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THE LEAST CORE, KERNEL, AND BARGAINING SETS OF LARGE GAMES †

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Abstract

We study the least core, the kernel, and bargaining sets of coalitional games with a countable set of players. We show that the least core of a continuous superadditive game with a countable set of players is a non-empty (norm-compact) subset of the space of all countable additive measures. Then we show that in such games the intersection of the prekernel and least core is non-empty. Finally, we show that this intersection is contained in the Aumann-Maschler and the Mas-Colell bargaining sets.

Key Words: Coalitional games, Least Core, Kernel, Bargaining Sets.

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1. Introduction

Solution concepts for coalitional games with an infinite set of players have been studied in many works. Most of these works deal with the core and the Shapley value. Bargaining sets and related solution concepts for games with a finite set of players have been studied intensively (for a comprehensive survey see Maschler[10]). There are a few works concerning bargaining sets (and related concepts) of games with an infinite set of players. Wesley[20] deals with the kernel of games with a countable set of players. Bird[3] studies the nucleolus-like solutions for games with a measurable space of players. Mas-Colell[13] introduces his bargaining set in the context of pure exchange economies with a continuum of agents. Shitovitz[18] deals with the Mas-Colell bargaining set in mixed market games. Einy et al.[8] study the Mas-Colell bargaining set in convex games with a measurable space of players.

The present work deals with the least core, the kernel, and the Aumann-Maschler and Mas-Colell bargaining sets for superadditive games with a countable set of players. The least core was introduced by Maschler, Peleg and Shapley[12], where they study its relation to the kernel and the nucleolus. It is well known that the core of a continuous game with a measurable space of players, if it is non-empty, consists of countably additive payoff measures (e.g., Schmeidler[16]). Here we show that the least core of a continuous superadditive game with a countable set of players is a non-empty norm-compact subset of the set of all countably additive measures defined on the set of coalitions (see Theorem A).

The kernel of a cooperative game was introduced by Davis and Maschler[5]. Since then it has been the subject of many studies . Originally it was regarded as an auxiliary solution concept whose main task was to illuminate the properties of the Aumann-Maschler bargaining set. Nevertheless, the kernel possesses interesting mathematical properties, and reflects in many ways the structure of the game. The prekernel is a simplified version of the kernel which is not restricted to individually rational payoffs. We show that in continuous superadditive games with a countable set of players the prekernel (and hence the kernel) and the least core have a non-empty intersection. The proof of this result uses finite approximations. Wesley[20] proved by using non-standard analysis that under some conditions the kernel of a superadditive game with a countable set of players is non-empty (for every coalition structure). We show that Wesley's conditions imply that the game is continuous. We also give an example of a continuous game which does not satisfy one of Wesley's conditions (see Lemma 4.3 and Example 4.4). Thus, Wesley's result is a special case of our result when the coalition structure includes only the grand coalition.

The first definition of a bargaining set for cooperative games was given in Aumann and Maschler[1]. Recently, several new concepts of bargaining set have been introduced (see Maschler[10] for a survey). Davis and Maschler[4] and Peleg[14] proved that the Aumann-Maschler bargaining set is non-empty in a coalitional game with a finite set of players. We show that in continuous superadditive games with a countable set of players, the Aumann-Maschler bargaining set contains the intersection of the prekernel and the least core, and thus it always contains a countably additive payoff measure (see Theorem C).

Mas-Colell[13] proposed a bargaining set which is a modification of the Aumann-Maschler bargaining set. One of the advantage of the Mas-Colell bargaining set is that it can be defined for games with a continuum of players. Mas-Colell[13] showed that in atomless pure exchange economies his bargaining set coincides with the set of competitive equilibria, and he pointed out that in finite coalitional games, the prekernel is always contained in his bargaining set (see also Vohra[19]). In the definition of the Mas-Colell bargaining set it is not assumed that payoffs are individually rational. We show that in continuous superadditive games with a countable set of players the Mas-Colell bargaining set contains the intersection of the prekernel and the least core, and thus it always contains an individually rational countably additive payoff measure.

The paper is organized as follows. Section 2 contains the basic definitions and the preliminary results which are relevant to our work. In Section 3 we prove that the least core of a continuous superadditive game with a countable set of players is a non-empty norm-compact subset of the space of all countably additive measures on the set of coalitions. In Section 4 we show that in continuous superadditive games with a countable set of players the least core and the prekernel have a non-empty intersection. In Section 5 we prove that in continuous superadditive games the bargaining set of Aumann and Maschler and that of Mas-Colell contain the intersection of the prekernel and the least core.

2. Basic Definitions and Preliminary Results

In this section we define some basic notions we use throughout and prove some preliminary results.

2.1. Mathematical Preliminaries

Let N be the set of natural numbers. The set of subsets of N is denoted by 2^N. The set of all functions $f : N \to \{0,1\}$ is denoted by $\{0,1\}^N$. Note that $\{0,1\}^N$ is a compact metric space (as the product of countable metric spaces), and convergence of sequences in this metric is identical to pointwise convergence. Every $S \in 2^N$ can be naturally identified with its indicator function $1_S \in \{0,1\}^N$. The correspondence $S \longleftrightarrow 1_S$ induces a metric on 2^N under which it is compact. It is well known that a sequence $\{S_n\}_{n=1}^{\infty}$ converges to S (written $\lim_{n\to\infty} S_n = S$) in this metric if and only if

$$S = \underline{\lim}_{n\to\infty} S_n = \overline{\lim}_{n\to\infty} S_n,$$

where $\underline{\lim}_{n\to\infty} S_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} S_k$, and $\overline{\lim}_{n\to\infty} S_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} S_k$.

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of sets in 2^N such that $S_n \subset S_{n+1}$, and $\bigcup_{n=1}^{\infty} S_n = S$ we write $S_n \nearrow S$. Similarly, if $\{S_n\}_{n=1}^{\infty}$ is a sequence of sets in 2^N such that $S_{n+1} \subset S_n$, and $\bigcap_{n=1}^{\infty} S_n = S$, then we write $S_n \searrow S$. Note that if $\{S_n\}_{n=1}^{\infty}$ is a sequence such that $S_n \nearrow S$ or $S_n \searrow S$, then $\lim_{n\to\infty} S_n = S$. A function $v : 2^N \to \Re$ is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$. A function v on 2^N is *continuous* if it is continous with respect to the natural topology on 2^N defined above. For monotonic functions we have the following characterization of continuity.

Lemma 2.1. *Let $v : 2^N \to \Re$ be a monotonic function. Then v is continuous if and only if for every $S \in 2^N$ we have $\lim_{n\to\infty} v(S_n) = v(S)$ whenever $S_n \nearrow S$ or $S_n \searrow S$.*

Proof: We prove the non-obvious part of the lemma. Let $S \in 2^N$. Assume $\lim_{n\to\infty} v(S_n) = v(S)$ whenever $S_n \nearrow S$ or $S_n \searrow S$. We show that v is continuous at S. Let $\{S_n\}_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} S_n = S$; that is $S = \underline{\lim}_{n\to\infty} S_n = \overline{\lim}_{n\to\infty} S_n$. We show that $\lim_{n\to\infty} v(S_n) = v(S)$. For each n let $A_n = \bigcap_{k=n}^{\infty} S_k$, and $B_n = \bigcup_{k=n}^{\infty} S_k$. As v is monotonic and $A_n \subset S_n \subset B_n$ for each n, then $v(A_n) \leq v(S_n) \leq v(B_n)$ for each n. Now,

$A_{n+1} \supset A_n$ and $S = \underline{\lim} S_n = \bigcup_{n=1}^{\infty} A_n$. Also $B_{n+1} \subset B_n$, and $S = \overline{\lim} S_n = \bigcap_{n=1}^{\infty} B_n$. Thus, $A_n \nearrow S$ and $B_n \searrow S$. Hence by our assumption

$$\lim_{n\to\infty} v(A_n) = \lim_{n\to\infty} v(B_n) = v(S),$$

and therefore $\lim_{n\to\infty} v(S_n) = v(S)$.□

The Banach space of all bounded finitely additive measures on 2^N with the variation norm is denoted by ba. Its closed subspace consisting of all countable additive measures is denoted by ca. It is well known that ba is naturally identified with the dual space of $l_\infty = l_\infty(N)$, which is the Banach space of all bounded sequences of real numbers. Also, ca can be naturally identified with $l_1 = l_1(N)$; i.e., every $x \in l_1$ corresponds to $\mu_x \in ca$, where

$$\mu_x(S) = \sum_{i\in S} x_i,$$

for every $S \in 2^N$.

The following lemma plays an important role in our work.

Lemma 2.2. *Let K be a weak*-compact subset of ba that contains only countably additive measures (i.e., $K \subset ca$). Then K is norm-compact.*

Proof: Since $K \subset ca$, we can view K as a subset of ℓ_1. As ba is the norm-dual of ℓ_∞, which in turns is the norm-dual of ℓ_1, the weak* topology induced by ba on K coincides with the weak topology induced by ℓ_1 on K. Hence K is a weakly compact subset of ℓ_1. Since ℓ_1 is a separable Banach space, the weak topology on its weakly compact subsets is metrizable (see e.g., Theorem 3.V.6.3 in Dunford-Schwartz[7]). Therefore K is weakly sequentially compact in ℓ_1. By Schur's Theorem (see Diestel[6], page 85, and Corollary 14 page 296 of Dunford-Schwartz[7]), weak convergence and norm convergence of sequences are equivalent in ℓ_1; hence K is norm-compact in ℓ_1.□

2.2. Game Theoretic Preliminaries

We refer to the members of N as *players*, and to each subset of N as a *coalition*. A *game* is a bounded function $v : 2^N \to \Re$ satisfying $v(\emptyset) = 0$ which is *superadditive;* that is, v satisfies $v(S \cup T) \geq v(S) + v(T)$ for every two disjoint subsets $S, T \in 2^N$, and for which there is $\lambda \in ca$ such that $v(S) \geq \lambda(S)$ for every $S \in 2^N$. (Note that the later requirement is satisfied for every non-negative set function v with $\lambda \equiv 0$.) Note that every $\lambda \in ca$ is continuous, and that if $v(S) \geq \lambda(S)$ for all S, then $v - \lambda$ is a monotonic function. Hence by Lemma 2.1 a game v is continuous on 2^N if and only if $\lim_{n\to\infty} v(S_n) = v(S)$ whenever $S_n \nearrow S$ or $S_n \searrow S$. Note that our definition of continuity coincides with that of Schmeidler[16] (see also Aumann and Shapley[2]).

For every game v and every coalition $S \in 2^N$, we define

$$\sigma_v(S) = \inf \sum_{i=1}^{n} v(S_i),$$

where infimum is taken over all finite partitions $\{S_1, S_2, ..., S_n\}$ of S. Note that $\sigma_v(i) = v(\{i\})$ for every $i \in N$. Note also that $\sigma_v \in ba$. Moreover, σ_v is the largest measure in ba such that $\sigma_v(S) \leq v(S)$ for all $S \in 2^N$; that is, if $\lambda \in ba$ and $\lambda(S) \leq v(S)$ for all $S \in 2^N$, then $\lambda \leq \sigma_v$ (pointwise). Also, if v is continuous then $\sigma_v \in ca$.

A *payoff measure* (or a *preimputation*) is a measure $\lambda \in ba$ such that $\lambda(N) = v(N)$. A payoff measure λ is *individually rational* if $\lambda \geq \sigma_v$. An *imputation* is an individually rational payoff measure. The set of all imputations is denoted by $I(v)$.

3. The Least Core

Recall that the core of a game v, denoted by $C(v)$, is the set off all imputations $\mu \in I(v)$ such that $\mu(S) \geq v(S)$ for each $S \in 2^N$. For every $\lambda \in ba$ let $f(\lambda) = \sup\{v(S) - \lambda(S) \mid S \subset N\}$. The *least core* of the game v, denoted $LC(v)$, is the set of all imputations $\mu \in I(v)$ for which f attains its minimal value on $I(v)$. Note that since f is the supremum of affine weak*-continuous functions, it is a convex and lower semicontinuous function on ba. As $I(v)$ is a weak*-compact subset of ba, f attains its minimal value ϵ_v on $I(v)$. Therefore $LC(v)$ is a non-empty convex weak*-compact subset of $I(v)$. We note that $\epsilon_v \geq 0$, and $\epsilon_v = 0$ if only if $C(v) \neq \emptyset$. In this case $C(v) = LC(v)$.

Theorem A. *Let v be a continuous game. Then $LC(v)$ is a non-empty convex norm-compact subset of ca.*

Proof: By Lemma 2.2 it suffices to prove that $LC(v) \subset ca$. Without loss of generality (w.l.o.g.) assume that σ_v is identically zero. (Alternatively, we could replace the game v with the game $w = v - \sigma_v$, which is also continuous and satisfies $\mu \in LC(w)$ if and only if $\mu + \sigma_v \in LC(v)$.) Also w.l.o.g. assume that $v(N) = 1$. Finally, we may assume that $\epsilon_v > 0$, as otherwise the result follows from Schmeidler[16].

Let $\mu \in LC(v)$. By Theorem 1.23 of Yosida and Hewitt[21], μ can be uniquely decomposed into a sum of a non-negative countably additive measure μ_c and a non-negative purely finitely additive measure μ_p (i.e., a measure μ_p satisfying $\mu_p(S) = 0$, for each finite $S \in \Sigma$). We must show that μ_p is identically zero. As μ_p is non-negative, it suffices to show that $\mu_p(N) = 0$. Assume that $\mu_p(N) > 0$. We show that this implies that $\max\left\{v(S) - \mu_c(S) \mid S \in 2^N\right\} > \epsilon_v$. This leads to a contradiction since $\max\left\{v(S) - \mu_c(S) \mid S \in 2^N\right\}$ (which exists because $v - \mu_c$ is continuous) is bounded above by ϵ_v. Indeed, let $S \in 2^N$, and for every $n \in N$ define $S_n = \{1, ..., n\}$. Then

$$\begin{aligned} v(S) - \mu_c(S) &= \lim_{n\to\infty} \left(v(S \cap S_n) - \mu_c(S \cap S_n)\right) \\ &= \lim_{n\to\infty} \left(v(S \cap S_n) - \mu(S \cap S_n)\right) \\ &\leq \epsilon_v. \end{aligned}$$

In the remaining of the proof of Theorem A assume that $\mu_p(N) > 0$. The proof proceeds in several steps.

STEP 1: We show that $\mu_c(N) > 0$.

Assume by way of contradiction that $\mu_c(N) = 0$. Then $\mu = \mu_p$. As $\mu \in LC(v)$, $v(S) - \mu(S) \leq \epsilon_v$, for every $S \in 2^N$. As v is continuous, we have $\lim_{n\to\infty} v(S_n) = v(N) = 1$. Since $\mu = \mu_p$ and $\mu_p(S_n) = 0$, we have $v(S_n) - \mu(S_n) = v(S_n)$, for each n. . Therefore $\epsilon_v \geq v(S_n)$ for each n, and thus $\epsilon_v = 1$. Let $\xi \in I(v)$ be given for each $S \in 2^N$ by $\xi(S) = \sum_{i \in S} 2^{-i}$. As $v - \xi$ is continuous, there is $S^* \in 2^N$ such that $v(S^*) - \xi(S^*) = \max\left\{v(S) - \xi(S) \mid S \in 2^N\right\}$. From the definition of ϵ_v it is clear that $1 = \epsilon_v \leq v(S^*) - \xi(S^*)$; hence $S^* \neq \emptyset$. Also $v(S) \leq 1$, and $\xi(S) > 0$ for each $S \neq \emptyset$ in 2^N. Summing up these inequalities we have

$$1 = \epsilon_v \leq v(S^*) - \xi(S^*) \leq 1 - \xi(S^*) < 1,$$

which is a contradiction. Hence $\mu_c(N) > 0$.

STEP 2: We show that the payoff measure $\lambda = \frac{1}{\mu_c(N)}\mu_c$ is a member of $LC(v)$.

Let $\alpha = \frac{1}{\mu_c(N)}$. As $\mu_p(N) > 0$, $\alpha > 1$. For each $S \in 2^N$ we have

$$v(S) - \lambda(S) = v(S) - \alpha\mu_c(S) = v(S) - \mu_c(S) - (\alpha - 1)\mu_c(S) \leq v(S) - \mu_c(S).$$

Recall that $S_n = \{1, ..., n\}$. Then for all $S \in 2^N$ we have

$$\begin{aligned} v(S) - \lambda(S) &= \lim_{n\to\infty}(v(S \cap S_n) - \lambda(S \cap S_n)) \\ &\leq \lim_{n\to\infty}(v(S \cap S_n) - \mu_c(S \cap S_n)) \\ &= \lim_{n\to\infty}(v(S \cap S_n) - \mu(S \cap S_n)) \\ &\leq \epsilon_v. \end{aligned}$$

Therefore $\lambda \in LC(v)$.

Step 3: We show that $\max\left\{v(S) - \mu_c(S) \mid S \in 2^N\right\} > \epsilon_v$, and we will get the desired contradiction.

For every $S \in 2^N$ we have $v(S) - \mu_c(S) \geq v(S) - \mu(S)$. As $\mu \in LC(v)$, we have $\sup\left\{v(S) - \mu(S) \mid S \in 2^N\right\} = \epsilon_v$. Hence $\max\left\{v(S) - \mu_c(S) \mid S \in 2^N\right\} \geq \epsilon_v$. We show that this inequality is strict. Assume to the contrary that $\max\left\{v(S) - \mu_c(S) \mid S \in 2^N\right\} = \epsilon_v$. Let $\lambda = \frac{1}{\mu_c(N)}\mu_c$; by Step 2, $\lambda \in LC(v)$. Now if $S \in 2^N$ is such that $v(S) - \lambda(S) = \epsilon_v$, then $\lambda(S) = 0$. For otherwise we would have $v(S) - \mu_c(S) > v(S) - \lambda(S) = \epsilon_v$. Let $S_0 = \{i \in N \mid \lambda(\{i\}) = 0\}$. Because $\lambda \in ca$, $\lambda(S_0) = 0$. Let $\bar{S} \in 2^N$ be such that $v\left(\bar{S}\right) - \lambda\left(\bar{S}\right) = \epsilon_v$ ($\bar{S} \neq \emptyset$ because $\epsilon_v > 0$); hence $\lambda\left(\bar{S}\right) = 0$, and therefore $\bar{S} \subset S_0$. Thus, $S_0 \neq \emptyset$.

Let $j \in N \backslash S_0$. Then $\lambda(\{j\}) > 0$. Let $\mathcal{C}_j = \left\{S \in 2^N \mid j \in S\right\}$. The set $\{1_S \mid S \in \mathcal{C}_j\}$ is a closed subset of $\{0,1\}^N$ in the topology of pointwise convergence, and therefore it is compact. Let $Q \in \mathcal{C}_j$ be such that $v(Q) - \lambda(Q) = \max\{v(S) - \lambda(S) \mid S \in \mathcal{C}_j\}$. Let $\delta = v(Q) - \lambda(Q)$. If $\delta = \epsilon_v$, then $\lambda(Q) = 0$, which is impossible since $j \in Q$, and therefore $\lambda(Q) \geq \lambda(\{j\}) > 0$. Hence $\delta < \epsilon_v$. Let $0 < \epsilon < \min(\lambda(\{j\}), \epsilon_v - \delta)$, and for each $i \in S_0$ let $\epsilon_i = \frac{\epsilon 2^{-i}}{\sum_{l \in S_0} 2^{-l}}$. Also for each $i \in N$ define

$$\hat{\lambda}(\{i\}) = \begin{cases} \epsilon_i & i \in S_0 \\ \lambda(\{j\}) - \epsilon & i = j \\ \lambda(\{i\}) & i \notin S_0 \cup \{j\} \end{cases}.$$

Note that $\hat{\lambda} \in ca_+$. Moreover, since $\hat{\lambda}(S_0) = \epsilon$, we have $\hat{\lambda}(N) = 1$, and therefore $\hat{\lambda} \in I(v)$. We show that $\max\left\{v(S) - \hat{\lambda}(S) \mid S \in 2^N\right\} < \epsilon_v$, and this will contradict the definition of ϵ_v. Let $S \in 2^N$, $S \neq \emptyset$. We distinguish two cases.

(a) $S \subset S_0$. In this case we have $v(S) - \hat{\lambda}(S) < v(S) - \lambda(S) \leq \epsilon_v$.

(b) $S = S_1 \cup S_2$, where $S_1 \subset S_0$, and $S_2 \subset N \backslash S_0$ satisfies $S_2 \neq \emptyset$. Since $S_2 \neq \emptyset$, we have $\lambda(S) > 0$; hence $v(S) - \lambda(S) < \epsilon_v$. If $j \notin S$, then $v(S) - \hat{\lambda}(S) \leq v(S) - \lambda(S) < \epsilon_v$. If $j \in S$, then $v(S) - \lambda(S) \leq \delta$. Thus

$$\begin{aligned} v(S) - \hat{\lambda}(S) &= v(S) - \lambda(S) + \lambda(S) - \hat{\lambda}(S) \\ &\leq \delta + \lambda(S) - \hat{\lambda}(S) \\ &= \lambda(S_1) - \hat{\lambda}(S_1) + \delta + \lambda(\{j\}) - \hat{\lambda}(\{j\}) \\ &\leq \delta + \epsilon < \epsilon_v. \square \end{aligned}$$

The following example taken from Kannai[9] (Example 2.1) shows that when a game is not continuous its least core may contain only purely finitely additive payoff measures.

Example 3.4. Define a game v by

$$v(S) = \begin{cases} 1 & N \backslash S \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the core of v is not empty, and therefore it coincides with the least core. Since a payoff measure $\mu \in ba$ is in the core of v if and only if $\mu(S) = 0$ for each finite $S \in 2^N$, the least core of v contains only purely finitely additive measures.

Remark: The least core for games with a finite set of players was defined by Maschler, Peleg and Shapley[12] as the set of all payoff measures μ for which

$$\min_{\lambda \in I^*(v)} \max_{S \neq \emptyset, S \neq N} (v(S) - \lambda(S))$$

is attained at $\lambda = \mu$, where $I^*(v)$ is the set of preimputations. It can be easily verified that for games with an empty core our definition applied to a game with a finite set of players coincides with the above definition. However,if $C(v) \neq \emptyset$ then $LC(v) = C(v)$ according to our definition, whereas the original definition may yield a strict subset of the core.

4. The Kernel

The set of all payoff measures in a game v is denoted by $I^*(v)$; that is

$$I^*(v) = \{\mu \in ba \mid \mu(N) = v(N)\}.$$

For $i, j \in N$, $i \neq j$, define the set

$$\mathcal{T}_{ij} = \left\{ S \in 2^N \mid i \in S,\ j \notin S \right\}.$$

Also for $i, j \in N$, $i \neq j$, and $\mu \in ba$ define

$$s_{ij}(\mu) = \sup \{ v(S) - \mu(S) \mid S \in \mathcal{T}_{ij} \}.$$

The *prekernel* of v is the set

$$PK(v) = \{ \mu \in I^*(v) \mid s_{ij}(\mu) = s_{ji}(\mu),\ \forall i, j \in N,\ i \neq j \}.$$

The *kernel* of v is the set

$$K(v) = \{ \mu \in I(v) \mid (s_{ij}(\mu) - s_{ji}(\mu)) (\mu(\{j\}) - \sigma_v(\{j\})) \leq 0,\ \forall i, j \in N,\ i \neq j \}.$$

The notion of the kernel of a coalitional game with a finite set of players was introduced by Davis and Maschler[5]. It is well-known that if v is a superadditive game with a finite set of players, then $K(v) = PK(v)$ (see Theorem 2.7 in Maschler, Peleg and Shapley[11]). It is also well-known that for such games $K(v) \cap LC(v) \neq \emptyset$. In fact, the nucleolus of v is contained in this intersection (see Corollary 6.7 in Maschler, Peleg and Shapley[12], and Theorem 3 in Schmeidler[15]). Theorem B below establishes that the prekernel (and hence the kernel) and the least core have a non-empty intersection for continuous games.

Theorem B. *Let v be a continuous game. Then $PK(v) \cap LC(v) \neq \emptyset$. In particular, $PK(v)$ contains an individually rational countably additive payoff measure.*

We need the following lemmas.

Lemma 4.1. *Let v be a continuous game. Then each function s_{ij}, $i, j \in N,$, $i \neq j$, is norm-continuous on ca.*

Proof: Let $\mu \in ca$ and let $\{\mu_n\}_{n=1}^{\infty} \subset ca$ be such that $\lim_{n \to \infty} \|\mu_n - \mu\| = 0$. We show that for each $i, j \in N$, $i \neq j$, $\lim_{n \to \infty} s_{ij}(\mu_n) = s_{ij}(\mu)$. Note that the set $\{1_S \mid S \in \mathcal{T}_{ij}\}$ is a closed (and therefore a compact) subset of $\{0,1\}^N$ in the topology of pointwise convergence. Since v is continuous and each $\mu_n \in ca$, $v - \mu_n$ attains its maximum on $\mathcal{T}_{ij}$. For each $n \in N$ let $S_n \in \mathcal{T}_{ij}$ be such that $s_{ij}(\mu_n) = v(S_n) - \mu_n(S_n)$, and similarly let $S \in \mathcal{T}_{ij}$ satisfying $s_{ij}(\mu) = v(S) - \mu(S)$. Since $v(S) - \mu(S) \geq v(S_n) - \mu(S_n)$ for every $n \in N$, we have

$$\begin{aligned} v(S_n) - \mu_n(S_n) &\leq v(S_n) - \mu_n(S_n) + (v(S) - \mu(S)) - (v(S_n) - \mu(S_n)) \\ &= v(S) - \mu(S) + \mu(S_n) - \mu_n(S_n) \\ &\leq v(S) - \mu(S) + \|\mu - \mu_n\|. \end{aligned}$$

Also we have $v(S_n) - \mu_n(S_n) \geq v(S) - \mu_n(S)$ for each $n \in N$.

As $\|\mu - \mu_n\| \to 0$, $\lim_{n\to\infty}(v(S_n) - \mu_n(S_n)) = v(S) - \mu(S)$. Hence,

$$\lim_{n\to\infty} s_{ij}(\mu_n) = \lim_{n\to\infty}(v(S_n) - \mu_n(S_n)) = v(S) - \mu(S) = s_{ij}(\mu). \square$$

Lemma 4.2. *Let w be a game with a finite support (i.e., for some $n \geq 1$, $w(S \cap \{1,...,n\}) = w(S)$, for all $S \in 2^N$). Then $PK(w) \cap LC(w) \neq \emptyset$.*

Proof: W.l.o.g. assume that $w \geq 0$. Let $\Sigma = \left\{S \in 2^N \mid S \subset \{1,...,n\}\right\}$, and let u be the restriction of w to Σ. Since w is superadditive , u is a superadditive game with a finite set of players. Therefore by Theorem 2.7 in Maschler, Peleg and Shapley[11], we have $PK(u) = K(u)$, and by Corollary 6.7 in Maschler, Peleg and Shapley[12], we have $PK(u) \cap LC(u) \neq \emptyset$.

Let $\mu \in PK(u) \cap LC(u)$. Define

$$\bar{\mu}(\{i\}) = \begin{cases} \mu(\{i\}), & 1 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}.$$

Obviously $\bar{\mu} \in ca$, and therefore there is $S^* \in 2^N$ such that

$$w(S^*) - \bar{\mu}(S^*) = \max\left\{w(S) - \bar{\mu}(S) \mid S \in 2^N\right\}.$$

Now,

$$\begin{aligned} \epsilon_w &\leq w(S^*) - \bar{\mu}(S^*) \\ &= u(S^* \cap \{1,...,n\}) - \mu(S^* \cap \{1,...,n\}) \\ &\leq \epsilon_u. \end{aligned}$$

Let $\lambda \in LC(w)$; since λ minimizes the function $f(\xi) = \sup\left\{w(S) - \xi(S) \mid S \in 2^N\right\}$ on $I(w)$, we have $\lambda(S) = 0$ for each $S \subset N \setminus \{1, ..., n\}$. Thus

$$\begin{aligned} \epsilon_w &= \max\left\{w(S) - \lambda(S) \mid S \in 2^N\right\} \\ &= \max\{u(Q) - \lambda(Q) \mid Q \in \Sigma\} \\ &\geq \epsilon_u. \end{aligned}$$

Hence $w(S^*) - \bar{\mu}(S^*) = \epsilon_w$, and therefore $\bar{\mu} \in LC(w)$.

Now,

$$\max\{w(S) - \bar{\mu}(S) \mid S \in \mathcal{T}_{ij}\} = \max\{u(Q) - \mu(Q) \mid Q \in \Sigma \cap \mathcal{T}_{ij}\}, \tag{4.1}$$

and

$$\max\{w(S) - \bar{\mu}(S) \mid S \in \mathcal{T}_{ji}\} = \max\{u(Q) - \mu(Q) \mid Q \in \Sigma \cap \mathcal{T}_{ji}\}. \tag{4.2}$$

As $\mu \in PK(u) \cap LC(u)$, we get from (4.1) and (4.2) that $\bar{\mu} \in PK(w) \cap LC(w)$.□

With these results at hand we can now complete the proof of Theorem B.

PROOF OF THEOREM B:

Assume, w.l.o.g., that σ_v is identically zero and $v(N) = 1$. For every $n \in N$ let $S_n = \{1, ..., n\}$, and let v_n be the game given by $v_n(S) = v(S \cap S_n)$. By Lemma 4.2, for each $n \in N$ there is $\mu_n \in PK(v_n) \cap LC(v_n)$. As the games v_n are continuous, then by Theorem A, $\mu_n \in ca$ for each n. Let $\bar{M}$ be the weak*-closure in ba of the set $M = \{\mu_n \mid n \in N\}$. We show that $\bar{M}$ is a norm-compact subset of ca. As $\bar{M}$ is a weak*-compact subset of ba, then by Lemma 2.2 it suffices to show that $\bar{M} \subset ca$. Moreover, since $M \subset ca$, it suffices to show that each $\mu \in \bar{M} \setminus M$ is a member of ca. Let $\mu \in \bar{M} \setminus M$. Then there exists a net $\left\{\mu_{n(\alpha)} \mid \alpha \in D\right\}$ in M which converges to μ. Let $\lambda \in LC(v)$. By Theorem A, $\lambda \in ca$. We distinguish four cases (note that $\lambda \geq 0$ and $\lambda(N) = 1$):

(1) The net $\left\{\lambda\left(S_{n(\alpha)}\right) \mid \alpha \in D\right\}$ does not converges to 1. In this case there is a subnet that converges to a number $a \in [0, 1)$. Assume, w.l.o.g., that $\left\{\lambda\left(S_{n(\alpha)}\right) \mid \alpha \in D\right\}$ itself converges to a. Let $\epsilon > 0$ be such that $a + \epsilon < 1$; then there is $\alpha_0 \in D$ such that for each $\alpha \succeq \alpha_0$ (here $\succeq$ denotes the order relation on D) we have $\lambda\left(S_{n(\alpha)}\right) < a + \epsilon < 1$. As $\lim_{n\to\infty} \lambda(S_n) = 1$, there is $n_0 \in N$ such that $\lambda(S_n) > a + \epsilon$ for every $n > n_0$.

Thus, for each $\alpha \succeq \alpha_0$ we have $n(\alpha) \leq n_0$. Let $S \in 2^N$, As $\{\mu_{n(\alpha)}\}$ converges in the weak*-topology to μ, the net $\{\mu_{n(\alpha)}(S)\}$ converges to $\mu(S)$. Hence there is $\beta_0 \in D$ such that for each $\beta \succeq \beta_0$ we have $\left|\mu_{n(\beta)}(S) - \mu(S)\right| < \epsilon$. From the definition of a net it follows that there is $\gamma \in D$, such that $\gamma \succeq \alpha_0$, and $\gamma \succeq \beta_0$. Therefore $n(\gamma) \leq n_0$, and $\left|\mu_{n(\gamma)}(S) - \mu(S)\right| < \epsilon$.Thus we have shown that if $0 < \epsilon < 1 - a$ is given, then for each $S \in 2^N$ there is $1 \leq n \leq n_0$ such that $|\mu_n(S) - \mu(S)| < \epsilon$. Since the set $\{1, ..., n_0\}$ is finite and ϵ can be chosen arbitrarily small, there is $1 \leq n < n_0$ such that $\mu_n(S) = \mu(S)$.

We now show that this implies that $\mu \in ca$. Assume that $\{T_l\}_{l=1}^{\infty} \subset 2^N$ is a non-decreasing sequence of coalitions such that $\bigcup_{l=1}^{\infty} T_l = N$. Then for every $l \in N$ there is $1 \leq n \leq n_0$ such that $\mu_n(T_l) = \mu(T_l)$. Therefore there is a subsequence $\{l_k\}_{k=1}^{\infty} \subset N$, and $1 \leq n \leq n_0$, such that $\mu(T_{l_k}) = \mu_n(T_{l_k})$ for each $k \in N$. As $\mu_n \in I(v_n) \cap ca_+$, $\lim_{k\to\infty} \mu_n(T_{l_k}) = 1$. Therefore $\lim_{k\to\infty} \mu(T_{l_k}) = 1$. Now $\{\mu(T_l)\}_{l=1}^{\infty}$ is a non-decreasing and bounded sequence of real numbers, and therefore it converges. As $\{\mu(T_{l_k})\}_{k=1}^{\infty}$ is a subsequence of $\{\mu(T_l)\}_{l=1}^{\infty}$ which converges to 1, we have $\lim_{l\to\infty} \mu(T_l) = 1$. Thus, $\mu \in ca$.

(2) The net $\left\{v\left(S_{n(\alpha)}\right)\right\}$ does not converge to 1, then an argument similar to the one given in (1) yields that $\mu \in ca$.

(3) There exists a coalition S such that the net $\left\{v_{n(\alpha)}(S)\right\}$ does not converge to $v(S)$. In this case we use the argument we used in (1) to obtain that $\mu \in ca$.

(4) Cases (1)-(3) do not hold. As $\lambda \in LC(v)$, then for each $\alpha \in D$ and all $S \subset N$ we have

$$v\left(S \cap S_{n(\alpha)}\right) - \lambda\left(S \cap S_{n(\alpha)}\right) \leq \epsilon_v.$$

Since $\lambda\left(S_{n(\alpha)}\right)$converges to 1, we assume, w.l.o.g., that $\lambda\left(S_{n(\alpha)}\right) > 0$ for each $\alpha \in D$. For any $S \subset N$ and $\alpha \in D$ let

$$\lambda_{n(\alpha)}(S) = \frac{v\left(S_{n(\alpha)}\right)}{\lambda\left(S_{n(\alpha)}\right)}\lambda\left(S \cap S_{n(\alpha)}\right),$$

and

$$a_{n(\alpha)} = 1 - \frac{v\left(S_{n(\alpha)}\right)}{\lambda\left(S_{n(\alpha)}\right)}.$$

Then $\lambda_{n(\alpha)} \in I\left(v_{n(\alpha)}\right)$, and the net $\left\{a_{n(\alpha)}\right\}$ converges to zero. Also for every $S \subset N$ we have

$$v_{n(\alpha)}(S) - \lambda_{n(\alpha)}(S) \leq v\left(S \cap S_{n(\alpha)}\right) - \lambda\left(S \cap S_{n(\alpha)}\right) + a_{n(\alpha)}.$$

Therefore

$$\epsilon_{v_{n(\alpha)}} \leq \epsilon_v + a_{n(\alpha)}.$$

As $\mu_{n(\alpha)} \in LC\left(v_{n(\alpha)}\right)$, for every $S \subset N$ we have

$$v_{n(\alpha)}(S) - \mu_{n(\alpha)}(S) \leq \epsilon_v + a_{n(\alpha)}.$$

Hence for every $S \subset N$ we have

$$v(S) - \mu(S) \leq \epsilon_v.$$

Thus $\mu \in LC(v)$. As v is continuous, by Theorem A, $LC(v) \subset ca$. Therefore $\mu \in ca$.

We have shown that in any case $\mu \in ca$. Thus, $\bar{M} \subset ca$, and therefore it is norm-compact in ca. As the sequence $\{\mu_n \mid n \in N\} \subset \bar{M}$, there is a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ which converges in the norm to a member μ of ca. As $\lim_{k\to\infty} \mu(S_{n_k}) = \lim_{k\to\infty} v(S_{n_k}) = 1$, and $\lim_{k\to\infty} v_{n_k}(S) = v(S)$ for each $S \in 2^N$, an argument identical to the one given in (4) above yields $\mu \in LC(v)$. Let $i,j \in N$, $i \neq j$. Since $\mu_{n_k} \in PK(v_{n_k})$ for each $k \in N$, we have $s_{ij}(\mu_{n_k}) = s_{ji}(\mu_{n_k})$. By Lemma 4.1, $\lim_{k\to\infty} s_{ij}(\mu_{n_k}) = s_{ij}(\mu)$, and $\lim_{k\to\infty} s_{ji}(\mu_{n_k}) = s_{ji}(\mu)$. Therefore $s_{ij}(\mu) = s_{ji}(\mu)$, and thus $\mu \in PK(v)$. Hence, $PK(v) \cap LC(v) \neq \emptyset$.□

Wesley[20] showed by using non-standard analysis that a non-negative game v has a non-empty kernel (for every coalition structure) if it satisfied the following conditions:

(4.3) For each $S \in 2^N$, $\lim_{n\to\infty}(v(S) - v(S \cap \{1,...n\})) = 0$; and

(4.4) $\sum_{i=1}^{\infty} r_i < \infty$, where $r_i = \sup\{v(S \cup \{i\}) - v(S) \mid S \subset N \backslash \{i\}\}$.

We show that (4.3) and (4.4) imply that v is continuous, and then we give an example of a continuous game which does not satisfy (4.4). Thus, Wesley's Theorem is a special case of Theorem B when the coalition structure is $\{N\}$.

Lemma 4.3. *Assume that v is a non-negative game which satisfies* (4.3) *and* (4.4). *Then v is continuous.*

Proof: Let $S \in 2^N$. We first show that if $\{S_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of coalitions such that $\bigcup_{n=1}^{\infty} S_n = S$, then $\lim_{n\to\infty} v(S_n) = v(S)$. Indeed, if $\{S_n\}_{n=1}^{\infty}$ is such a sequence then for each $n \in N$ there is $\bar{m}(n)$ such that $m \geq \bar{m}(n)$ implies $S \cap \{1,...,n\} \subset S_m$. As v is monotonic we have

$$v(S \cap \{1,...,n\}) \leq v(S_m) \leq v(S),$$

for every $m \geq \bar{m}(n)$. Thus by (4.3), $\lim_{n\to\infty} v(S_n) = v(S)$.

Assume now that $\{S_n\}_{n=1}^{\infty}$ is a non-increasing sequence of coalitions such that $\bigcap_{n=1}^{\infty} S_n = S$. We show that $\lim_{n\to\infty} v(S_n) = v(S)$. For $A \subset N$ we define $r(A) = \sum_{i\in A} r_i$. Then $r \in ca_+$. Let $n \in N$, and let $B \in 2^N$ be such that $B \cap S = \emptyset$. We show that $v(S \cup B) - v(B) \leq r(B)$. Define $B_n = B \cap \{1,...,n\}$. Then by (4.4)

$$v(S \cup B_n) - v(S) \leq r(B_n).$$

As $B_{n+1} \supset B_n$, and $\bigcup_{n=1}^{\infty} B_n = B$, by what we have just shown, $\lim_{n\to\infty} v(S \cup B_n) = v(S \cup B)$. Since $\lim_{n\to\infty} r(B_n) = r(B)$, we have

$$v(S \cup B) - v(S) \leq r(B).$$

As B was an arbitrary coalition, for each $n \in N$ we have

$$0 \leq v(S_n) - v(S) = v(S \cup (S_n \backslash S)) - v(S) \leq r(S_n \backslash S).$$

As $\lim_{n\to\infty} r(S_n \backslash S) = 0$, we have $\lim_{n\to\infty} v(S_n) = v(S)$.□

We give an example of a non-negative continuous game which does not satisfy (4.4).

Example 4.4. For every $0 \leq x \leq 1$ let $f(x) = 1 - \sqrt{1-x}$. Define a measure $\mu \in ca$ by $\mu(S) = c\sum_{i\in S} \frac{1}{i^2}$, where $c = \left(\sum_{i=1}^{\infty} \frac{1}{i^2}\right)^{-1}$. For every $S \subset N$ let $v(S) = f(\mu(S))$. Since f is continuous on $[0,1]$, v is continuous. Moreover, as f is convex on [0,1], v is convex; that is,

$$v(S_1 \cup S_2) + v(S_1 \cap S_2) \geq v(S_1) + v(S_2)$$

for each $S_1, S_2 \subset N$ (see Shapley[17]), and in particular, v is superadditive. For every $i \in N$ we have

$$r_i = \sup\{v(S \cup \{i\}) - v(S) \mid S \subset N\backslash\{i\}\} \geq v(N) - v(N\backslash\{i\}).$$

As

$$v(N) - v(N \backslash \{i\}) = \frac{\sqrt{c}}{i},$$

we have

$$\sum_{i=1}^{\infty} r_i \geq \sqrt{c} \sum_{i=1}^{\infty} \frac{1}{i}.$$

Thus $\sum_{i=1}^{\infty} r_i = \infty$, and (4.4) is not satisfied.

As it was mentioned above, Maschler, Peleg, and Shapley[12] showed that for superadditive games with a finite set of players the kernel coincides with the prekernel. By Theorems B there are always countably additive payoffs measures in the kernel and the prekernel of a continuous superadditive game on 2^N. Using the compactness of 2^N the same proofs as those of Theorem 2.4 and Theorem 2.7 in Maschler, Peleg, and Shapley[12] yield

Proposition 4.5. *Let v be a continuous (superadditive) game. Then every countably additive payoff measure in $PK(v)$ is individually rational.*

Proposition 4.6. *Let v be a continuous (superadditive) game. Then*

$$K(v) \cap ca = PK(v) \cap ca.$$

5. Bargaining Sets

In this section we establish that the Aumann-Maschler and the Mas-Colell bargaining sets of a continuous game with a countable set of players are non-empty sets which contain the intersection of the prekernel and the least core.

Let v be a game, let $\mu \in I(v)$, and let $i, j \in N$, $i \neq j$, be two players. An *objection* of i to j in μ is a pair (A, λ) such that $\lambda \in ba$, $\lambda \geq \sigma_v$, $A \subset N$, $i \in A$, $j \notin A$, $\lambda(A) \leq v(A)$, and $\lambda(\{k\}) > \mu(\{k\})$ for each $k \in A$. A *counterobjection* of j to (A, λ) is a pair (B, ξ) such that $\xi \in ba$, $\xi \geq \sigma_v$, $B \subset N$, $j \in B$, $i \notin B$, $\xi(B) \leq v(B)$, $\xi(\{k\}) \geq \mu(\{k\})$ for $k \in B \backslash A$, and $\xi(\{k\}) \geq \lambda(\{k\})$ for $k \in A \cap B$. A *justified objection* of $i \in N$ to j in μ is an objection which does not have a counterobjection. The *Aumann-Maschler bargaining* *set* (see Aumann and Maschler[1]) of v is the set $B(v)$ of all imputations $\mu \in I(v)$ such that no player i has a justified objection to another player in μ.

Davis and Maschler[4], and Peleg[14] showed that if v is a game with a finite set of players, then $B(v)$ is a non-empty set. For this class of games, Davis and Maschler[5] proved that $K(v)$ is a subset of $B(v)$. Theorem C below establishes that if v is a continuous game, then the intersection of the prekernel and the least core is contained $B(v)$.

Theorem C. *Let v be a continuous game. Then the $PK(v) \cap LC(v) \subset B(v)$. In particular, $B(v)$ contains a countably additive payoff measure.*

Proof: W.l.o.g. assume that σ_v is identically zero. By Theorem B, $PK(v) \cap LC(v) \neq \emptyset$. Let $\mu \in PK(v) \cap LC(v)$. By Theorem A, $\mu \in ca$. We show that $\mu \in B(v)$. Assume to the contrary that $\mu \notin B(v)$. Then there are $i, j \in N$, $i \neq j$, such that j has a justified objection (A, λ) to i in μ. By the Yosida and Hewitt Theorem[21], λ can be uniquely decomposed into a sum of a non-negative countably additive measure λ_c, and a non-negative purely finitely additive measure λ_p (note that $\lambda \geq \sigma_v$, and thus $\lambda \geq 0$). As λ_p vanishes on finite subsets of N, we have $\lambda_c(\{k\}) > \mu(\{k\})$ for each $k \in A$, and $\lambda_c(A) \leq \lambda(A) \leq v(A)$. Since any counterobjection to (A, λ_c) is also a counterobjection to (A, λ), (A, λ_c) is a justified objection. Since v is continuous and $\mu \in ca$, there is $B \in 2^N$ such that $i \in B$, $j \notin B$, and

$$s_{ij}(\mu) = v(B) - \mu(B).$$

As $\mu \in PK(v)$,

$$v(B) - \mu(B) = s_{ji}(\mu) \geq v(A) - \mu(A).$$

Since $\lambda_c(\{j\}) > \mu(\{j\})$,

$$\begin{aligned} v(A) - \mu(A) &\geq \sum_{k \in A} (\lambda_c(\{k\}) - \mu(\{k\})) \\ &> \sum_{k \in A \cap B} (\lambda_c(\{k\}) - \mu(\{k\})). \end{aligned}$$

Therefore

$$v(B) - \sum_{k \in A} \mu(\{k\}) > \sum_{k \in A \cap B} (\lambda_c(\{k\}) - \mu(\{k\}));$$

i.e.,

$$v(B) > \sum_{k \in A \cap B} \lambda_c(\{k\}) + \sum_{k \in B \setminus A} \mu(\{k\}).$$

For each $S \subset N$ let $\xi(S) = \lambda_c(S \cap A) + \mu(S \cap (B \setminus A))$. Then $\xi(B) \leq v(B)$ by the last inequality, and if $k \in A \cap B$, $\xi(\{k\}) = \lambda_c(\{k\})$, and if $l \in B \setminus A$, then $\xi(\{l\}) = \mu(\{l\})$. Therefore each member of B (and in particular player i) has a counterobjection to the objection (A, λ_c) of j, which contradicts the fact that (A, λ_c) is a justified objection.□

Mas Colell[13] proposed a notion of bargaining set different from that of Aumann and Maschler bargaining set, and showed that in the context of a market with a continuum of players, this new bargaining set coincides with the set of wallrasian allocations. The advantage of the Mas-Colell bargaining set is that it can be defined for games with an uncountable set of players.

The following definitions are taken from Einy et al.[8], who provide a straightforward generalization of the Mas-Colell[13] definition to games with an infinite set of players. Let v be a game, and let μ be a payoff measure in v. An *objection* to μ (in the sense of Mas-Colell) is a pair (A, λ) such that $A \in 2^N$, and $\lambda \in ba$ satisfies $\lambda(A) \leq v(A)$, $\lambda(A) > \mu(A)$, and $\lambda(B) \geq \mu(B)$ for every coalition $B \subset A$. A *counterobjection* (in the sense of Mas-Colell) to the objection (A, λ) is a pair (C, ξ) such that

(5.1) $\xi \in ba$, and $\xi(C) \leq v(C)$;

(5.2) For every $B \subset A \cap C$, $\xi(B) \geq \lambda(B)$, and for every $D \subset C \setminus A$, $\xi(D) \geq \mu(D)$; and

(5.3) $\xi(C) > \lambda(A \cap C) + \mu(C \setminus A)$.

A *justified objection* is an objection which has no counterobjection. The *Mas-Colell bargaining set* of v is the set $MB(v)$ of all payoff measures which have no justified objection.

Mas-Colell[13] in considering exchange economies with a continuum of agents, defines the bargaining set without restricting attention to individually rational allocations. Thus, his equivalence result holds for a large set. From the point of view of an existence result it will be interesting to show that the Mas-Colell bargaining set always contains an individually rational payoff measure. Theorem D below establishes that the Mas-Colell bargaining set of a continuous game contains the intersection of the prekernel and the least core of this game. As a consequence, for these games the Mas-Colell bargaining set contains an individually rational countably additive payoff measure.

Theorem D. *Let v be a continuous game. Then $PK(v) \cap LC(v) \subset MB(v)$. In particular, $MB(v)$ contains an individually rational countably additive payoff measure.*

The following lemma will be useful in the proof of Theorem D.

Lemma 5.1. *Let v be game, and let μ be a payoff measure in v. If (A, λ) is a justified objection to μ, then $v(B) \leq \lambda(B \cap A) + \mu(B \backslash A)$ for each $B \in \Sigma$.*

Proof: Assume to the contrary that there is $B \in \Sigma$ such that $v(B) > \lambda(B \cap A) + \mu(B \backslash A)$. Then $B \neq \emptyset$. Let $\epsilon = v(B) - (\lambda(B \cap A) + \mu(B \backslash A))$. Choose $t \in B$, and let δ_t be the probability measure concentrated on $\{t\}$. Define $\xi \in ba$ by

$$\xi(S) = \lambda(S \cap (B \cap A)) + \mu(S \cap (B \backslash A)) + \epsilon \delta_t(S).$$

Thus, if $S \in \Sigma$ satisfies $S \subset A \cap B$ we have $\xi(S) \geq \lambda(S)$, and if $S \in \Sigma$ satisfies $S \subset B \backslash A$ we have $\xi(S) \geq \mu(S)$. Also

$$\xi(B) = v(B) > \lambda(B \cap A) + \mu(B \backslash A).$$

Thus (B, ξ) is a counterobjection to (A, λ), which is a contradiction.□

PROOF OF THEOREM D:

W.l.o.g. assume that σ_v is identically zero. By Theorem B, $PK(v) \cap LC(v) \neq \emptyset$. Let $\mu \in PK(v) \cap LC(v)$. By Theorem A, $\mu \in ca$. We show that $\mu \in MB(v)$. Assume by way of contradiction that $\mu \notin MB(v)$. Then there is a justified objection (A, λ) to μ. Since μ is a payoff measure, $A \neq N$. Assume, w.l.o.g., that $\lambda(A) = v(A)$. Since (A, λ) is a justified objection to μ, by Lemma 5.1 for each $B \subset A$ we have $\lambda(B) \geq v(B)$. For each $S \in 2^N$ let $v_A(S) = v(S \cap A)$, and $\lambda_A(S) = \lambda(S \cap A)$. Then λ_A is in the core of the game v_A. Since v_A is continuous, $\lambda_A \in ca$. As $\lambda(A) > \mu(A)$, there is $j \in A$ such that $\lambda_A(\{j\}) > \mu(\{j\})$. Let $i \in N \backslash A$. As $\mu \in PK(v)$, there is $C \subset N$, $i \in C$, $j \notin C$, such that $v(C) - \mu(C) = s_{ij}(\mu) \geq v(A) - \mu(A)$. Since $\lambda_A(\{j\}) > \mu(\{j\})$, an argument identical to the one given in the proof of Theorem C yields

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$$\begin{aligned} v(C) &> \sum_{k \in A \cap C} \lambda_A(\{k\}) + \sum_{k \in C \setminus A} \mu(\{k\}) \\ &= \lambda(A \cap C) + \mu(C \setminus A). \end{aligned}$$

By Lemma 5.1 this contradicts the assumption that (A, λ) is a justified objection to μ.$\square$

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