

**EFFICIENCY, MONOTONICITY AND RATIONALITY IN
PUBLIC GOODS ECONOMIES***

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WP-AD 96-12

* We thank J. Alcalde, S. Chattopadhyay, L. Corchón and an anonymous referee for their comments. Financial support from the Instituto Valenciano de Investigaciones Económicas and from the Ministry of Education project no. PB94-1504 is gratefully acknowledged.

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ABSTRACT

In economies with public goods, we provide a necessary and sufficient condition for the existence of cost monotonic selections from the set of Pareto optimal and individually rational allocations. Such selections exist if and only if the preferences of the agents satisfy what we call the equal ordering property. This requirement is very restrictive in the context of more than one public good. However, whenever it holds any such mechanism must choose an egalitarian equivalent allocation.

KEYWORDS: Public goods, technological monotonicity, egalitarian equivalent allocations.

JEL: H41

1 INTRODUCTION

Consider the question of finding the optimal allocation of a bundle of public goods, and the way in which its cost should be shared by the agents who consume it. Most of the literature has concentrated on two aspects of this problem. Firstly, there is the controversy of designing mechanisms which induce the agents to reveal their utilities; one would expect that, in most cases, the agents have strong incentives to hide their true utility regarding the public goods. Secondly, as in the present work, there is the issue of selecting an optimal bundle of public goods and distributing the cost involved in financing the production plan among the members in the Economy.

To address this problem we adopt the normative approach: The solution is determined by considering some "equitable" properties which are agreed upon by the agents and express their sense of fairness. Once the relevant "ethical guidelines" have been acknowledged, one tries to pinpoint a solution complying with them. If one is found, then it is applied to the problem at hand.

A universally accepted property is Pareto optimality. Allocations for which it is possible to improve the welfare of some agents without making the rest worse off, should not be considered. However, Pareto efficiency by itself has one major drawback; it does not determine a unique allocation. Even worse, it contains proposals such as "one agent absorbs all the surplus" which are objectionable on grounds of non-equitability.

Individual rationality is another of the most widely accepted requirements for a solution to have. Since the technology is jointly owned by all members of the society, it seems reasonable to require that the optimal production plan and its financing should possess a certain degree of unanimity. In this framework, this corresponds to individual rationality; a possible allocation will be objected to by some member who operating the technology on his own, could improve his utility.

The same objection to the one considered above for Pareto optimality also applies to the latter solution concept: quite often, the set of individually rational allocations, even with the added requirement of Pareto optimality, turns out to be a very large set. And there is no obvious way of picking an appropriate selection from it because there seems to be no single universal solution which would satisfy everyone's sense of fairness. This naturally leads to the question of finding relevant situations in which there is a suitable one-point selection process.

Another property considered in the literature as being desirable is “cost (or technological) monotonicity,” i.e., if the publicly owned technology gets better, then no agent should be worse off. Technological monotonicity was introduced by J. Roemer ([12]) and has been subsequently used to study some solution concepts (see, for example, [7, 9, 13]).

In the case of just one public good, H. Moulin ([6]) has characterized the egalitarian-equivalent solution, proposed originally by E. A. Pazner and D. Schmeidler [11], as the only selection from the set of Pareto efficient allocations which satisfies cost monotonicity and the Core property. Due to the interest of this result, it seems very natural to ask whether it can be extended to wider contexts.

In the present work, we characterize when such an extension is possible. We find that in a setting very similar to the one in [6], but with several public goods, the axioms we have just discussed are not always compatible when taken together. As we prove in Section 4, under some mild assumptions, there is a mechanism satisfying the three properties above if and only if the preferences of the agents satisfy the equal ordering property. This latter condition, which is very natural in the context of one public and one private good (as in [6]), is severely restrictive in general. For example, with quasilinear, strictly increasing preferences in public goods, the equal ordering property is equivalent to the assertion that all the agents have the same ordinal (but not necessarily cardinal) preferences on public goods.

Thus, with several public goods, one needs to impose additional restrictions on the preferences of the consumers in order to find cost monotonic and individually rational selections from the set of Pareto efficient allocations.

On the other hand, when such a mechanism exists, then: (1) It must pick an egalitarian equivalent allocation; (2) the latter form a subset of the core; and (3) all cost monotonic mechanisms are equivalent, i.e., they provide the same utility profile to the agents. Thus, the results in [6] cover essentially all the cases for which a cost monotonic selection mechanism from the set of Pareto optimal and individually rational allocations is possible.

The difference between just one and several public goods is that, in the first case, there is no conflict of interests: everybody likes more of the public good. Nevertheless, with more than one public good to choose from, different agents might differ in their opinions about which should be given priority over the others creating, thus, a possible source of conflict. Clearly, under the equal ordering property, these discrepancies in priority do not arise.

2 THE MODEL

We consider economies with one private good and, possibly, more than one public good. The space of public goods is $X = \mathbb{R}_+^m$, with $m \geq 1$. These are produced at a cost which is financed by the members of the society. The technology available to produce the public goods is described by a cost function $c : X \rightarrow \mathbb{R}_+$. Given $y \in X$, the cost, in terms of the private good, needed to produce the bundle y of public goods is $c(y)$. In addition, the technology is jointly owned by all the agents and only one bundle of public goods is eventually produced.

We will assume that whatever technology we consider, it exhibits some bounded returns to scale when producing very large bundles of public goods. Of course, this does not preclude having arbitrarily large increasing returns to scale for public goods within some compact set.

Assumption 2.1 *The technology $c : X \rightarrow \mathbb{R}_+$ is continuous, nondecreasing, satisfies $c(0) = 0$ and*

$$\limsup_{\|y\| \rightarrow +\infty} \frac{\|y\|}{c(y)} < +\infty.$$

We use the Euclidean norm $\|y\| = \sqrt{\sum_{k=1}^m y_k^2}$. For the purposes of computing the lim sup we adopt the following convention: Consider the extended real line $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$. We also extend the usual ordering on \mathbb{R} to \mathbb{R}^* by defining $-\infty < x < +\infty$ for any real number $x \in \mathbb{R}$ and we let $\|y\|/c(y) = +\infty$ whenever $c(y) = 0$.

As usual, the ordering in Euclidean space is defined as follows. Given two vectors $x, z \in \mathbb{R}_+^m$, $x \geq z$ (resp. $x \gg z$) means that $x_i \geq z_i$ (resp. $x_i > z_i$) for every $i = 1, \dots, m$; the notation $x > z$ indicates that $x \geq z$ and $x \neq z$.

We let $N = \{1, \dots, n\}$ denote the set of agents. The initial endowment of private good for agent $i \in N$ is $\omega_i \in \mathbb{R}_+$ ($\omega_i = +\infty$ is allowed). Each agent $i \in N$ has preferences over public and private goods represented by a utility function $u_i : X \times Y_i \rightarrow \mathbb{R}$, where $Y_i \subset \mathbb{R}$ is his consumption set of private good. By abuse of notation and for simplicity, we will write $u_i(y, t)$ rather than $u_i(y, \omega_i - t)$. In other words, for each $i = 1, \dots, n$, $u_i(y, t)$ is the utility obtained by agent i when the bundle y of public goods is implemented and he has to contribute the amount t from his private endowment towards

its financing. The payment t could be negative, meaning that agent $i \in N$ receives some compensation from other agents for accepting the bundle y instead of another one he might have preferred to y .

We use the notation $x(S) = \sum_{i \in S} x_i$ for a non empty subset $S \subseteq N$ and a vector $x \in \mathbb{R}^Q$, with $S \subseteq Q \subseteq N$.

Assumption 2.2 *For each agent $i \in N$, the following assumptions hold:*

- (i) $Y_i = [-\omega(N), \omega_i]$.
- (ii) *The utility function $u_i : X \times Y_i \rightarrow \mathbb{R}$ is continuous non-decreasing in the first argument (public goods) and strictly decreasing in the second (the private good).*
- (iii) *Given a given bundle of public goods $y \in X$, there is one and only one $\varphi_i(y) \in Y_i$ such that $u_i(y, \varphi_i(y)) = u_i(0, 0) = 0$. The mapping $\varphi_i(y)$ verifies*

$$\limsup_{\|y\| \rightarrow +\infty} \frac{\varphi_i(y)}{\|y\|} = 0.$$

By (ii) and (iii) there are no public bads and the amount of private good which agents are willing to provide for the consumption of a fixed bundle of public goods is limited. Note that $u_i(y, t)$ is decreasing in the private good to indicate that t denotes a payment. For convenience, we have normalized $u_i(0, 0) = 0$ for each $i = 1, \dots, n$.

Part (i) says that no agent can contribute more than his own endowment to fund the construction of the public good; thus restricting total investment towards the construction of the public good to $\omega(N)$. In case $\omega_i = +\infty$, for some $i \in N$ then we take $\omega(N) = +\infty$. As mentioned above, agents are allowed to transfer part or all of their endowments to other agents to encourage them to accept a particular bundle of public goods. The results below will still hold whenever transfers of private good are not allowed, as long as the utility functions are (strictly) monotone in public goods. One needs only to modify the argument in Remark 5.1 in an easy way.

It follows from (ii) and (iii) that, for $i = 1, \dots, n$, the mappings φ_i are non decreasing and satisfy $\varphi_i(0) = 0$.

We let $Y = \prod_{i=1}^n Y_i$ and extend the utility functions of the agents to $X \times Y$, by $u_i(y; t) = u_i(y, t_i)$, with $(y; t) = (y, t_1, \dots, t_n) \in X \times Y$ and $i \in N$.

The utility profile of the agents is the mapping $u : X \times Y \longrightarrow \mathbb{R}^n$ given by $u(y; t) = (u_1(y, t_1), \dots, u_n(y, t_n))$. Likewise, we will also write $\varphi(y) = (\varphi_1(y), \dots, \varphi_n(y))$.

From now on, we fix the set Y and a profile of utilities satisfying assumptions 2.2. An economy is a pair (u, c) consisting of an utility profile and a technology. Since the utilities of the agents are fixed throughout the paper, we will use the notation c instead of (u, c) to denote an economy. An allocation is a pair $(z; t) \in X \times Y$. The allocation $(z; t)$ is feasible in the economy c for a non empty coalition $S \subset N$ if $c(z) \leq t(S)$ with $t \in \prod_{i \in S} Y_i$. We will simply say that $(z; t)$ is feasible whenever it is feasible for the grand coalition N .

Given a technology c , an allocation $(z; t)$ is said to be Pareto optimal in the economy c if it is feasible and $u(z; t) = u(y; s)$ for any other feasible allocation $(y; s)$ such that $u(z; t) \leq u(y; s)$. The set of Pareto optimal allocations is denoted by $P(c)$. A nonempty coalition $S \subset N$ can improve upon an allocation $(z; t)$ if there is another allocation $(y; s)$, feasible for S , such that $u_i(y; s) \geq u_i(z; t)$ for each $i \in S$ with at least some strict inequality. An allocation $(z; t)$ is individually rational (resp. in $\text{Core}(c)$) if no agent can improve upon it (resp. if no coalition can improve it).

3 Egalitarian equivalent allocations

One of the principles we will be considering to determine the allocation which is "optimal" for the society is cost monotonicity. As we will see, if a solution satisfying this requisite exists, then it has to select an egalitarian equivalent allocation. In the present section, we review this notion. Consider a technology $c : x \rightarrow \mathbb{R}_+$ satisfying assumption 2.1.

Definition 3.1 *The set of egalitarian equivalent allocations is defined to be*

$$EE(c) = \{(x; t) \in P(c) : \text{there is } z \in X \text{ with } u(x; t) = u(z; 0)\}.$$

The bundle of public goods z appearing in the definition of $EE(c)$ is the reference bundle. The egalitarian equivalent solution was proposed by E. A. Pazner and D. Schmeidler ([11]) and has been characterized in [2] and [6]. An alternative procedure to describe the set $EE(c)$ (more appropriate for the present set up) is given in the following construction.

Let $S^{m-1} = \{z \in \mathbb{R}^m : \|z\| = 1\}$ denote the $m - 1$ sphere and consider its positive orthant

$$S_+^{m-1} = S^{m-1} \cap X \quad \text{and} \quad S_{++}^{m-1} = S^{m-1} \cap \mathbb{R}_{++}^m.$$

Given $\alpha \in S_+^{m-1}$ define

$$F_\alpha(c) = \{\lambda \in \mathbb{R}_+ : \text{there is } (y; t) \in X \times Y \text{ with} \\ c(y) \leq t(N) \quad \text{and } u(y; t) = u(\lambda\alpha; 0)\}$$

and the application $H_\alpha : F_\alpha(c) \rightarrow \mathbb{R}^n$ by $H_\alpha(\lambda) = u(\lambda\alpha; 0)$. Note that $F_\alpha(c) \neq \emptyset$, since $0 \in F_\alpha(c)$. The function H_α is nondecreasing because u is nondecreasing as well.

The set of egalitarian levels along the ray $\alpha \in S_+^{m-1}$ in the economy c is defined as

$$\text{EL}_\alpha(c) = \{\lambda\alpha : \lambda \in \arg \max H_\alpha\}$$

and the set of egalitarian levels is

$$\text{EL}(c) = \bigcup_{\alpha \in S_+^{m-1}} \text{EL}_\alpha(c).$$

The feasible allocations supporting the egalitarian levels are the sets

$$\text{AEL}_\alpha(c) = \{(x; t) \in X \times Y : c(y) \leq t(N) \text{ and there is} \\ z \in \text{EL}_\alpha(c) \text{ with } u(x; t) = u(z; 0)\}.$$

The relationship between the two concepts presented above is given by the following observation.

Remark 3.2 Suppose assumptions 2.2 hold and let $c : X \rightarrow \mathbb{R}_+$ satisfy assumption 2.1. Then

$$\bigcup_{\alpha \in S_{++}^{m-1}} \text{AEL}_\alpha(c) \subset \text{EE}(c) \subset \bigcup_{\alpha \in S_+^{m-1}} \text{AEL}_\alpha(c).$$

The second inclusion might be strict whenever preferences are constant on the boundary of the positive orthant (e.g. Cobb-Douglas preferences). In such cases, the boundary of the positive orthant is contained in the set of egalitarian levels and typically will not be supported by Pareto optimal allocations. Remark 3.2 combined with the next straightforward result shows that under rather weak assumptions, egalitarian equivalent allocations exist. The proof of it is in the Appendix.

Proposition 3.3 *Suppose assumptions 2.2 hold. Then, $EL_\alpha(c) \neq \emptyset$, for any $c : X \rightarrow \mathbb{R}_+$ satisfying 2.1 and any $\alpha \in S_{++}^{m-1}$.*

As the following example shows, when the number of public goods is greater than one, there are two problems associated with the egalitarian equivalent allocations: firstly, in many cases there is a continuum of egalitarian equivalent allocations which are not individually rational. Secondly, the ones which are individually rational, form a continuum of allocations; furthermore, there does not seem to be a natural procedure for selecting any one of them, since they yield different utilities to the agents.

Example 3.4 The economy c consists of two public goods (so $X = \mathbb{R}_+^2$) and two consumers with quasi-linear preferences in money given by the utility functions

$$u_1(y; t) = 2\sqrt{y_1} + 2\sqrt{y_2} - t, \quad u_2(y; t) = 2\sqrt{y_1} - t$$

where $y = (y_1, y_2) \in X$. The cost of producing the bundle $y \in X$ of public goods is

$$c(y) = y_1 + y_2.$$

It is easy to compute $([6, 8])$ that the set of egalitarian levels is

$$EL(c) = \{y \in X : 4\sqrt{y_1} + 2\sqrt{y_2} = 5\}$$

Only a strict subset of the egalitarian equivalent allocations are individually rational. The set of utilities given by individually rational egalitarian equivalent allocations is

$$U = \{(v_1, v_2) : v_1 + v_2 = 5, v_1 \geq v_2, v_1 \geq 2, v_2 \geq 1\}.$$

Hence, not all the egalitarian levels provide individually rational allocations and, furthermore, there are several distributions of utilities in the set U .

4 Cost monotonic mechanisms

One way to overcome the difficulties mentioned at the end of the last section is to consider alternative properties to those of optimality and technological monotonicity in order to narrow down the solution to the cost allocation problem. In this section, we consider whether any obstructions exist to considering individual rationality as a third axiom compatible with the other two just mentioned.

A mechanism R will be defined below to be a mapping assigning to each economy c a feasible allocation. In order to give a precise definition we need to specify the domain of R . As a first step, we consider the following set for the admissible technologies.

$$E_0 = \{c : X \longrightarrow \mathbb{R}_+ / c \text{ satisfies assumption 2.1} \}$$

However, there are cost functions $c \in E_0$ for which the economy c does not have any allocation which is individually rational for the agents. In order to avoid such economies, we will consider only technologies satisfying the condition

$$E_1 = \{c \in E_0 : c(x \vee y) \leq c(x) + c(y)\}.$$

where, given any two vectors $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$ in \mathbb{R}^m we denote $x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_m, y_m\})$. The requirement $c(x \vee y) \leq c(x) + c(y)$ guarantees that agents can benefit from cooperating.

Remark 4.1 We observe that for any $c \in E_1$ the set of individually rational and feasible allocations of c is non-empty. Indeed, let $c \in E_1$ and suppose that for each agent $i = 1, \dots, n$, the vector $(x^i, t_i) \in X \times Y_i$ is a solution to the problem

$$\max\{u_i(z, r) : (z, r) \in X \times Y_i, \quad c(z) = r\}$$

Assumptions 2.1 and 2.2 guarantee that there is a solution to the maximization problem. Let $x = x^1 \vee x^2 \vee \dots \vee x^n$. Then

$$c(x) \leq \sum_{i=1}^n c(x^i) = t(N)$$

Hence, $(x; t)$ is feasible for the grand coalition in the economy c and every agent $i \in N$ will be at least as well off with $(x; t)$ as with (x^i, t_i) .

The domain E_1 still includes some technologies which might be considered unreasonable like cost functions which are constant on arbitrarily large sets. We will eventually show the non existence of mechanisms satisfying certain normative axioms. Of course, showing the non existence of such mechanisms for the smaller domain, immediately implies the same result for the larger one. Thus no loss of generality is entailed in restricting further the domain of R to

$$E = \{c \in E_1 : c(x) > c(y) \text{ whenever } x > y\}$$

Definition 4.2 A mechanism is a mapping

$$R : E \longrightarrow X \times \prod_{i \in N} Y_i$$

assigning to every technology $c \in E$ an allocation, $R(c)$, feasible in the economy c .

A mechanism R is Pareto efficient (resp. egalitarian equivalent) if $R(c) \in P(c)$ (resp. $R(c) \in EE(c)$) for every $c \in E$. It is individually rational if $R(c)$ is an individually rational allocation in the economy c . The mechanism R is said to be cost monotonic if, given two cost functions $c_1, c_2 \in E$ such that $c_1(y) \leq c_2(y)$ for every $y \in X$, it assigns allocations $R(c_j)$, for $j = 1, 2$, such that $u(R(c_1)) \geq u(R(c_2))$.

In the context of one public and one private good, H. Moulin ([6]) has proved the existence of cost monotonic, individually rational and Pareto efficient mechanisms. He also shows that such a mechanism must select an egalitarian equivalent allocation. It is easy to check that $c_1 \vee c_2 \in E$ whenever $c_1, c_2 \in E$, where $(c_1 \vee c_2)(y) = \max\{c_1(y), c_2(y)\}$. Using this remark, one can verify that the same proof that is used in [6], also applies here to obtain the following result.

Lemma 4.3 Let R be a Pareto optimal and cost monotonic mechanism. Then for each $c_1, c_2 \in E$ we have that either $u(R(c_1)) \geq u(R(c_2))$ or $u(R(c_1)) \leq u(R(c_2))$

We address now the main issue: given a fixed set of agents, are the axioms of cost monotonicity, Pareto efficiency and individual rationality compatible? The key to answering this question lies in the equal ordering property.

Definition 4.4 We say that the agents order the bundle of public goods equally (or that the profile of utilities u satisfies the equal ordering property) whenever for each bundle of public goods $y, z \in X$ if $u_i(y, 0) > u_i(z, 0)$ for some agent $i \in N$ then $u_j(y, 0) \geq u_j(z, 0)$ for every other agent $j \in N$.

In other words, the equal ordering property is fulfilled whenever given $y, z \in X$, either $u(y, 0) \geq u(z, 0)$ or else $u(z, 0) \geq u(y, 0)$. So, if a consumer prefers to take the bundle of public goods z for free, rather than choosing bundle y , then so do all the other agents. This property eliminates the possible sources of disagreement among players in ranking the bundles of public goods. It clearly holds in the case of one public good.

Example 4.5 We illustrate this notion in the case of quasilinear utility functions. The utility of each agent $i \in N$ is given by

$$u_i(y, t) = b_i(y) - t$$

where $b_i : X \rightarrow \mathbb{R}$ is the utility obtained by agent $i \in N$ whenever he enjoys the bundle of public goods $y \in X$ for free. The equal ordering property is equivalent to the following statement: for each pair of bundles of public goods $y, z \in X$, either $b_i(y) \geq b_i(z)$ for every agent $i \in N$ or else $b_i(z) \geq b_i(y)$ for every agent $i \in N$.

We can now answer the question posed above concerning the compatibility of cost monotonicity, Pareto efficiency and individual rationality. The result stated below makes precise the conditions under which there is a solution satisfying these three properties.

Theorem 4.6 *Let $N = \{1, \dots, n\}$ be a set of agents whose profile of utilities, $u = (u_1, \dots, u_n) : X \times Y \rightarrow \mathbb{R}$, satisfies assumptions 2.2. Then, there is a cost monotonic, Pareto efficient and individually rational mechanism $R : E \rightarrow X \times Y$ if and only if u verifies the equal ordering property.*

Furthermore, if such a mechanism $R : E \rightarrow X \times Y$ exists, then for every technology $c \in E$,

- (i) $R(c) \in EE(c)$.
- (ii) *The map $u(\cdot, 0)$ is constant on $EL(c)$. In fact, $u(\alpha, 0) = u(R(c))$ for any $\alpha \in EL(c)$.*
- (iii) $EE(c) \subset Core(c)$.

As a consequence, such a mechanism exists only if the agents have exactly the same ordinal preferences when the bundles of public goods are free. This condition, which clearly holds for economies with one public good, is very restrictive in the case of several types of public goods. Thus, the first part of Theorem 4.6 limits severely the existence of individually rational and cost monotonic selections from the set of Pareto optimal allocations.

The second part makes explicit that, whenever the equal property holds, we are back in the setting of [6]. Namely, part (i) implies that Pareto optimal, individually rational and cost monotonic mechanisms coincide with the egalitarian equivalent correspondence and by part (ii) all of them are

equivalent, since the agents are indifferent between them. Finally, it follows from (iii) that any such mechanism is also a selection from the Core of the economy and that the egalitarian allocations are individually rational and do not allow private transfers among the agents.

There is a related literature, in the context of monotonicity with respect to changes in resources ([14], [10]). The conclusion therein is that Pareto optimality and resource monotonicity are incompatible with other normative properties such as individual rationality from equal division or envy-free. The egalitarian equivalent solution has also been characterized by Pareto efficiency, monotonicity and a certain notion of fairness with respect to some commodity ([2]). These authors also show that the equity axiom cannot be imposed on more than one commodity; thus their results show the strength of the monotonicity axiom in another setting.

We finish this section with two remarks. Firstly, we could extend the domain of the mechanism to allow for changes in the number of agents. With this modification we could also consider the axiom of population monotonicity: roughly speaking, when the number of players increases, the cost of financing the optimal bundle of goods is shared among more agents. Thus, population monotonicity requires that by increasing the number of players everybody should be no worse off as before. It is easy to argue that the axioms of population monotonicity and Pareto optimality imply the Core property and, hence, individual rationality. Therefore, Theorem 4.6 also holds when we replace “individual rationality” with “population monotonicity.”

Secondly, if assumptions 2.2 hold, then there are plenty of cost monotonic, Pareto efficient (but of course, not individually rational) mechanisms defined on E_0 . It is straightforward to check the following result: Fix $\alpha \in S_{++}^{m-1}$. Then, the map R which assigns to each $c \in E_0$ any allocation $R(c) = (y; t) \in EE(c)$ such that $u(y; t) = u(\lambda\alpha; 0)$, with $\lambda \in \mathbb{R}$ satisfying $\lambda\alpha \in EL(c)$, is cost monotonic. The problem, as remarked above, is that for some technologies $c \in E$, the allocation $R(c) = (y; t)$ will not be individually rational. This example also shows that the axiom of cost monotonicity alone cannot discriminate among the different egalitarian equivalent allocations.

5 PROOFS

We first make a simple but useful observation.

Remark 5.1 Let $S \subset N$ be a non empty coalition, let $(y; t) \in X \times Y^S$ be an allocation feasible for S and let $v \in \mathbb{R}^S$ such that $v_i \geq u_i(0, 0)$ for each $i \in S$. Suppose that for every $i \in S$ we have $u_i(y, t_i) \geq v_i$, with some inequality being strict. Then, $r \in Y^S$ exists such that the allocation $(y; r)$ is feasible for S and, for every $i \in S$, $u_i(y, r_i) > v_i$.

Indeed, assume without loss of generality that $u_1(y, t_1) > v_1 \geq u_1(0, 0) = u_1(y, \varphi_1(y))$. Then, $t_1 < \varphi_1(y) \leq \omega_1$, so we can find $\varepsilon > 0$ small enough so that $r_1 = t_1 + \varepsilon < \varphi_1(y)$, $u_1(y, r_1) > v_1$ and for $i \in S \setminus \{1\}$, $r_i = t_i - \varepsilon/(|S| - 1) > -\omega(N)$. Therefore, $r \in Y^S$, $t(S) = r(S)$ and $u_i(y, r_i) > v_i$ for every $i \in S$. \square

The following Lemma shows that in the definition of $F_\alpha(c)$ one can replace $u(y; t) = u(\lambda\alpha; 0)$ by $u(y; t) \leq u(\lambda\alpha; 0)$. It will be used in the subsequent proofs.

Lemma 5.2 Let $c \in E_0$, v be a vector in \mathbb{R}^n such that $v \geq u(0; 0)$ and $(y; r) \in X \times Y$ be a feasible allocation such that

$$u(y; r) \geq v \geq u(0; 0)$$

Define $L : \mathbb{R} \longrightarrow Y$ by

$$L(\lambda) = \{t \in Y : c(\lambda y) \leq t(N) \text{ and } u(\lambda y; t) \geq v\}$$

and let $\lambda_* = \sup\{\lambda \in \mathbb{R} : L(\lambda) \neq \emptyset\}$. Then, $\lambda_* < +\infty$, $L(\lambda_*) \neq \emptyset$ and for any $s \in L(\lambda_*)$ we have that $u(\lambda_* y; s) = v$.

Proof

First, note that $L(1) \neq \emptyset$. Let $\{\lambda_k\}_{k=1}^\infty$ be an decreasing sequence converging to λ_* . For each $k = 1, \dots, \infty$, let $s^k \in L(\lambda_k)$ such that $(\lambda_k y; s^k)$ is feasible and

$$u(0; 0) = u(\lambda_k y; \varphi(\lambda_k y)) \leq v \leq u(\lambda_k y; s^k)$$

In particular, we have that $\varphi(\lambda_k y) \geq s^k$ for every $k = 1, \dots, \infty$. We claim that λ_* is finite. Otherwise, the sequence $\{\lambda_k\}_{k=1}^\infty$ diverges to infinity and, by assumption 2.2, for each $i = 1, \dots, n$

$$\limsup_{k \rightarrow \infty} \frac{s_i^k}{\|\lambda_k y\|} \leq \limsup_{k \rightarrow \infty} \frac{\varphi_i(\lambda_k y)}{\|\lambda_k y\|} = 0$$

Therefore,

$$0 \leq \limsup_{k \rightarrow \infty} \frac{c(\lambda_k y)}{\|\lambda_k y\|} \leq \limsup_{k \rightarrow \infty} \frac{\sum_{i \in N} s_i^k}{\|\lambda_k y\|} = 0$$

but this contradicts assumption 2.1.

We conclude that $\{\lambda_k\}_k$ is a bounded sequence and λ_* is finite. Since $s^k \leq \varphi(\lambda_k y) \leq \varphi(\lambda_* y)$ for every $k = 1, 2, \dots$, we also have that the sequence $\{s^k\}_{k=1}^\infty$ is bounded. Taking an appropriate subsequence we may assume that $(\lambda_k y; s^k)$ converges to, say $(\lambda_* y; s_*)$. Since Y is a closed set and c is continuous, then $(\lambda_* y; s_*) \in X \times Y$ is also a feasible allocation. In addition, $u(\lambda_* y; s_*) \geq v$, because u is continuous. Therefore, $s_* \in L(\lambda_*) \neq \emptyset$.

Finally, let $s \in L(\lambda_*)$ and suppose that $u(\lambda_* y; s) > v$. Say, $u_1(\lambda_* y; s_1) > v_1 \geq u_1(0, 0)$. Then, $s_1 < \varphi_1(\lambda_* y)$, so we can find $r_1 \in Y_1$ with $s_1 < r_1 < \varphi_1(\lambda_* y)$ and still $u_1(\lambda_* y; r_1) > v_1$. Take $t_1 = r_1$ and $t_i = s_i$, for $i = 2, \dots, n$. Then, $t \in Y$ and $t(N) > s(N) \geq c(\lambda_* y)$. By continuity and Assumption 2.1, we can find $\lambda > \lambda_*$ such that $c(\lambda y) \leq t(N)$. Then, $(\lambda y; t) \in X \times Y$ is feasible and $u(\lambda y; t) \geq v$ contradicting the definition of λ_* . Therefore, $u(\lambda_* y; s) = v$.

□

Proof of Remark 3.2: The second inclusion is clear. To prove the first one, let $\alpha \in S_{++}^{m-1}$ and let $(z; r) \in \text{AEL}_\alpha(c)$ be a feasible allocation supporting the egalitarian level $\lambda_0 \alpha \in \text{EL}_\alpha(c)$ with $\lambda_0 \in \mathbb{R}_+$; that is $u(z; r) = u(\lambda_0 \alpha, 0)$. We only need to show that $(z; r) \in \text{P}(c)$. Suppose not. Then, there is another feasible allocation $(y; t) \in X \times Y$ such that

$$u(y; t) > u(\lambda_0 \alpha; 0) \geq u(0; 0)$$

By Remark 5.1 above, we can find $r \in Y$ such that $c(y) = r(N)$ and

$$u(y; r) >> u(\lambda_0 \alpha; 0)$$

Let $\bar{\lambda} = \sup\{\lambda \in \mathbb{R} : u(y; r) >> u(\lambda \alpha; 0)\}$. Since, $r(N) = c(y) \geq 0$, there is some $i_0 \in N$ such that $r_{i_0} \geq 0$. Hence, $u_{i_0}(\lambda \alpha, 0) \geq u_{i_0}(y, r_{i_0})$, whenever $\lambda \alpha \geq y$. Therefore, $\lambda_0 < \bar{\lambda} < \infty$.

It cannot be the case that $u(y; r) \gg u(\bar{\lambda}\alpha; 0)$ because if it were we may find $\mu > \bar{\lambda}$ such that $u(y; r) \gg u(\mu\alpha; 0)$. Hence, we must have that

$$u(y; r) \geq u(\bar{\lambda}\alpha; 0) > u(\lambda_0\alpha; 0)$$

Applying now lemma 5.2 with $v = u(\bar{\lambda}\alpha; 0)$ we find $\lambda_* \in \mathbb{R}$ and $s^* \in Y$ such that $(\lambda_* y; s^*)$ is feasible and $u(\lambda_* y; s^*) = u(\bar{\lambda}\alpha; 0)$. But this contradicts that $\lambda_0\alpha \in \text{EL}_\alpha(c)$.

□

We are now in a position to prove the existence of egalitarian equivalent allocations.

Proof of Proposition 3.3:

Let $c \in E$ and $\alpha \in S_{++}^{m-1}$. Consider a non decreasing sequence $\{\lambda_k\}_{k=1}^\infty \subset F_\alpha(c)$ such that

$$\lim_{k \rightarrow \infty} \lambda_k = \sup(F_\alpha(c)).$$

For each $k \in \mathbb{N}$ there is an allocation $(y^k; t^k) \in X \times Y$ feasible in c such that $u(\lambda_k\alpha; 0) = u(y^k; t^k)$.

Assume that the sequence $\{y^k\}_k$ is unbounded. We may find an increasing unbounded subsequence which, for simplicity, we also denote by $\{y^k\}_k$. Using that u is non decreasing in public goods, we have that

$$u(y^k; t^k) = u(\lambda_k\alpha; 0) \geq u(0; 0) = u(y^k; \varphi(y^k)).$$

Hence, $t^k \leq \varphi(y^k)$ for every $k \in \mathbb{N}$ and assumption 2.2 implies that for any agent $i = 1, \dots, n$,

$$\limsup_{k \rightarrow \infty} \frac{t_i^k}{\|y^k\|} \leq \limsup_{k \rightarrow \infty} \frac{\varphi_i(y^k)}{\|y^k\|} = 0 \quad (5.1)$$

by adding these equations we get

$$0 \leq \limsup_{k \rightarrow \infty} \frac{c(y^k)}{\|y^k\|} = \limsup_{k \rightarrow \infty} \frac{\sum_{i \in N} t_i^k}{\|y^k\|} = 0 \quad (5.2)$$

which contradicts assumption 2.1

Hence, $\{y^k\}_k$ must be bounded. It follows now from Equation 5.1 that for each $i \in N$, the sequence $\{t_i^k\}_k$ is bounded above and, thus from Equation 5.2, there is also an infinite subsequence of $\{t_i^k\}_k$ bounded below.

By taking appropriate subsequences we may assume $\{y^k\}_k$ converges to a limit, say y and $\{t^k\}_k$ converges to, say t^* . By continuity, the allocation $(y; t^*)$ is feasible.

Suppose first that $\sup(F_\alpha(c)) = +\infty$. Then the sequence $\{\lambda_k\}_{k=1}^\infty$ is unbounded. Fix $k_0, \lambda \in \mathbb{R}$ such that $\lambda_{k_0}\alpha \geq y$ and $\lambda \geq \lambda_{k_0}$. Given $k \in \mathbb{R}$ large enough so that $\lambda_k \geq \lambda$, we have

$$u(y; 0) \leq u(\lambda\alpha; 0) \leq u(\lambda_k\alpha; 0) = u(y^k; t^k)$$

By taking limits as k tends to infinity, we obtain the inequality $u(y; 0) \leq u(\lambda\alpha; 0) \leq u(y; t^*)$. Let $t_{i_0}^* = \max\{t_1^*, \dots, t_n^*\}$. Since $0 \leq c(y) \leq t^*(N)$, we must have $t_{i_0}^* \geq 0$. On the other hand, since preferences are decreasing in the amount paid and $u_{i_0}(y; 0) \leq u_{i_0}(y; t_{i_0}^*)$, we must also have that $t_{i_0}^* \leq 0$. Therefore, $t_{i_0}^* = 0$ and $t^* = 0$.

It follows that $u(y; 0) = u(\lambda\alpha; 0)$ whenever $\lambda \geq \lambda_{k_0}$. Hence, the mapping H_α is continuous and constant for $\lambda \geq \lambda_{k_0}$, so it must attain its maximum.

If $\sup(F_\alpha(c)) = \lim_{k \rightarrow \infty} \lambda_k = \lambda$ a finite number, then

$$u(\lambda\alpha; 0) = \lim_{k \rightarrow \infty} u(\lambda_k\alpha; 0) = \lim_{k \rightarrow \infty} u(y^k; t^k) = u(y; t^*),$$

so $\lambda \in F_\alpha(c)$. □

Next, we start the preliminaries to prove Theorem 4.6.

Lemma 5.3 *Let $i \in N$ and $y \in X$. Then, there is a continuous, non-decreasing function $d_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

- (i) *If $\lambda \leq 1$, then, $d_i(\lambda) = 0$.*
- (ii) *If $\lambda \geq 1$, then, $u_i(\lambda y, d_i(\lambda)) = u_i(y, 0)$.*

Proof

Fix $\lambda \geq 1$ and let $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be defined by $h(s) = u_i(\lambda y, \varphi_i(sy))$. Then,

$$h(\lambda) = u_i(\lambda y, \varphi_i(\lambda y)) = u_i(0, 0) \leq u_i(y, 0)$$

and

$$h(0) = u_i(\lambda y, \varphi_i(0)) = u_i(\lambda y, 0) \geq u_i(y, 0)$$

Since h is continuous, there is $s(\lambda) \in [0, \lambda]$ such that $h(s(\lambda)) = u_i(\lambda y, \varphi_i(s(\lambda)y)) = u_i(y, 0)$. We define $d_i(\lambda) = \varphi_i(s(\lambda)y)$. Note that $d_i(\lambda)$ is uniquely defined because u is strictly increasing in the second argument. Also, $0 = d_i(1) \leq d_i(\lambda) \leq \varphi_i(\lambda y)$, since φ_i is increasing.

In addition, d_i is continuous and non-decreasing, since is defined implicitly by the equation

$$u_i(y, 0) = u_i(\lambda y, d_i(\lambda))$$

Thus, we can extend $d_i(\lambda)$ continuously to $[0, 1]$ by requiring that it vanishes in that interval. \square

Lemma 5.4 *Let $y \in \mathbb{R}_{++}^m$ be a bundle of public goods in the interior of X . Then, a technology $c_y \in E_1$ exists such that $u(z; t) = u(y; 0)$ for any $(z; t) \in X \times Y$ which is an individually rational and Pareto efficient allocation in the economy c_y .*

Proof

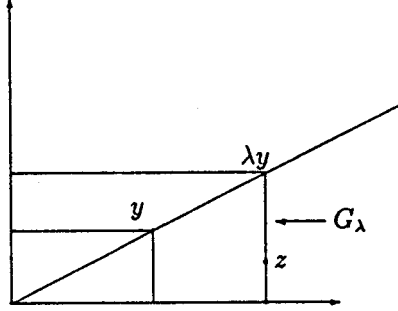
Let $G_\lambda = \partial([0, \lambda y])$ be the boundary of the set $\{x \in X : 0 \leq x \leq \lambda y\}$. (The boundary is taken as a topological subset of X .) For each $i \in N$ choose d_i as in Lemma 5.3 and let

$$d = d_1 \vee \dots \vee d_n$$

Define now

$$c_y(x) = 2n \max\{d(\lambda), \lambda - 1\}$$

where λ is the unique point in \mathbb{R}_+ such that $x \in G_\lambda$. Note that $c_y(x) = 0$ if $x \leq y$ so, in particular, $(y; 0, \dots, 0)$ is feasible in c_y . In addition, $c(x) > 0$ if $x \in G_\lambda$ with $\lambda > 1$.



For each $x \in G_\lambda$ we have $\|x\| \leq \lambda \|y\|$ and $c_y(x) \geq (\lambda - 1)$. So,

$$\limsup_{\|x\| \rightarrow +\infty} \frac{\|x\|}{c_y(x)} \leq \limsup_{\lambda \rightarrow +\infty} \frac{\lambda \|y\|}{\lambda - 1} < +\infty$$

Let $x \in G_{\lambda_1}$, $z \in G_{\lambda_2}$. Assume, without loss of generality, that $\lambda_1 \geq \lambda_2$. Then $x \leq \lambda_1 y$ and $z \leq \lambda_2 y \leq \lambda_1 y$. Hence, $x \vee z \leq \lambda_1 y$ so $x \vee z \in G_{\lambda_1}$ and $c(x \vee z) = c(x) \leq c(x) + c(z)$. Therefore, $c_y \in E_1$.

Let now $(z; t) = (z; t_1, \dots, t_n)$ be an allocation which is individually rational and Pareto efficient in c_y with $z \in G_\lambda$. Denote by $t_{i_0} = \max\{t_1, \dots, t_n\}$. Suppose that $z \leq y$ and $t_{i_0} = 0$. Then, $t = 0$ and $u(z; t) \leq u(y; 0)$. Hence, by Pareto optimality, $u(z; t) = u(y; 0)$ and the proposition is proved. If $z \leq y$ and $t_{i_0} > 0$, then $u_{i_0}(z; t) < u_{i_0}(y, 0)$ with y feasible for i_0 . But, this contradicts that $(z; t)$ is individually rational.

Otherwise, $z \notin [0, y]$, so $\lambda > 1$. Again, by feasibility, $0 \leq c_y(z) \leq t(N)$. Therefore,

$$t_{i_0} \geq \frac{c_y(z)}{n} \geq 2d(\lambda)$$

It follows that, $t_{i_0} \geq 2d(\lambda) \geq 2d_{i_0}(\lambda)$. Assume that $d_{i_0}(\lambda) > 0$, then $t_{i_0} > d_{i_0}(\lambda)$ and

$$u_{i_0}(z, t_{i_0}) < u_{i_0}(z, d_{i_0}(\lambda)) \leq u_{i_0}(\lambda y, d_{i_0}(\lambda)) = u_{i_0}(y, 0)$$

so $(z; t)$ cannot be individually rational because agent i_0 would be strictly better off by deviating to the allocation $(y; 0)$, which is feasible for him.

Hence, we must have that $d_{i_0}(\lambda) = 0$. But then $u_{i_0}(\lambda y, 0) = u_{i_0}(y, 0)$, so

$$u_{i_0}(y, 0) = u_{i_0}(\lambda y, 0) \geq u_{i_0}(z, 0) \geq u_{i_0}(z, t_{i_0})$$

By individual rationality, $u_{i_0}(y, 0) = u_{i_0}(z, t_{i_0}) = u_{i_0}(z, 0)$. Hence, $t_{i_0} = 0$ and it follows that $c_y(z) = 0$. But this contradicts that $z \in G_\lambda$ with $\lambda > 1$.

□

We say that sequence $\{c^k\}_{k=1}^\infty \subset E_0$ converges to $c \in E_0$, whenever for every $z \in X$, we have $\lim_{k \rightarrow \infty} c^k(z) = c(z)$.

Lemma 5.5 *Let $\{c^k\}_{k=1}^\infty \subset E_0$ be a sequence converging to $c \in E_0$ and let $\{(z^k; t^k)\}_{k=1}^\infty \subset X \times Y$ be a sequence of allocations such that for every $k \in \mathbb{N}$, $u(z^k; t^k) \geq u(0, 0)$ and $(z^k; t^k)$ is individually rational and Pareto optimal in the economy c^k . Suppose that $\{(z^k; t^k)\}_{k=1}^\infty$ converges to $(z; t)$. Then, the allocation $(z; t)$ is individually rational and Pareto optimal in the economy c .*

Proof

The allocation $(z; t)$ is feasible because for each $k \in \mathbb{N}$ we have that $c^k(z^k) \leq t^k(N)$ with $t^k \in Y$. Since the latter set is closed, by taking limits we have that $c(z) \leq t(N)$ and $t \in Y$.

Note also that, by continuity, $u(z; t) \geq u(0; 0)$. Suppose $(z; t)$ is not Pareto optimal. Then, there is another allocation $(x, r) \in X \times Y$ which satisfies $c(x) = r(N)$ and $u(x; r) > u(z; t)$. By remark 5.1 we may assume there is another feasible allocation $(x; s) \in X \times Y$ such that $u(x; s) >> u(z; t)$. By increasing s slightly, we may also assume that $s(N) > c(x)$.

Since the sequence $\{(z^k; t^k)\}_{k=1}^\infty$ converges to $(z; t)$, there is $N_0 \in \mathbb{N}$ such that whenever $k \geq N_0$ we have

$$u(x; s) >> u(z^k; t^k)$$

The sequence $c^k(x)$ converges to $c(x) < s(N)$ so, by a continuity argument, we can take $N_1 \in \mathbb{N}$, $N_1 \geq N_0$ such that for every $k \geq N_1$ we have $c^k(x) \leq s(N)$ and $u(x; s) >> u(z^k; t^k)$. But this contradicts that $(z^k; t^k)$ is Pareto optimal in c^k .

A similar, but simpler argument shows that $(z; t)$ is individually rational in the economy c . □

Remark 5.6 Let $c \in E_0$ and consider the set

$$A(c) = \{(y; t) \in X \times Y : c(y) = \sum_{i=1}^n t_i, \quad u(y; t) \geq 0\} \quad (5.3)$$

From assumptions 2.1 and 2.2 we have that for any technology $c \in E_0$ and any agent $i \in N$,

$$\limsup_{\|y\| \rightarrow \infty} \frac{\varphi_i(y)}{c(y)} = 0$$

Thus, if $c \in E_0$, $(y; t)$ is feasible and $\|y\|$ is large enough, there must be some agent $i_0 \in N$ such that $t_{i_0} \geq c(y)/n > \varphi_{i_0}(y)$. Therefore, $u_{i_0}(y, t_{i_0}) < u_{i_0}(y, \varphi_{i_0}(y)) = u_{i_0}(0, 0) = 0$

It follows that there is $M \in \mathbb{R}$ such that $\|y\| \leq M$ whenever $(y; t) \in A(c)$. In addition, since $c(y) = \sum_{i=1}^n t_i \geq 0$ and $t_i \leq \varphi_i(y)$ for each $i = 1, \dots, n$, we conclude that the set $A(c)$ is bounded and has compact closure. Note that the set of feasible and individually rational allocations is a subset of $A(c)$.

Now we can prove the “only if” part of Theorem 4.6.

Proposition 5.7 *Let R be a cost monotonic, Pareto efficient and individually rational mechanism. Then,*

- (i) *For any function $c \in E$, $R(c) \in \text{EE}(c)$.*
- (ii) *For every technology c and $\alpha \in \text{EL}(c)$ we have that $u(\alpha, 0) = u(R(c))$, i.e. the map $u(\cdot, 0)$ is constant on $\text{EL}(c)$.*
- (iii) *The utility profile $u = (u_1, \dots, u_n)$ satisfies the equal ordering property.*

Proof

(i) Fix a technology $c \in E$ and suppose $R(c) = (x; t)$. Let $z \in \mathbb{R}_{++}^m$ be bundle of public goods which is strictly positive. We will prove that $u(R(c)) = u(\lambda_0 z; 0)$ for some $\lambda_0 \in \mathbb{R}_+$.

Given $\lambda \in \mathbb{R}_+$ we may apply Lemma 5.4 with $y = \lambda z$ to construct $c_{\lambda z} \in E_1$. Clearly, for each $\varepsilon \in (0, 1]$, the technologies

$$c_{\lambda z}^\varepsilon = \varepsilon c + (1 - \varepsilon) c_{\lambda z}$$

belong to E .

The set $A(c_0)$, defined by equation 5.3, with $c_0(\theta) = \min\{c(\theta), c_{\lambda z}(\theta)\}$ is a bounded subset of $X \times Y$. The set of feasible and individually rational allocations of all the economies $c_{\lambda z}^\varepsilon$ with $\varepsilon \in [0, 1]$, being a subset of $A(c_0)$, is also bounded. By a compactness argument, there is a sequence $\{\varepsilon_k\}_{k=1}^\infty$ contained in $(0, 1)$, converging to 0 and such that the sequence $\{R(c_{\lambda z}^{\varepsilon_k})\}_{k=1}^\infty$ converges to a feasible allocation, say $(\bar{z}; \bar{l})$, in the economy $c_{\lambda z}$.

Since R is Pareto efficient and individually rational, so is $(\bar{z}; \bar{l})$. By Lemma 5.4 we have that

$$\lim_{k \rightarrow \infty} u(R(c_{\lambda z}^{\varepsilon_k})) = u(\bar{z}; \bar{l}) = u(\lambda z; 0).$$

For each $k \in \mathbb{N}$, we may apply now Lemma 4.3 to obtain that either $u(R(c)) \geq u(R(c_{\lambda z}^{\varepsilon_k}))$ or else $u(R(c)) \leq u(R(c_{\lambda z}^{\varepsilon_k}))$. By a limiting argument we conclude that for each $\lambda \in \mathbb{R}_+$

$$\text{either } u(R(c)) \geq u(\lambda z; 0) \text{ or } u(R(c)) \leq u(\lambda z; 0). \quad (5.4)$$

Let λ be large enough so that $\lambda z \geq x$. Since $0 \leq c(x) \leq \sum_{i=1}^n t_i$, there must be some agent, say $i_0 \in N$, such that $t_{i_0} \geq 0$. Therefore, $u_{i_0}(\lambda z, 0) \geq u_{i_0}(x, t_{i_0})$ and hence $u(\lambda z; 0) \geq u(R(c))$. Observe also that $u(0; 0) \leq u(R(c))$. Let

$$\lambda_0 = \inf\{\lambda : u_i(R(c)) \leq u_i(\lambda z, 0) \text{ for all } i = 1, \dots, n\}$$

By continuity, $u_i(R(c)) \leq u_i(\lambda_0 z, 0)$, for each $i = 1, \dots, n$. Suppose some inequality is strict, say

$$u_1(R(c)) < u_1(\lambda_0 z, 0).$$

Again, by continuity, there is $\lambda' < \lambda_0$ close enough to λ_0 such that we still have $u_1(R(c)) < u_1(\lambda' z, 0)$. On the other hand, recalling the definition of infimum, there must be some index, say $i = 2$, such that

$$u_2(R(c)) > u_2(\lambda' z, 0).$$

The last two equations contradict equation 5.4. Therefore, $u_i(R(c)) = u_i(\lambda_0 z, 0)$ for all $i = 1, \dots, n$.

(ii) Let now $z \in \text{EL}(c)$ and $\lambda \in \mathbb{R}_+$; by taking $y = \lambda z$ in Lemma 5.4, we may construct $c_{\lambda z}$ as in part (i) above and we may find $\lambda \in \mathbb{R}_+$ such that

$$u(R(c)) = u(\lambda z; 0)$$

Since $z \in \text{EL}(c)$ and $R(c)$ is feasible, then $u(z; 0) \geq u(R(c))$. But $R(c)$ is Pareto optimal, so $u(z; 0) = u(R(c))$. This proves (ii) for bundles of public goods in $\text{EL}(c)$. A simple continuity arguments can be used to extend the result to all bundles of public goods in $\text{EL}(c)$.

(iii) Let $i \in N$ be an agent, and let $y, z \in \mathbb{R}_{++}^m$ be two bundles of public goods. Suppose that $u_i(y, 0) > u_i(z, 0)$. Construct, as above, $c_y^{f_k}, c_z^{f_k}$ such that $\lim_{k \rightarrow \infty} u(R(c_y^{f_k})) = u(y; 0)$ and $\lim_{k \rightarrow \infty} u(R(c_z^{f_k})) = u(z; 0)$.

By Lemma 4.3, for each $k \in \mathbb{N}$, either $u(R(c_y^{f_k})) \geq u(R(c_z^{f_k}))$ or $u(R(c_z^{f_k})) \leq u(R(c_y^{f_k}))$. By assumption,

$$\lim_{k \rightarrow \infty} u_i(R(c_y^{f_k})) = u_i(y, 0) > u_i(z, 0) = \lim_{k \rightarrow \infty} u_i(R(c_z^{f_k})).$$

Thus, for large enough k , $u(R(c_y^{f_k})) \geq u(R(c_z^{f_k}))$ and, taking limits we obtain

$$u(y; 0) \geq u(z; 0).$$

Hence, the equal ordering property holds for $y, z \in \mathbb{R}_{++}^m$. A simple continuity argument extends this property to $y, z \in X = \mathbb{R}_+^m$.

□

To finish we prove the converse of Theorem 4.6. The equal ordering property is a sufficient condition for the existence of cost monotonic mechanisms. This property by itself also guarantees that the egalitarian equivalent allocations are in the Core of the economy.

Proposition 5.8 *Suppose the equal ordering property holds. Then,*

- (i) *Any egalitarian equivalent mechanism is cost monotonic.*
- (ii) $EE(c) \subset \text{Core}(c)$.

Proof

(i) Let R be an egalitarian equivalent mechanism and let $c_1 \leq c_2$ be two technologies in E . For each $i = 1, 2$, let $R(c_i) = (y^i; x^i) \in EE(c_i)$ with $u(z^i; 0) = u(y^i; x^i)$ for some $z^i \in X$.

By the equal ordering property, either $u(R(c_1)) = u(z^1; 0) \geq u(R(c_2)) = u(z^2; 0)$ or else $u(R(c_1)) \leq u(R(c_2))$. But, $c_1(y^2) \leq c_2(y^2) = x^2(N)$, so the allocation $R(c_2)$ is feasible in the economy c_1 . Since $R(c_1)$ is Pareto optimal in c_1 , we cannot have that $u(R(c_1)) < u(R(c_2))$. Therefore, $u(R(c_1)) \geq u(R(c_2))$.

(ii) Let $x \in EL(c)$ and $(y; t) \in EE(c)$ so that $u(y; t) = u(x; 0)$. Suppose that there is a nonempty coalition $S \subset N$ and an allocation $(z; r') \in X \times Y^S$ such that $c(z) = r'(S)$ and $u_i(z, r'_i) \geq u_i(x, 0)$ for $i \in S$ with some inequality being strict. Consider the allocation $(z; r) \in X \times Y$, where $r_i = 0$ if $i \in N \setminus S$ and $r_i = r'_i$ if $i \in S$. Then, $c(z) = r(N)$, so $(z; r)$ is feasible for N .

By the equal ordering property, either $u(z; 0) \geq u(x; 0)$ or else $u(z; 0) \leq u(x; 0)$. However, $u(z; 0) \geq u(x; 0)$ is not possible, because we would have that $u(z; r) > u(x; 0)$ contradicting that the egalitarian equivalent allocations are Pareto optimal.

Hence, we must have that $u(z; 0) \leq u(x; 0)$. Then, for each $i \in S$ we have the inequalities $u_i(z; 0) \leq u_i(x, 0) \leq u_i(z, r_i)$. Therefore, $r_i \leq 0$ for every $i \in S$. But $0 \leq c(z) = r(S)$, so $r_i = 0$ for every $i \in S$. Hence, $c(z) = 0$, $z = 0$ and $u_i(0, 0) = u_i(z, r_i) \geq u_i(x, 0) \geq u_i(0, 0)$ for $i \in S$ with some inequality strict, which is a contradiction. \square

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