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STABILITY IN ONE-SIDED MATCHING MARKETS

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Abstract

The stable roommates problem may be unsolvable for some instances, therefore we study a relaxation, when it is allowed to form groups of any size (the stable partition problem). Two extensions of preferences over individuals to preferences over sets are suggested. For the first one, derived from the most prefered member of a set, it is shown that a stable partition always existis if the original preferences are strict and a simple algorithm for its computation is derived. This algorithm turns out to be strategy proof. The second extension, based on the least prefered member of a set, produces solutions very similar to those for the stable roommates problem.

Keywords: Matching markets, stable partition, digraphs, algorithms.

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1. Introduction

The problem of two-sided matchings is very well studied in the game theoretical literature. Its generalized version, with only one kind of agents (the stable roommates problem) is in some sense not so elegant, since there are instances where no stable matching exists. One possibility for avoiding this difficulty is to restrict the set of considered preferences, as was studied by Alcalde [1]. Another approach was suggested by Romero-Medina [3] in the sense that not only pairs, but also groups of different cardinality are allowed. Since the set of agents is now *partitioned* into several disjoint subsets, we shall also use the term *stable partition* instead of 'matching'.

There are several practical situations, that could be modelled as the stable partition problem. Suppose e.g. that a group of graduates is contemplating to start their own new firms, towns plan to join their efforts in establishing a certain kind of services, or, in international politics, countries are forming military coalitions, etc. Stable partitions are similar to coalition formation games, but compared to games given by their characteristic function, our approach is more suitable when it is difficult to evaluate the worth of a coalition.

At the beginning, each agent has preferences over other agents. We consider two possibilities of extending the preferences over individuals to preferences over groups of agents. In the first case, an agent decides according to the most prefered member of a set and to the size of the set. We show that for such preferences a stable partition exists for any instance of the problem, if all the preferences are strict, and propose a simple mechanism for finding a stable partition. Moreover, this mechanism turns out to be strategy proof. In the second case, the preferences over sets are derived from preferences over the worst member of a set. We show that again, if all the preferences are strict, the problem is very similar to the stable roommates problem.

2. Definitions

$N = \{1, 2, ..., n\}$ is the set of *agents*. The *preferences* of agent i over the set N are denoted by $P(i)$. Notation $j \prec_i k$ means that i prefers k over j, $j \sim_i k$ means that i is indifferent between j and k and by $j \preccurlyeq_i k$ we mean that i prefers k over j or is indifferent between them. The same symbols will later be used to denote preferences over sets. A *preference profile* $(P(1), P(2), ...P(n))$ will be denoted by $\mathcal{P}$. We shall consider *partitions* of the set N, like $\mathcal{M} = (M_1, M_2, \ldots, M_r)$, where M_p and M_q are disjoint if $p \neq q$ and the union of $M_1, \ldots, M_r$ is equal to N. The set in this partition, containing i, will be denoted by $M(i)$.

In writing the preference orderings of the agents we shall understand that the agents that are written earlier in the list $P(i)$ of i are prefered by i over those that appear later in this list. If the agent i is indifferent between some agents, they appear in brackets. For technical reasons, in Section 3, each preference list $P(i)$ will end with agent i herself. Agents that do not appear in the preference list of i are for her *unacceptable agents.*

There are several possible ways of extending the preferences of agents over individuals to preferences over groups. We shall consider two possibilities, for which we need the following notation. Let $i \in N, M \subseteq N$. $\mathcal{B}_i(M)$ is any agent $j \in M$, such that $j \succcurlyeq_i k$ for all $k \in M$. On the other hand, $\mathcal{W}_i(M)$ is any agent $j \in M$, such that $j \preccurlyeq_i k$ for all $k \in M, k \neq i$. ($\mathcal{B}$ standing for 'best' and $\mathcal{W}$ for 'worst', respectively.)

Definition 1. A set S is *$\mathcal{B}$-prefered* by i over a set T (written $S \succ_{\mathcal{B},i} T$) if

(1) $\mathcal{B}_i(S) \succ_i \mathcal{B}_i(T)$ or

(2) $\mathcal{B}_i(S) \sim_i \mathcal{B}_i(T)$ and $|S| < |T|$.

$\{i\}$ is $\mathcal{B}$-prefered by i over sets containing only unacceptable agents and is $\mathcal{B}$-prefered less than sets containing at least one acceptable agent.

Definition 2. A set S is *$\mathcal{W}$-prefered* by i over a set T (written $S \succ_{\mathcal{W},i} T$) if $\mathcal{W}_i(S) \succ_i \mathcal{W}_i(T)$. $\{i\}$ is $\mathcal{W}$-prefered by i over sets containing at least one unacceptable agent and is $\mathcal{W}$-prefered less than sets containing only acceptable agents.

Definition 1, considered by Romero-Medina [3], represents in some sense an 'optimistic' approach, when an agent looks only at her most prefered member of a set and does not care about the rest explicitly, only through the size of the set. Definition 2 is on the other hand a 'pesimistic' approach, representing agents who are trying to avoid people whom they do not like. We say that a partition $\mathcal{M}$ is *individually rational*, if $M(i) \succcurlyeq_i \{i\}$ for every i.

A partition $\mathcal{M}$ is *$\mathcal{B}$-stable*, if there exists no set Z of agents, such that for each $i \in Z$ it holds $Z \succcurlyeq_{\mathcal{B},i} M(i)$ and for at least one $j \in Z$ we have $Z \succ_{\mathcal{B},j} M(j)$. In this case we say that Z *$\mathcal{B}$-blocks* $\mathcal{M}$. Similarly, *$\mathcal{W}$-stable* partitions and *$\mathcal{W}$-blocking* is defined.

Let us recall that in the stable roommates problem only blocking by pairs is considered, namely a pair $Z = \{i, j\}$ blocks a solution (i.e. partition into pairs) if $Z \succ_i M(i)$ and $Z \succ_j M(j)$.

Example 1. (This example is taken from [3].) Let n=10 and the agents have the following preferences:

P(1) =10,2,3,4,7,1

P(2) =4,3,1,8,2

P(3) =2,1,5,9,3

P(4) =1,9,2,6,4

P(5) =3,6,7,5

P(6) =4,7,5,6

P(7) =1,5,6,7

P(8) =2,5,6,8

P(9) =5,6,3,10,4,9

P(10) =7,5,9,1,10

Let us consider the following partitions:

$\mathcal{M}_1$ ={1,4},{2,3},{5,6,7},{8},{9,10}

$\mathcal{M}_2$ ={1,3},{2,4},{5,6,7},{8},{9,10}

$\mathcal{M}_3$ ={1,10},{2,4},{5,6,7},{8},{9,3}

$\mathcal{M}_4$ ={1,7,10},{2,3,4,5,9},{6},{8}

It can be easily checked that $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are stable against blocking by pairs of agents with respect to Definition 1, i.e. there is no pair $\{i,j\}$ such that $j \succ_i \mathcal{B}_i(M(i))$ and $i \succ_j \mathcal{B}_j(M(j))$. However, they are not $\mathcal{B}$-stable. The first two of them are namely $\mathcal{B}$-blocked by the set {1,7,10}, because in this set each agent has her most prefered choice unlike in the above partitions. That means, that 1 prefers 10 to 4 and 3 she is grouped with in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, 7 prefers

1 to both 5 and 6 who are her partners in $\mathcal{M}_1$ and $\mathcal{M}_2$ and 10 prefers 7 to 9. A similar analysis shows that $\mathcal{M}_3$ is blocked by {9,4,6}. On the other hand, $\mathcal{M}_4$ is $\mathcal{B}$-stable. However, if we use Definition 2, $\mathcal{M}_4$ is not even individually rational, since in the set {1,7,10} agent 7 has an unacceptable partner, namely agent 10. $\mathcal{M}_4$ is $\mathcal{W}$-blocked by the pair {1,7}, because 1 is now indifferent between {1,7,10} and {1,7} and $\{1,7\} \succ_7 \{1,7,10\}$. If we look at $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, we can see that all of them are $\mathcal{W}$-blocked by $\{5,7\}$, since $\mathcal{W}_5(\{5,6,7\}) = 7 \sim_5 \mathcal{W}_5(\{5,7\})$ and $\mathcal{W}_7(\{5,6,7\}) = 6 \prec_7 \mathcal{W}_7(\{5,7\})$.

Later we shall see that there is no $\mathcal{W}$-stable partition for this instance of the stable partition problem.

In Section 3 we show that there exists a $\mathcal{B}$-stable partition for every instance of the stable partition problem if all the preferences are strict and propose a simple mechanism that finds such a partition. In section 4 it is shown that the $\mathcal{W}$-stable partition problem is closely related to the stable roommates problem.

In what follows, we shall use some notions and facts from graph theory.

A *digraph* is a pair $G = (V, E)$, where V is the set of vertices and E is the set of arcs, i.e. ordered pairs of vertices. If $e = (i, j) \in E$, we say that j is the head of e and i its tail; both vertices are said to be *incident* to e. Cardinality of the set $\{j; (i, j) \in E\}$ is called the *outdegree* of vertex i. A sequence $(x_0, x_1, \ldots, x_r)$ is a path, if $(x_0, x_1), (x_1, x_2), \ldots, (x_{r-1}, x_r)$ are arcs in G; a path with $x_0 = x_k$ is a *cycle*. A cycle consisting of the single arc (i, i) is a *loop*. A path leading to a cycle C is called a *tail* of this cycle, the vertices on the cycle C and on all its tails are said to be the *attraction set* of C.

3. Properties of $\mathcal{B}$-stable partitions.

Let us define the *first preferences digraph* (FPD for short) $G(\mathcal{P}) = (V, E)$ for the preference profile $\mathcal{P}$ by $V = N$ and $(i, j) \in E$ if and only if $j = \mathcal{B}_i(N)$. If all the preferences are strict, then in the first preferences digraph all the vertices have outdegree 1. Such a digraph is then a collection of disjoint attraction sets of several cycles.

Theorem 1. *For every preference profile, if all the preferences are strict, there exists a $\mathcal{B}$-stable partition.*

Proof. We shall proceed by induction on the number of agents. The first preferences digraph of $\mathcal{P}$ contains at least one cycle $\mathcal{C} = (i_0, i_1, \ldots, i_{r-1})$. Let $\mathcal{M}$ be any partition of N such that $\mathcal{C} \in \mathcal{M}$ and suppose that Z is a blocking set such that $\mathcal{C} \cap Z \neq \emptyset$. Take an arbitrary $i_k \in Z \cap \mathcal{C}$. Then $Z \succeq_{\mathcal{B}, i_k} \mathcal{C}$, which is only possible when $i_{k+1} \in \mathcal{C}$ (indices are understood modulo r, when necessary). This implies that then $\mathcal{C} \subseteq Z$. However, when $\mathcal{C}$ is a proper subset of Z, then members of $\mathcal{C}$ strictly prefer $\mathcal{C}$ to Z, hence we obtain that $\mathcal{C} \cap Z \neq \emptyset$ implies $\mathcal{C} = Z$, which is a contradiction. Hence the members of $\mathcal{C} \in \mathcal{M}$ cannot belong to any set $\mathcal{B}$-blocking $\mathcal{M}$.

That means that now it is possible to reduce the original problem $\mathcal{P}$ to the problem $\mathcal{P}'$, which is obtained when all the members of $\mathcal{C}$ are deleted from N as well as from the preference lists of the remaining agents. Since the number of agents in $\mathcal{P}'$ is smaller than the number of agents in $\mathcal{P}$, the induction hypothesis applies and for $\mathcal{P}'$ there exists a $\mathcal{B}$-stable partition $\mathcal{M}'$. When $\mathcal{C}$ is added as another set to $\mathcal{M}'$, a $\mathcal{B}$-stable partition for $\mathcal{P}$ arises. □

The previous assertion leads to the following algorithm. In this algorithm, each cycle of execution of Step 2 to Step 4 is called a *round*, the variable r counts the number of rounds. The partition that is obtained is denoted by $\mathcal{M} = (M_1, M_2, \ldots, M_r)$.

Algorithm BSTABLE.

Step 1. Set $r := 0, V := N$.

Step 2. If $V = \emptyset$, end. Else choose an agent i_0 randomly from V.

Step 3. i_0 proposes to the first agent in her preference list, i_1. i_1 proposes to the first agent in her preference list, i_2 etc, until an agent in this chain proposes to somebody, who has already made a proposal in this round. That means, that among proposing agents a cycle C has emerged.

Step 4. Set $r := r + 1, M_r := C, V := V - C$ and from all the preference lists of agents in $V - C$ omit the agents from C. Go to Step 2.

Theorem 2. *Algorithm BSTABLE generates the same $\mathcal{B}$-stable partition independently from the random choices in Step 2 for every preference profile $\mathcal{P}$, if all the preferences are strict.*

Proof. We shall show that the partition generated by Algorithm BSTABLE (let us call it the FPD-partition) has the following structure: First, it contains all the cycles of $G(\mathcal{P})$. To see this, it is enough to realize that if C is a cycle in $G(\mathcal{P})$, then it will not be destroyed when the members of cycles reached by a particular execution of Algorithm BSTABLE before C, are deleted, since the attraction sets in the FPD are disjoint, when all the preferences are strict. Hence as soon as an agent in the attraction set of C is chosen to make the first proposal in some round, C will be reached. Moreover, C may also be reached from some vertex, that was in the attraction set of another cycle in $G(\mathcal{P})$ (but not directly on the cycle itself), and is in the attraction set of C after some rounds, when further entries in the preference lists are used in construction of FPD. Hence, all the agents lying on cycles in $G(\mathcal{P})$ can be deleted, and a reduced problem $\mathcal{P}'$ is obtained. Again the first preferences digraph $G(\mathcal{P}')$ is created and the partition contains all the cycles in it etc.

Since the structure of the partition generated by Algorithm BSTABLE is uniquelly determined by the structure of preferences, the result of Algorithm BSTABLE is independent from the order in which the proposals are made.

We still have to show that the FPD-partition is $\mathcal{B}$-stable. But the generated partition is the one defined in the proof of Theorem 1, where it was shown it is stable. □

Example 1 (continued). The first preferences digraph for this problem is:

$$\begin{array}{ccccccccc} 9 \rightarrow & 5 \rightarrow & 3 \rightarrow & 2 \rightarrow & 4 \rightarrow & 1 \rightarrow & 10 \rightarrow & 7 \rightarrow & 1 \\ & & & \uparrow & \uparrow & & & & \\ & & & 8 & 6 & & & & \end{array}$$

Cycle (1,10,7) is deleted and the new FPD is

$$\begin{array}{cccccc} 9 \rightarrow & 5 \rightarrow & 3 \rightarrow & 2 \rightarrow & 4 \rightarrow & 9 \\ & & & \uparrow & \uparrow & \\ & & & 8 & 6 & \end{array}$$

Now cycle (9,5,3,2,4) is deleted and only two vertices, 8 and 6 are left, each having a loop in the new defined FPD. In this way, $\mathcal{M}_4$ is obtained.

Example 2. Let us illustrate the algorithm by one more example. The preference profile is:

P(1) =2,3,15,10,9,1;

P(2) =3,1,4,11,8,5,2;

P(3) =4,2,16,1,7,6,8,9,3;

P(4) =1,3,9,10,5,8,4;

P(5) =1,9,10,15,14,13,2,5;

P(6) =3,7,1,15,4,5,11,6;

P(7) =2,5,6,16,10,8,3,1,7;

P(8) =7,10,11,16,15,4,3,5,8;

P(9) =3,8,1,2,6,7,9;

P(10) =11,1,2,3,4,5,10;

P(11) =12,1,3,4,2,16,11;

P(12) =13,10,11,15,16,12;

P(13) =10,5,4,3,12,11,13;

P(14) =15,16,13,14;

P(15) =13,16,2,6,9,10,15;

P(16) =15,13,10,8,9,16.

The FPD is in the following picture:

$$\begin{array}{ccccccccc} 5 & \rightarrow & 1 & \rightarrow & 2 & \leftarrow & 7 & \leftarrow & 8 \\ & & \uparrow & & \downarrow & & & & \\ & & 4 & \leftarrow & 3 & \leftarrow & 6 & & \\ & & & & \uparrow & & & & \\ & & & & 9 & & & & \end{array} \qquad \begin{array}{ccccccc} 10 & \leftarrow & 13 & \leftarrow & 15 & \leftarrow & 14 \\ \downarrow & & \uparrow & & \uparrow & & \\ 11 & \rightarrow & 12 & & 16 & & \end{array}$$

Here are two cycles, (1,2,3,4) and (10,11,12,13). After they are deleted, we get the following FPD:

$$\begin{array}{ccc} 7 & \leftarrow & 8 \\ \downarrow & & \uparrow \\ 5 & \rightarrow & 9 \end{array} \qquad 16 \leftrightarrows 15 \leftarrow 14$$

We get two new cycles, (7,5,9,8) and (15,16). In the end 14 is left proposing to herself, so the resulting partition is

$$\mathcal{M} = \{\{1,2,3,4\},\{10,11,12,13\},\{7,5,9,8\},\{15,16\},\{14\}\}.$$

If the preferences are not strict, however, the above result does no longer hold. It may happen that the problem has no solution, or there may be several solutions, as is illustrated by the following examples.

Example 3. Suppose we are given the following preference profile.

P(1) =(2,4),1

P(2) =3,2

P(3) =1,2,3

P(4) =5,4

P(5) =1,4,5

Consider two partitions, $\mathcal{M}_1 = \{\{1,2,3\},\{4,5\}\}$ and $\mathcal{M}_2 = \{\{1,4,5\},\{2,3\}\}$. In the proof of $\mathcal{B}$-stability of $\mathcal{M}_1$ we show that there is no agent i and a set $Z \subseteq N$ such that $Z \succ_i M(i)$ and $Z \succcurlyeq_j M(j)$ for all the agents $j \in Z$.

First realize that 4 strictly prefers {4,5} to every other set. Hence for 1, who now cannot count with 4, the only set remains, prefered before {1,2,3}, namely {1,2}. However, 2 would not support 1 in her effort, since she prefers {1,2,3}. 2 would be happier with {2,3}, which is in turn not prefered by 3 over {1,2,3}. 3 wants to be with 1 alone, but 1 would not accept this. 5 prefers sets containing 1, but 1 does not like it. Hence $\mathcal{M}_1$ is $\mathcal{B}$-stable, because no agent can find a partner to join him in breaking the present partition. Similarly it can be shown that $\mathcal{M}_2$ is $\mathcal{B}$-stable too.

Example 4. Suppose we are given the following preference profile.

P(1) =(5,3),4,1

P(2) =3,(1,4),2

P(3) =1,5,3

P(4) =(2,5),3,4

P(5) =2,4,5

For agent 3 the most prefered set is $\{1,3\}$. For agent 1, there are two most prefered sets, $\{1,3\}$ and $\{1,5\}$. However, $\{1,5\}$ is not $\mathcal{B}$-stable, because $\{5\}$ blocks it. Hence any $\mathcal{B}$-stable partition must contain the set $\{1,3\}$. Altogether there are three possibilities, taking into account that 5 is unacceptable for 2:

$\mathcal{M}_1$ ={1,3},{4,5},{2}

$\mathcal{M}_2$ ={1,3},{2,4},{5}

$\mathcal{M}_3$ ={1,3},{2,4,5}

However, $\mathcal{M}_3$ is blocked by {2,4}, $\mathcal{M}_2$ is blocked by {4,5} and $\mathcal{M}_1$ is blocked by {2,4}.

4. Strategic issues

A partition algorithm may be considered as a function ϕ assigning to each preference profile $\mathcal{P}$ a stable partition $\mathcal{M}$. Could an agent i obtain a more prefered result, if she submits to the partition algorithm a preference list $P'(i)$ different from her true preference list $P(i)$? To be able to formulate this question more formally, let $\mathcal{P}' = (P(1), P(2), \ldots, P'(i), \ldots, P(n))$ and let $\phi(\mathcal{P}) = \mathcal{M}$ and $\phi(\mathcal{P}') = \mathcal{M}'$. We say that a partition algorithm is *manipulable* by an agent i if $M'(i) \succ_i M(i)$, where $\succ_i$ is taken with respect to the true preference list $P(i)$. A partition algorithm is *strategy-proof* is it not manipulable by any agent.

Now we can state the following result:

Theorem 3. *Algorithm BSTABLE is strategy proof.*

Proof. Suppose that Algorithm BSTABLE produces partition $\mathcal{M}$ with the set $M(i) = \{i = i_0, i_1, \dots i_k\}$ when submitted the preference profile $\mathcal{P}$ and it gave $\mathcal{M}'$ with $M'(i) = \{i = j_0, j_1, \dots, j_m\}$ having as input P'. (The agents are written in $M(i)$ and $M'(i)$ in the same order in which they appear in a cycle in the corresponding first preferences digraph with respect to $\mathcal{P}$ and $\mathcal{P}'$ respectively.) Suppose that $M'(i) \succ_i M(i)$. This could have happened in two different ways.

a) $B_i(M'(i)) \succ_i B_i(M(i))$. This means however, that $j_1 \succ_i i_1$ and hence when performing BSTABLE according to original preferences, i proposed to j_1 before she could propose to i_1. Due to the way BSTABLE works, as soon as the proposals ran into a cycle, a new partition set is created. Hence BSTABLE could not have created a partition containing set $M(i)$, a contradiction.

b) $B_i(M'(i) \sim_i B_i(M(i)$, but $|M'(i)| < |M(i)|$. Since all the preferences are strict, $i_1 = j_1$ and since all the agents different form i behave in the same way under both preference profiles, BSTABLE reached the same cycle in both cases, again a contradiction. □

5. Properties of $\mathcal{W}$-stable partitions

Theorem 4. *Let all the preferences of the agents over individuals be strict. Let $\mathcal{M}$ be any partition containg a set with cardinality more than 2. Then $\mathcal{M}$ cannot be a $\mathcal{W}$-stable partition.*

Proof. Let $M \in \mathcal{M}$ be such that $|M| > 2$. Take an arbitrary $i \in M$. Then the pair $\{i, \mathcal{B}_i(M)\}$ blocks $\mathcal{M}$, since i now strictly prefers this pair to the set she has been in and $\mathcal{B}_i(M)$ is not worse than before. □

Theorem 5. *Let a preference profile $\mathcal{P}$ be given. Then all the solutions of the stable roommates problem with respect to $\mathcal{P}$ are solutions of the $\mathcal{W}$-stable partitions.*

Proof. We want to show that if $\mathcal{M}$ is a solution of the stable roommates problem with respect to $\mathcal{P}$, then $\mathcal{M}$ is also a $\mathcal{W}$-stable partition. Suppose that $\mathcal{M}$ is not $\mathcal{W}$-stable. Then there exists a $\mathcal{W}$-blocking set Z. Let us consider two cases:

a) $Z = \{i\}$. Let us denote the partner of i in the stable roommates solution as j. Then $\{i\} \succ_{\mathcal{W},i} \{i,j\}$, which means that j is unacceptable for i, a contradiction.

b) $|Z| \geq 2$. Let us take $i \in Z$ such that $Z \succ_{\mathcal{W},i} M(i)$. Denote $j = \mathcal{B}_i(Z)$ and consider $Z' = \{i,j\}$. Z' is obviously also $\mathcal{W}$-blocking. Let $\{i,k\},\{j,l\} \in \mathcal{M}$. Since Z' is $\mathcal{W}$-blocking, $Z' \succ_{\mathcal{W},i} \{i,k\}$ and $Z' \succcurlyeq_{\mathcal{W},j} \{j,l\}$. Definition 2 now implies $j \succ_i k$ and $i \succcurlyeq_j l$, which can happen in one of two ways. The case $i \sim_j l$ would mean (since the preferences are strict) $i = l$, and hence Z' is one set of the partition $\mathcal{M}$, thus it cannot be $\mathcal{W}$-blocking. If $i \succ_j l$, then Z' fulfills the definition of a blocking pair for the stable roommates problem, again we arrived at a contradiction. □

Theorem 5 does not hold conversely, as can be seen e.g. from Example 1, which is insoluble as the stable roommates problem. However, if from N the agents are omitted, that are single in a $\mathcal{W}$-stable partition, the rest form a solution to the stable roommates problem.

In the search for the $\mathcal{W}$-stable partitions, the first natural step will be to delete an agent i from $P(j)$ if j is not acceptable for i, since in this case $\{i,j\}$ will never be a stable pair. The preference profile obtained after all possible reductions of this type, is called *consistent* (and only such preference profiles are considered as inputs for the Stable roommates algorithm, SR algorithm for short, described in [2], Chapter 4.. Let us first look at our previous examples, now i is no longer needed as an entry in $P(i)$.

Example 3 (continued). The reduced preference profile is

$$P(1) = \emptyset; P(2) = 3; P(3) = 2; P(4) = 5; P(5) = 4.$$

The resulting unique $\mathcal{W}$-stable partition is $\mathcal{M} = \{\{4,5\},\{2,3\},\{1\}\}$.

Example 4 (continued). The reduced preference list is

$$P(1) = 3; P(2) = 4; P(3) = 1; P(4) = (2,5); P(5) = 4.$$

Hence $\{1,3\}$ must be in any $\mathcal{W}$-stable partition, but for the remaining agents, $\{2,4\}$ blocks any partition $\mathcal{M}$ such that $\{2,4\} \notin \mathcal{M}$ and $\{4,5\}$ behaves similarly.

Noe we describe the modifications of the SR algorithm, needed for solving the $\mathcal{W}$-stable partition problem. The SR algorithm has two phases. In both of them successively those pairs are deleted, which can never become a stable pair. The first phase deletes a pair $\{i',j\}$ for every i' such that $j \succ_i i'$ and i is the first entry in j's preference list. Phase 1 ends with a preference profile such that for each agent i, either i's preference list is empty (which means that $\{i\}$ is stable), or i is somebody's (whom she accepts) first choice. The first phase may lead already to some stable pairs. If the reduced preference list of i contains only j and conversely, $\{i,j\}$ is stable.

In the second phase the so called rotations are eliminated. After this process either somebody's preference list is empty (and hence the stable partition does not exist) or in the positive case, all the stable roommates pairs become sets of a stable partition.

The above observations are summarized in the following theorem:

Theorem 6. *If the preference list of agent i remains empty at the end of Phase 1 of the Stable roommates algorithm, then $\{i\}$ is a stable set. If however, the preference list of some agent becomes empty at the end of Phase 2, the $\mathcal{W}$-stable partition problem is unsolvable.*

In the end, let us work out the example from [3].

Example 1 (continued). The consistent preference 'subprofile' is:

P(1) =10,2,3,4,7

P(2) =4,3,1,8

P(3) =2,1,5,9

P(4) =1,9,2,6

P(5) =3,6,7

P(6) =4,7,5

P(7) =1,5,6

P(8) =2

P(9) =3,10,4

P(10) =9,1

Since 2 is the first choice of 3, the pairs {1,2} and {2,8} are deleted. It is now clear that 8 will not take part in the continuation of the game, hence she can be omitted. Further reduction will be achieved e.g. by $3 = \mathcal{B}_5(N)$, which causes the pair {3,9} to disappear. Now $9 = \mathcal{B}_{10}(N)$ and also $10 = \mathcal{B}_9(N)$, which leads to $\{9, 10\} \in \mathcal{M}$ and they are also omitted from N and form the remaining preference lists. We get

P(1) =3,4,7

P(2) =4,3

P(3) =2,1,5

P(4) =1,2,6

P(5) =3,6,7

P(6) =4,7,5

P(7) =1,5,6

After some more reductions, in the end of Phase 1 the following preference profile will be obtained:

P(1) =3,4

P(2) =4,3

P(3) =2,1

P(4) =1,2

P(5) =6,7

P(6) =7,5

P(7) =5,6

and the second phase of the stable roommates algorithm starts. We avoid here explanations of how to delete rotations and use a simple argument for this example. For agents from the set {5,6,7} only the members of this set are acceptable. However, since each of them is somebody's first choice, a partition containing a singleton will not be stable, neither a partition containing all three of them. So we see that there is no $\mathcal{W}$-stable partition for the example given by Romero Medina.

5. Open problems and directions for further research

In spite of the positive result about the existence of a $\mathcal{B}$-stable partition, the question of how to describe all the $\mathcal{B}$-stable partitions remains open. In fact, the uniqueness is not ensured even in the case of strict preferences. Consider the following example: Let the first choice of each agent i from the set $\{1, 2, \ldots, n\}$ be the agent $i+1$ and her second choice $i-1$ (modulo n). Then in addition to the grand coalition also the partition consisting of pairs $\{i, i+1\}$ is $\mathcal{B}$-stable, if n is even. The search for stable partitions is complicated by the fact, that at the moment it is not clear how to check a given partition for stability efficiently. Since every set of agents can now be blocking, the brute force approach would require an exponential number of operations. Also the question whether a $\mathcal{B}$-stable partition in the case of indiferences exists, has not been answered; the authors suspects that it may be NP-complete.

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