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## NONREPRESENTATIVE REPRESENTATIVE CONSUMERS

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### Abstract

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Representative consumers can be very Pareto inconsistent. We describe a community, with equal income distribution, where all consumers require 56% higher aggregate income than the representative consumer requires in order to be compensated for the doubling of a price. Such large inconsistencies are ruled out if the representative consumer is homothetic, or if the consumers' income shares are fixed and all goods are normal. We show that optimality of the income distribution rule is not necessary for Pareto consistency of the representative consumer, and we give a weaker sufficient condition for Pareto consistency in communities with two goods and two consumers.

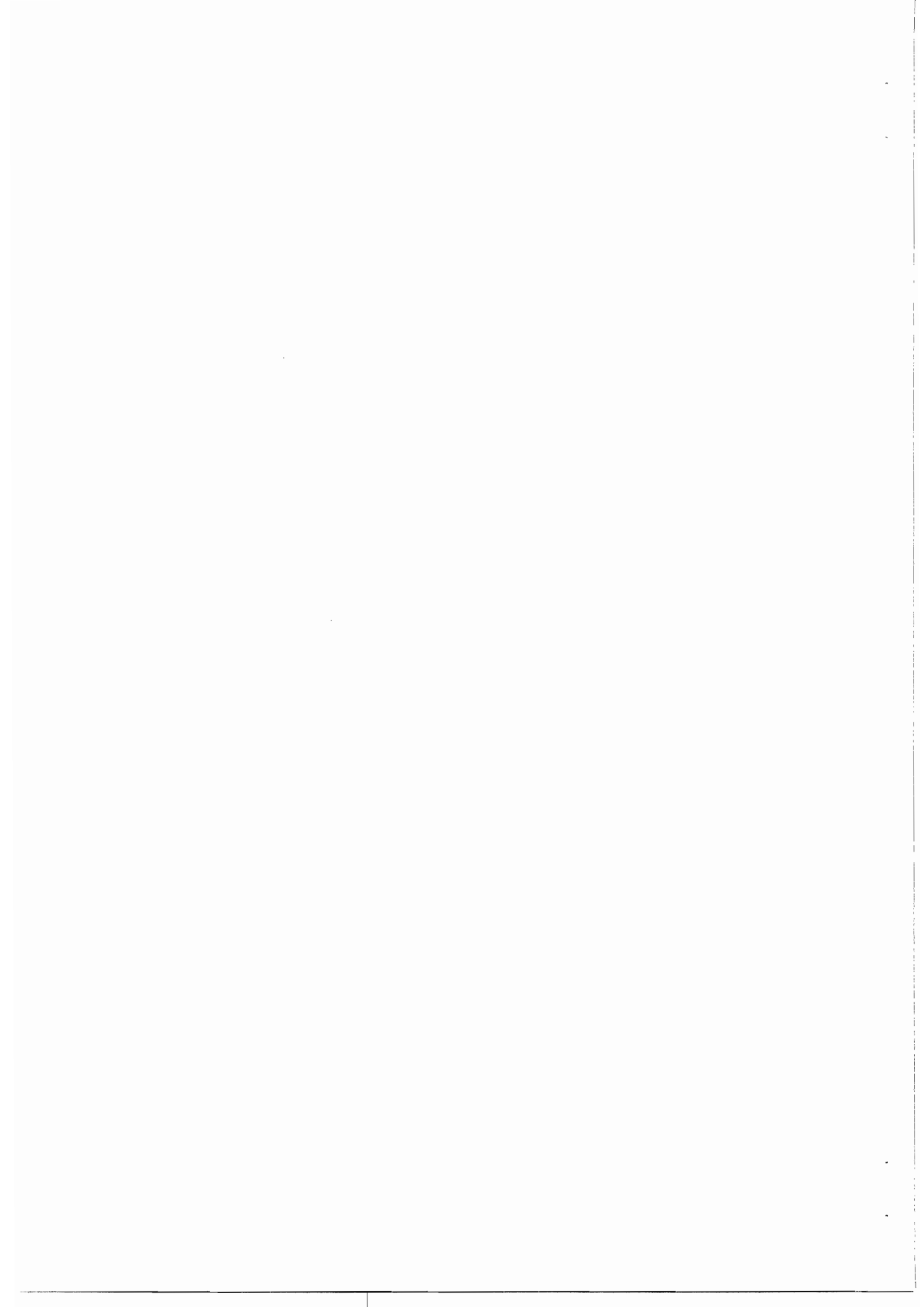
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## 1. Introduction

A great deal of economic analysis treats aggregate community demand as if it were the demand of a single competitive “representative” consumer. Representative consumer models allow analysts to focus on economic efficiency, leaving equity considerations aside. However, the aggregation across consumers that is implicit in these models is often problematic. Aggregate community demand might violate revealed preference axioms that would be satisfied if there were just one consumer. As a result, representative consumer models could misrepresent the effects of changes in endowments, technology or policy on prices and aggregate consumption. But this is not the only problem. Even when aggregate community demand satisfies the strong axiom and therefore is indistinguishable from the demand of a single competitive consumer, the single-consumer model might not be adequate for evaluating efficiency. The representative consumer can be Pareto inconsistent, preferring an aggregate situation A to B even though all the actual consumers in the community prefer B to A, cf. Jerison (1984) and Dow and Werlang (1988).

Should representative consumers be banished from economic analysis because they might be Pareto inconsistent? That would be going too far. The inconsistencies might be very small or might arise only in unrealistic settings. In Jerison’s (1984) example of inconsistency, a payment of less than 0.5% of aggregate income is enough to bring the representative consumer into agreement with the actual consumers in the community. Dow and Werlang (1988) give an example of a larger Pareto inconsistency, but it is not robust. It requires a special discontinuity in the way that the consumers’ incomes vary depending on prices and aggregate income.

This paper examines the sizes of possible Pareto inconsistencies and the conditions under which they can occur. Since the goal is to study models that are used to analyze changes in endowments, technology and policy, we must allow aggregate income to vary independently of prices. At the same time, we restrict attention to representative consumers who are perfectly representative in the positive sense. These “positive” representative consumers demand the aggregate demand vector no matter what aggregate income and prices prevail. The reasons for restricting attention to positive representative consumers are discussed below; for now, we note that any Pareto inconsistencies exhibited by positive representative consumers could only be amplified if the representative consumers’ demands were allowed to differ from the aggregate demand.

The adequacy of a representative consumer model depends on which policies or events and which communities are to be analyzed. If all the actual consumers are identical then there is a positive representative consumer model that makes accurate predictions and welfare judgments for all possible policies and events. But this is certainly a limited case, considering varied consumer behavior we observe. In order to apply the conclusions from representative

consumer models to broader classes of communities, it is necessary to restrict the range of possible policies and events.

Positive representative consumers exist only for extremely implausible communities if there is no restriction on the distribution of income. Existence of a positive representative consumer requires aggregate demand to be determined by aggregate income and prices at almost all price vectors. Therefore, redistribution of income (at fixed prices) must have no effect on aggregate demand. This implies that for each good, at all prices and income levels, all consumers' marginal propensities to consume are equal, cf. Antonelli (1886) and Gorman (1953). In order to avoid making such an unrealistic assumption, we restrict attention to events and policies that change relative prices. Such events and policies are different from pure redistributions of income, so pure redistributions are ruled out a priori. It follows that we can treat the consumers' incomes as functions of the aggregate income and prices. Then aggregate demand is a well-defined function of aggregate income and prices without any further assumptions about consumer preferences. A (positive) representative consumer is simply a utility function that generates the aggregate demand function.

A striking example of a representative consumer that fits the framework we are considering is in Lucas (1987). He considers an infinite horizon model with a Cobb-Douglas consumer who discounts future consumption at a rate chosen to match certain features of aggregate U.S. demand. He shows that this consumer would be willing to reduce its initial consumption by 42% if it could raise its growth rate in consumption from 3% to 6%. Policies capable of generating such changes in the equilibrium consumption growth rate appear in the endogenous growth literature, cf. Stokey and Rebelo (1995). In the framework of the present paper, these policies can be represented through their effects on intertemporal prices and the consumers' income or wealth levels. One might ask if the surprisingly strong preference for growth in Lucas' example could be caused by Pareto inconsistency of the representative consumer. Could the representative consumer's preferences differ substantially from the preferences of the actual consumers in the community? We will see in section 4 that the answer is no.

The main contributions of this paper are as follows. We describe a simple way to construct robust examples of representative consumers with large Pareto inconsistencies. In one example, the actual consumers require 56% more income than the representative consumer requires in order to be compensated for the doubling of a price. But in this example there is a Giffen good for all the consumers. We show, on the other hand, that if all goods are normal and the consumers receive fixed shares of the aggregate income, then there cannot be large Pareto inconsistencies, given the amount of price variation arising in typical applications. Even when there are inferior goods, Pareto inconsistencies must be small if the representative consumer has homothetic preferences, as it has in Lucas' (1987) example and

most macroeconomic models.

Pareto consistency is necessary in order for the representative consumer's preferences to have a social welfare interpretation. Dow and Werlang (1988) show that it is sufficient as well. If the representative consumer is Pareto consistent, then its preferences coincide with preferences derived from a particular Bergson-Samuelson social welfare function. It is therefore worthwhile trying to characterize the communities with Pareto consistent representative consumers.

A well-known sufficient condition for existence of a Pareto consistent representative consumer is for income to be distributed optimally according to some social welfare function no matter what prices prevail, cf. Samuelson (1956), Chipman and Moore (1979). We show that this sufficient condition is not necessary. Optimality of the distribution rule is essentially locally equivalent to symmetry and positive semidefiniteness of a matrix of covariances of the consumers' marginal propensities to consume and their marginal utility vectors with respect to prices, cf. Jerison (1994). For a representative consumer to exist, this matrix of covariances must be symmetric, but not necessarily positive semidefinite. We give an example of a Pareto consistent representative consumer in a community in which the matrix of covariances is not positive semidefinite (hence the distribution rule is not optimal with respect to any social welfare function). We also show that when there are only two goods and two consumers, a Pareto consistent representative consumer exists if the matrix of covariances is nonzero everywhere.

We already noted that this paper restricts attention to positive representative consumers, or, more precisely, to communities for which there is a single consumer whose demand is the aggregate demand at all aggregate incomes and prices. Users of representative consumer models can always say that their analyses meant to apply only to such communities. And such communities always exist. Each competitive consumer has a class of communities for which it is a positive representative consumer. It is interesting to know in which of these communities the representative consumer can be to evaluate efficiency. Another reason for considering only positive representative consumers is that aggregate time series demand data rarely violate the strong axiom of revealed preference. Thus the limited aggregate data we have are often consistent with existence of positive representative consumers, cf. Landsburg (1981), Varian (1982).

The framework and notation are presented in the next section. Section 3 introduces the matrix of covariances and uses it to characterize economies with representative consumers when the actual consumers' incomes are determined by prices and the aggregate income. Section 4 presents examples of Pareto inconsistent representative consumers along with bounds on the inconsistencies. Section 5 examines conditions under which a Pareto consistent representative consumer exists.

## 2. Notation

We consider a group of  $m \geq 2$  competitive consumers in an  $n$ -good economy. Each consumer  $i$  has a column-vector valued demand function  $X^i(y^i, p)$  generated by a utility function  $u^i$ . The corresponding indirect utility function  $v^i(y^i, p)$  is assumed to be twice continuously differentiable with strictly positive marginal utility of income,  $\partial v^i(y^i, p)/\partial y^i > 0$ . The expenditure function of consumer  $i$  is  $e^i(u, p)$ . At price vector  $p$ , with income  $y^i$ , consumer  $i$  has the *marginal propensity to consume*  $M^i(y^i, p) \equiv \partial X^i(y^i, p)/\partial y^i$ , the *average propensity to consume*  $A^i(y^i, p) \equiv X^i(y^i, p)/y^i$  and the *Slutsky matrix*

$$S^i(y^i, p) \equiv \partial X^i(y^i, p)/\partial p + M^i(y^i, p)X^i(y^i, p)^T,$$

where the superscript  $T$  denotes the transpose.

An economic *situation* is represented by  $(y, p)$ , where  $y$  is aggregate income (the sum of the consumers' incomes) and  $p \gg 0$  is a vector of  $n$  prices.<sup>2</sup> A *mean situation* is a vector  $(z, p)$  where  $z$  represents the average of the consumers' incomes and  $p$  is a price vector. We allow aggregate income to vary independently of prices so that the model can apply to comparative static analyses of changes in endowments, technology or policy. The group members might be thought of as members of a family or as citizens of a small country so that their consumption has no effect on prices.

We will use the term *community* to mean the group of consumers along with a "distribution rule" specifying their incomes as functions of aggregate income and prices. A *distribution rule* is a continuously differentiable function  $D = (D^1, \dots, D^m) \geq 0$ , homogeneous of degree 1, satisfying  $\sum D^i(y, p) = y$  and  $D_y^i(y, p) > 0$ . Here and below, subscripts denote partial derivatives.  $D^i(y, p)$  is the income of consumer  $i$  in situation  $(y, p)$ . The *income share* of consumer  $i$  in situation  $(y, p) \gg 0$  is  $D^i(y, p)/y$ . A distribution rule need not be determined by private ownership. It can incorporate the effects of redistributive policies. Since there are no consumption externalities in our model, allocation by means of distribution rules is essentially equivalent to allocation that is Pareto efficient in every situation, cf. Bourguignon and Chiappori (1992).<sup>3</sup>

A distribution rule  $D$  determines aggregate demand as a function of aggregate income and prices:

$$X^D(y, p) = \sum X^i[D^i(y, p), p].$$

It also determines the vector of consumer utilities  $V^D(y, p)$  with  $i$ th component  $V^{D^i}(y, p) \equiv v^i[D^i(y, p), p]$ . We say that there is a *representative consumer* for  $D$  if the aggregate demand function  $X^D$  is generated by a utility function. When such a utility function exists, it determines preferences over situations and also over mean situations. We say that the

representative consumer is *Pareto consistent* if it prefers one situation to another whenever all the consumers prefer the former to the latter. Formally, the representative consumer with indirect utility function  $v^D(y, p)$  is *Pareto consistent* if  $V^D(y, p) \gg V^D(z, q)$  implies  $v^D(y, p) > v^D(z, q)$ .

The distribution rule  $D$  is *optimal for*  $w : \mathbb{R}^m \rightarrow \mathbb{R}$  if, for each  $(y, p) \gg 0$  and each vector of incomes  $(y^1, \dots, y^m) \geq 0$  satisfying  $\sum y^i \leq y$ , we have

$$w[V^D(y, p)] \geq w[v^1(y^1, p), \dots, v^m(y^m, p)]. \quad (1)$$

The distribution rule is optimal for  $w$  if in every situation there is no alternative distribution of the aggregate income that yields a higher value of  $w$ . A *social welfare function* (in the given community) is a nondecreasing real-valued function on  $\mathbb{R}^m$  that is strictly increasing on the set of attainable utility vectors  $\{V^D(y, p) | (y, p) \gg 0\}$ .<sup>4</sup> We call  $D$  *optimal* if it is optimal for *some* social welfare function. Note that a constant function cannot be a social welfare function, so distribution rules are not necessarily optimal. In fact, Jerison (1994) shows that a typical (i.e., generic) distribution rule is not optimal with respect to any social welfare function.

### 3. Existence of a Representative Consumer

As noted in the introduction, existence of a representative consumer requires the Slutsky matrix of the aggregate demand function to be symmetric and negative semidefinite. These conditions are not satisfied automatically because the Slutsky matrix of aggregate demand generally differs from the sum of the individual consumers' Slutsky matrices. The difference matrix can be interpreted as the covariance matrix of two vector valued random variables (or as part of the covariance matrix of a single vector-valued random variable). Under certain conditions the matrix can be estimated from cross section or time series data. This section defines the "covariance matrix" and shows that when it is symmetric and positive semidefinite a representative consumer exists. Symmetry of the covariance matrix is necessary for existence of a representative consumer whereas positive semidefiniteness is not. On the other hand, both conditions are necessary in order for the distribution rule to be optimal with respect to some social welfare function, cf. Jerison (1994). Section 5, below, shows that if the covariance matrix is positive definite on a hyperplane and if there are only two goods and two consumers, then there is a Pareto consistent representative consumer.

It seems then that the covariance matrix should be of special interest to any user of representative consumer models for normative analysis. For this reason we offer a number of interpretations for the above restrictions on the covariance matrix along with examples of communities in which the properties are satisfied.

The  $jk$  component of the covariance matrix will be defined to be the covariance of the consumers' marginal propensities to consume good  $j$  and their "adjusted demands" for good  $k$ . The *adjusted demand* of consumer  $i$  at situation  $(y, p)$  is

$$X^{Di}(y, p) \equiv [1/D_y^i(y, p)][X^i(D^i(y, p), p) - D_p^i(y, p)],$$

where the subscripts on  $D^i$  denote partial derivatives. When aggregate income is fixed, a consumer's adjusted demand vector is orthogonal to the consumer's indifference curve in price space (taking account of the way the consumer's income  $D^i$  is affected by price changes). Thus, the adjusted demand vector  $X^{Di}(1, p)$  is parallel to the vector of price derivatives  $\partial V^{Di}(1, p)/\partial p$ . (This is easily verified using Roy's identity.) The homogeneity of  $D$  implies that  $p \cdot X^{Di}(y, p) = y$ , so the adjusted demand vector lies in the frontier of the aggregate budget set. The adjusted demand for good  $k$  is approximately equal to the change in *aggregate* income  $y$  required to compensate consumer  $i$  for a unit change in the price of good  $k$  taking account of the effect of the price change on the consumer's income.

The *covariance matrix* is defined to be

$$C^D(y, p) \equiv \sum_i D_y^i(y, p) M^i(D^i(y, p), p) [X^{Di}(y, p) - X^D(y, p)]^T.$$

It is the covariance matrix of the consumers' marginal propensities to consume and their adjusted demands, with consumers weighted by their marginal income shares  $D_y^i(y, p)$ . (This uses the fact that the mean of the adjusted demands is the aggregate demand:  $\sum D_y^i(y, p) X^{Di}(y, p) = \sum [X^i(D^i(y, p), p) - D_p^i(y, p)] = X^D(y, p)$ .)

Under the smoothness assumptions of our model, there is a representative consumer for the distribution rule  $D$  if and only if the Slutsky matrix of aggregate demand is symmetric and negative semidefinite, cf. Richter (1979, Theorem 12). As a consequence we have

**Proposition 1.** *There is a representative consumer if the covariance matrix  $C^D(y, p)$  is symmetric and positive semidefinite at each  $(y, p)$ . Symmetry of this covariance matrix is necessary for existence of a representative consumer.*

*Proof.* Let  $y^i \equiv D^i(y, p)$ . The Slutsky matrix of aggregate demand is

$$\begin{aligned} S^D(y, p) &\equiv X_p^D(y, p) + X_y^D(y, p) X^D(y, p)^T \\ &= \sum_i [M^i(y^i, p) D_p^i(y, p) + X_p^i(y^i, p)] + \sum_i M^i(y^i, p) D_y^i(y, p) X^D(y, p)^T \\ &= \sum_i [X_p^i(y^i, p) + M^i(y^i, p) X^i(y^i, p)] \\ &\quad - \sum_i M^i(y^i, p) [X^i(y^i, p) - D_p^i(y, p)] + \sum_i D_y^i(y, p) M^i(y^i, p) X^D(y, p)^T \\ &= \sum_i S^i(y^i, p) - C^D(y, p). \end{aligned}$$



Since each consumer's Slutsky matrix  $S^i(y^i, p)$  is symmetric and negative semidefinite,  $C^D$  must be symmetric for  $S^D$  to be symmetric. If in addition  $C^D$  is positive semidefinite, then  $S^D$  is negative semidefinite. □

**Remark 1.** Requiring the covariance matrix  $C^D(y, p)$  to be symmetric and positive semidefinite does not restrict the form of the demand function of any consumer. For example if the consumers have the same arbitrary demand function and equal income shares, then the covariance restrictions are satisfied with  $C^D = 0$ . On the other hand, *symmetry of the covariance matrix is not robust*. It is lost under perturbation of the consumers' preferences when there are at least three goods, cf. Jerison (1994). Thus, in a typical economy with more than two goods there is no representative consumer.

**Remark 2.** To illustrate the covariance restrictions, consider a consumption sector in which the consumers' demands have Muellbauer's (1976) PIGLOG form,

$$X^i(y^i, p) = y^i[a^i(p) + b^i(p) \ln y^i], \quad (2)$$

where  $a^i$  and  $b^i$  are functions from  $\mathbb{R}^n$  into  $\mathbb{R}^n$ . The class (2) includes as special cases nonidentical homothetic preferences (with  $b^i = 0$ ), the AID system of Deaton and Muellbauer (1980) and demands generated by commonly used indirect translog utility functions, cf. Christensen, et. al. (1975). If the consumers' income shares are fixed, with  $D(y, p) = y\theta$ , the covariance matrix becomes  $C^D(1, p) = \sum \theta_i(b^i + c^i)(c^i)^T - \sum \theta_i(b^i + c^i) \sum \theta_i(c^i)^T$ , where  $b^i$  and  $c^i$  are evaluated at  $p$  with  $c^i(p) \equiv a^i(p) + b^i(p) \ln \theta_i$ . This is the covariance matrix of the consumers'  $b^i$  and  $c^i$  vectors. It is symmetric and positive semidefinite if the  $b^i$ 's or if the  $c^i$ 's are identical across consumers, though neither of these restrictions is necessary for symmetry or positive semidefiniteness.

**Remark 3.** If the consumers have fixed income shares  $\theta_i$ , then the covariance matrix is

$$C^D(y, p) = \sum \theta_i M^i [(X^i/\theta_i) - X^D]^T = y \sum \theta_i M^i (A^i - A^D)^T, \quad (3)$$

aggregate income times the covariance matrix of the consumers' marginal and average propensities to consume, with consumers weighted by their income shares. Thus, positive semidefiniteness of the covariance matrix with fixed income shares implies that consumers with larger than average budget shares for a good also tend to have higher marginal propensities to consume that good.

**Remark 4.** If the consumers' income shares are fixed, positive semidefiniteness of the covariance matrix is equivalent to "increasing dispersion," the requirement that the consumers' demands become more dispersed when their incomes rise by equal amounts. To make this

precise, consider a situation  $(y, p)$  and imagine a thought experiment in which all consumers are given an additional income transfer  $\Delta$ . The demand of consumer  $i$  is then  $X^i(y^i + \Delta, p)$ , where  $y^i = \theta_i y$ . Letting  $x$  be a vector of length 1, the dispersion in the consumers' demand vectors in the direction  $x$  can be measured by the variance of the scalars  $x \cdot X^i(y^i + \Delta, p)$ , with consumers weighted equally:

$$\frac{1}{m} \sum x^T X^i(y^i + \Delta, p) X^i(y^i + \Delta, p)^T x - \frac{1}{m^2} \sum x^T X^i(y^i + \Delta, p) \sum X^i(y^i + \Delta, p)^T x.$$

To find the effect of a small income transfer, we differentiate this variance with respect  $\Delta$  and evaluate at  $\Delta = 0$ . We obtain  $x^T [\tilde{C}(y, p) + \tilde{C}(y, p)^T] x$ , where

$$\tilde{C}(y, p) \equiv \frac{1}{m} \sum M^i(y^i, p) X^i(y^i, p)^T - \frac{1}{m^2} \sum M^i(y^i, p) X^D(y, p)^T.$$

We say that the community exhibits *increasing dispersion* if for each  $(y, p)$ , the matrix  $\tilde{C}(y, p)$  is positive semidefinite on the hyperplane orthogonal to the aggregate demand vector  $X^D(y, p)$ . This means that the consumers' demand vectors become more dispersed (or at least not less dispersed) in all directions orthogonal to the aggregate demand when the consumers' incomes all rise by the same small amount. A community with fixed income shares exhibits increasing dispersion if and only if its covariance matrix  $C^D(y, p)$  is positive semidefinite for each  $(y, p)$ , cf. Jerison (1994, Remark 3).

The hypothesis of increasing dispersion can be tested using the nonparametric statistical method of average derivatives. Härdle, et. al. (1991), Hildenbrand (1994), Kneip (1993) show that French, U.K. and U.S. consumer expenditure data are consistent with the hypothesis. In testing for increasing dispersion, the number of commodities matters. With very narrowly defined commodity categories there are likely to be inferior goods. If a good is bought only by poorer households then the dispersion in household demands is likely to fall when incomes rise. In spite of this, French data are consistent with increasing dispersion when commodities are grouped in 60 aggregated categories.

**Remark 5.** It is easy to verify that  $C^D(y, p)p = C^D(y, p)^T p = 0$ . In addition, the  $n \times n$  symmetrized covariance matrix  $C^D + (C^D)^T$  is the sum of  $2m$  rank 1 matrices, so it cannot have rank greater than  $\max\{2m, n - 1\}$ . When it has this maximal rank, positive semidefiniteness of the covariance matrix is a robust condition that is preserved under perturbation of the consumers' preferences.

#### 4. Nonrepresentative Representative Consumers

We have noted that representative consumers need not be Pareto consistent. But to what extent can the representative consumer's preferences differ from the preferences of the

actual members of the community? In this section we show how to construct examples of consumption sectors in which the difference is substantial. On the other hand, we show that the difference must be considerably smaller if the representative consumer has homothetic preferences. The following result is useful for constructing examples of Pareto inconsistency.

**Remark 6.** *If all the consumers are indifferent between two situations but the representative consumer is not, then the representative consumer is Pareto inconsistent.*

To see this, we note first that if a representative consumer exists, it can be assumed to have a  $C^1$  indirect utility function  $v$ . This follows from the fact that the aggregate demand function is  $C^1$ . Homogeneity of the consumers' indirect utility functions implies that without loss of generality we can let aggregate income equal 1. Suppose that all members of the consumption sector are indifferent between the price vectors  $p$  and  $q$ , but the representative consumer is not. Then  $V^D(p) = V^D(q)$  and  $v(1, p) < v(1, q)$ . This last inequality is preserved if all components of  $q$  are proportionally decreased by a small enough amount. But then every consumer prefers  $p$  to  $q$  even though the representative consumer prefers  $q$  to  $p$ .

#### 4.1 A Representative Consumer with a Large Pareto Inconsistency

We will consider consumption sectors with two goods and two consumers with equal income shares. If the consumers' preferences are homothetic, then there is a Pareto consistent representative consumer. We will start with a pair of homothetic consumers and modify their preferences in order to obtain a consumption sector with a Pareto inconsistent representative consumer. Let the price of good 2 be fixed at 1. We start with a homothetic consumer  $i$  (for  $i = 1, 2$ ) with an indirect utility function  $v^i$ , expenditure function  $e^i$  and demand function  $x^i(y^i, p)$  for good 1. The homothetic consumers 1 and 2 are indifferent between the mean situations  $A = (1, q)$  and  $B = (z, \bar{p})$ , but their indifference curves through  $A$  and  $B$  do not intersect at any other points in the space of income and the price of good 1. Figure 1 shows the consumers' indifference curves in that space. Letting  $u_i$  be the utility of consumer  $i$  in the mean situations  $A$  and  $B$ , we see that  $e^1(u_1, q) = e^2(u_2, q) = 1$  and  $e^1(u_1, p) = e^2(u_2, p) = z$ . The corresponding representative consumer is also indifferent between the mean situations  $A$  and  $B$ .

In a two good economy, a consumer's (smooth) demand function  $x(y, p)$  for good 1 completely determines the consumer's indirect preferences as long as the demand function satisfies the Slutsky condition

$$\frac{\partial x(y, p)}{\partial p_1} + x(y, p) \frac{\partial x(y, p)}{\partial y} \leq 0.$$

Using this fact, we can specify new consumer preferences by changing consumer  $i$ 's demand function for good 1 to  $\hat{x}^i(y^i, p) = \phi_i(v^i(y^i, p) - u_i)x^i(y^i, p)$  for  $i = 1, 2$ , where  $\phi_i : \mathbb{R} \rightarrow (0, 1)$  is a smooth function with  $\phi_i'(t) > 0$  for  $t < 0$  and  $\phi_i(t) = 1$  for  $t \geq 0$ . The new demand for good 1 is the same as the old demand at points above the consumer's indifference curve through  $A$  and is strictly less than the old demand at points below that indifference curve. It is easy to verify that the new demand function for good 1 satisfies the Slutsky condition if the old one does. This implies that the new demand functions  $x^i$  determine a new consumption sector.

In the income and price space of Figure 1, consumer  $i$ 's indifference curve of utility level  $u_i^*$  is the graph of the expenditure function  $e^i(u_i^*, \cdot, 1)$ . Since consumer  $i$ 's demand for good 1 at mean situation  $(y, p)$  equals  $\partial e^i(u_i^*, p)/\partial p_i$  the modification of  $i$ 's demand described above flattens  $i$ 's indifference curves below the curve through  $A$  while preserving  $i$ 's indifference between  $A$  and  $B$ . If a representative consumer exists in the new consumption sector then its indifference curve through  $A$  is flatter than that of the original representative consumer. Therefore it passes below  $B$  and so, by Remark 6, the representative consumer is Pareto inconsistent.

In order for the new consumption sector to have a representative consumer, the Slutsky matrix of aggregate demand must be symmetric and negative semidefinite. Symmetry holds automatically when there are only two goods. Slutsky negative semidefiniteness requires

$$\begin{aligned} 0 &\geq (1/2)(\phi_1' v_p^1 x^1 + \phi_2' v_p^2 x^2 + \phi_1 x_p^1 + \phi_2 x_p^2) \\ &\quad + (1/4)(\phi_1 x^1 + \phi_2 x^2)(\phi_1' v_y^1 x^1 + \phi_2' v_y^2 x^2 + \phi_1 x_y^1 + \phi_2 x_y^2) \\ &= -(1/2)[\phi_1' v_y^1 (x^1)^2 + \phi_2' v_y^2 (x^2)^2 + \phi_1 x_p^1 + \phi_2 x_p^2] \\ &\quad + (1/4)(\phi_1 x^1 + \phi_2 x^2)(\phi_1' v_y^1 x^1 + \phi_2' v_y^2 x^2 + \phi_1 x_y^1 + \phi_2 x_y^2). \end{aligned}$$

Here,  $v^i$  and  $x^i$  and their derivatives are evaluated at  $(y, p)$ , and the subscripts  $y$  and  $p$  denote partial derivatives with respect to the first and second arguments,  $y$  and  $p_1$ . Both  $\phi_i$  and  $\phi_i'$  are evaluated at  $v^i(y, p) - u_i$  for each  $i$ .

When each  $\phi_i$  is sufficiently close to 1, the Slutsky condition above is satisfied. This follows from the fact that there is a representative consumer when the actual consumers have homothetic preferences. So the argument above shows that there can be a Pareto inconsistent representative consumer. The question is how inconsistent. If each  $\phi_i$  is close to 1, then the modified consumer preferences are nearly homothetic and the representative consumer's indifference curve through  $A$  passes close to  $B$ . Thus the inconsistency is small. To obtain a large inconsistency we must make  $\phi_i(t)$  approach 0 rapidly as  $t$  decreases starting from 0. But the Slutsky condition above is violated if either  $\phi_i'$  is too large. For this reason, the representative consumer's indifference curve through  $A$  cannot be made arbitrarily flat. Still, it can be made to pass substantially below  $B$ , as shown by the following example.

**Example A.** We start with two homothetic consumers characterized by the indirect utility functions  $v^1(y, p) = yp_1^{-a}p_2^{a-1}$  and  $v^2(y, p) = y/(p_1 + \beta p_2)$ , where  $a = [\ln(7/4)]/\ln 2$  and  $b = 1/3$ . Let the consumers receive equal income shares. Then they are indifferent between the mean situations  $(1, q)$  and  $(1.75, p)$ , where  $q = (1, 1)$  and  $p = (2, 1)$ . Let  $x^i$  be the demand function for good 1 corresponding to the indirect utility function  $v^i$ . As described above, we let consumer  $i$  have a demand for good 1 that is less than  $x^i$  at mean situations that yield utility less than  $v^i(1, q)$ . To be precise, let  $\phi_i(t) = \exp(-10^4 t^2)$  for  $t < 0$  and  $\phi_i(t) = 1$  for  $t \geq 0$ . Let consumer  $i$ 's demand function for good 1 be  $\hat{x}^i \equiv \phi_i(v^i - v^i(1, q))x^i$ , for  $i = 1, 2$ . It can be verified that  $\phi_i$  is  $C^\infty$  and satisfies  $\phi_i'(t) > 0$  for  $t < 0$ . It can be checked numerically that aggregate demand satisfies Slutsky negative semidefiniteness, so the modified consumption sector has a representative consumer.

In the modified consumption sector the representative consumer's indifference curve through  $(1, q)$  is the graph of the function  $c(t)$  satisfying  $c(1) = 1$  and the differential equation  $c'(t) = \hat{x}(c(t), t, 1)$ , where  $\hat{x} \equiv (1/2)(\hat{x}^1 + \hat{x}^2)$  is the mean demand function. Numerical solution of this differential equation shows that  $c(2) < 1.1203$ . Thus, consumers 1 and 2 require 75% more income in order to be compensated for a doubling of the price of good 1. But the representative consumer requires only 12% more income. The inconsistency ratio is more than 1.56. In order to be compensated for the price rise, the actual consumers require over 56% more than the representative consumer requires.

□

In Example A, the consumer's preferences are transformed without changing their indifference curves through the points  $A$  and  $B$ . Such a transformation is possible because in the initial consumption sector the set of mean situations that yield the consumer utility vector  $(u_1, u_2)$  (with the price of good 2 fixed) is disconnected. Jerison (1984) shows that as long as the mean situations that yield a given vector of consumer utilities form a connected set, the representative consumer must be indifferent among them. The disconnectedness in Figure 1 comes from the fact that for consumer 1 the goods are perfect complements (a large change in relative prices has little effect on the ratio of demands), whereas for consumer 2 the goods are substitutes. There is no obvious reason for ruling out such preference profiles. On the other hand, they are not completely normal. In the modified economy, good 2 is inferior for both consumers. We will consider the case in which all goods are normal for all consumers below.

Example A shows that there are consumption sectors with representative consumers whose preferences differ greatly from the preferences of the actual consumers. A similar argument can be used to show that every utility function can be viewed as the utility of a Pareto inconsistent representative consumer for some consumption sector. To be precise, for every smooth indirect utility function  $v$  there exists a consumption sector with a Pareto

inconsistent representative consumer whose indirect utility function is  $v$ . This can be shown by perturbing a consumption sector in which all consumers have the indirect utility function  $v$ .

## 4.2 A Bound on the Pareto Inconsistency of a Representative Consumer

Representative consumers' Pareto inconsistencies cannot be arbitrarily large. This is implied by the following fact, which is of independent interest. It is impossible for all the consumers to prefer situation A to B if B is revealed preferred to A for the representative consumer.

**Lemma 1.** *If  $(y, p)$  is strictly revealed preferred to  $(z, q)$  for the representative consumer then at least one consumer prefers  $(y, p)$  to  $(z, q)$ .*

*Proof.* Suppose that  $pX^i(D^i(z, q), q) \geq D^i(y, p)$  for every  $i$ . Summing over  $i$  yields  $pX(z, q) \geq y$ . Thus if  $(y, p)$  is strictly revealed preferred to  $(z, q)$  by the representative consumer ( $pX(z, q) < y$ ) then  $pX^i(D^i(z, q), q) < D^i(y, p)$  and  $V^i(z, q) < V^i(y, p)$  for some  $i$ .

□

If all the consumers are at least as well off at  $(z, q)$  as at  $(y, p)$  then aggregate income  $y$  cannot be too high. Lemma 1 implies that  $y \leq pX(z, q)$ . Similarly, if all the consumers are at least as well off at  $(y, p)$  as at  $(z, q)$  then there is a lower bound on  $y$  determined by  $qX(y, p) \geq z$ . Let  $y^*(q, p)$  be the minimum  $y$  satisfying  $qX(y, p) \geq z$ . It follows that the inconsistency ratio for a move from  $(1, q)$  to the price vector  $p$  lies in the interval  $[y^*(q, p)/e^*(q, p), pX(1, q)/e^*(q, p)]$ , where  $e^*(q, p) \equiv e(v(1, q), p)$ .

The interval determines bounds that are not tight. Better bounds can probably be found. But for commonly used models and normal price variation, the above interval is rather small. Consider, for example, a Cobb-Douglas representative consumer in a two-good consumption sector. Let the representative consumer's utility function be  $u(x_1, x_2) = x_1^\alpha x_2^\beta$ , with  $\alpha + \beta = 1$ , and let  $q = (1, 1)$ . Then  $pX(1, q) = \alpha p_1 + \beta p_2$ ,  $e^*(q, p) = p_1^\alpha p_2^\beta$  and  $y^*(q, p) = p_1 p_2 / (\alpha p_2 + \beta p_1)$ . The weakest bound on the inconsistency ratio occurs when  $\alpha = \beta = 1/2$ . (This is the case in which  $pX(1, q)/y^*(q, p)$  is maximized.) In this case, the inconsistency ratio lies in the interval  $[2\sqrt{p_1 p_2} / (p_1 + p_2), (p_1 + p_2) / (2\sqrt{p_1 p_2})]$ . If, starting at the price vector  $q = (1, 1)$ , the price of good 1 doubles and the price of good 2 does not change, then the inconsistency ratio lies in the interval  $[\.9428, 1.0607]$ . So the compensation required by the Cobb-Douglas representative consumer will not differ by more than about 6% from that required by the actual consumers when the price of good 1 doubles.

### 4.3 Nonrepresentative Representative Consumers in Macroeconomics

The nonrepresentative representative consumer in Example A has preferences that are far from homothetic. This is not an accident. The bounds on the Pareto inconsistency obtained above are particularly confining when the representative consumer is homothetic. We will illustrate this with examples of CES representative consumers with utility functions that often appear in macroeconomic applications.

Representative consumers in macroeconomics are commonly assumed to have homothetic, stationary, completely separable preferences for flows of consumption expenditures over an infinite time horizon. The utility function of such a consumer has the form

$$u(x) = \left( \sum_{t=0}^{\infty} \delta^t x_t^{1-\sigma} \right)^{1/(1-\sigma)},$$

where  $x = \{x_t\}_{t=0}^{\infty}$  and where  $x_t$  is consumption expenditure in period  $t$ . The corresponding indirect utility and expenditure functions are

$$v(y, p) = y \left( \sum_{t=0}^{\infty} \delta^{t/\sigma} p_t^\epsilon \right)^{-1/\epsilon} \quad \text{and} \quad e(\bar{u}, p) = \bar{u} \cdot \left( \sum_{t=0}^{\infty} \delta^{t/\sigma} p_t^\epsilon \right)^{1/\epsilon},$$

where  $\epsilon \equiv (\sigma - 1)/\sigma$ . In this case,  $y$  is interpreted as (lifetime) wealth. If the consumer can borrow or save at a constant interest rate  $r$  then the price of consumption expenditure in period  $t$  can be taken to be  $p_t = (1 + r)^{-t}$ , and optimal consumption (when it exists) grows at the rate  $g$ , where

$$\delta(1 + r) = (1 + g)^\sigma. \quad (4)$$

As in the examples above, there are only two consumers and their income shares are equal. It can be shown that in this case, the consumers' indirect utilities can be expressed as functions of the mean income and two price indices. We focus on the case in which these indices are for consumption in nonoverlapping time intervals. The price indices can be defined by:

$$\alpha(p) = \left( \sum_{t=0}^T \delta^{t/\sigma} p_t^\epsilon \right)^{1/\epsilon} \quad \text{and} \quad \beta(p) = \left( \sum_{t=T+1}^{\infty} \delta^{t/\sigma} p_t^\epsilon \right)^{1/\epsilon},$$

and interpreted as indices for early and late consumption respectively. The expenditure function of the representative consumer can be written as a function of these price indices:  $e(\bar{u}, p) = \tilde{e}(\bar{u}, \alpha(p), \beta(p))$ , where  $\tilde{e}(\bar{u}, a, b) \equiv (a^\epsilon + b^\epsilon)^{1/\epsilon}$ .

The preferences of the representative and the two actual consumers are determined by their preferences over bundles of two commodity aggregates whose price indices are defined

above. The utility function of consumer 2 (for the two aggregate commodities) is  $u^2(x_1, x_2) \equiv (x_1 + x_2)/(s - 1)$  if  $sx_1 + x_2 < s$  and  $u^2(x_1, x_2) \equiv [1/(s - 1)] + (x_2/s)$  otherwise, where  $s > 1$  is a fixed scalar. It is easy to verify that  $u^2$  is continuous, quasiconcave and nondecreasing. Consumer 2 demands strictly positive amounts of both commodity aggregates whenever  $sb > y > a > b$ , where  $y$  is the consumer's income and  $a$  and  $b$  are the price indices of early and late consumption. On this region consumer 2 has the indirect utility function  $v^2(y, a, b) = [(y - a)/(sb - a)] + s$  and the demand function for good 1  $x_2 = (sb - y)/(sb - a)$ . At prices  $a$  and  $b$  for the aggregate commodities the demand for good 1 by consumer 1 is  $x_1(y, a, b) = 2y[a^{\epsilon-1}/(a^\epsilon + b^\epsilon)] - [(sb - y)/(sb - a)]$ . Suppose that we fix  $\bar{u}$  and  $a_L$  and let  $y_L \equiv e^1(\bar{u}, a_L, 1)$  and  $y(a) \equiv e^1(\bar{u}, a, 1)$ , where  $e^1$  is the expenditure function for consumer 1 in terms of the aggregated commodities. Then  $y(\cdot)$  is the income compensation function that solves the differential equation  $y'(a) = P(a)y(a) - Q(a)$ , with the initial condition  $y(a_L) = y_L$ , where  $P(a) = 2a^{\epsilon-1}/(a^\epsilon + 1)$  and  $Q(a) = s/(s - a)$ . The graph of  $y(\cdot)$  is the indifference curve for consumer 1 of utility level  $\bar{u}$  in the space of income and the price of "early consumption" when the price of late consumption is fixed at 1. The solution to this differential equation is

$$y(a; y_L) = \frac{(a^\epsilon + 1)^{2/\epsilon}}{(s - a)} \left[ y_L \frac{s - a_L}{(a_L^\epsilon + 1)^{2/\epsilon}} - s \int_{a_L}^a (\alpha^\epsilon + 1)^{-2/\epsilon} d\alpha \right]. \quad (5)$$

In the case of Cobb-Douglas preferences,  $\sigma = 1$  and  $\epsilon = 0$ . Then equation (5) is replaced by

$$y(a; y_L) = \frac{1}{s - a} \left( \frac{a}{a_L} \right)^{2m} \left[ (s - a_L)y_L + \frac{sa_L}{1 - 2m} \right] - \frac{sa}{(1 - 2m)(s - a)}. \quad (6)$$

To construct the examples of inconsistency, we fix  $T$ , the date separating early and late consumption, and specify alternative values of the interest rate. These interest rates determine alternative prices of early and late consumption  $(a_j, b_j)$ , for  $j = L, H$ . We then find  $y_L$  and  $y_H$  such that both consumers 1 and 2 are indifferent between the budget situations  $(y_L, a_L/b_L, 1)$  and  $(y_H, a_H/b_H, 1)$ . The  $y_L$  and  $y_H$  are the unique solutions to the equations  $y_H = y(a_H/b_H; y_L)$  and  $v^2(y_L, a_L/b_L, 1) = v^2(y_H, a_H/b_H, 1)$ . The solution  $y_H$  is the wealth required by both consumers to compensate for a rise in the relative price of early consumption from  $a_L/b_L$  to  $a_H/b_H$ . The wealth required by the representative consumer is  $y_R = y_L [((a_H/b_H)^\epsilon + 1)/((a_L/b_L)^\epsilon + 1)]^{1/\epsilon}$ . If  $y_R \neq y_H$  then the representative consumer is Pareto inconsistent, and the inconsistency ratio is  $y_H/y_R$ .

We consider Pareto inconsistencies that can arise in the example of Lucas (1987) who compared a budget situation in which the representative consumer's consumption expenditure grows at 3% to a situation in which the consumption growth rate is 6%.

**Example B.** Let  $\delta = .97$  and  $\sigma = 2$ , so that  $\epsilon = 1/2$ . These are parameters taken as the base case by Stokey and Rebelo (1995) in their analysis of growth under alternative capital



and labor tax rates. The optimal (constant) rate of consumption growth for the representative consumer changes from 3% to 6% if the interest rate changes from approximately 9.37% to 15.8%. The interest rate determines relative prices of early and late consumption for each value of  $T$ . It turns out that the inconsistency ratio is maximized when  $T = 59$ . For this  $T$ , a rise in the interest rate from 9.37% to 15.8% raises the relative price of early consumption,  $a/b$ , from 5.43 to 14.67. Taking  $s = 1.001(a_H/b_H) \approx 14.68$ , the inconsistency ratio is approximately 1.0217.

**Example C.** Let  $\delta = .95$  and  $\sigma = 1$ . This is the case of Cobb-Douglas utility. The rate of consumption growth for the representative consumer rises from 3% to 6% when the interest rate changes from 8.4% to 11.58%. Again, taking  $s = 1.001(a_H/b_H)$ , the largest inconsistency ratio is slightly below 1.034, and it occurs with  $T = 30$ .

The inconsistencies in these examples are rather small (less than 3.4%). This is well below the inconsistency bounds implied by Lemma 1 (9.6% for Example B and 11.8% for Example C). But those bounds are not tight, whereas the examples themselves appear to be close to the worst possible given the range of relative price variation. In particular, the specification of the preferences of consumer 2 (through the choice of  $s$ ) depends on the range of price variation. Over this range, good 1 is inferior for consumer 2. Still, the degree of Pareto inconsistency is quite small. If both goods were normal for both consumers, the inconsistency would be even smaller.

## 5. Normative Representative Consumers

The Pareto inconsistencies illustrated above show that representative consumer models might not be adequate for normative analysis. On the other hand, Dow and Werlang (1988) show that if a representative consumer is Pareto consistent, its preferences have a welfare interpretation: they are the same as preferences generated by a particular Bergson-Samuelson social welfare function. To state this result precisely, we say that a representative consumer (for  $D$ ) has a *welfare interpretation* if its preferences over situations are represented by  $w[V^D(y, p)]$  for some nondecreasing function  $w$  that is strictly increasing on the set of attainable consumer utility vectors  $\{V^D(y, p) | y \geq 0, p \gg 0\}$ .

**Proposition 2.** *A representative consumer has a welfare interpretation if and only if it is Pareto consistent.*

For a proof, see Dow and Werlang (1988) or Jerison (1994). Note that a social planner might have a social welfare function that is different from the  $w$  in the previous paragraph. Then the planners' preferences would differ from those of the representative consumer even if the

latter had a welfare interpretation.

It is an open question under what conditions a consumption sector has a Pareto consistent representative consumer. One well known sufficient condition is that the income distribution rule is optimal with respect to a social welfare function.

**Proposition 3.** *If the income distribution rule is optimal then the consumption sector has a Pareto consistent representative consumer.*

Chipman and Moore (1979) and Dow and Sonnenschein (1986) show that when  $D$  is optimal, there is a utility function that generates a correspondence that contains the aggregate demand function as a selection. Jerison (1994) shows that since the aggregate demand function is smooth, the utility function generates the aggregate demand *function* itself. The distribution rule need not be uniquely optimal, but every other optimal rule determines the same aggregate demand.

We will show that the distribution rule need not be optimal in order for a Pareto consistent representative consumer to exist. To do so, we need the following property of consumption sectors with optimal income distribution, proved by Jerison (1994).

**Proposition 4.** *If the income distribution rule  $D$  is optimal then for each situation  $(y, p)$  the covariance matrix  $C^D(y, p)$  is symmetric and positive semidefinite.*

Thus if the covariance matrix fails to be symmetric or positive semidefinite, then there is *no* social welfare function for which the distribution rule is optimal.

**Example D.** Pareto Consistency Without Optimal Income Distribution

In this example the consumption sector has a Pareto consistent representative consumer, but the covariance matrix is not positive semidefinite, so the income distribution rule is not optimal with respect to any social welfare function. There are two goods and two consumers with equal shares of aggregate income. Consumer 1 has Cobb-Douglas utility and demand function

$$X_1^1(y, p) = y/(2p_1),$$

$$X_2^1(y, p) = y/(2p_2).$$

Consumer 2 has the demand function

$$X_1^2(y, p) = (y/2p_1) + (p_1/2y),$$

$$X_2^2(y, p) = (y/2p_2) - (p_1^2/2p_2y).$$

The demand vector  $X^2(y, p)$  of consumer 2 is nonnegative whenever  $y \geq p_1$ . We will restrict attention to mean incomes and prices in this region.

The consumers' indirect utility functions are

$$v^1(y, p) \equiv y^2 / (p_1 p_2) \text{ and}$$

$$v^2(y, p) \equiv (y^2 / p_1 p_2) - (p_1 / p_2).$$

The two consumers receive equal shares of aggregate income, so when they have income  $y$  at prices  $p$ , aggregate demand is

$$\begin{aligned} X(2y, p) &= X^1(y, p) + X^2(y, p) \\ &= \left( \frac{y}{p_1} + \frac{p_1}{2y}, \frac{y}{p_2} - \frac{p_1^2}{2p_2 y} \right). \end{aligned}$$

An indirect utility function for this aggregate demand function is

$$v(y, p) \equiv \frac{y^2}{p_1 p_2} - \frac{p_1}{p_2}.$$

It is easy to show that  $v^1$ ,  $v^2$  and  $v$  are quasiconvex in  $p$  over the region where  $X^2 \gg 0$ .

To show that the distribution rule is not optimal with respect to any social welfare function it suffices to show that the covariance matrix of average and marginal properties to consume is not positive semidefinite. By (3), the upper left component of this covariance matrix is a positive multiple of  $(M_1^2 - M_1^1)(A_1^2 - A_1^1)$  with each function evaluated at  $(y, p)$ . But we have

$$M_1^2 - M_1^1 = \frac{1}{2p_1} - \frac{p_1}{2y^2} - \frac{1}{2p_1} = -\frac{p_1}{2y^2} < 0$$

$$A_1^2 - A_1^1 = \frac{1}{2p_1} + \frac{p_1}{2y^2} - \frac{1}{2p_1} = \frac{p_1}{2y^2} > 0.$$

Thus the covariance matrix is not positive semidefinite, and the income distribution cannot be optimal.

We next show that the representative consumer is Pareto consistent. Let each consumer  $i$  have income  $y$ , and suppose that the price vector  $p$  is Pareto superior to  $q$ . Then

$$\frac{y^2}{p_1 p_2} \geq \frac{y^2}{q_1 q_2} \quad \text{and}$$

$$\frac{y^2}{p_1 p_2} - \frac{p_1}{p_2} \geq \frac{y^2}{q_1 q_2} - \frac{q_1}{q_2}$$

with at least one strict inequality. Adding the second inequality to three times the first, we see that

$$v(2y, p) = \frac{4y^2}{p_1 p_2} - \frac{p_1}{p_2} > \frac{4y^2}{q_1 q_2} - \frac{q_1}{q_2} = v(2y, q),$$

so the representative consumer prefers the price vector  $p$  to  $q$  at aggregate income  $2y$ . This shows that the representative consumer is Pareto consistent.

It remains an open question how to characterize consumption sectors with Pareto consistent representative consumers. The next result provides a sufficient condition in a special case.

**Proposition 5.** *In a community with two goods, two consumers and a representative consumer, the representative consumer is Pareto consistent if for each  $p \gg 0$  the matrix of covariances  $C^D(1, p)$  is nonzero.*

In a two-good economy, the matrix of covariances is symmetric. If it is positive semidefinite, then a representative consumer exists. If in addition the matrix is nonzero, then, by Proposition 5, the representative consumer is Pareto consistent.

Figure 2 shows the main idea in the proof. In the figure, the actual consumers, 1 and 2, prefer A to B, but the representative consumer prefers B to A. By minimizing the utility of the representative consumer along the indifference curve of consumer 2 through A, we obtain a point C at which all three consumers' indifference curves are tangent. At such a point, the matrix of covariances is 0, so as long as that matrix is nonzero there cannot be Pareto inconsistency.

*Proof of Proposition 5:* Suppose that all the consumers in the community prefer  $(z, q)$  to  $(\bar{y}, \bar{p})$ , but the representative consumer prefers  $(\bar{y}, \bar{p})$  to  $(z, q)$ . We will show that this leads to a contradiction. By the homogeneity of  $D$  and of each consumer's indirect utility function, there is no loss of generality in assuming that  $q_2 = \bar{p}_2$ . If  $q_1 = \bar{p}_1$  then there cannot be a Pareto inconsistency, since the utility of each consumer (including the representative) is strictly increasing in aggregate income. Assume that  $\bar{p}_1 > q_1$ .

We begin by showing that  $X_1^{D1}(z, q) \neq X_1^{D2}(z, q)$ . Recall that  $\sum_i D_y^i = 1$  and  $\sum_i D_y^i X^{Di} = X^D$ , with  $D_y^i > 0$  for each  $i$ . Also, for each  $i$ ,  $X^{Di}(p, y)$  and  $X^D(p, y)$  satisfy the same budget identity,  $px = y$ . Since there are only two goods, if  $X_1^{D1}(z, q) = X_1^{D2}(z, q)$  then  $X^{D1}(z, q) = X^{D2}(z, q) = X^D(z, q)$ . But then  $C^D(z, q) = 0$ , contradicting the hypothesis. This proves that  $X_1^{D1}(z, q) \neq X_1^{D2}(z, q)$ , and without loss of generality we can assume that  $X_1^{D1}(z, q) > X_1^{D2}(z, q)$ . It follows that  $X_1^D(z, q) > X_1^{D2}(z, q)$ , since  $\sum D_y^i X^{Di} = X^D$ .

Consider the minimization problem

$$\min_{y, p} v(y, p) \quad \text{s.t.} \quad y \leq \bar{y} + 1, \quad \bar{p}_1 \geq p_1 \geq q_1, \quad p_2 = \bar{p}_2,$$

$$\text{and} \quad V^{D2}(y, p) = V^{D2}(z, q).$$

The constraint set is compact and contains  $(z, q)$ , so the problem has a solution denoted  $(y^*, p^*)$ . If  $p_1^* = \bar{p}_1$  then  $p^* = \bar{p}$  and  $y^* > \bar{y}$ , since  $V^{D2}(\bar{y}, \bar{p}) < V^{D2}(z, q) = V^{D2}(y^*, p^*)$ . But

then  $v(y^*, p^*) > v(\bar{y}, \bar{p}) > v(z, q)$ , which contradicts the assumption that  $(y^*, p^*)$  solves the minimization problem. Thus  $p_1^* < \bar{p}_1$ . Since  $v$  is nonincreasing in prices and nondecreasing in aggregate income, the first two constraints in the minimization problem hold with strict inequality. With these constraints not binding, the necessary first order conditions are  $v_y - \lambda V_y^{D2} = 0$  and  $v_{p_1} - \lambda V_{p_1}^{D2} - \mu = 0$ , for nonnegative scalars  $\lambda$  and  $\mu$ , with  $\mu(p_1^* - q_1) = 0$ , where the subscripts denote partial derivatives and all functions are evaluated at  $(y^*, p^*)$ .

Note that  $-v_p/v_y = X^D$  and  $-V_p^{D2}/V_y^{D2} = X^{D2}$ . So the first order conditions above imply that  $X_1^D(y^*, p^*) \leq X_1^{D2}(y^*, p^*)$ , with equality if  $p_1^* > q_1$  (since  $p_1^* > q_1$  implies  $\mu = 0$ ). It follows that  $(y^*, p^*) \neq (z, q)$  since  $X_1^D(z, q) > X_1^{D2}(z, q)$ , and therefore  $p_1^* > q_1$  and  $X^D(y^*, p^*) = X^{D2}(y^*, p^*)$ . This implies that  $X^{D1} = X^{D2} = X^D$  at  $(y^*, p^*)$ , since  $\sum D_y^i X^{Di} = X^D$ . But then  $C^D(y^*, p^*) = 0$ , which contradicts the hypothesis. This proves that there cannot be a Pareto inconsistency of the form described above with  $\bar{p}_1 > q_1$ .

If  $\bar{p}_1 < q_1$  we arrive at a similar contradiction by showing that  $X_1^D(\bar{y}, \bar{p})$  is strictly between  $X_1^{D1}(\bar{y}, \bar{p})$  and  $X_1^{D2}(\bar{y}, \bar{p})$ , and then by minimizing  $V^{Di}(y, p)$  subject to  $v(y, p) = v(\bar{y}, \bar{p})$ ,  $\bar{p}_1 \leq p_1 \leq q_1$  and  $p_2 = \bar{p}_2$ , where consumer  $i$  has the larger  $X_1^{Di}(\bar{y}, \bar{p})$ . □

## 6. Conclusion

The problem of characterizing communities with Pareto consistent representative consumers remains open. In the examples in section 4 above, the inconsistencies arise because the degree of substitutability among commodities is significantly different for different consumers. Figures 1 and 2 suggest that a “single crossing” property for the consumers’ indifference curves might ensure Pareto consistency. This is the type of condition implied by the nonzero matrix of covariances in Proposition 5, but that proposition does not generalize to the case of more than two goods and two consumers.

If the consumers receive fixed shares of aggregate income, then positive semidefiniteness of the matrix of covariances is essentially equivalent to the requirement that the consumers’ demand vectors become more dispersed when their incomes all rise by the same amount. This “increasing dispersion” is plausible and has empirical support, cf. Härdle, et. al. (1991), Hildenbrand (1994). If the consumers’ income shares are fixed, then increasing dispersion is necessary in order for the fixed distribution rule to be optimal with respect to a social welfare function. It remains to be seen whether increasing dispersion along with existence of a positive representative consumer implies Pareto consistency.

We have shown that in important classes of communities (ones in which all goods are

normal or there is a positive representative consumer with homothetic preferences) there cannot be large Pareto inconsistencies for the degree of price variation in most applications. This does not extend the applicability of representative consumer models very far. There are many communities without positive representative consumers. And even if there is a Pareto consistent representative consumer, its preferences need not be appropriate for planning. The representative consumer's preferences are locally like a compensation criterion, cf. Jerison (1990). They attach greater weight to richer consumers who consume more. It might not be desirable to identify the representative consumer's preferences with social welfare even when this entails no logical inconsistency.

## Footnotes

1. Kirman (1992) presents an example of Pareto inconsistency in which a "representative individual" has preferences that differ greatly from those of the actual consumers. But the indifference curves of the representative individual generate the aggregate demand vectors only at the two budget situations being compared, not at others. In the present paper, as in Samuelson (1956), Jerison (1984), Dow and Werlang (1988), we require that the representative consumer's preferences generate the entire aggregate demand function. The representative individual in Kirman's example is not a representative consumer in this sense.

2. For vectors  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  we write  $x \gg y$  [resp.  $x \geq y$ ] if  $x_j > y_j$  [resp.  $x_j \geq y_j$ ] for  $j = 1, \dots, n$ .

3. In order for consumption to be Pareto efficient the distribution rule need not be differentiable or homogeneous. We impose these rather weak restrictions in order to consider the cases most favorable for the existence of a representative consumer.

4. A function  $w$  is *nondecreasing* if  $u \geq r$  implies  $w(u) \geq w(r)$  for every  $u$  and  $r$  in the domain of  $w$ . The function  $w$  is *strictly increasing on a set* if  $w(u) \gg w(r)$  whenever  $u \gg r$  for  $u$  and  $r$  in the set.

3. A matrix  $M$  is positive definite on a set  $X$  if  $x^T M x > 0$  for every  $x \neq 0$  in  $X$ .

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Figure 1

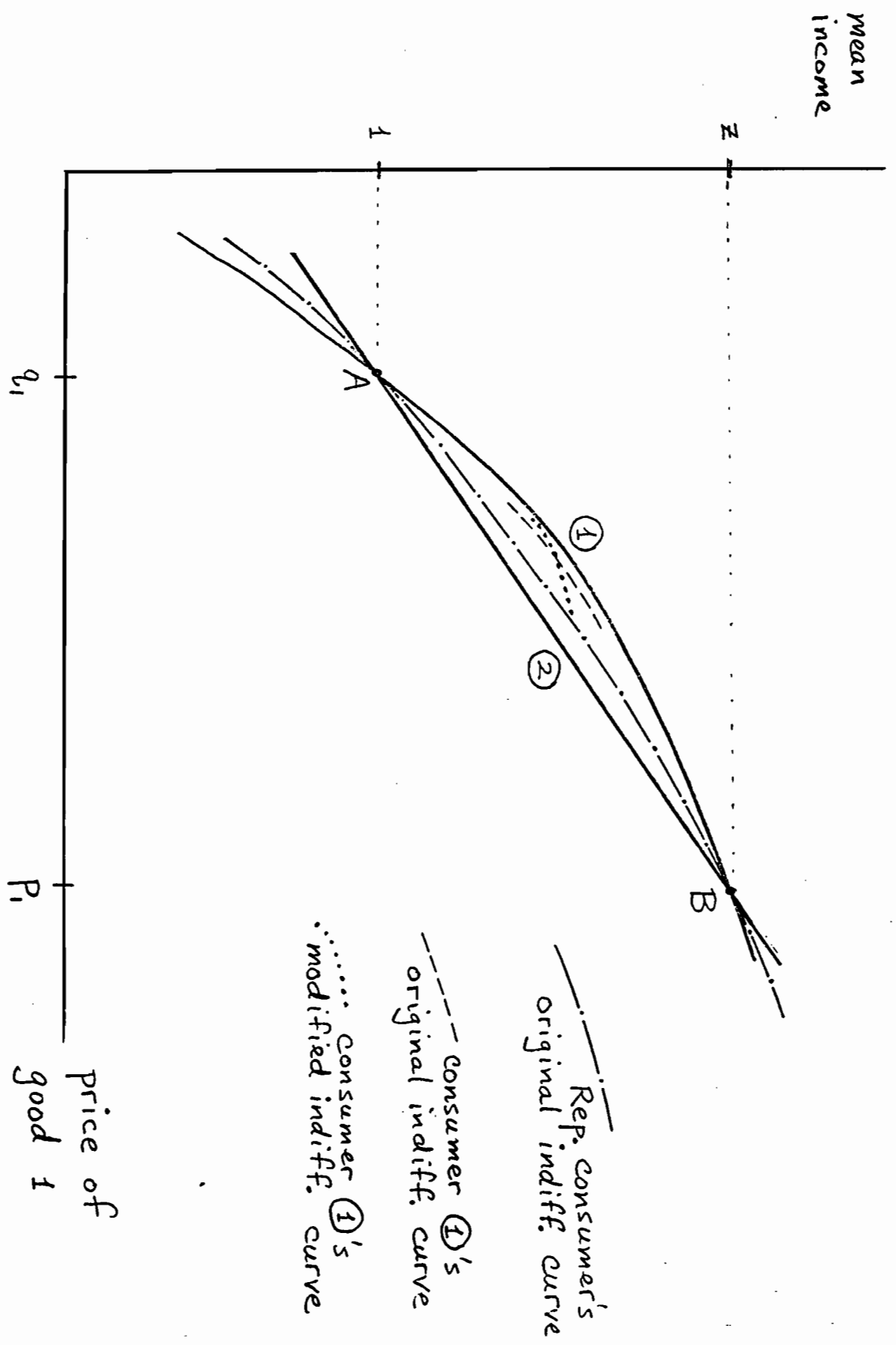


Figure 2

