega; \theta_0))\, d\omega$$

is 0 *for all* $\lambda \in [0, \pi]$*. The reason to prefer* $\widetilde{S}_n(\vartheta, \theta_0)$ *instead of that in* (11) *is because, see for instance Brillinger's (1981) Theorem 7.6.1, the limiting covariance structure of* (11) *times* $n^{1/2}$ *has a term of the form* $(2\pi)\int_0^{\pi\vartheta} f^2(\lambda)\, d\lambda$ *which is only finite if* $f(\lambda) \in \mathbb{L}_2[0, \pi]$*, that is, if* $d < 1/4$ *in* (3)*. In contrast, when* $d \ge 1/4$*,* $\int_0^{\pi\vartheta} f^2(\lambda)\, d\lambda$ *is not defined, that is it is not finite, which will imply, among other matters, that the normalization factor needed in* (11) *is of a smaller order of magnitude than* $n^{1/2}$*, and thus, it will affect the (local) power of the test. Specifically, the corresponding tests, constructed from* (11)*, would only be able to detect contiguous alternatives which converge to the null at a rate of order* $n^{-\alpha}$ *with* $\alpha < 1/2$ *when* $d \ge 1/4$*. So, the asymptotic relative efficiency of tests based on* (11) *compared to those based on* (10) *would be* 0*. Therefore, in our framework, the approach given in* (10) *becomes very desirable and important.*

As is known, from related literature involving $\widetilde{S}_n(\vartheta, \theta_0)$, see for instance Anderson (1993) for a later reference, the asymptotic covariance structure of $\widetilde{S}_n(\vartheta, \theta_0)$ depends on the fourth cumulant κ_4 of the innovation process ε_t in (3), which may be difficult or inaccurately estimated. Because of that, Anderson (1993), see also Klüppelberg and Mikosch (1996), suggests to transform $\widetilde{S}_n(\vartheta, \theta_0)$ to avoid the dependency of the limiting covariance structure on the fourth cumulant of the innovations ε_t. To that end, consider

$$S_n(\vartheta, \theta_0) = \widetilde{S}_n(\vartheta, \theta_0) - \vartheta \widetilde{S}_n(1, \theta_0), \ \vartheta \in [0,1]. \tag{12}$$

>From a straightforward extension of some results in Robinson (1995b), see Proposition 1 in Section 5, to the region $[0, \pi]$, that is for all $\vartheta \in [0,1]$,

$$S_n(\vartheta, \theta_0) = \frac{1}{n}\sum_{j=1}^{[n/2]} \left(\mathcal{I}(j \le [n\vartheta/2]) - \vartheta\right)\left(\left(\frac{2\pi}{\sigma_\varepsilon^2}\right) I_{\varepsilon,j} - 1\right) + o_p\left(n^{-1/2}\right), \tag{13}$$

where $I_{\varepsilon,j}$ denotes the periodogram of the innovation process ε_t at the frequency λ_j, that is

$$I_{\varepsilon,j} = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} \varepsilon_t e^{-it\lambda_j}\right|^2 .$$

But the first term on the right of (13) is a well known marked empirical process whose limiting behaviour was studied by, say, Brillinger (1975) Theorem 7.6.1, who showed that it converges weakly to a Brownian bridge in the Skorohod space $\mathbb{D}[0,1]$, and therefore the process $n^{1/2}S_n(\vartheta,\theta_0)$ as well. Thus, $n^{1/2}S_n(\vartheta,\theta_0)$ could form the basis for test statistics when H_0 in (5) is confronted against H_1 in (6) and θ_0 is known.

In practice, θ_0 is not known, and thus to make $n^{1/2}S_n(\vartheta;\theta_0)$ feasible, the value of the parameter θ_0 has to be replaced by some reasonable estimator, like the Whittle estimator defined as

$$\widehat{\theta}_n = \arg\min_{\theta\in\Theta} \sum_{j=-[n/2]+1}^{[n/2]} \left\{\log\left(f_j(\theta)\right) + \frac{I_{nj}}{f_j(\theta)}\right\}, \tag{14}$$

or equivalently, based on (7) and the symmetry around the origin of I_{nj} and $f_j(\theta)$, by

$$\widehat{\beta}_n = \arg\min_{\beta\in\Theta_1\times\Theta_2} \sum_{j=1}^{[n/2]} \frac{I_{nj}}{|A(\lambda_j;\beta)|^2} \text{ and } \widehat{\sigma}^2_{\varepsilon n} = \frac{2}{n}\sum_{j=1}^{[n/2]} \frac{(2\pi)I_{nj}}{\left|A\left(\lambda_j;\widehat{\beta}_n\right)\right|^2} n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right), \tag{15}$$

so that tests for H_0 can be implemented from any continuous functional of the statistic $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$.

That is, denote φ a continuous functional, $\varphi:\mathbb{R}\to\mathbb{R}^+$, of $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$. The test statistic is based on $\varphi\left(n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)\right)$. Among these functionals, we have the supremum and the $\mathbb{L}_2$ norm in the space $[0,1]$. That is, the Barlett's T_p−test

$$B_n = \sup_{\{j:j=1,\ldots,n\}} \left|n^{1/2}S_n\left(\frac{j}{n},\widehat{\theta}_n\right)\right| \tag{16}$$

which is of the Kolmogorov-Smirnov type, or the normalized ω−statistic of Cràmer-Von Mises test given by

$$C_n = \frac{1}{n}\sum_{j=1}^{n}\left(n^{1/2}S_n\left(\frac{j}{n},\widehat{\theta}_n\right)\right)^2. \tag{17}$$

Under a correct specification of the model, the statistical properties of $\hat{\beta}_n$ and $\widehat{\sigma}^2_{\varepsilon n}$ in (15) are well known. See among others, Yajima (1985), Fox and Taqqu (1986) or Dahlhaus (1989), who assumed that the innovations ε_t are Gaussian, or Hannan (1973) and Giraitis and Surgailis (1990) or Hosoya (1997) for a general covariance stationary linear process, where the innovations ε_t may not be Gaussian, although in the former the process x_t is short range dependent. In particular, the above authors have shown that, under H_0 and A.1 and A.2,

$$\widehat{\theta}_n = \theta_0 + (2\pi)\, \mathcal{A}^{-1} b_n + o\left(n^{-1/2}\right) \ a.s., \tag{18}$$

where

$$b_n = \frac{1}{n} \sum_{j=1}^{[n/2]} \phi_j(\theta_0) \left[f_j(\theta_0)^{-1} I_{nj} - 1 \right]$$

and

$$\mathcal{A} = \int_0^\pi \phi(\lambda, \theta_0)\, \phi(\lambda, \theta_0)'\, d\lambda,$$

with $\phi(\lambda, \theta) = f^{-1}(\lambda, \theta)\, \dot{f}(\lambda, \theta)$, $\dot{f}(\lambda, \theta) = \partial f(\lambda, \theta) / \partial \theta$ and $\phi_j(\theta) = \phi(\lambda_j, \theta)$ and $\dot{f}_j(\theta) = \partial f_j(\theta) / \partial \theta$.

As was mentioned in the introduction, one possible criticism of the parametric estimator $\widehat{\theta}_n$ is that its statistical properties are very sensitive to a correct specification of the model. In particular, if the model has not been correctly specified, say, it is not a $FARIMA(p, d, q)$ model or we have underspecified the order of the polynomials AR and/or MA, the Whittle estimator may lead to inconsistent estimation of the important parameter d, and so, to inadequate statistical inferences and predictions of future values of x_t. Therefore, it seems convenient to decide whether or not there is any statistical justification to employ the $FARIMA(p, d, q)$ model.

Introduce

$$\mathcal{G}(\vartheta) = \int_0^\pi \left(\mathcal{I}(\lambda \le \pi\vartheta) - \vartheta \right) \phi(\lambda; \theta_0)\, d\lambda.$$

Theorem 1 *Assume A.1 and A.2. Then, under H_0 and $\widehat{\theta}_n$ given by* (14),

$$n^{1/2} S_n\left(\vartheta, \widehat{\theta}_n\right) \text{ converges weakly to } S_\infty(\vartheta) \text{ in } \mathbb{D}[0,1],$$

endowed with the Skorohod metric, and where S_∞ is a Gaussian process centered at zero with covariance structure

$$K(\vartheta_1, \vartheta_2) = \frac{1}{2}\left(\min(\vartheta_1, \vartheta_2) - \vartheta_1 \vartheta_2\right) - \frac{1}{2\pi} \mathcal{G}(\vartheta_1)'\, \mathcal{A}^{-1} \mathcal{G}(\vartheta_2). \tag{19}$$

>From the asymptotic covariance of $n^{1/2} S_n\left(\vartheta, \widehat{\theta}_n\right)$ given in Theorem 1, that is (19), we observe that the covariance structure of $S_\infty(\vartheta)$ is different than that obtained in Ibragimov (1963) or Anderson (1993), say, where only the first term on the right of (19) appears. However, this difference does not come as a surprise, since instead of evaluating the function $S_n(\vartheta, \cdot)$ at θ_0, as is done in the

aforementioned papers, we do it at $\widehat{\theta}_n$. This difference of the limiting behaviour of $n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)$, compared to that of $n^{1/2}S_n(\vartheta,\theta_0)$, is expected when the null hypothesis changes from simple to a composite one, as it has appeared in related problems, see Durbin (1973) or more recently Stute (1997), Andrews (1997) or Bierens and Ploberger (1997).

>From the results of Theorem 1, it is expected that tests based on $\varphi\left(n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right)\right)$, say, those given in (16) and/or (17), should be able to detect contiguous alternatives which converge to the null H_0 at the rate $n^{-1/2}$. To this end, consider the contiguous alternatives,

$$H_{1n}: f(\lambda) = f(\lambda,\theta)\left(1+\frac{g(\lambda)}{n^{1/2}}\right) \text{ for some } \theta\in\Theta \text{ and for all } \lambda\in[-\pi,\pi],$$

where $g(\lambda)$ is some symmetric, around the origin, positive non-constant integrable function in $[-\pi,\pi]$. This type of alternatives has also been considered, in related specification testing problems, by Andrews (1997), Bierens and Ploberger (1997) and Stute (1997) among others.

Introduce

$$R(\vartheta) = \int_0^\pi [\mathcal{I}(\lambda\le\pi\vartheta)-\vartheta]\,g(\lambda)\,d\lambda - \mathcal{G}(\vartheta)'\mathcal{A}^{-1}\int_0^\pi \phi(\lambda,\theta)\,g(\lambda)\,d\lambda,$$

which is different than zero. If $g(\lambda)$ were constant, then, recall (7) and (8), $R(\vartheta)$ would be 0, which is expected since when $g(\lambda)$ is a constant function, H_{1n} becomes a member in the class of models defined under H_0. Thus, we have the following Corollary, which shows the limiting behaviour of any continuous functional φ of the process $n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)$ under H_{1n}.

Corollary 1 *Assuming that $f(\lambda,\theta)$ satisfies A.1 and A.2, and under H_{1n}, for any continuous functional φ,*

$$\varphi\left(n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)\right) \xrightarrow{d} \varphi\left(S_\infty(\vartheta)+R(\vartheta)\right).$$

As an example the above corollary implies that for the functionals φ defined in (16) and (17),

$$B_n \xrightarrow{d} \sup_{\vartheta\in[0,1]}\left|S_\infty(\vartheta)+R(\vartheta)\right| \text{ and } C_n \xrightarrow{d} \int_0^1 \left(S_\infty(\vartheta)+R(\vartheta)\right)^2 d\vartheta.$$

Remark 2 *Corollary 1 implies that the test based on $\varphi\left(n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)\right)$ is consistent. That observation comes from the fact that under*

$$H_1: f(\lambda) = f(\lambda,\theta)(1+g(\lambda)) \text{ for some } \theta\in\Theta \text{ and for all } \lambda\in[0,\pi],$$

the function $R(\vartheta)$ would be proportional to

$$n^{1/2}\left(\int_0^\pi [\mathcal{I}(\lambda\le\pi\vartheta)-\vartheta]\,g(\lambda)\,d\lambda - \mathcal{G}(\vartheta)'\mathcal{A}^{-1}\int_0^\pi \phi(\lambda,\theta)\,g(\lambda)\,d\lambda\right)$$

which increases (decreases) to $+\infty(-\infty)$ *as* $R(\vartheta) \neq 0$ *in a set* $C \subset [0,\pi]$ *with positive Lebesgue measure.*

Turning our attention to the issue of how to implement tests based on the asymptotic distribution of $\varphi\left(n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)\right)$, Theorem 1 indicates that it can be difficult, in general, to find a transformation of the test statistic such that its limiting distribution is a known one. Velilla (1994) has proposed a distribution free test for finite $ARMA\,(p,q)$ models based on a transformation of the standardized sample autocorrelations. However, such a method requires the choice of a bandwidth parameter, and in addition, his conditions are not satisfied for the $FARIMA$ process with $d \neq 0$. Anderson (1997) has proposed a method to tabulate the limiting distribution of a Cràmer-V. Mises statistic for the goodness-of-fit test for finite $AR\,(p)$ models, by estimating (approximating) the eigenvalues of the covariance function of the limiting process. However, its implementation into more general structures, as the one considered here, that is $FARIMA$ process, does not seem straightforward. We should emphasize that Anderson and Velilla's (1997) approach is case by case. Therefore, the application of bootstrap methods to approximate the distribution function of the test statistic seem an appealing and convenient approach. In the next section, a bootstrap test is proposed which is easy to implement and, as shown in Section 4, enjoys much better accuracy level than tests based on their asymptotic distribution, when they are known.

3. BOOTSTRAP TESTS

Once a continuous functional $\varphi : \mathbb{R} \to \mathbb{R}^+$ of $n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)$ is designed to test for H_0, an α significance level test is based on the critical value, $c^f_{n(1-\alpha)}$, such that $\Pr\left(\hat{\eta}_n > c^f_{n(1-\alpha)}\right) = \alpha$ where $\hat{\eta}_n = \varphi\left(n^{1/2}S_n\left(\vartheta;\widehat{\theta}_n\right)\right)$. In general, the finite sample distribution of $\hat{\eta}_n$ is unknown, so that the critical value $c^f_{n(1-\alpha)}$ is approximated by the asymptotic critical value $c^a_{1-\alpha}$, where $\Pr\left(\eta_\infty > c^a_{1-\alpha}\right) = \alpha$ with $\hat{\eta}_n \to_d \eta_\infty = \varphi\left(S_\infty\right)$. However, as was mentioned in the previous section, critical values of the asymptotic distribution of $\hat{\eta}_n$, and thus of η_∞, are difficult to tabulate, if possible. Therefore, bootstrap tests are an attractive and necessary alternative.

We propose to estimate the distribution of $\hat{\eta}_n$ by the conditional distribution, given the sample, of its bootstrap analog $\hat{\eta}^*_n = \varphi\left(n^{1/2}S^*_n\left(\vartheta;\widehat{\theta^*_n}\right)\right)$, where

$$S^*_n\left(\vartheta,\hat{\theta}^*_n\right) = \widetilde{S}^*_n\left(\vartheta,\hat{\theta}^*_n\right) - \vartheta\widetilde{S}^*_n\left(1,\hat{\theta}^*_n\right),\ \vartheta \in [0,1],$$

with

$$\widetilde{S}_n^*\left(\vartheta,\widehat{\theta}_n^*\right)=\frac{1}{n}\sum_{j=1}^{[n/2]}\mathcal{I}\left(\lambda_j\leq\pi\vartheta\right)\left(\frac{(2\pi)\,I_{nj}^*}{\widehat{\sigma}_{\varepsilon n}^{2*}\left|A\left(\lambda_j;\widehat{\beta}_n^*\right)\right|^2}-1\right),$$

and $\widehat{\theta}_n^*=\left(\widehat{\beta}_n^{*\prime},\widehat{\sigma}_{\varepsilon}^{2*}\right)'$ defined by the bootstrap analog of (15), that is

$$\widehat{\beta}_n^*=\arg\min_{\beta\in\Theta_1\times\Theta_2}\sum_{j=1}^{[n/2]}\frac{I_{nj}^*}{\left|A\left(\lambda_j;\beta\right)\right|^2}\text{ and }\widehat{\sigma}_{\varepsilon n}^{2*}=\frac{2}{n}\sum_{j=1}^{[n/2]}\frac{(2\pi)\,I_{nj}^*}{\left|A\left(\lambda_j;\widehat{\beta}_n^*\right)\right|^2},$$

where

$$I_{nj}^*=\frac{1}{2\pi n}\left|\sum_{t=1}^{n}x_t^*e^{-it\lambda_j}\right|^2,$$

and $\underset{\sim}{x}^*=(x_1^*,x_2^*,...,x_n^*)'$ is an artificial sample, generated by resampling from $\underset{\sim}{x}=(x_1,x_2,...,x_n)'$, as are described below.

First, the parameters θ_0 of the $FARIMA\,(p,d,q)$ process specified under H_0 are estimated from (14) or (15), obtaining $\widehat{\theta}_n$. Once $\widehat{\theta}_n$ is computed, the bootstrap sample $\underset{\sim}{x}^*=\{x_t^*,\,t=1,...,n\}$ is generated according to

$$\Phi\left(L,\widehat{\phi}_n\right)(1-L)^{\widehat{d}_n}x_t^*=\Theta\left(L,\widehat{\psi}_n\right)\widehat{\sigma}_{\varepsilon n}\varepsilon_t^*,\quad t=1,...,n,$$

where $\underset{\sim}{\varepsilon}^*=(\varepsilon_1^*,\varepsilon_2^*,...,\varepsilon_n^*)'$ is a random sample with replacement from the empirical distribution function of $\underset{\sim}{\tilde{\varepsilon}}=(\tilde{\varepsilon}_1,\tilde{\varepsilon}_2,...,\tilde{\varepsilon}_n)'$, where

$$\tilde{\varepsilon}_t=\frac{\widehat{\varepsilon}_t-n^{-1}\sum_{t=1}^n\widehat{\varepsilon}_t}{\tilde{\sigma}_{\varepsilon n}},\ \tilde{\sigma}_{\varepsilon n}^2=\frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\varepsilon}_t-\frac{1}{n}\sum_{t=1}^{n}\widehat{\varepsilon}_t\right)^2,$$

and $\underset{\sim}{\hat{\varepsilon}}=(\hat{\varepsilon}_1,\hat{\varepsilon}_2,...,\hat{\varepsilon}_n)'$ is such that $\underset{\sim}{x}=\hat{L}_n'\,\underset{\sim}{\widehat{\varepsilon}}$, where $\hat{L}_n'\hat{L}_n=\Omega\left(\widehat{\theta}_n\right)$ with $\Omega\left(\theta\right)=\left[\gamma\left(|i-j|\,,\theta\right)\right]_{i,j=1,...,n}$. Thus, $E^*\left(\underset{\sim}{\varepsilon}^*\right)=0$ and $E^*\left(\underset{\sim}{\varepsilon}^*\underset{\sim}{\varepsilon}^{*\prime}\right)=I_n$, where $E^*\left(\cdot\right)=E\left(\cdot\mid\mathcal{X}_n\right)$, and under our resampling scheme, $\underset{\sim}{x}^*=\hat{L}_n'\,\underset{\sim}{\varepsilon}^*$ and $E^*\left(\underset{\sim}{x}^*\underset{\sim}{x}^{*\prime}\right)=\Omega\left(\widehat{\theta}_n\right)$, so that the spectral density function of x_t^*, conditional on $\underset{\sim}{x}$, is $f\left(\lambda,\widehat{\theta}_n\right)$.

The resampling method must be such that, given the sample $\underset{\sim}{x}$, the conditional distribution of the bootstrap statistic $\widehat{\eta}_n^*$ consistently estimates the distribution of η_∞ under H_0. That is, the bootstrap test is consistent if under H_0, $\widehat{\eta}_n^*\rightarrow_{d^*}\eta_\infty$ in probability, where "$\rightarrow_{d^*}$in probability" means convergence in bootstrap distribution according to the following definition.

Definition 1 *Let $\underset{\sim}{x}^*$ denote the bootstrap sample drawn from $\underset{\sim}{x}$ using some given resampling scheme. Let $\widehat{\eta}_n^*$ a test statistic computed from $\underset{\sim}{x}^*$. We say that $\widehat{\eta}_n^*$ converges weakly in bootstrap distribution to* *the random variable* η_∞ *(with distribution function* $G(z)$*), and denoted as* $\hat{\eta}_n^* \to_{d^*} \eta_\infty$ *in probability, whenever the sequence of random variables* $\Pr\left(\hat{\eta}_n^* \leq z \mid \underset{\sim}{x}\right)$ *converges to* $G(z)$ *in probability for every continuity point* z *of* $G(z)$.

A second feature for the bootstrap test statistic to be valid is that under the alternative hypothesis, the bootstrap statistic must also converge in bootstrap distribution, possibly to a different distribution than under the null hypothesis when the statistic is not pivotal, as is our case. Finally, under contiguous alternatives H_{1n}, the bootstrap statistic $\hat{\eta}_n^*$ must also converge, in bootstrap distribution, to η_∞.

The following theorem provides the consistency of the bootstrap test under contiguous alternatives H_{1n}.

Theorem 2 *Under* H_{1n},

$$\varphi\left(n^{1/2} S_n^*\left(\vartheta; \widehat{\theta}_n^*\right)\right) \xrightarrow{d^*} \varphi\left(S_\infty(\vartheta)\right) \quad \textit{in probability.}$$

Thus, under H_0 and H_{1n}, the bootstrap test statistic converges, in bootstrap distribution, to the same limiting random variable, that is $\varphi(S_\infty)$. However, under fixed alternatives H_1, proceeding as with the proof of Theorem 2, it is straightforwardly shown that the bootstrap statistic converges to $\varphi(S'_\infty)$, where S'_∞ is a Gaussian process centered at zero and with covariance structure (19) with θ_0 replaced by θ_1, where $\widehat{\theta}_n \xrightarrow{P} \theta_1$ under H_1.

Theorem 2 justifies the bootstrap test statistic. More specifically, the critical value of the test at, say, α significance level, is $c^*_{n(1-\alpha)}$, where $\Pr\left(\hat{\eta}_n^* > c^*_{n(1-\alpha)} \middle| \underset{\sim}{x}\right) = \alpha$. Because the conditional distribution of $\hat{\eta}_n^*$, and thus the bootstrap critical value $c^*_{n(1-\alpha)}$, is computationally impossible to obtain, it has to be approximated via Monte Carlo, as accurate as desired. We can use the quantiles obtained from the empirical distribution of the Monte Carlo sample of $\hat{\eta}_n^*$ as estimators of the corresponding quantiles of $\hat{\eta}_n$. That is, consider B bootstrap samples of size n, $\underset{\sim}{x}^{*(k)} = \left(x_1^{*(k)}, x_2^{*(k)}, ..., x_n^{*(k)}\right)'$, $k = 1, ..., B$, each of which have a corresponding test statistic value $\hat{\eta}_n^{*(k)}$. Then, the critical value $c^*_{n(1-\alpha)}$ is approximated by $c^*_{n(1-\alpha),B}$, where $c^*_{n(1-\alpha),B}$ satisfies

$$B^{-1} \sum_{k=1}^{B} \mathcal{I}\left(\hat{\eta}_n^{*(k)} > c^*_{n(1-\alpha),B}\right) = \alpha.$$

That is, $c^*_{n(1-\alpha),B}$ is the $(1-\alpha)\,th$ quantile of the Monte Carlo sample $\left(\hat{\eta}_n^{*(k)}, k = 1, ..., B\right)$, so that the null hypothesis is rejected when $\hat{\eta}_n > c^*_{n(1-\alpha),B}$.

4. MONTE CARLO EXPERIMENTS

In all the experiments, we have generated 5000 Monte Carlo samples and we have used 2000 replications in the bootstrap tests, that is, we have chosen $B = 2000$. We have considered sample sizes of $n = 25, 50, 100$ and 150.

To compare the performance of the bootstrap test with respect to the asymptotic one, when this is feasible, we performed the test when the null hypothesis is that the data x_t follows a white noise process. To this end, the observations x_t were generated as *iid* $N(0,1)$ and $Uniform(-0.5, 0.5)$. The empirical level of the Monte Carlo experiments is reported in Table 1. We observe that the bootstrap tests exhibit an excellent accuracy level for the two distributions considered, even for sample sizes as small as $n = 25$. In contrast, the performance of the tests based on their asymptotic distribution is much worst for the smallest sample sizes. In addition, we observe that the Kolmogorov-Smirnov's test, B_n, works very badly compared with the Cràmer-v. Mises, C_n, a well known fact (see e.g. D'Agostino and Stephens, 1986). This illustrates that even in situations as in Velilla (1994) or Anderson (1995), where the limiting distribution of the statistic can be tabulated, the bootstrap test seems more accurate and preferable.

In Table 2, we study the performance of the level of the bootstrap test when the null hypothesis is an $AR(1)$ process with parameter 0.5 and the innovations ε_t were *iid* $N(0,1)$ or $Uniform(-0.5, 0.5)$. In both situations, the bootstrap tests exhibit excellent accuracy level, even for sample sizes of $n = 25$.

In Table 3, we have also examined the accuracy level of the test when testing that the model is a $FARIMA(0, d, 0)$ process. For that purpose, we have generated the observations x_t according to a $FARIMA(0, d, 0)$ with $d = 0.2, 0.3$, and 0.4, and where the innovations ε_t are *iid* $N(0,1)$. As could be expected, larger sample sizes, at least of $n = 100$, are needed to obtain a reasonable accuracy level than when testing for a short memory specification, across the spectrum of values of d.

In Tables 4 and 5, we illustrate the power of the tests. In Table 4, we describe the empirical power when testing that the model is an $AR(1)$ process, but the true model is a $FARIMA(0, d, 0)$ process with parameter $d = 0.2, 0.3$, or 0.4, while in Table 5, we report the power of the tests when testing that the data follows a $FARIMA(0, d, 0)$ process but the true model is an $AR(1)$ with parameter 0.5. In both cases, the innovations ε_t were generated as *iid* $N(0,1)$. Not surprisingly, the power increases with the sample size and it seems that to achieve a "good" power level, larger sample sizes are needed than to obtain a good size of the tests, as it is illustrated in Tables 1 to 3. Of course, this is what one can expect, as, in finite samples, the power depends very much on how far away the true model is from the hypothetical one. This fact is illustrated when testing an $AR(1)$ model and the alternative is a $FARIMA$ process, the greater is the parameter d and, thus, the far away the model is from the $AR(1)$ structure, the smaller the sample sizes are required to achieve a reasonable power behaviour.

5. PROOFS

In this and next sections, for notational simplicity, we write $[n/2]$ as $n/2$.

Proof of Theorem 1

Put $\mathcal{J}(\vartheta, j) = (\mathcal{I}(j \leq [n\vartheta/2]) - \vartheta)$. By definition

$$
\begin{aligned}
S_n\left(\vartheta, \widehat{\theta}_n\right) &= \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\left(\frac{I_{nj}}{f_j(\theta_0)} - 1\right) + \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\, I_{nj}\left(\frac{1}{f_j\left(\widehat{\theta}_n\right)} - \frac{1}{f_j(\theta_0)}\right) \\
&= S_n(\vartheta, \theta_0) + \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\, I_{nj}\left(\frac{1}{f_j\left(\widehat{\theta}_n\right)} - \frac{1}{f_j(\theta_0)}\right). \qquad (20)
\end{aligned}
$$

By standard linearization of $f_j^{-1}\left(\widehat{\theta}_n\right) - f_j^{-1}(\theta_0)$ and the asymptotic expansion in (18), the right side of (20) is

$$
\begin{aligned}
& S_n(\vartheta, \theta_0) - \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0)'\frac{I_{nj}}{f_j(\theta_0)}\left(\widehat{\theta}_n - \theta_0\right)\left(1 + O_p\left(n^{-1/2}\right)\right) \\
= \; & S_n(\vartheta, \theta_0) - \frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0)'\left(\widehat{\theta}_n - \theta_0\right) \\
& -\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0)'\left(\frac{I_{nj}}{f_j(\theta_0)} - 1\right)\left(\widehat{\theta}_n - \theta_0\right)\left(1 + O_p\left(n^{-1/2}\right)\right).
\end{aligned}
$$

But

$$
\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0) - \frac{1}{(2\pi)}\mathcal{G}(\vartheta) = O\left(n^{-1}\log n\right)
$$

because by A.1 and A.2, the function $\phi(\lambda; \theta_0)$ is continuously differentiable outside any neighbourhood containing the origin, so that by Brillinger (1981, p.15), for any $\delta > 0$,

$$
\frac{1}{n}\sum_{j=\delta n/2}^{n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0) - \frac{1}{(2\pi)}\int_{\delta\pi}^{\pi}\left(\mathcal{I}(\lambda \leq \pi\vartheta) - \vartheta\right)\phi(\lambda; \theta_0)\, d\lambda = O\left(n^{-1}\right)
$$

while, by Lemma 3 of Robinson (1995b) and A.1 and A.2,

$$
\frac{1}{n}\sum_{j=1}^{\delta n/2}\mathcal{J}(\vartheta, j)\,\phi_j(\theta_0) - \frac{1}{(2\pi)}\int_{0}^{\delta\pi}\left(\mathcal{I}(\lambda \leq \pi\vartheta) - \vartheta\right)\phi(\lambda; \theta_0)\, d\lambda = O\left(n^{-1}\log n\right).
$$

Therefore, the right of (20) is

$$\begin{aligned} S_n\left(\vartheta,\widehat{\theta}_n\right) &= S_n(\vartheta,\theta_0)-\frac{1}{(2\pi)}\mathcal{G}(\vartheta)'\left(\widehat{\theta}_n-\theta_0\right) \\ &\quad -\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\,\phi_j(\theta_0)'\left(\frac{I_{nj}}{f_j(\theta_0)}-1\right)\left(\widehat{\theta}_n-\theta_0\right)+o_p\left(n^{-1/2}\right). \end{aligned}$$

By Propositions 1 to 3, the third term on the right of the above equation is $o_p\left(n^{-1/2}\right)$, so that to finish the proof, it suffices to study the behaviour of

$$S_n(\vartheta,\theta_0)-\frac{1}{(2\pi)}\mathcal{G}(\vartheta)'\left(\widehat{\theta}_n-\theta_0\right)$$

which, by the asymptotic properties of the Whittle estimator, that is (18), is

$$\begin{aligned} &\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f_j(\theta_0)}-1\right) \\ &-\mathcal{G}(\vartheta)'\mathcal{A}^{-1}\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\phi_j(\theta_0)\left(\frac{I_{nj}}{f_j(\theta_0)}-1\right)+o_p(1). \end{aligned} \tag{21}$$

Next, by Proposition 1, (21) is

$$\mathcal{Z}_n(\vartheta)=\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\left\{\left(\mathcal{J}(\vartheta,j)-\mathcal{G}(\vartheta)\mathcal{A}^{-1}\phi_j(\theta_0)\right)\left((2\pi)I_{\varepsilon,j}-1\right)\right\}+o_p(1).$$

Now apply Propositions 2 and 3, and that $\int_{-\pi}^{\pi}\phi(\lambda;\theta_0)\,d\lambda=0$, that is (8), to conclude that the first term on the right of $\mathcal{Z}_n(\vartheta)$ converges to a Gaussian process with covariance structure given by (19). Notice that the function $\psi_j(\vartheta)$ in those propositions, in our case, is $\mathcal{J}(\vartheta,j)-\mathcal{G}(\vartheta)\mathcal{A}^{-1}\phi_j(\theta_0)$, which satisfies that it integrates to zero. Thus, the first term on the right of (26), in Proposition 2, is zero. That is, by Proposition 2, we have that the covariance structure is given by

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n/2}\left\{\left(\mathcal{J}(\vartheta_1,j)-\mathcal{G}(\vartheta_1)\mathcal{A}^{-1}\phi_j(\theta_0)\right)\left(\mathcal{J}(\vartheta_2,j)-\mathcal{G}(\vartheta_2)\mathcal{A}^{-1}\phi_j(\theta_0)\right)\right\},$$

which is that in (19). This concludes the proof of the convergence of the process $S_n\left(\vartheta;\widehat{\theta}_n\right)$ to $S_\infty(\vartheta)$ and the theorem. □

In the remainder of this and in next sections, let $\psi_j(\vartheta;\theta)=\psi(\lambda_j,\vartheta;\theta)$, where for all $\vartheta\in[0,1]$, $\psi(\lambda,\vartheta;\theta)$ is a function defined in $[0,\pi]$ squared integrable, that it is continuous from the right, for instance that one of the terms in $\psi(\cdot)$ is the indicator function, and that $n^{-1}\sum_{j=1}^{n/2}\psi_j(\vartheta;\theta) \to \int_0^{1/2}\psi(2\pi u,\vartheta;\theta)\,du$. Also, put $\psi_j(\vartheta)=\psi_j(\vartheta;\theta_0)$.

We now prove three Propositions employed in the proof of Theorem 1. To that end, let

$$\begin{aligned} R_n(\vartheta) &= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j(\vartheta)\left(\frac{I_{nj}}{f_j(\theta_0)}-1\right) \\ &= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j(\vartheta)\left(\frac{I_{nj}}{f_j(\theta_0)}-(2\pi)\frac{I_{\varepsilon,j}}{\sigma_{0\varepsilon}^2}\right)+\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j(\vartheta)\left((2\pi)\frac{I_{\varepsilon,j}}{\sigma_{0\varepsilon}^2}-1\right) \\ &= R_n^1(\vartheta)+R_n^2(\vartheta). \end{aligned} \tag{22}$$

Specifically, in Proposition 1, we will show that $R_n^1(\vartheta)$ is asymptotically negligible, whereas in Propositions 2 and 3, we show the weak convergence of $R_n^2(\vartheta)$ to a Gaussian process, by showing the convergence of its finite dimensional distributions, and the tightness condition of the process, respectively. Also, to simplify the notation, we assume, without loss of generality, that $\sigma_{0\varepsilon}^2=1$.

Proposition 1 *Assume A1 and A2. Then, for all* $\vartheta\in[0,1]$,

$$R_n^1(\vartheta)=O_p\left(n^{-1/6}\log^{2/3}n\right). \tag{23}$$

Proof. By definition of $R_n^1(\vartheta)$, the left side of (23) is

$$\frac{1}{n^{1/2}}\sum_{j=1}^{q}\psi_j(\vartheta)\left(\frac{I_{nj}}{f_j(\theta_0)}-(2\pi)I_{\varepsilon,j}\right)+\frac{1}{n^{1/2}}\sum_{j=q+1}^{n/2}\psi_j(\vartheta)\left(\frac{I_{nj}}{f_j(\theta_0)}-(2\pi)I_{\varepsilon,j}\right),$$

where q is a number to be determined later. Because, by A.2, $E\left|(2\pi)I_{\varepsilon,j}\right|<K$ and, by Theorem 2 of Robinson (1995a), $E\left|f_j^{-1}(\theta_0)I_{nj}\right|<K$, then by Markov's inequality and that $\psi(\lambda;\vartheta)$ is an integrable function, the first term of the above expression is $O_p\left(n^{-1/2}q\right)$, while the second moment of the second term is

$$\frac{1}{n}(a_1+a_2+b_1+b_2) \tag{24}$$

where

$$\begin{aligned} a_1 &= \sum_{j=q+1}^{n/2}\psi_j(\vartheta)^2\Big\{2\left(E|u_j|^2\right)^2+\left|Eu_j^2\right|^2-2\left|E(u_jv_j)\right|^2-2\left|E(u_j\overline{v}_j)\right|^2 \\ &\qquad -2E|u_j|^2E|v_j|^2+2\left(E|v_j|^2\right)^2+\left|E\left(v_j^2\right)\right|^2\Big\}, \\ a_2 &= \sum_{j=q+1}^{n/2}\psi_j(\vartheta)^2\left\{cum(u_j,u_j,\overline{u}_j,\overline{u}_j)-2cum(u_j,v_j,\overline{u}_j,\overline{v}_j)+cum(v_j,v_j,\overline{v}_j,\overline{v}_j)\right\}, \end{aligned}$$

$$
\begin{aligned}
b_1 &= 2\sum_{q+1=j<k}^{n/2} \psi_j(\vartheta)\psi_k(\vartheta)\left\{\left(E|u_j|^2 E|u_k|^2\right) + |E(u_j u_k)|^2 + |E(u_j\overline{u}_k)|^2\right. \\
&\quad -2E|u_j|^2 E|v_k|^2 - |E(u_j v_k)|^2 - |E(u_j\overline{v}_k)|^2 - E|u_k|^2 E|v_j|^2 - |Eu_k v_j|^2 \\
&\quad \left. - |Eu_k\overline{v}_j|^2 + E|u_j|^2 E|v_k|^2 + 2|E(v_j v_k)|^2 + |E(v_j v_k)|^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
b_2 &= 2\sum_{q+1=j<k}^{n/2} \psi_j(\vartheta)\psi_k(\vartheta)\left\{cum(u_j, u_k, \overline{u}_j, \overline{u}_k) - cum(u_j, v_k, \overline{u}_j, \overline{v}_k)\right. \\
&\quad \left. -cum(u_k, v_j, \overline{u}_k, \overline{v}_k) + cum(v_j, v_k, \overline{v}_j, \overline{v}_k)\right\},
\end{aligned}
$$

where $u_j = A_j^{-1}w_j$, $v_j = w_{\varepsilon,j}$, $A_j = A(\lambda_j;\beta_0)$, $\overline{c}$ denotes the conjugate of the complex number c, and

$$
w_j = w(\lambda_j) = (2\pi n)^{-1/2}\sum_{t=1}^{n} x_t e^{-it\lambda_j} \text{ and } w_{\varepsilon,j} = w_\varepsilon(\lambda_j) = (2\pi n)^{-1/2}\sum_{t=1}^{n} \varepsilon_t e^{-it\lambda_j}.
$$

Let us examine each of the terms in (24). Proceeding as in page 1649-1651 of Robinson (1995b), but applying a straightforward extension of such a theorem given in Lemmas 3 and 4 of Giraitis et al. (1998) to all frequencies in $[0,\pi]$ and that $\psi(\lambda;\vartheta)$ is a squared integrable for all $\vartheta \in [0,1]$, then, by Markov's inequality, it is

$$
O_p\left(\frac{\log^2 n}{n} + \frac{\log^2 n}{q} + \frac{1}{n^{1/2}} + \frac{\log n}{n^{1/2}}\right),
$$

and thus, the left side of (23) is

$$
O_p\left(\frac{q}{n^{1/2}} + \frac{\log^2 n}{n} + \frac{\log^2 n}{q} + \frac{1}{n^{1/2}} + \frac{\log n}{n^{1/2}}\right).
$$

It is worth noticing that the term $r_2^{\beta+1}/n^\beta$, that appears in the proof of (4.8) in Robinson (1995b), is not included, since in contrast to him, we employ the true spectral density $f_j(\theta_0)$ instead of its approximation $C\lambda_j^{-2d}$ for frequencies $\lambda_j \to 0$, which is obtained from A.1, as Robinson (1995b) did.

Now, choose q to be $O\left(n^{1/3}(\log n)^{2/3}\right)$ to conclude the proof of the proposition. □

Define $g(\vartheta_1, \vartheta_2)$ as

$$
\int_0^{1/2} \psi(2\pi u, \vartheta_1)\psi(2\pi u, \vartheta_2)\,du - 2\Phi(\vartheta_1)\Phi(\vartheta_2),
$$

with $\Phi(\vartheta) = \int_0^{1/2}\psi(2\pi u, \vartheta)\,du$ and put

$$
c_s(\vartheta) = 2n^{-3/2}\sum_{j=1}^{n/2}\psi_j(\vartheta)\cos(s\lambda_j). \tag{25}
$$

Proposition 2 *Assuming A.1 and A.2, the finite dimensional distributions of* $R_n^2(\vartheta)$ *converge to those of a Gaussian process with covariance structure* $g(\vartheta_1, \vartheta_2) + (\kappa_4 + 2)\Phi(\vartheta_1)\Phi(\vartheta_2)$.

Proof. Fix $a_1, ..., a_q$ and $\vartheta_1, ..., \vartheta_q$. By Crâmer-Wold device, it suffices to investigate the limiting distribution of, recall that for notational convenience we have assumed that $\sigma_{0\varepsilon}^2 = 1$,

$$\sum_{p=1}^{q} a_p R_n^2(\vartheta_p) = \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \left\{ \left(\sum_{p=1}^{q} a_p \psi_j(\vartheta_p) \right) ((2\pi) I_{\varepsilon,j} - 1) \right\}.$$

By definition of $I_{\varepsilon,j}$, the right side of the above equation is

$$\begin{aligned} & \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \left(\sum_{p=1}^{q} a_p \psi_j(\vartheta_p) \right) \frac{1}{n} \sum_{t=1}^{n} (\varepsilon_t^2 - 1) \\ & + \frac{2}{n^{1/2}} \sum_{j=1}^{n/2} \left\{ \left(\sum_{p=1}^{q} a_p \psi_j(\vartheta_p) \right) \frac{1}{n} \sum_{t=2}^{n} \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s \cos((t-s)\lambda_j) \right\} \\ = & \frac{1}{n} \sum_{j=1}^{n/2} \left(\sum_{p=1}^{q} a_p \psi_j(\vartheta_p) \right) \frac{1}{n^{1/2}} \sum_{t=1}^{n} (\varepsilon_t^2 - 1) + \sum_{t=2}^{n} z_t, \end{aligned} \tag{26}$$

where

$$z_t = \varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s \left(\sum_{p=1}^{q} a_p c_{t-s}(\vartheta_p) \right),$$

suppressing any reference to n in z_t and $c_{t-s}(\vartheta_p)$, $p = 1, ..., q$.

The first and second terms on the right of (26) are uncorrelated since, by A.2, $E(\varepsilon_t \varepsilon_s (\varepsilon_r^2 - 1)) = 0$ for all $t < s$. Moreover, by standard CLT for martingale differences, the first term on the right of (26) converges in distribution to a normal random variable with variance $(\kappa_4 + 2)\left(\sum_{p=1}^{q} a_p \Phi(\vartheta_p)\right)^2$, by the properties of $\psi(\cdot)$. Thus, it suffices to examine the behaviour of the second term on the right of (26). Because z_t forms a triangular array of a martingale difference sequence then, see for instance Hall and Heyde (1980), it suffices to check

$$(a) \quad \sum_{t=2}^{n} E(z_t^2 | \mathcal{F}_{t-1}) - \sum_{p_1=1}^{q} \sum_{p_2=1}^{q} a_{p_1} g(\vartheta_{p_1}, \vartheta_{p_2}) a_{p_2} \xrightarrow{P} 0$$

$$(b) \quad \sum_{t=2}^{n} E(z_t^2 \mathcal{I}(|z_t| > \delta)) \xrightarrow{P} 0 \quad \text{for all } \delta > 0.$$

First we prove (a), whose left side is

$$\sum_{t=2}^{n} \sum_{s=1}^{t-1} \varepsilon_s^2 \left(\sum_{p=1}^{q} a_p c_{t-s}(\vartheta_p) \right)^2 - \sum_{p_1=1}^{q} \sum_{p_2=1}^{q} a_{p_1} g(\vartheta_{p_1}, \vartheta_{p_2}) a_{p_2} \tag{27}$$

$$+ \sum_{t=2}^{n} \sum_{1=s_1 \neq s_2}^{t-1} \varepsilon_{s_1} \varepsilon_{s_2} \left\{ \left(\sum_{p=1}^{q} a_p c_{t-s_1}(\vartheta_p) \right) \left(\sum_{p=1}^{q} a_p c_{t-s_2}(\vartheta_p) \right) \right\}. \tag{28}$$

First, we examine (27), which is

$$\sum_{t=1}^{n-1}\left(\varepsilon_t^2-1\right)\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2$$
$$+\left(\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2-\sum_{p_1=1}^{q}\sum_{p_2=1}^{q}a_{p_1}g\left(\vartheta_{p_1},\vartheta_{p_2}\right)a_{p_2}\right).$$

By Lemma 1 in Section 6, the second term converges to zero while, by A.2, the first term has mean zero and variance

$$\left(2+\kappa_4\right)\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2\right)^2.$$

Now, for $\vartheta\in[0,1]$,

$$\left|c_s\left(\vartheta\right)\right|\leq n^{-1/2} \tag{29}$$

while, by summation by parts, it is also $O\left(n/s\right)$, because, by Zygmund (1977),

$$\left|\sum_{\ell=1}^{j}\psi_\ell\left(\vartheta\right)\cos\left(s\lambda_\ell\right)\right|=O\left(\frac{n}{s}\right) \tag{30}$$

if $1\leq s\leq n/2$, while for $[n/2]\leq s\leq n-1$ because $\cos\left(s\lambda_\ell\right)=\left(-1\right)^\ell\cos\left(\left(s-[n/2]\right)\lambda_\ell\right)$ and $\psi\left(2\pi u,\vartheta\right)$ is an integrable function for all ϑ. So, we can restrict ourselves to the sum for those $s\leq[n/2]$. But,

$$\begin{aligned}\sum_{s=1}^{n/2}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2 &= \sum_{s=1}^{n/m}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2+\sum_{s=n/m+1}^{n/2}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2 \\ &= O\left(\frac{1}{n}\frac{n}{m}+\frac{1}{n}\sum_{s=n/m}s^{-2}\right)=O\left(\frac{1}{m}+\frac{m}{n^2}\right)\end{aligned} \tag{31}$$

because $\sum_{p=1}^{q}c_s\left(\vartheta_p\right)=\sum_{p=1}^{q}c_{n-s}\left(\vartheta_p\right)$, and where for the first and second terms on the right of (31) we have used (29) and (30) respectively, and the definition of $c_s\left(\vartheta_p\right)$. Therefore,

$$\begin{aligned}\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2\right)^2 &= O\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n/2}\left(\sum_{p=1}^{q}a_pc_s\left(\vartheta_p\right)\right)^2\right)^2\right) \\ &= O\left(\frac{n}{m^2}+\frac{m^2}{n^3}\right)=o\left(1\right)\end{aligned}$$

choosing $m=n^\xi$ with $1/2>\xi$. Then, by Markov's inequality, (27) $=o_p\left(1\right)$.

To complete the proof of part (a), we need to examine (28), whose expectation is zero and its second moment has a typical element equal to

$$\begin{aligned}
&\sum_{t,u=2}^{n}\sum_{s_1,s_2=1}^{\min(t-1,u-1)} c_{t-s_1}\left(\vartheta_{p_1}\right) c_{u-s_1}\left(\vartheta_{p_2}\right) c_{t-s_2}\left(\vartheta_{p_3}\right) c_{u-s_2}\left(\vartheta_{p_4}\right) \\
= &\sum_{t=2}^{n}\sum_{s_1\neq s_2=1}^{t-1} c_{t-s_1}\left(\vartheta_{p_1}\right) c_{t-s_1}\left(\vartheta_{p_2}\right) c_{t-s_2}\left(\vartheta_{p_3}\right) c_{t-s_2}\left(\vartheta_{p_4}\right) \\
&+\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s_1\neq s_2=1}^{u-1} c_{t-s_1}\left(\vartheta_{p_1}\right) c_{u-s_1}\left(\vartheta_{p_2}\right) c_{t-s_2}\left(\vartheta_{p_3}\right) c_{u-s_2}\left(\vartheta_{p_4}\right).
\end{aligned}$$

The first term on the right of the above equation is $o(1)$ proceeding as in the proof of (31), while the second term, by the Cauchy-Schwarz inequality, is bounded by

$$\begin{aligned}
&\sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{s=1}^{u-1} c_{t-s}\left(\vartheta_{p_1}\right) c_{t-s}\left(\vartheta_{p_3}\right)\sum_{s=1}^{u-1} c_{u-s}\left(\vartheta_{p_2}\right) c_{u-s}\left(\vartheta_{p_4}\right)\right) \qquad (32)\\
\leq &\left(\sum_{t=1}^{n} c_t\left(\vartheta_{p_1}\right) c_t\left(\vartheta_{p_3}\right)\right)\left(\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=t-u+1}^{t-1} c_s\left(\vartheta_{p_2}\right) c_s\left(\vartheta_{p_4}\right)\right).
\end{aligned}$$

The expression in the second brackets on the right of (32) is

$$\begin{aligned}
\sum_{s=1}^{n-2} s(n-s-1)\, c_s\left(\vartheta_{p_2}\right) c_s\left(\vartheta_{p_4}\right) &\leq 2n\sum_{s=1}^{n/2} s c_s\left(\vartheta_{p_2}\right) c_s\left(\vartheta_{p_4}\right) \\
&\leq 2n\left(\sum_{s=2}^{n/m} s c_s\left(\vartheta_{p_2}\right) c_s\left(\vartheta_{p_4}\right) + \sum_{s=n/m+1} s c_s\left(\vartheta_{p_2}\right) c_s\left(\vartheta_{p_4}\right)\right) \\
&= O\left(\frac{nn^2}{nm^2} + \log\left(\frac{n}{m}\right)\right) \\
&= O\left(\frac{n^2}{m^2} + \log\left(\frac{n}{m}\right)\right),
\end{aligned}$$

using (29) and (30) for the first and second terms on the right of the above inequality, respectively. Therefore,

$$(32) = O\left(\frac{n^2}{m^3} + \frac{1}{m}\right) = o(1),$$

by choosing $m = n^{\xi}$ with $\xi > 2/3$ and because by (29) and (30) the first factor on the right of (32) is $O\left(m^{-1} + mn^{-2}\right)$. Observe that this choice of m is also valid for (31). Thus, by Markov's inequality, $(28) = o_p(1)$, which concludes the proof of part (a).

To prove part (b), it suffices to show the sufficient condition

$$\sum_{t=2}^{n} E\left(z_t^4\right) \to 0,$$

whose proof is easier and similar to that given in Robinson (1995b), so is omitted. That concludes the proof of this proposition. □

Next, we examine the tightness condition of the process given in (22).

Proposition 3 *Assuming A.1 and A.2, the process* $R_n(\vartheta)$ *equipped with the Skorohod's metric in* $\mathbb{D}[0,1]$ *is tight.*

Proof. The right side of (22) and Proposition 1 imply that it suffices to prove the tightness condition for the second term on the right of (22), which is

$$R_n^2 = \left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta)\right)\left(\frac{1}{n^{1/2}}\sum_{t=2}^{n}\left(\varepsilon_t^2-1\right)\right)+\sum_{t=2}^{n}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s}(\vartheta). \tag{33}$$

That the first term on the right of (33) is tight follows because $\left|n^{-1}\sum_{j=1}^{n/2}\left(\psi_j(\vartheta_1)-\psi_j(\vartheta_2)\right)\right| \le |\vartheta_1-\vartheta_2|^{\zeta}$ for $\zeta > 1/2$, by the properties of $\psi(\lambda,\vartheta)$, and that $E\left(n^{-1/2}\sum_{t=2}^{n}\left(\varepsilon_t^2-1\right)\right)^2$ is finite, by A.2. Thus, it suffices to examine the tightness condition for the second term on the right of (33). To this end, put

$$\mathcal{E}_n(\vartheta) = \sum_{t=2}^{n}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s}(\vartheta).$$

First, by the properties of $\psi_j(\vartheta)$, $\mathcal{E}_n(\vartheta)$ is a process which belongs to $\mathbb{D}[0,1]$, so that, by Billingsley's (1968) Theorem 15.6, it suffices to show the moment condition

$$E\left|\mathcal{E}_n(\vartheta_2)-\mathcal{E}_n(\vartheta_1)\right|^4 \le D\left(F(\vartheta_2)-F(\vartheta_1)\right)^{1+\delta}$$

for some $\delta > 0$ and where $F(\vartheta)$ is a nondecreasing function in ϑ.

Writing $\widetilde{c}_t = c_t(\vartheta_2)-c_t(\vartheta_1)$, the left side of the above inequality is, except constants,

$$E\left[\sum_{2=t_1\le t_2\le t_3\le t_4}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\varepsilon_{t_3}\varepsilon_{t_4}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=1}^{t_4-1}\varepsilon_{s_4}\widetilde{c}_{t_4-s_4}\right)\right].$$

By A.2, the above expectation is clearly zero if $t_3 < t_4$, so that it is

$$E\left[\sum_{2=t_1\le t_2\le t_3}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\varepsilon_{t_3}^2\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=1}^{t_3-1}\varepsilon_{s_4}\widetilde{c}_{t_3-s_4}\right)\right]$$

$$= E\left[\sum_{2=t_1\le t_2\le t_3}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=1}^{t_2-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)^2\right] \tag{34}$$

$$+2E\left[\sum_{2=t_1\le t_2\le t_3}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_2-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)\left(\sum_{s_4=t_2+1}^{t_3-1}\varepsilon_{s_4}\widetilde{c}_{t_3-s_4}\right)\right] \tag{35}$$

$$+E\left[\sum_{2=t_1\le t_2\le t_3}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=t_2+1}^{t_3-1}\varepsilon_{s_3}\widetilde{c}_{t_3-s_3}\right)^2\right]. \tag{36}$$

Since s_4 is greater than t_2 and so is than s_1, s_2, s_3 and t_1, then, by A.2, (35) $= 0$.

Because $s_3 > t_2$ and by A.2, (36) is

$$E\left[\sum_{2=t_1\le t_2\le t_3}^{n}\varepsilon_{t_1}\varepsilon_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon_{s_1}\widetilde{c}_{t_1-s_1}\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon_{s_2}\widetilde{c}_{t_2-s_2}\right)\left(\sum_{s_3=t_2+1}^{t_3-1}\widetilde{c}^2_{t_3-s_3}\right)\right],$$

which is zero unless $t_1 = t_2$, in which case, (36) becomes

$$\sum_{2=t_1\le t_3}^{n}\left(\sum_{s_1=1}^{t_1-1}\widetilde{c}^2_{t_1-s_1}\right)\left(\sum_{s_3=t_1+1}^{t_3-1}\widetilde{c}^2_{t_3-s_3}\right)\le\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}^2_{t-s}\right)\right)^2 \tag{37}$$

because the quantities $\widetilde{c}^2_{t-s}$ are nonnegative, and by the Cauchy-Schwarz's inequality.

Next (34), which is zero unless $t_1 = t_2$, and thus it is

$$E\left(\varepsilon_t^4\right)\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}^2_{t_1-s}\widetilde{c}^2_{t_3-s}\right)+\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}^2_{t_1-s}\right)\left(\sum_{s=1}^{t_1-1}\widetilde{c}^2_{t_3-s}\right)$$
$$+2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}_{t_1-s}\widetilde{c}_{t_3-s}\right)^2. \tag{38}$$

The first term of (38) is

$$E\left(\varepsilon_t^4\right)\sum_{t_1=2}^{n}\left(\sum_{s=1}^{t_1-1}\widetilde{c}^2_{t_1-s}\right)\left(\sum_{t_3=t_1+1}^{n}\widetilde{c}^2_{t_3-s}\right)\le\frac{E\left(\varepsilon_t^4\right)}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}^2_{t-s}\right)$$

because $\widetilde{c}^2_{t-s}\le n^{-1}\left|t-s\right|^{-2}$, by (30). The second term of (38) is bounded by

$$\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\widetilde{c}^2_{t-s}\right)\right)^2$$

because the quantities $\tilde{c}_{t-s}^2$ are nonnegative. Finally, the third term of (38), by the Cauchy-Schwarz inequality, is bounded by

$$2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}_{t_1-s}^2\right)\left(\sum_{s=1}^{t_1-1}\tilde{c}_{t_3-s}^2\right)\le 2\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}_{t-s}^2\right)\right)^2.$$

Thus (34) + (36) is bounded by

$$4\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}_{t-s}^2\right)\right)^2+\frac{E\left(\varepsilon_t^4\right)}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}_{t-s}^2\right),$$

and proceeding as in Lemma 1, the above expression is further bounded by

$$\begin{aligned}
&4\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)^2\\
&+\frac{E\left(\varepsilon_t^4\right)}{n}\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)\\
\le\ & D\left(\vartheta_2-\vartheta_1\right)^{2\zeta}
\end{aligned}$$

because $\psi(\lambda,\vartheta)$ is squared integrable and such that $\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du$ is ζ-Liptchitz continuous with $\zeta>1/2$. That concludes the proof of the proposition and the tightness of the process given in (22). □

Proof of Corollary 1

By definition, $S_n\left(\vartheta, \widehat{\theta}_n\right)$ is

$$\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f_j(\theta_0)}-1\right)+\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)I_{nj}\left(\frac{1}{f_j\left(\widehat{\theta}_n\right)}-\frac{1}{f_j(\theta_0)}\right).$$

Proceeding as in the proof of Theorem 1, and under H_{1n}, the above expression is

$$\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)g(\lambda_j)$$
$$-\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\phi_j(\theta_0)\frac{I_{nj}}{f_j(\theta_0)}\left(\widehat{\theta}_n-\theta_0\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)g(\lambda_j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right).$$

Next, from the proof of Theorem 1, the above expression is

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)g(\lambda_j)\\
&-\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\phi_j(\theta_0)\left(1+\frac{1}{n^{1/2}}g(\lambda_j)\right)\left(\widehat{\theta}_n-\theta_0\right)+o_p\left(n^{-1/2}\right)\\
=\;&\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right)+\frac{1}{n^{3/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)g(\lambda_j)\\
&-\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\phi_j(\theta_0)\left(\widehat{\theta}_n-\theta_0\right)+o_p\left(n^{-1/2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
n^{1/2}S_n\left(\vartheta,\widehat{\theta}_n\right) &= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right)+\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)g(\lambda_j)\\
&\quad-\frac{1}{n}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\phi_j(\theta_0)n^{1/2}\left(\widehat{\theta}_n-\theta_0\right)+o_p(1)\\
&= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\mathcal{J}(\vartheta,j)\left(\frac{I_{nj}}{f(\lambda_j)}-1\right)+\int_0^{\pi}\left(\mathcal{I}(\lambda\le\pi\vartheta)-\vartheta\right)g(\lambda)\,d\lambda\\
&\quad-\mathcal{G}(\vartheta)'n^{1/2}\left(\widehat{\theta}_n-\theta_0\right)+o_p(1).
\end{aligned}$$

But, under H_{1n},

$$n^{1/2}\left(\widehat{\theta}_n-\theta_0\right)\xrightarrow{d}N\left(\mathcal{A}^{-1}\int_0^{\pi}\phi(\lambda)g(\lambda)\,d\lambda;(4\pi)\mathcal{A}^{-1}\right).$$

>From here, the conclusion of the proof of the corollary is standard. □

Before we give the proof of Theorem 2, let us introduce the following

Definition 2 *We say that* $Z_n^* = o_{p^*}(1)$, *if for all* $\delta > 0$, $\Pr\left\{ |Z_n^*| > \delta |\underset{\sim}{x}\right\} \xrightarrow{P} 0$.

Proof of Theorem 2

The technique of the proof used arguments in Stute et al. (1998) and those of Theorem 1, but instead of applying Propositions 1 to 3, we apply Propositions 4 to 7 below. First, by Lemma 4 of Section 6, and the continuity of $\phi(\lambda, \theta)$ in θ,

$$
\begin{aligned}
\widehat{\theta}_n^* &= \widehat{\theta}_n - \left(\frac{1}{n} \sum_{j=1}^{n/2} \left(\phi_j \left(\widehat{\theta}_n \right) \phi_j \left(\widehat{\theta}_n \right)' \right) \right)^{-1} \frac{1}{n} \sum_{j=1}^{n/2} \left(\frac{I_{n,j}^*}{f_j\left(\widehat{\theta}_n\right)} - 1 \right) (1 + o_{p^*}(1)) \\
&= \widehat{\theta}_n - (2\pi) \mathcal{A}^{-1} \frac{1}{n} \sum_{j=1}^{n/2} \left(\frac{I_{n,j}^*}{f_j\left(\widehat{\theta}_n\right)} - 1 \right) (1 + o_{p^*}(1)) \qquad (39)
\end{aligned}
$$

where for the second equality, we have used the consistency of $\widehat{\theta}_n$ to θ_0 and the definition of $\mathcal{A}$. Then, proceeding as in Theorem 1, we have that $n^{1/2} S_n^*\left(\vartheta, \widehat{\theta}_n^*\right)$ is

$$
\begin{aligned}
& n^{1/2} S_n^* \left(\vartheta, \widehat{\theta}_n\right) - \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j) \phi_j \left(\widehat{\theta}_n\right)' \frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n\right)} \left(\widehat{\theta}_n^* - \widehat{\theta}_n\right) (1 + o_{p^*}(1)) \\
= \; & n^{1/2} S_n^* \left(\vartheta, \widehat{\theta}_n\right) - \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j) \phi_j \left(\widehat{\theta}_n\right)' \left(\widehat{\theta}_n^* - \widehat{\theta}_n\right) \qquad (40) \\
& - \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j) \phi_j \left(\widehat{\theta}_n\right)' \left(\frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n^*\right)} - 1 \right) \left(\widehat{\theta}_n^* - \widehat{\theta}_n\right) (1 + o_{p^*}(1)).
\end{aligned}
$$

>From Propositions 4 to 7 below and Lemma 4 in Section 6, the last term on the right of the above equation is $o_{p^*}(1)$. So, we are left with the first two terms. Because $\left(\widehat{\theta}_n - \theta_0\right) = o_p(1)$ and the arguments given in Theorem 1

$$
\frac{1}{n} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j) \phi_j \left(\widehat{\theta}_n\right) \xrightarrow{P} \frac{1}{2\pi} \mathcal{G}(\vartheta)
$$

so that, by (39), the first two terms on the right of (40) are equal to

$$
\frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \mathcal{J}(\vartheta, j) \left(\frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n\right)} - 1 \right) - \mathcal{G}(\vartheta)' \mathcal{A}^{-1} \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \phi_j \left(\widehat{\theta}_n\right) \left(\frac{I_{nj}^*}{f_j\left(\widehat{\theta}_n\right)} - 1 \right) + o_{p^*}(1). \qquad (41)
$$

Next, by Proposition 4, (41) is

$$\mathcal{Z}_n^*(\vartheta) = \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \left\{ \left(\mathcal{J}(\vartheta, j) - \mathcal{G}(\vartheta) \mathcal{A}^{-1} \phi_j \left(\widehat{\theta}_n \right) \right) \left((2\pi) I_{\varepsilon^*,j}^* - 1 \right) \right\} + o_{p^*}(1),$$

where

$$I_{\varepsilon^*,j}^* = I_{\varepsilon^*}^*(\lambda_j) = \frac{1}{(2\pi n)} \left| \sum_{t=1}^{n} \varepsilon_t^* e^{-it\lambda_j} \right|^2.$$

Now apply Propositions 5 to 7, that $\int_{-\pi}^{\pi} \phi(\lambda; \theta) \, d\lambda = 0$, that is (8), and that $n^{1/2} \left(\widehat{\theta}_n^* - \widehat{\theta}_n \right)$ has the same asymptotic distribution function as that of $n^{1/2} \left(\widehat{\theta}_n - \theta_0 \right)$, as it can be seen from (39) and Propositions 4 to 7 below, to conclude that the first term on the right of $\mathcal{Z}_n^*(\vartheta)$ converges to a centered Gaussian process with covariance structure given by (19). Notice that, as in the proof of Theorem1, in our case the function $\psi_j \left(\vartheta, \widehat{\theta}_n \right)$, in Propositions 4 to 7 below, is $\mathcal{J}(\vartheta, j) - \mathcal{G}(\vartheta) \mathcal{A}^{-1} \phi_j \left(\widehat{\theta}_n \right)$, whose sum from $j = 1$ to $n/2$ is zero. So, the second term in (47) is zero. This concludes the proof of the theorem. □

As was done in the proofs of Propositions 1 to 3, consider

$$R_n^*(\vartheta) = \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j \left(\vartheta; \widehat{\theta}_n \right) \left(\frac{I_{nj}^*}{f_j \left(\widehat{\theta}_n \right)} - 1 \right). \tag{42}$$

The outline of Propositions 4 to 7 is as follows. In Proposition 4, we show that $f_j^{-1} \left(\widehat{\theta}_n \right) I_{nj}^*$ in (42) can be replaced by $(2\pi) I_{\varepsilon^*,j}^*$. This result, together with Proposition 5 shows that the asymptotic covariance structure of (42). In Proposition 6, we show the convergence of the finite dimensional distributions of (42) to those of Proposition 2, and finally, in Proposition 7, we show the tightness condition of (42).

Proposition 4 *Under the same conditions of Proposition 1, for all* $\vartheta \in [0, 1]$

$$E^* \left| \frac{1}{n^{1/2}} \sum_{j=1}^{n/2} \psi_j \left(\vartheta; \widehat{\theta}_n \right) \left(\frac{I_{nj}^*}{f_j \left(\widehat{\theta}_n \right)} - (2\pi) I_{\varepsilon^*,j}^* \right) \right| = o_p(1). \tag{43}$$

Proof. By construction, conditional on the sample $\underset{\sim}{x} = (x_1, ..., x_n)'$, $\underset{\sim}{x}^* = (x_1^*, ..., x_n^*)'$ is a sample of size n from a process whose spectral density function is $f \left(\lambda; \widehat{\theta}_n \right)$. The left side of (43) is, by the triangle inequality, bounded by

$$E^* \left| \frac{1}{n^{1/2}} \sum_{j=1}^{q} \psi_j \left(\vartheta; \widehat{\theta}_n \right) \left(\frac{I_{nj}^*}{f_j \left(\widehat{\theta}_n \right)} - (2\pi) I_{\varepsilon^*,j}^* \right) \right| + E^* \left| \frac{1}{n^{1/2}} \sum_{j=q+1}^{n/2} \psi_j \left(\vartheta; \widehat{\theta}_n \right) \left(\frac{I_{nj}^*}{f_j \left(\widehat{\theta}_n \right)} - (2\pi) I_{\varepsilon^*,j}^* \right) \right|, \tag{44}$$

where $q = o\left(n^{1/2}\right)$. Because $E^*\left|f_j^{-1}\left(\widehat{\theta}_n\right)I^*_{nj}\right|$ and $E^*\left|I_{\varepsilon^*,j}\right|$ are both bounded in probability, then by the triangle inequality and because

$$\begin{aligned}\psi_j\left(\vartheta;\widehat{\theta}_n\right) &= \psi_j\left(\vartheta;\theta_0\right)+\left(\widehat{\theta}_n-\theta_0\right)\psi_j^{(1)}\left(\vartheta;\overline{\theta}_n\right)\\ &\xrightarrow{P}\psi_j\left(\vartheta;\theta_0\right),\end{aligned}$$

the first term of (44) is $o_p(1)$. Conditional on the sample $\underset{\sim}{x}=(x_1,...,x_n)'$, the second term of (44) has a second moment, by Jensen's inequality, bounded by the square root of

$$\begin{aligned}&E^*\left|\frac{1}{n^{1/2}}\sum_{j=q+1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\left(\frac{I^*_{nj}}{f_j\left(\widehat{\theta}_n\right)}-(2\pi)\,I^*_{\varepsilon^*,j}\right)\right|^2 \qquad (45)\\ &= \frac{1}{n}\left(a_1^*\left(\widehat{\theta}_n\right)+a_2^*\left(\widehat{\theta}_n\right)+b_1^*\left(\widehat{\theta}_n\right)+b_2^*\left(\widehat{\theta}_n\right)\right),\end{aligned}$$

where

$$\begin{aligned}a_1^*\left(\widehat{\theta}_n\right) &= \sum_{j=q+1}^{n/2}\psi_j^2\left(\vartheta;\widehat{\theta}_n\right)\left\{2\left(E^*\left|u_j^*\right|^2\right)^2+\left|E^*\left(u_j^{*2}\right)\right|^2-2\left|E^*\left(u_j^*v_j^*\right)\right|^2-2\left|E^*\left(u_j^*\overline{v}_j^*\right)\right|^2\right.\\ &\qquad\left.-2E^*\left|u_j^*\right|^2E^*\left|v_j^*\right|^2+2\left(E^*\left|v_j^*\right|^2\right)^2+\left|E^*\left(v_j^{*2}\right)\right|^2\right\},\\ a_2^*\left(\widehat{\theta}_n\right) &= \sum_{j=q+1}^{n/2}\psi_j^2\left(\vartheta;\widehat{\theta}_n\right)\left\{cum^*\left(u_j^*,u_j^*,\overline{u}_j^*,\overline{u}_j^*\right)-2cum^*\left(u_j^*,v_j^*,\overline{u}_j^*,\overline{v}_j^*\right)\right.\\ &\qquad\left.+cum^*\left(v_j^*,v_j^*,\overline{v}_j^*,\overline{v}_j^*\right)\right\},\end{aligned}$$

$$\begin{aligned}b_1^*\left(\widehat{\theta}_n\right) &= 2\sum_{q+1=j<k}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\psi_k\left(\vartheta;\widehat{\theta}_n\right)\left\{\left(E^*\left|u_j^*\right|^2E\left|u_k^*\right|^2\right)+\left|E^*\left(u_j^*u_k^*\right)\right|^2+\left|E^*\left(u_j^*\overline{u}_k^*\right)\right|^2\right.\\ &\quad -2E^*\left|u_j^*\right|^2E^*\left|v_k^*\right|^2-\left|E^*\left(u_j^*v_k^*\right)\right|^2-\left|E^*\left(u_j^*\overline{v}_k^*\right)\right|^2-E^*\left|u_k^*\right|^2E^*\left|v_j^*\right|^2-\left|E^*\left(u_k^*v_j^*\right)\right|^2\\ &\quad\left.-\left|E^*\left(u_k^*\overline{v}_j^*\right)\right|^2+E^*\left|u_j^*\right|^2E^*\left|v_k^*\right|^2+2\left|E^*\left(v_j^*v_k^*\right)\right|^2+\left|E^*\left(v_j^*v_k^*\right)\right|^2\right\}\end{aligned}$$

and

$$\begin{aligned}b_2^*\left(\widehat{\theta}_n\right) &= 2\sum_{q+1=j<k}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\psi_k\left(\vartheta;\widehat{\theta}_n\right)\left\{cum^*\left(u_j^*,u_k^*,\overline{u}_j^*,\overline{u}_k^*\right)-cum^*\left(u_j^*,v_k^*,\overline{u}_j^*,\overline{v}_k^*\right)\right.\\ &\quad\left.-cum^*\left(u_k^*,v_j^*,\overline{u}_k^*,\overline{v}_k^*\right)+cum^*\left(v_j^*,v_k^*,\overline{v}_j^*,\overline{v}_k^*\right)\right\},\end{aligned}$$

where $u_j^*=A_j^{-1}\left(\widehat{\theta}_n\right)w_j^*$, $v_j^*=w_{\varepsilon^*,j}^*$, $A_j\left(\widehat{\theta}_n\right)=\sum_{\ell=0}^{\infty}\alpha_\ell\left(\widehat{\theta}_n\right)e^{i\ell\lambda_j}$,

$$w_j^*=w^*\left(\lambda_j\right)=(2\pi n)^{-1/2}\sum_{t=1}^{n}x_t^*e^{-it\lambda_j},\ w_{\varepsilon^*,j}^*=w_{\varepsilon^*}^*\left(\lambda_j\right)=(2\pi n)^{-1/2}\sum_{t=1}^{n}\varepsilon_t^*e^{-it\lambda_j}$$

and $cum^*\left(u_j^*, u_k^*, \overline{u}_j^*, \overline{u}_k^*\right)$, say, denotes the cumulant conditional on the sample $\underset{\sim}{x} = (x_1, ..., x_n)'$.

Let us examine each of the terms on the right of (45). The proof of the behaviour of $a_1^*\left(\widehat{\theta}_n\right)$ and $b_1^*\left(\widehat{\theta}_n\right)$ is exactly the same as in Proposition 1, because the order of magnitude does not depend on any parameter or unknown quantity, while for the terms $a_2^*\left(\widehat{\theta}_n\right)$ and $b_2^*\left(\widehat{\theta}_n\right)$, we apply instead of Robinson's (1995b) Lemma 3, Lemma 2 of Section 6. After noting that $\psi(\lambda;\vartheta) = \psi(\lambda;\vartheta;\theta_0)$ is squared integrable, we have that the left side of (45) is $o_p(1)$, which completes the proof of the proposition. □

Put

$$S_{\varepsilon^*}^*(\vartheta) = \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\left((2\pi)I_{\varepsilon^*,j}^* - 1\right).$$

Proposition 5 *Under the same conditions of Proposition 4,*

$$E^*\left(S_{\varepsilon^*}^*(\vartheta_1)S_{\varepsilon^*}^*(\vartheta_2)\right) \xrightarrow{P} g(\vartheta_1,\vartheta_2) + (\kappa_4+2)\Phi(\vartheta_1)\Phi(\vartheta_2), \tag{46}$$

where $\Phi(\vartheta)$ and $g(\vartheta_1,\vartheta_2)$ as were defined before Proposition 2.

Proof. First, by definition of $I_{\varepsilon^*,j}^*$, it is straightforward to observe that $E^*\left(S_{\varepsilon^*}^*(\vartheta)\right) = 0$. Next, as in Proposition 2, conditional on the sample, the left side of (46) is

$$\begin{aligned} &4E^*\left(\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^*\cos\left((t-s)\lambda_j\right)\right. \\ &\left.\times\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^*\cos\left((t-s)\lambda_j\right)\right) \\ &+\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\right)E^*\left(\frac{1}{n}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)\right).\end{aligned} \tag{47}$$

By Lemma 3 in Section 6 and the properties of ε_t^*, the second term of (47) converges in probability to $(\kappa_4+2)\Phi(\vartheta_1)\Phi(\vartheta_2)$, because the empirical distribution function of ε_t^* converges uniformly to the distribution function of ε_t, so that $\kappa_4^* - \kappa_4 = o_p(1)$. Thus, we are left with the behaviour of the first term of (47). To this end, put

$$c_s\left(\vartheta;\widehat{\theta}_n\right) = 2n^{-3/2}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\cos(s\lambda_j)$$

and

$$z_t^*(\vartheta) = \varepsilon_t^*\sum_{s=1}^{t-1}\varepsilon_s^* c_{t-s}\left(\vartheta;\widehat{\theta}_n\right),$$

where, for notational convenience, the reference to n and $\widehat{\theta}$ in $z_t^*(\vartheta)$ and to n in $c_{t-s}\left(\vartheta;\widehat{\theta}_n\right)$ has been suppressed. First, observe that conditional on the sample, $z_t^*(\vartheta)$ forms a triangular array of a martingale difference sequence. Let $\mathcal{F}_t^*$ be the smallest sigma algebra generated by $\{\varepsilon_s^*, s \leq t\}$. Then,

$$E^*\left(\sum_{t=2}^{n} z_t^*(\vartheta_1)\sum_{t=2}^{n} z_t^*(\vartheta_2)\middle|\mathcal{F}_{t-1}^*\right) = \sum_{t=2}^{n}\sum_{s=1}^{t-1}\varepsilon_s^{*2}\left\{c_{t-s}\left(\vartheta_1;\widehat{\theta}_n\right)c_{t-s}\left(\vartheta_2;\widehat{\theta}_n\right)\right\} + \sum_{t=2}^{n}\sum_{1=s_1\neq s_2}^{t-1}\left(\varepsilon_{s_1}^*\varepsilon_{s_2}^* c_{t-s_1}\left(\vartheta_1;\widehat{\theta}_n\right)c_{t-s_2}\left(\vartheta_2;\widehat{\theta}_n\right)\right) \quad (48)$$

since $E^*(\varepsilon_t^*) = 0$ and $E^*\left(\varepsilon_t^{*2}\right) = 1$.

Let us examine the first term on the right of (48), which is

$$\sum_{t=1}^{n-1}\left(\varepsilon_t^{*2}-1\right)\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right) + \sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right).$$

By Lemma 3 in Section 6, the second term converges in probability to $g(\vartheta_1,\vartheta_2)$, while, the first term, conditional on the sample $\underset{\sim}{x} = (x_1,...,x_n)'$, has mean zero and variance

$$(2+\kappa_4)\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2,$$

because $cum^*(\varepsilon_t^*,\varepsilon_t^*,\varepsilon_t^*,\varepsilon_t^*) \xrightarrow{P} \kappa_4$.

Next, because for all $\vartheta \in [0,1]$, $\psi_j\left(\vartheta;\widehat{\theta}_n\right)$ is continuously differentiable in θ, $\partial/\partial\theta\left(\psi_j(\vartheta;\theta)\right) = \psi_j^{(1)}(\vartheta;\theta)$ satisfies all the conditions of $\psi_j(\vartheta;\theta)$ and $\left(\widehat{\theta}_n - \theta_0\right) = O_p\left(n^{-1/2}\right)$, then

$$\left|c_s\left(\vartheta;\widehat{\theta}_n\right)\right| = n^{-1/2}\left(1+o_p(1)\right), \quad (49)$$

while, by summation by parts, it is also $(n/s)(1+o_p(1))$, because, by Zygmund (1977),

$$\left|\sum_{\ell=1}^{j}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\cos(s\lambda_\ell)\right| = O_p\left(\frac{n}{s}\right) + o_p\left(\frac{n}{s}\right), \quad (50)$$

if $1 \leq s \leq n/2$, while for $[n/2] \leq s \leq n-1$ because $\cos(s\lambda_\ell) = (-1)^\ell\cos\left((s-[n/2])\lambda_\ell\right)$ and $\psi(2\pi u,\vartheta) = \psi(2\pi u,\vartheta;\theta_0)$ is an integrable function for all $\vartheta \in [0,1]$. So, we can restrict ourselves to the sum for those $s \leq [n/2]$. But,

$$\sum_{s=1}^{n/2}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right) = \sum_{s=1}^{n/m}c_s\left(\vartheta_1;\widehat{\theta}_n\right)c_s\left(\vartheta_2;\widehat{\theta}_n\right)$$

$$+\sum_{s=n/m+1}^{n/2} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right) \tag{51}$$

$$= O_p\left(\frac{1}{n}\frac{n}{m}+\frac{1}{n}\sum_{s=n/m+1} s^{-2}\right) = O_p\left(\frac{1}{m}+\frac{m}{n^2}\right),$$

because $c_s\left(\vartheta;\widehat{\theta}_n\right) = c_{n-s}\left(\vartheta;\widehat{\theta}_n\right)$, and by (49) and (50). Therefore,

$$\begin{aligned}\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2 &= O\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n/2} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right)\right)^2\right)\\ &= O_p\left(\frac{n}{m^2}+\frac{m^2}{n^3}\right) = o_p(1),\end{aligned}$$

choosing $m = n^{\zeta}$ with $1/2 > \zeta$. That finishes the proof that the first term on the right of (48) converges in probability to $g(\vartheta_1, \vartheta_2)$.

To complete the proof of the proposition, we are left to prove that the second term on the right of (48) $= o_{p^*}(1)$. Conditional on the sample $\underset{\sim}{x}$, the first moment of the term is 0, while its second moment is

$$\begin{aligned}&\sum_{t,u=2}^{n}\sum_{s_1,s_2=1}^{\min(t-1,u-1)} c_{t-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{t-s_2}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_2}\left(\vartheta;\widehat{\theta}_n\right)\\ =& \sum_{t=2}^{n}\sum_{s_1\neq s_2=1}^{t-1} c^2_{t-s_1}\left(\vartheta;\widehat{\theta}_n\right) c^2_{t-s_2}\left(\vartheta;\widehat{\theta}_n\right)\\ &+\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s_1\neq s_2=1}^{u-1} c_{t-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_1}\left(\vartheta;\widehat{\theta}_n\right) c_{t-s_2}\left(\vartheta;\widehat{\theta}_n\right) c_{u-s_2}\left(\vartheta;\widehat{\theta}_n\right).\end{aligned}$$

The first term on the right of the above equation is $o_p(1)$, by (51), while the second term on the right, by the Cauchy-Schwarz inequality, is bounded by

$$\sum_{t=3}^{n}\sum_{u=2}^{t-1}\left(\sum_{s=1}^{u-1} c^2_{t-s}\left(\vartheta;\widehat{\theta}_n\right)\sum_{s=1}^{u-1} c^2_{u-s}\left(\vartheta;\widehat{\theta}_n\right)\right) \leq \left(\sum_{t=1}^{n} c_t^2\left(\vartheta;\widehat{\theta}_n\right)\right)\left(\sum_{t=3}^{n}\sum_{u=2}^{t-1}\sum_{s=t-u+1}^{t-1} c_s^2\left(\vartheta;\widehat{\theta}_n\right)\right). \tag{52}$$

The expression in the second brackets on the right of (52) is

$$\begin{aligned}\sum_{s=1}^{n-2} s(n-s-1)\, c_s^2\left(\vartheta;\widehat{\theta}_n\right) &\leq 2n\sum_{s=1}^{n/2} s c_s^2\left(\vartheta;\widehat{\theta}_n\right)\\ &\leq 2n\sum_{s=2}^{n/m} s c_s^2\left(\vartheta;\widehat{\theta}_n\right) + 2n\sum_{s=n/m+1} s c_s^2\left(\vartheta;\widehat{\theta}_n\right)\end{aligned}$$

$$
\begin{aligned}
&= O_p\left(\frac{nn^2}{nm^2}+\log\left(\frac{n}{m}\right)\right) \\
&= O_p\left(\frac{n^2}{m^2}\right),
\end{aligned}
$$

because $c_s^2\left(\vartheta;\widehat{\theta}_n\right)=n^{-1}s^{-2}\left(1+o_p\left(1\right)\right)$. Therefore, together with (51), it implies that

$$
(52)=O_p\left(\frac{n^2}{m^3}+\frac{1}{m}\right)=o_p\left(1\right),
$$

by choosing $m=n^{\zeta}$ with $\zeta>2/3$. Note that this choice of m is also valid for (51). That concludes the proof of Proposition 5. □

Proposition 6 *Under the same conditions of Proposition 4, the finite dimensional distributions converge in bootstrap law to those of a centered Gaussian process.*

Proof. Fix $a_1,...,a_q$ and $\vartheta_1,...,\vartheta_q$. By Cràmer-Wold device, it suffices to investigate the limiting distribution of

$$
\begin{aligned}
&\sum_{p=1}^{q}a_p\left(\frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_p;\widehat{\theta}_n\right)\left(\left(2\pi\right)I_{\varepsilon^*,j}^*-1\right)\right) \\
&= \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\left(\sum_{p=1}^{q}\left(a_p\psi_j\left(\vartheta_p;\widehat{\theta}_n\right)\right)\left(\left(2\pi\right)I_{\varepsilon^*,j}^*-1\right)\right),
\end{aligned}
$$

which is

$$
\frac{1}{n}\sum_{j=1}^{n/2}\left(\sum_{p=1}^{q}a_p\psi_j\left(\vartheta_p;\widehat{\theta}_n\right)\right)\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)+\sum_{t=2}^{n}z_t^*\left(\underset{\sim}{\vartheta}\right), \tag{53}
$$

where

$$
z_t^*\left(\underset{\sim}{\vartheta}\right)=\varepsilon_t^*\frac{2}{n^{3/2}}\sum_{j=1}^{n/2}\left(\sum_{p=1}^{q}\left(a_p\psi_j\left(\vartheta_p;\widehat{\theta}_n\right)\right)\left(\sum_{s=1}^{t-1}\varepsilon_s^*\cos\left(\left(t-s\right)\lambda_j\right)\right)\right)
$$

with $\underset{\sim}{\vartheta}=\left(\vartheta_1,...,\vartheta_p\right)'$. Proceeding as in the proof of Proposition 2, the two terms in (53) are uncorrelated. Moreover, it is straightforward to show that the second moment of the first term of (53) converges in probability to $\left(\kappa_4+2\right)\left(\sum_{p=1}^{q}a_p\Phi\left(\vartheta_p\right)\right)^2$.

So, we are left to examine the second term of (53). Proceeding as in the proof of the previous proposition, we have that

$$
\sum_{j=1}^{n/2}E^*\left(z_t^{*2}\left(\underset{\sim}{\vartheta}\right)|\mathcal{F}_{t-1}^*\right)\overset{P}{\to}\sum_{p_1=1}^{q}\sum_{p_2=1}^{q}a_{p_1}g\left(\vartheta_{p_1},\vartheta_{p_2}\right)a_{p_2},
$$

then, it suffices to verify the Lindeberg's condition, that is $\forall\delta > 0$,

$$\sum_{t=2}^{n} E^{*}\left[z_t^{*2}\left(\underset{\sim}{\vartheta}\right)\mathcal{I}\left(\left|z_t^{*}\left(\underset{\sim}{\vartheta}\right)\right| > \delta\right)\right] \xrightarrow{P} 0,$$

or the sufficient condition that

$$\sum_{t=2}^{n} E^{*}\left[z_t^{*4}\left(\underset{\sim}{\vartheta}\right)\right] \xrightarrow{P} 0.$$

But, the proof is essentially as that of Robinson (1995b) proceeding as in Proposition 2, and thus is omitted. □

Finally, in the next proposition we will show the tightness condition.

Proposition 7 *Under the same conditions of Proposition 4, $S_{\varepsilon^*}^{*}(\vartheta)$, as defined before Proposition 5, is tight.*

Proof. By definition

$$S_{\varepsilon^*}^{*}(\vartheta) = \left(\frac{1}{n^{1/2}}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)\right)\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\right) + \mathcal{E}_n^{*}\left(\vartheta;\widehat{\theta}_n\right), \tag{54}$$

where

$$\mathcal{E}_n^{*}\left(\vartheta;\widehat{\theta}_n\right) = \frac{1}{n^{1/2}}\sum_{j=1}^{n/2}\psi_j\left(\vartheta;\widehat{\theta}_n\right)\frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^{*}\sum_{s=1}^{t-1}\varepsilon_s^{*}\cos\left((t-s)\lambda_j\right).$$

That the first term on the right of (54) is tight follows because

$$\left|\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right) - \psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\right| \xrightarrow{P} v\left(\vartheta_1,\vartheta_2\right) \le \left|\vartheta_1-\vartheta_2\right|^{\varsigma}$$

with $\varsigma > 1/2$, and that $n^{-1/2}\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-1\right)$ is stochastically bounded. For the second term on the right of (54), because $\mathcal{E}_n^{*}\left(\vartheta;\widehat{\theta}_n\right)$ is a process belonging to the space $\mathbb{D}[0,1]$, recall the definition of $\psi(\cdot)$, to show tightness, it suffices to examine the moment condition (see Theorem 15.6 of Billingsley (1968)),

$$\begin{aligned} E^{*}\left|\mathcal{E}_n^{*}\left(\vartheta_2;\widehat{\theta}_n\right)-\mathcal{E}_n^{*}\left(\vartheta_1;\widehat{\theta}_n\right)\right|^4 &\le D_1\left[H_n\left(\vartheta_2;\widehat{\theta}_n\right)-H_n\left(\vartheta_1;\widehat{\theta}_n\right)\right]^{1+\delta} \\ &\xrightarrow{P} D_1\left(H\left(\vartheta_2;\theta_0\right)-H\left(\vartheta_1;\theta_0\right)\right)^{1+\delta}, \end{aligned}$$

and where D_1 is a finite constant and $H\left(\vartheta_2;\theta_0\right)$ is a nondecreasing function in ϑ.

Denote $\widetilde{c}_t\left(\widehat{\theta}_n\right) = c_t\left(\vartheta_2;\widehat{\theta}_n\right) - c_t\left(\vartheta_1;\widehat{\theta}_n\right)$. Then, the left side of the above inequality is bounded by a constant times

$$E^*\left[\sum_{2=t_1\le t_2\le t_3\le t_4}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\varepsilon^*_{t_3}\varepsilon^*_{t_4}\left(\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_3-1}\varepsilon^*_{s_3}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_4=1}^{t_4-1}\varepsilon^*_{s_4}\widetilde{c}_{t_4-s_4}\left(\widehat{\theta}_n\right)\right)\right].$$

Since ε^*_t, conditional on the sample, is iid with mean 0, variance 1 and finite fourth moments, then the above expectation is clearly zero if $t_3 < t_4$, and thus it is

$$E^*\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\varepsilon^{*2}_{t_3}\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_3-1}\varepsilon^*_{s_3}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_4=1}^{t_3-1}\varepsilon^*_{s_4}\widetilde{c}_{t_3-s_4}\left(\widehat{\theta}_n\right)\right)\right]$$

$$= E^*\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_2-1}\varepsilon^*_{s_3}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)^2\right] \tag{55}$$
$$+2E^*\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\right.$$
$$\left.\times\left(\sum_{s_3=1}^{t_2-1}\varepsilon^*_{s_3}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_4=t_2+1}^{t_3-1}\varepsilon^*_{s_4}\widetilde{c}_{t_3-s_4}\left(\widehat{\theta}_n\right)\right)\right] \tag{56}$$

$$+E^*\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\right.$$
$$\left.\times\left(\sum_{s_3=t_2+1}^{t_3-1}\varepsilon^*_{s_3}\widetilde{c}_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)^2\right]. \tag{57}$$

Since s_4 is greater than t_2 and so is than s_1, s_2, s_3 and t_1, then, by the properties of ε^*_t, (56) = 0.

Because $s_3 > t_2$, and the properties of ε^*_t conditional on the sample, (57) is

$$E^*\left[\sum_{2=t_1\le t_2\le t_3}^{n} \varepsilon^*_{t_1}\varepsilon^*_{t_2}\left(\sum_{s_1=1}^{t_1-1}\varepsilon^*_{s_1}\widetilde{c}_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_2=1}^{t_2-1}\varepsilon^*_{s_2}\widetilde{c}_{t_2-s_2}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_3=t_2+1}^{t_3-1}\widetilde{c}^2_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\right],$$

which is 0 unless $t_1 = t_2$, in which case, (57) becomes

$$\sum_{2=t_1\le t_3}^{n}\left(\sum_{s_1=1}^{t_1-1}\tilde{c}^2_{t_1-s_1}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s_3=t_1+1}^{t_3-1}\tilde{c}^2_{t_3-s_3}\left(\widehat{\theta}_n\right)\right)\le\left(\sum_{2=t}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right)\right)^2 \tag{58}$$

since the quantities $\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)$ are nonnegative and by the Cauchy-Schwarz's inequality.

Next (55), which is zero unless $t_1 = t_2$, and thus it is

$$E^*\left(\varepsilon_t^{*4}\right)\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_1-s}\left(\widehat{\theta}_n\right)\tilde{c}^2_{t_3-s}\left(\widehat{\theta}_n\right)\right)+\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_1-s}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_3-s}\left(\widehat{\theta}_n\right)\right)$$
$$+2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}_{t_1-s}\left(\widehat{\theta}_n\right)\tilde{c}_{t_3-s}\left(\widehat{\theta}_n\right)\right)^2. \tag{59}$$

The first term of (59) is, except the constant $E^*\left(\varepsilon_t^{*4}\right)$,

$$\sum_{t_1=2}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_1-s}\left(\widehat{\theta}_n\right)\right)\left(\sum_{t_3=t_1+1}^{n}\tilde{c}^2_{t_3-s}\left(\widehat{\theta}_n\right)\right)\le\frac{1}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right)$$

because $\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)=n^{-1}\left|t-s\right|^{-2}\left(1+o_p\left(1\right)\right)$, by (30) and that $\widehat{\theta}_n-\theta_0=o_p\left(1\right)$. The second term of (59) is bounded by

$$\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right)\right)^2$$

because the quantities (random variables) $\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)$ are nonnegative. Finally, the third term of (59) is bounded, by the Cauchy-Schwarz inequality, by

$$2\sum_{2=t_1\le t_3}^{n}\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_1-s}\left(\widehat{\theta}_n\right)\right)\left(\sum_{s=1}^{t_1-1}\tilde{c}^2_{t_3-s}\left(\widehat{\theta}_n\right)\right)\le 2\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right)\right)^2.$$

Thus (55) + (57), except multiplicative constants, is bounded by

$$4\left(\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right)\right)^2+\frac{E^*\left(\varepsilon_t^{*4}\right)}{n}\sum_{t=2}^{n}\left(\sum_{s=1}^{t-1}\tilde{c}^2_{t-s}\left(\widehat{\theta}_n\right)\right),$$

which proceeding as in the proof of Lemma 3 of Section 6, it converges a.s. to

$$\begin{aligned}
&4\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)^2\\
&+\frac{E^*\left(\varepsilon_t^{*4}\right)}{n}\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)^2du-2\left(\int_0^{1/2}\left(\psi\left(2\pi u,\vartheta_2\right)-\psi\left(2\pi u,\vartheta_1\right)\right)du\right)^2\right)\\
\le\;& D_1\left(\vartheta_2-\vartheta_1\right)^{2\varsigma}
\end{aligned}$$

because the function $\psi(\lambda,\vartheta)=\psi(\lambda,\vartheta;\theta_0)$ is squared integrable and such that $\int_0^{1/2}(\psi(2\pi u,\vartheta_2)-\psi(2\pi u,\vartheta_1))\,du$ is ζ-Liptchitz continuous with $\zeta>1/2$. That concludes the proof of the proposition. □

6. TECHNICAL LEMMAS

Lemma 1 *Let $c_s(\vartheta)$ be as in (25). Then, uniformly in ϑ_1 and $\vartheta_2\in[0,1]$,*

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}c_s(\vartheta_1)\,c_s(\vartheta_2)=g(\vartheta_1,\vartheta_2)(1+o(1)) \tag{60}$$

and where $g(\vartheta_1,\vartheta_2)$ is as defined before Proposition 2.

Proof. The left side of (60) is

$$\begin{aligned}
&4n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{j_2=1}^{n/2}\psi_{j_2}(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s\lambda_{j_1})\cos(s\lambda_{j_2})\\
=\;&4n^{-3}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\,\psi_j(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j)\\
&+4n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s\lambda_{j_1})\cos(s\lambda_{j_2})
\end{aligned}$$

$$\begin{aligned}
=\;&4n^{-3}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\,\psi_j(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) \qquad (61)\\
&+2n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}(\vartheta_2)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s(\lambda_{j_1}+\lambda_{j_2}))+\cos(s(\lambda_{j_1}-\lambda_{j_2})).
\end{aligned}$$

Because, see for instance Robinson (1995b),

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j)=\frac{(n-1)^2}{4} \tag{62}$$

and for $\lambda_{j_1}\neq\lambda_{j_2}$

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s(\lambda_{j_1}+\lambda_{j_2}))+\cos(s(\lambda_{j_1}-\lambda_{j_2}))=-n, \tag{63}$$

the right side of (61) is

$$\left(\frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta_1)\,\psi_j(\vartheta_2)\right)-2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}(\vartheta_2)\right)$$

$$= \quad g(\vartheta_1, \vartheta_2)(1 + o(1)),$$

by the properties, i.e. the continuity, of $\psi(u, \vartheta)$, which concludes the proof of the lemma. □

Lemma 2 *Assuming A1 and A2, as $n \to \infty$,*

$$\int_{-\pi}^{\pi} \left| \frac{\alpha\left(\lambda; \widehat{\theta}_n\right)}{\alpha\left(\lambda_j; \widehat{\theta}_n\right)} - 1 \right|^2 K(\lambda - \lambda_j) \, d\lambda = O_p\left(\frac{1}{j}\right),$$

where $K(\lambda)$ is the Fèjer kernel.

Proof. The proof is essentially the same as that of Lemma 3 of Robinson (1995b). We split the integral up as follows:

$$\int_{-\pi}^{-\delta} + \int_{-\delta}^{-\lambda_j/2} + \int_{-\lambda_j/2}^{\lambda_j/2} + \int_{\lambda_j/2}^{2\lambda_j} + \int_{2\lambda_j}^{\delta} + \int_{\delta}^{\pi},$$

for $\delta \in (2\lambda_j, \pi)$. Proceeding as in Robinson's (1995b) Lemma 3, conditional on the sample, the first integral is bounded by

$$\frac{1}{\pi f\left(\lambda_j; \widehat{\theta}_n\right)} \left\{ \frac{1}{n\delta^2} \int_{-\pi}^{\pi} f\left(\lambda; \widehat{\theta}_n\right) d\lambda + f\left(\lambda_j; \widehat{\theta}_n\right) \frac{2\pi}{n\delta^2} \right\}$$

$$= \frac{1}{\pi f\left(\lambda_j; \widehat{\theta}_n\right)} \left\{ \frac{\widehat{\sigma}_\varepsilon^2}{n\delta^2} + f\left(\lambda_j; \widehat{\theta}_n\right) \frac{2\pi}{n\delta^2} \right\}$$

since by construction $\int_{-\pi}^{\pi} f\left(\lambda; \widehat{\theta}_n\right) d\lambda = \widehat{\sigma}_\varepsilon^2$. Because, by A.1, $f(\lambda; \theta)$ is differentiable in θ and that $\left(\widehat{\theta}_n - \theta_0\right) = O_p\left(n^{-1/2}\right)$, then the right side of the above equation is $O_p\left(j^{-1}\right)$, also $\left|\int_{\delta}^{\pi}\right|$ has the same bound by the same arguments. Proceeding as in Robinson (1995b), conditional on the sample, the second integral is bounded by

$$\frac{1}{2\pi n f\left(\lambda_j; \widehat{\theta}_n\right)} \left\{ \int_{\lambda_j/2}^{\pi} \lambda^{2\widehat{d}_n} d\lambda + f\left(\lambda_j; \widehat{\theta}_n\right) \int_{\lambda_j/2}^{\pi} \lambda^{-2} d\lambda \right\}.$$

But, again by A1 and that $\left(\widehat{\theta}_n - \theta_0\right) = O_p\left(n^{-1/2}\right)$, the above expression is also $O_p\left(j^{-1}\right)$, and $\left|\int_{2\lambda_j}^{\delta}\right|$ has also the same bound. Finally, the third integral is, as in Lemma 3 of Robinson (1995b), $O_p\left(j^{-1}\right)$. That completes the proof. □

Lemma 3 *Put*

$$c_s\left(\vartheta; \widehat{\theta}_n\right) = 2n^{-3/2} \sum_{j=1}^{n/2} \psi_j\left(\vartheta; \widehat{\theta}_n\right) \cos(s\lambda_j),$$

where $\psi_j(\vartheta;\theta) = \psi(j/n,\vartheta;\theta)$, *where* $\psi(u,\vartheta;\theta)$ *is a continuous differentiable function in* θ, *and Liptchitz continuous in its first argument. Then,for all* ϑ_1 *and* $\vartheta_2 \in [0,1]$,

$$\sum_{t=1}^{n-1}\sum_{s=1}^{n-t} c_s\left(\vartheta_1;\widehat{\theta}_n\right) c_s\left(\vartheta_2;\widehat{\theta}_n\right) - g(\vartheta_1,\vartheta_2) \xrightarrow{P} 0.$$

Proof. The left side is

$$\begin{aligned}
& 4n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{j_2=1}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s\lambda_{j_1})\cos(s\lambda_{j_2}) \\
= \; & 4n^{-3}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) \\
& +4n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s\lambda_{j_1})\cos(s\lambda_{j_2})
\end{aligned}$$

$$\begin{aligned}
= \; & 4n^{-3}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos^2(s\lambda_j) \qquad (64)\\
& +2n^{-3}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\sum_{t=1}^{n-1}\sum_{s=1}^{n-t}\cos(s(\lambda_{j_1}+\lambda_{j_2}))+\cos(s(\lambda_{j_1}-\lambda_{j_2})).
\end{aligned}$$

Again, proceeding as in Lemma 1, c.f. (62) and (63), the left side of (64) is

$$\left(\frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j\left(\vartheta_1;\widehat{\theta}_n\right)\psi_j\left(\vartheta_2;\widehat{\theta}_n\right)\right) - 2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}\left(\vartheta_1;\widehat{\theta}_n\right)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}\left(\vartheta_2;\widehat{\theta}_n\right)\right). \qquad (65)$$

Because $\psi_j(\vartheta;\theta)$ is continuously differentiable in θ, then by the Mean Value Theorem, continuity of the derivative and that, by Giraitis and Surgailis (1990), $\left(\widehat{\theta}_n - \theta_0\right) \xrightarrow{P} 0$,

$$(65) - \frac{(n-1)^2}{n^2}\left(\frac{1}{n}\sum_{j=1}^{n/2}\psi_j(\vartheta_1;\theta_0)\psi_j(\vartheta_2;\theta_0)\right) + 2n^{-2}\sum_{j_1=1}^{n/2}\psi_{j_1}(\vartheta_1;\theta_0)\sum_{\substack{j_2=1\\ j_1\neq j_2}}^{n/2}\psi_{j_2}(\vartheta_2;\theta_0) \xrightarrow{P} 0.$$

But, proceeding as in Lemma 1, the last two terms on the left of the above equation converges to $-g(\vartheta_1,\vartheta_2)$, which concludes the proof of the lemma. □

Lemma 4 *Let $\widehat{\theta}_n$ be such that it converges almost surely to $\theta_1 \in \Theta$. Then*

$$\theta_n^* - \widehat{\theta}_n = o_{p^*}(1).$$

Proof. Conditional on the sample $\underset{\sim}{x} = (x_1, ..., x_n)'$, x_t^* is, by construction, a linear covariance stationary process with spectral density $f\left(\lambda; \widehat{\theta}_n\right)$, where the innovations ε_t^* are iid with mean 0 and variance 1. Moreover, because $f\left(\lambda; \widehat{\theta}_n\right)$ satisfies

$$\int_{-\pi}^{\pi} \log\left(f\left(\lambda; \widehat{\theta}_n\right)\right) d\lambda > -\infty,$$

it implies that the sequence x_t^* is ergodic, because it posses a spectral distribution function which does not have atom at frequency 0. Then proceeding as in the proof of Lemma 1 of Hannan (1973),

$$\frac{1}{n} \sum_{j=1-n/2}^{n/2} \frac{I_{nj}^*}{f_j(\theta)} - \int_{-\pi}^{\pi} \frac{f\left(\lambda; \widehat{\theta}_n\right)}{f(\lambda; \theta)} d\lambda \xrightarrow{P} 0$$

uniformly in $\theta \in \Theta$. Now proceed as in the proof of Theorem 1 of Hannan (1973) to conclude. □

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TABLE 1

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is white noise. Observations generated according to a $N(0,1)$ and a $Uniform(-0.5, 0.5)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		Asymptotic				Bootstrap			
		Normal		*Uniform*		*Normal*		*Uniform*	
		C_n	B_n	C_n	B_n	C_n	B_n	C_n	B_n
$n=25$	$\alpha=0.01$	0.0064	0.0026	0.0094	0.0034	0.0102	0.0108	0.0126	0.0124
	$\alpha=0.05$	0.0368	0.0148	0.0452	0.0172	0.0492	0.0478	0.0534	0.0510
	$\alpha=0.10$	0.0852	0.0330	0.0894	0.0408	0.0960	0.0976	0.0982	0.0978
$n=50$	$\alpha=0.01$	0.0172	0.0104	0.0184	0.0092	0.0120	0.0120	0.0116	0.0110
	$\alpha=0.05$	0.0594	0.0296	0.0662	0.0340	0.0476	0.0486	0.0514	0.0538
	$\alpha=0.10$	0.1140	0.0624	0.1210	0.0734	0.0976	0.0930	0.1010	0.0988
$n=100$	$\alpha=0.01$	0.0118	0.0064	0.0122	0.0070	0.0960	0.0100	0.0096	0.0098
	$\alpha=0.05$	0.0592	0.0346	0.0584	0.0366	0.0536	0.0540	0.0512	0.0542
	$\alpha=0.10$	0.1100	0.0732	0.1152	0.0772	0.1010	0.1010	0.1042	0.1042
$n=150$	$\alpha=0.01$	0.0120	0.0064	0.0112	0.0070	0.0108	0.0108	0.0098	0.0102
	$\alpha=0.05$	0.0596	0.0370	0.0610	0.0380	0.0580	0.0518	0.0568	0.0516
	$\alpha=0.10$	0.1136	0.0796	0.1148	0.0818	0.1076	0.1064	0.1066	0.1102

TABLE 2

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is an $AR(1)$. Observations generated as $x_t = 0.5x_{t-1} + \varepsilon_t$, $\varepsilon_t \sim iid\ N(0,1)$ and $\varepsilon_t \sim iidUniform(-0.5, 0.5)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$Normal\ \varepsilon_t$		$Uniform\ \varepsilon_t$	
		C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0066	0.0070	0.0056	0.0064
	$\alpha = 0.05$	0.0438	0.0396	0.0432	0.0408
	$\alpha = 0.10$	0.0840	0.0832	0.0856	0.0844
$n = 50$	$\alpha = 0.01$	0.0092	0.0104	0.0098	0.0122
	$\alpha = 0.05$	0.0518	0.0496	0.0498	0.0470
	$\alpha = 0.10$	0.0952	0.0964	0.0944	0.0964
$n = 100$	$\alpha = 0.01$	0.0088	0.0080	0.0086	0.0082
	$\alpha = 0.05$	0.0458	0.0466	0.0490	0.0484
	$\alpha = 0.10$	0.0944	0.0954	0.0946	0.1000
$n = 150$	$\alpha = 0.01$	0.0116	0.0130	0.0108	0.0116
	$\alpha = 0.05$	0.0482	0.0524	0.0516	0.0564
	$\alpha = 0.10$	0.0962	0.0996	0.1030	0.1016

TABLE 3

Proportion of rejections, in 5000 Monte Carlo experiments, under H_0 when testing that the process is a $FARIMA\,(0,d,0)$ process with $d = 0.2, 0.3, 0.4$ and the innovations ε_t are $N\,(0,1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$d = 0.2$		$d = 0.3$		$d = 0.4$	
		C_n	B_n	C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0028	0.0044	0.0034	0.0048	0.0064	0.0070
	$\alpha = 0.05$	0.0290	0.0334	0.0332	0.0362	0.0460	0.0500
	$\alpha = 0.10$	0.0680	0.0758	0.0766	0.0810	0.0952	0.0968
$n = 50$	$\alpha = 0.01$	0.0046	0.0064	0.0056	0.0070	0.0068	0.0074
	$\alpha = 0.05$	0.0340	0.0376	0.0366	0.0408	0.0448	0.0464
	$\alpha = 0.10$	0.0766	0.0808	0.0854	0.0864	0.0942	0.0958
$n = 100$	$\alpha = 0.01$	0.0080	0.0094	0.0100	0.0108	0.0088	0.0102
	$\alpha = 0.05$	0.0408	0.0452	0.0448	0.0464	0.0438	0.0442
	$\alpha = 0.10$	0.0882	0.0892	0.0926	0.0938	0.0862	0.0912
$n = 150$	$\alpha = 0.01$	0.0072	0.0074	0.0082	0.0080	0.0054	0.0058
	$\alpha = 0.05$	0.0480	0.0466	0.0498	0.0476	0.0414	0.0430
	$\alpha = 0.10$	0.0952	0.0972	0.0968	0.1004	0.0890	0.0914

TABLE 4

Proportion of rejections, in 5000 Monte Carlo experiments under H_1, when testing that the process is an $AR(1)$ and the observations are generated according to a $FARIMA(0,d,0)$ process with $d = 0.2, 0.3, 0.4$, and the innovations ε_t are $N(0,1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		$d = 0.2$		$d = 0.3$		$d = 0.4$	
		C_n	B_n	C_n	B_n	C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0334	0.0280	0.0538	0.0442	0.0514	0.0398
	$\alpha = 0.05$	0.1112	0.0990	0.1634	0.1444	0.1684	0.1404
	$\alpha = 0.10$	0.1786	0.1714	0.2454	0.2264	0.2614	0.2292
$n = 50$	$\alpha = 0.01$	0.0602	0.0412	0.1106	0.0856	0.1412	0.0994
	$\alpha = 0.05$	0.1476	0.1374	0.2536	0.2212	0.3402	0.2816
	$\alpha = 0.10$	0.2218	0.2048	0.3552	0.3180	0.4594	0.4050
$n = 100$	$\alpha = 0.01$	0.0982	0.0680	0.2536	0.1978	0.4278	0.3426
	$\alpha = 0.05$	0.2224	0.1958	0.4344	0.3766	0.6410	0.5792
	$\alpha = 0.10$	0.3202	0.2898	0.5344	0.4914	0.7290	0.6840
$n = 150$	$\alpha = 0.01$	0.1328	0.0968	0.3440	0.2786	0.5998	0.5188
	$\alpha = 0.05$	0.2816	0.2344	0.5356	0.4668	0.7662	0.7042
	$\alpha = 0.10$	0.3786	0.3368	0.6360	0.5740	0.8350	0.7878

TABLE 5

Proportion of rejections, in 5000 Monte Carlo experiments, under H_1 when testing that the process is a $FARIMA\,(0,d,0)$ and the observations are generated according to an $AR\,(1)$ with parameter 0.5 and the innovations ε_t are $N\,(0,1)$. Bootstrap critical values are computed based on 2000 bootstrap samples.

		C_n	B_n
$n = 25$	$\alpha = 0.01$	0.0260	0.0318
	$\alpha = 0.05$	0.1344	0.1358
	$\alpha = 0.10$	0.2448	0.2316
$n = 50$	$\alpha = 0.01$	0.0560	0.0538
	$\alpha = 0.05$	0.2082	0.1892
	$\alpha = 0.10$	0.3402	0.3116
$n = 100$	$\alpha = 0.01$	0.1350	0.1156
	$\alpha = 0.05$	0.3890	0.3412
	$\alpha = 0.10$	0.5436	0.4862
$n = 150$	$\alpha = 0.01$	0.2540	0.2122
	$\alpha = 0.05$	0.5518	0.4792
	$\alpha = 0.10$	0.6982	0.6340