UNIVERSIDAD CARLOS III DE MADRID

working papers

Working Paper 09-21
Statistics and Econometrics Series 08
March 2009

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

ON THE CONJECTURE OF KOCHAR AND KORWAR*

Nuria Torrado[a], Rosa E. Lillo[b], Michael P. Wiper[b]

Abstract

In this paper, we solve for some cases a conjecture by Kochar and Korwar (1996) in relation with the normalized spacings of the order statistics related to a sample of independent exponential random variables with different scale parameter. In the case of a sample of size n=3, they proved the ordering of the normalized spacings and conjectured that result holds for all n. We give the proof of this conjecture for n=4 and for both spacing and normalized spacings. We also generalize some results to n>4.

Keywords: Heterogeneous exponential distribution, Hazard rate order, Normalized spacing.

* The research of two of the authors (Nuria Torrado and Rosa E. Lillo) was supported by Comunidad de Madrid, project CCG07-UC3M/HUM-3260.

[a] Universidad Carlos III de Madrid, Department of Statistics, Escuela Politécnica Superior, Campus de Leganés, Madrid, Spain. e-mail address: nuria.torrado@uc3m.es

[b] Universidad Carlos III de Madrid, Department of Statistics, Facultad de Ciencias Sociales y Jurídicas, Campus de Getafe, Madrid, Spain. e-mail addresses: rosaelvira.lillo@uc3m.es (Rosa E. Lillo) and michael.wiper@uc3m.es (Michael P. Wiper).

1 Introduction

Given a set of independent random variables, $X_1, X_2, \ldots, X_n$, let the order statistics of these variables be $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Then, the random variables

$$D_{i:n} = X_{i:n} - X_{i-1:n}$$

and

$$D^*_{i:n} = (n - i + 1)\left(X_{i:n} - X_{i-1:n}\right)$$

for $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, are called spacings and normalized spacings respectively.

Spacings and their functions are important in statistics, in general, and in particular in the context of life testing and reliability models. See the book edited by Balakrishnan and Basu [1] and two volumes of papers on this topic by Balakrishnan and Rao [2, 3].

Many authors have studied the stochastic properties of spacings of independent and identically distributed (i.i.d.) random variables. In particular, the exponential distribution shows no ageing over time and has constant failure rates, and spacings correspond to times elapsed between successive failures of components in a system. A remarkable property of the exponential distribution, first reported by Sukhatme [11], in 1937, is that the normalized spacings of a random sample from an exponential distribution are i.i.d. random variables with same exponential distribution. More generally, if $X_1, X_2, \ldots, X_n$ is an i.i.d. random sample from a *decreasing failure rate* DFR distribution, then it has been proved in [4] that the successive normalized spacings are stochastically increasing. Kochar and Kirmani [6] strengthened this result from stochastic ordering to hazard rate ordering, that is, for $i = 1, \ldots, n-1$

$$D^*_{i:n} \preccurlyeq_{hr} D^*_{i+1:n}.$$

Spacings of non identically distributed variables have also been considered in the literature. Pledger and Proschan [9] proved that if the scale parameters of the exponential distributions are not all equal then the i'th normalized spacing is stochastically smaller than the $(i+1)$'th normalized spacing. They also considered the problem of stochastically comparing the order statistics and the spacings of nonidentical independent exponential random variables with those corresponding to i.i.d. exponential random variables with the mean of all scale parameters of nonidentical independent exponential random variables as parameter. Kochar and Korwar [7] strengthened this last result from stochastic ordering to likelihood ratio ordering and conjectured that the successive normalized spacings are increasing in hazard rate ordering in the case when $X_1, X_2, \ldots, X_n$ are independent exponential random variables with X_i having hazard rates λ_i for $i = 1, \ldots, n$. They proved this conjecture for

$n = 3$. In a recent paper, Wen et al.[12] established likelihood ratio ordering between consecutive spacings from a multiple-outlier exponential model and conjectured that this result can be strengthened to spacings from heterogeneous exponential variables. They also noticed that the result of Kochar and Korwar [7] continues being a conjecture. The reader is referred to Khaledi and Kochar [5] for a review of some results in the area of stochastic comparisons of order statistics and spacings.

The purpose of this paper is to investigate the hazard rate ordering of spacings and normalized spacings of heterogeneous exponential random variables. In particular, we prove the conjecture of Kochar and Korwar [7] for $n = 4$ and we show that the successive spacings are increasing in hazard rate ordering. We also generalize some earlier results, that is we show that $D_{2:n} \preccurlyeq_{hr} D_{3:n}$ and $D^*_{2:n} \preccurlyeq_{hr} D^*_{3:n}$ for any n.

The rest of the paper is organized as follows. In Section 2, we recall the definitions of some stochastic orders and of the probability density function (p.d.f.) of normalized spacings, and give two useful lemmas which will be used in the following sections. We prove the conjecture of Kochar and Korwar [7] for $n = 4$ in Section 3 and strengthen this result for generalized spacings in Section 4. Section 5 establishes some generalizations of the previous results for both spacings and normalized spacings. Section 6 makes some concluding remarks and conjectures. Finally, some proofs are given in the Appendix.

2 Preliminaries and notation

In this section we review some definitions and well-known notions of stochastic orders, and also give some useful lemmas which will be used later.

Let X and Y be univariate random variables with cumulative distribution functions (c.d.f.'s) F and G, survival functions $\overline{F}\,(= 1 - F)$ and $\overline{G}\,(= 1 - G)$, p.d.f.'s f and g, and failure rate functions $r_F\left(= f/\overline{F}\right)$ and $r_G\left(= g/\overline{G}\right)$, respectively. Let $l_X(l_Y)$ and $u_X(u_Y)$ be the left and the right endpoints of the support of $X(Y)$. The following definitions introduce the stochastic orders we consider in this article.

Definition 2.1. X is said to be smaller than Y in the usual stochastic order, denoted by $X \preccurlyeq_{st} Y$, if $\overline{F}(t) \leq \overline{G}(t)$ for all t.

Definition 2.2. X is said to be smaller than Y in the hazard rate order, denoted by $X \preccurlyeq_{hr} Y$, if $G(t)/F(t)$ is increasing in t for which the ratio $G(t)/F(t)$ is well defined.

When the failure rate function exists, it is easy to see that $X \preccurlyeq_{hr} Y$, if and only if $r_G(t) \leq r_F(t)$ for all t.

Definition 2.3. X is said to be smaller than Y in likelihood ratio ordering, denoted by $X \preccurlyeq_{lr} Y$, if $g(t)/f(t)$ is increasing in $t \in (l_X, u_X) \cup (l_Y, u_Y)$.

Likelihood ratio ordering implies hazard rate ordering which in turn implies stochastic ordering. For more details on stochastic orderings see Shaked and Shanthikumar [10].

For heterogeneous but independent exponential random variables, Kochar and Korwar [7] proved that, for $i \in \{2, \dots, n\}$, the distribution of D_i^* is a mixture of independent exponential random variables with p.d.f.:

$$f_i(t) = \sum_{\mathbf{r}_n} \frac{\prod_{k=1}^n \lambda_k}{\prod_{k=1}^n \left(\sum_{j=k}^n \lambda(r_j)\right)} \cdot \frac{\sum_{j=i}^n \lambda(r_j)}{n-i+1} \cdot exp\left\{-t\,\frac{\sum_{j=i}^n \lambda(r_j)}{n-i+1}\right\} \tag{2.1}$$

where $\mathbf{r}_n = (r_1, \dots, r_n)$ is a permutation of $(1, \dots, n)$ and $\lambda(i) = \lambda_i$. They also showed that $D_{1:n}^*$ is independent of $(D_{2:n}^*, \dots, D_{n:n}^*)$ and due to this result, they could show that $D_{1:n}^* \preccurlyeq_{lr} D_{i:n}^*$ for $i = 2, \dots, n$.

Observe that in equation (2.1) the term $\frac{\sum_{j=i}^n \lambda(r_j)}{n-i+1}$ coincides for all permutations $\mathbf{r}_n$ that have the same groups of λ_k's in the last $n-i+1$ positions. This permits us to simplify the notation as follows. Let

$$\beta_{m_j}^i = \frac{\sum_{l=i}^n \lambda(r_l)}{n-i+1} \tag{2.2}$$

where m_j indicates a group of indices of size $n-i+1$. Then, (2.1) can be written as

$$f_i(t) = \sum_{j=1}^{M_i} \Delta(\beta_{m_j}^i, n)\, \beta_{m_j}^i\, e^{-t\beta_{m_j}^i} \tag{2.3}$$

where $M_i = \binom{n}{n-i+1}$ and

$$\Delta(\beta_{m_j}^i, n) \;=\; \sum_{\mathbf{r}_{i-1,m_j}} \left(\prod_{k\in H_{m_j}} \lambda_k\right) \left[\prod_{l=1}^{i-1} \left\{ \sum_{\substack{u=l \\ r(u)\in H_{m_j}}}^{i-1} \lambda(r(u)) + (n-i+1)\beta_{m_j}^i \right\}\right]^{-1}, \tag{2.4}$$

where $H_{m_j} = \{1, \dots, n\} - m_j$ and the outer summation is being taken over all permutations of the elements of H_{m_j}. Note that equation (2.4) and equation (2.3) of Kochar and Korwar [7] are equivalent, although with different notation.

Before proceeding to our main results, we recall two lemmas, which will be used repeatedly in the following sections.

Lemma 2.4 (Lemma 3.1., in Kochar and Korwar [7])**.** *Let $\Delta(\beta^i_{m_j}, n)$ be as defined in (2.4). Suppose that m_1 and m_2 are two subsets of $\{1,\ldots,n\}$ of size $n-i+1$ $(1 < i \le n)$ and having all but one element in common. Denote the uncommon element in m_1 by a_1 and that in m_2 by a_2. Then:*

$$\lambda(a_1)\Delta(\beta^i_{m_1}, n) \ge \lambda(a_2)\Delta(\beta^i_{m_2}, n) \qquad \text{if} \quad \lambda(a_2) \ge \lambda(a_1).$$

Lemma 2.5 (Chebyshev inequality, Theorem 1, in Mitrinovic [8])**.** *Let $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$ be two increasing sequences of real numbers. Then*

$$n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

3 Normalized spacings

In this section we prove some results on hazard rate ordering between the normalized spacings. Kochar and Korwar [7] conjectured that

$$D^*_{i:n} \preccurlyeq_{hr} D^*_{i+1:n} \qquad \text{for} \quad i = 1,\ldots,n$$

for heterogeneous exponential random variables. We give below the proof of this result for $n = 4$. They established likelihood ratio ordering between the first normalized spacings and the others, in particular $D^*_{1:4} \preccurlyeq_{hr} D^*_{2:4}$, so we have to show $D^*_{2:4} \preccurlyeq_{hr} D^*_{3:4}$ and $D^*_{3:4} \preccurlyeq_{hr} D^*_{4:4}$. Observing equation (2.3), note that $D^*_{i:n} \preccurlyeq_{hr} D^*_{i+1:n}$ if and only if

$$r_{i+1}(t) = \frac{\sum_{j=1}^{M_{i+1}} \Delta(\beta^{i+1}_{m_j}, n)\, \beta^{i+1}_{m_j}\, e^{-t\beta^{i+1}_{m_j}}}{\sum_{j=1}^{M_{i+1}} \Delta(\beta^{i+1}_{m_j}, n)\, e^{-t\beta^{i+1}_{m_j}}} \le \frac{\sum_{j=1}^{M_i} \Delta(\beta^{i}_{m_j}, n)\, \beta^{i}_{m_j}\, e^{-t\beta^{i}_{m_j}}}{\sum_{j=1}^{M_i} \Delta(\beta^{i}_{m_j}, n)\, e^{-t\beta^{i}_{m_j}}} = r_i(t),$$

which can be rewritten as

$$\sum_{j=1}^{M_{i+1}} \sum_{k=1}^{M_i} \Delta(\beta^i_{m_k}, n)\Delta(\beta^{i+1}_{m_j}, n)\, e^{-t\left(\beta^i_{m_k} + \beta^{i+1}_{m_j}\right)} \left(\beta^i_{m_k} - \beta^{i+1}_{m_j}\right) \ge 0 \tag{3.5}$$

Throughout this paper we suppose without loss of generality that the λ_i's are in increasing order.

Theorem 3.1. *Let $X_1,\ldots,X_n$ be independent, exponential random variables with X_i having survival function $\overline{F}_i(t) = exp(-\lambda_i t)$, $t \ge 0$, for $i = 1,\ldots,n$. Then*

$$D^*_{3:4} \preccurlyeq_{hr} D^*_{4:4}.$$

Proof. We have to show that

$$\sum_{j=1}^{M_4} \sum_{k=1}^{M_3} \Delta(\beta^3_{m_k}, 4)\Delta(\beta^4_{m_j}, 4)\, e^{-t\left(\beta^3_{m_k}+\beta^4_{m_j}\right)} \left(\beta^3_{m_k} - \beta^4_{m_j}\right) \geq 0 \tag{3.6}$$

First, let us examine the values of $\beta^3_{m_k} - \beta^4_{m_j}$ where $M_3 = 6$ and $M_4 = 4$.

$$\begin{pmatrix} \frac{\lambda_3+\lambda_4}{2} - \lambda_1 & \frac{\lambda_3+\lambda_4}{2} - \lambda_2 & \frac{\lambda_3+\lambda_4}{2} - \lambda_3 & \frac{\lambda_3+\lambda_4}{2} - \lambda_4 \\ \frac{\lambda_2+\lambda_4}{2} - \lambda_1 & \frac{\lambda_2+\lambda_4}{2} - \lambda_2 & \frac{\lambda_2+\lambda_4}{2} - \lambda_3 & \frac{\lambda_2+\lambda_4}{2} - \lambda_4 \\ \frac{\lambda_2+\lambda_3}{2} - \lambda_1 & \frac{\lambda_2+\lambda_3}{2} - \lambda_2 & \frac{\lambda_2+\lambda_3}{2} - \lambda_3 & \frac{\lambda_2+\lambda_3}{2} - \lambda_4 \\ \frac{\lambda_1+\lambda_4}{2} - \lambda_1 & \frac{\lambda_1+\lambda_4}{2} - \lambda_2 & \frac{\lambda_1+\lambda_4}{2} - \lambda_3 & \frac{\lambda_1+\lambda_4}{2} - \lambda_4 \\ \frac{\lambda_1+\lambda_3}{2} - \lambda_1 & \frac{\lambda_1+\lambda_3}{2} - \lambda_2 & \frac{\lambda_1+\lambda_3}{2} - \lambda_3 & \frac{\lambda_1+\lambda_3}{2} - \lambda_4 \\ \frac{\lambda_1+\lambda_2}{2} - \lambda_1 & \frac{\lambda_1+\lambda_2}{2} - \lambda_2 & \frac{\lambda_1+\lambda_2}{2} - \lambda_3 & \frac{\lambda_1+\lambda_2}{2} - \lambda_4 \end{pmatrix} \tag{3.7}$$

This construction was motivated in the proof of Theorem 3.6 of Kochar and Korwar [7]. Our main idea is to find coefficients of the matrix (3.7) which sum to zero. We can divide these values into two types:

$$\begin{aligned} \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_j\right) + \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_k\right) &= 0 \\ \left(\frac{\lambda_k+\lambda_l}{2} - \lambda_j\right) + \left(\frac{\lambda_j+\lambda_l}{2} - \lambda_k\right) + \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_l\right) &= 0 \end{aligned}$$

for $j, k, l = 1, \ldots, 4$ and $j < k < l$. To simplify the notation let $\beta^3_{m_k} = \frac{\lambda_j+\lambda_l}{2}$ and $\beta^4_{m_j} = \lambda_j$ be, if $m_k = (j, l)$ and $m_j = j$, respectively. Then, inequality (3.6) can be written as

$$\sum_{j=1}^{4} \sum_{k=j+1}^{4} A_{(j,k)} + \sum_{u=1}^{4} B_u \geq 0$$

where

$$\begin{aligned} A_{(j,k)} \;=\;& \Delta(\beta^3_{(j,k)}, 4)\, e^{-t\left(\frac{\lambda_j+\lambda_k}{2}\right)} \\ & \left[\Delta(\beta^4_j, 4)\, e^{-t\lambda_j} \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_j\right) + \Delta(\beta^4_k, 4)\, e^{-t\lambda_k} \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_k\right)\right] \end{aligned}$$

and

$$\begin{aligned} B_u =\; & \Delta(\beta^3_{(k,l)}, 4)\Delta(\beta^4_j, 4)\, e^{-t\left(\frac{\lambda_k+\lambda_l}{2}+\lambda_j\right)} \left(\frac{\lambda_k+\lambda_l}{2} - \lambda_j\right) \\ + \; & \Delta(\beta^3_{(j,l)}, 4)\Delta(\beta^4_k, 4)\, e^{-t\left(\frac{\lambda_j+\lambda_l}{2}+\lambda_k\right)} \left(\frac{\lambda_j+\lambda_l}{2} - \lambda_k\right) \\ + \; & \Delta(\beta^3_{(j,k)}, 4)\Delta(\beta^4_l, 4)\, e^{-t\left(\frac{\lambda_j+\lambda_k}{2}+\lambda_l\right)} \left(\frac{\lambda_j+\lambda_k}{2} - \lambda_l\right) \end{aligned}$$

where $u \notin (j,k,l)$. We divide the proof into two parts according to different types of addition. First, we will see that $A_{(j,k)}$ are positive for all $j<k$. After some manipulations we can see that

$$A_{(j,k)} = \Delta(\beta^3_{(j,k)},4)\, e^{-t\left(\frac{\lambda_j+\lambda_k}{2}\right)} \left(\frac{\lambda_k-\lambda_j}{2}\right) \left[\Delta(\beta^4_j,4)\, e^{-t\lambda_j} - \Delta(\beta^4_k,4)\, e^{-t\lambda_k}\right].$$

Lemma 2.4 and $\lambda_j \le \lambda_k$ imply that

$$1 \le \frac{\lambda_k}{\lambda_j} \le \frac{\Delta(\beta^4_j,4)}{\Delta(\beta^4_k,4)}.$$

Then $A_{(j,k)} \ge 0$ since $\Delta(\beta^4_j,4) \ge \Delta(\beta^4_k,4)$ and $e^{-t\lambda_j} \ge e^{-t\lambda_k}$.

Next, we are interested in proving that B_u are positive for all u. Notice that

$$a_{u,1} = \frac{\lambda_k+\lambda_l}{2} - \lambda_j, \quad a_{u,2} = \frac{\lambda_j+\lambda_l}{2} - \lambda_k, \quad a_{u,3} = \frac{\lambda_j+\lambda_k}{2} - \lambda_l$$

and

$$e^{-t\left(\frac{\lambda_k+\lambda_l}{2}+\lambda_j\right)}, \quad e^{-t\left(\frac{\lambda_j+\lambda_l}{2}+\lambda_k\right)}, \quad e^{-t\left(\frac{\lambda_j+\lambda_k}{2}+\lambda_l\right)}$$

are decreasing.

Now, if $u = 1$ or 2, using Lemma A.1 in the Appendix, we find that

$$\Delta(\beta^3_{(3,4)},4)\Delta(\beta^4_j,4) \ge \Delta(\beta^3_{(j,4)},4)\Delta(\beta^4_3,4) \ge \Delta(\beta^3_{(j,3)},4)\Delta(\beta^4_4,4) \tag{3.8}$$

From this, we conclude that

$$\begin{aligned} b_{u,1} &= \Delta(\beta^3_{(3,4)},4)\Delta(\beta^4_{3-u},4)\, e^{-t\left(\frac{\lambda_3+\lambda_4}{2}+\lambda_{3-u}\right)} \\ b_{u,2} &= \Delta(\beta^3_{(3-u,4)},4)\Delta(\beta^4_3,4)\, e^{-t\left(\frac{\lambda_{3-u}+\lambda_4}{2}+\lambda_3\right)} \\ b_{u,3} &= \Delta(\beta^3_{(3-u,3)},4)\Delta(\beta^4_4,4)\, e^{-t\left(\frac{\lambda_{3-u}+\lambda_3}{2}+\lambda_4\right)} \end{aligned}$$

are decreasing in $h = 1,2,3$. Note that B_u can be written as $\sum_{h=1}^3 a_{u,h} b_{u,h}$. Finally, by Lemma 2.5,

$$B_u = \sum_{h=1}^3 a_{u,h} b_{u,h} \ge \left(\sum_{h=1}^3 a_{u,h}\right)\left(\sum_{h=1}^3 b_{u,h}\right)/3 = 0$$

since $\sum_{h=1}^3 a_{u,h} = 0$. Now, if $u = 3$ or 4, we have that

$$\Delta(\beta^3_{(2,l)},4)\Delta(\beta^4_1,4) \quad \ge \quad \Delta(\beta^3_{(1,2)},4)\Delta(\beta^4_l,4) \tag{3.9}$$

$$\Delta(\beta^3_{(1,l)},4)\Delta(\beta^4_2,4) \quad \geq \quad \Delta(\beta^3_{(1,2)},4)\Delta(\beta^4_l,4) \tag{3.10}$$

and if $\beta^3_{1,l} - \beta^4_2 < 0$

$$\Delta(\beta^3_{(2,l)},4)\Delta(\beta^4_1,4) \quad \geq \quad \Delta(\beta^3_{(1,l)},4)\Delta(\beta^4_2,4) \tag{3.11}$$

The proof of (3.9)-(3.11) is given in Lemma A.2 in the Appendix.

It is easy to check that $a_{u,3} = -(a_{u,1} + a_{u,2}) < 0$ and if $\beta^3_{1,l} - \beta^4_2 > 0$ then

$$\begin{aligned} B_u = \sum_{h=1}^{3} a_{u,h} b_{u,h} \quad &\geq \quad min\{b_{u,1}, b_{u,2}\}(a_{u,1} + a_{u,2}) + a_{u,3} b_{u,3} = \\ &-a_{u,3}(min\{b_{u,1}, b_{u,2}\} - b_{u,3}) \geq 0, \end{aligned}$$

where

$$\begin{aligned} b_{u,1} &= \Delta(\beta^3_{(2,l)},4)\Delta(\beta^4_1,4)\, e^{-t\left(\frac{\lambda_2+\lambda_l}{2}+\lambda_1\right)} \\ b_{u,2} &= \Delta(\beta^3_{(1,l)},4)\Delta(\beta^4_2,4)\, e^{-t\left(\frac{\lambda_1+\lambda_l}{2}+\lambda_2\right)} \\ b_{u,3} &= \Delta(\beta^3_{(1,2)},4)\Delta(\beta^4_l,4)\, e^{-t\left(\frac{\lambda_1+\lambda_2}{2}+\lambda_l\right)} \end{aligned}$$

and $min\{b_{u,1}, b_{u,2}\} \geq b_{u,3}$ by (3.9) and (3.10). However, if $\beta^3_{1,l} - \beta^4_2 < 0$, $b_{u,1} \geq b_{u,2} \geq b_{u,3}$ and again by Lemma 2.5 $B_u \geq 0$. Hence (3.6) holds, which implies that $D^*_{3:4} \preccurlyeq_{hr} D^*_{4:4}$ and the proof is complete. □

Theorem 3.2. *Under the same assumptions that in Theorem 3.1*

$$D^*_{2:4} \preccurlyeq_{hr} D^*_{3:4}.$$

Proof. We have to show

$$\sum_{j=1}^{M_3} \sum_{k=1}^{M_2} \Delta(\beta^2_{m_k},4)\Delta(\beta^3_{m_j},4)\, e^{-t\left(\beta^2_{m_k}+\beta^3_{m_j}\right)} \left(\beta^2_{m_k} - \beta^3_{m_j}\right) \geq 0 \tag{3.12}$$

To do this, we consider the values of $\beta^2_{m_k} - \beta^3_{m_j}$ which add zero for each $k = 1, \ldots, 4$ and $j = 1, \ldots, 6$ in the next matrix transpose.

$$\begin{pmatrix} \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_1+\lambda_2}{2} \\ \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_1+\lambda_3}{2} \\ \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_1+\lambda_4}{2} \\ \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_3}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_2+\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_2+\lambda_3}{2} \\ \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_4}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_2+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_2+\lambda_4}{2} \\ \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_3+\lambda_4}{2} & \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_3+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_3+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_3+\lambda_4}{2} \end{pmatrix} \tag{3.13}$$

To simplify the notation let $\beta_u^2 = \frac{\lambda_j+\lambda_k+\lambda_l}{3}$ where $u \notin \{j,k,l\}$ and $\beta_{(j,k)}^3 = \frac{\lambda_j+\lambda_k}{2}$ be. There are three elements in each row of the transpose of the matrix (3.13) that sum to zero, that is

$$\begin{array}{ccccccc} (\beta_u^2 - \beta_{(j,k)}^3) & + & (\beta_u^2 - \beta_{(j,l)}^3) & + & (\beta_u^2 - \beta_{(k,l)}^3) & = & \\ \left(\frac{\lambda_j+\lambda_k+\lambda_l}{3} - \frac{\lambda_j+\lambda_j}{2}\right) & + & \left(\frac{\lambda_j+\lambda_k+\lambda_l}{3} - \frac{\lambda_j+\lambda_l}{2}\right) & + & \left(\frac{\lambda_j+\lambda_k+\lambda_l}{3} - \frac{\lambda_k+\lambda_l}{2}\right) & = & 0 \end{array}$$

for $j,k,l = 1,\dots,4$ and $j<k<l$. We are interested in proving

$$\begin{aligned} \Delta(\beta_u^2,4)\, e^{-t\left(\frac{\lambda_j+\lambda_k+\lambda_l}{3}\right)} \quad & \Big[\Delta(\beta_{(j,k)}^3,4)\, e^{-t\left(\frac{\lambda_j+\lambda_k}{2}\right)} \left(\beta_u^2 - \beta_{(j,k)}^3\right) \\ & + \Delta(\beta_{(j,l)}^3,4)\, e^{-t\left(\frac{\lambda_j+\lambda_l}{2}\right)} \left(\beta_u^2 - \beta_{(j,l)}^3\right) \\ & + \Delta(\beta_{(k,l)}^3,4)\, e^{-t\left(\frac{\lambda_k+\lambda_l}{2}\right)} \left(\beta_u^2 - \beta_{(k,l)}^3\right) \Big] \geq 0 \end{aligned} \tag{3.14}$$

for $u = 1,\dots,4$.
Notice that

$$a_1 = \beta_u^2 - \beta_{(j,k)}^3, \quad a_2 = \beta_u^2 - \beta_{(j,l)}^3, \quad a_3 = \beta_u^2 - \beta_{(k,l)}^3$$

and

$$e^{-t\left(\frac{\lambda_j+\lambda_k}{2}\right)}, \quad e^{-t\left(\frac{\lambda_j+\lambda_l}{2}\right)}, \quad e^{-t\left(\frac{\lambda_k+\lambda_l}{2}\right)}$$

are decreasing in $h = 1,2,3$. It follows from Lemma 2.4

$$\Delta(\beta_{(j,k)}^3,4) \geq \Delta(\beta_{(j,l)}^3,4) \geq \Delta(\beta_{(k,l)}^3,4) \tag{3.15}$$

Then

$$\begin{aligned} b_1 &= \Delta(\beta_{(j,k)}^3,4)\, e^{-t\left(\frac{\lambda_j+\lambda_k}{2}\right)} \\ b_2 &= \Delta(\beta_{(j,l)}^3,4)\, e^{-t\left(\frac{\lambda_j+\lambda_l}{2}\right)} \\ b_3 &= \Delta(\beta_{(k,l)}^3,4)\, e^{-t\left(\frac{\lambda_k+\lambda_l}{2}\right)} \end{aligned}$$

are decreasing in $h = 1,2,3$. Finally, by Lemma 2.5, we conclude that (3.14) holds since $\sum_{h=1}^3 a_i = 0$. We group twelve remaining values of the matrix (3.13) in four diagonals.

$$\begin{pmatrix} a_{1,2} = \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_2}{2} \geq a_{1,3} = \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_1+\lambda_3}{2} \geq a_{1,4} = \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_1+\lambda_4}{2} \\ a_{2,1} = \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_2}{2} \geq a_{2,3} = \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_2+\lambda_3}{2} \geq a_{2,4} = \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_2+\lambda_4}{2} \\ a_{3,1} = \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_3}{2} \geq a_{3,2} = \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_3}{2} \geq a_{3,4} = \frac{\lambda_1+\lambda_2+\lambda_3}{3} - \frac{\lambda_3+\lambda_4}{2} \\ a_{4,1} = \frac{\lambda_2+\lambda_3+\lambda_4}{3} - \frac{\lambda_1+\lambda_4}{2} \geq a_{4,2} = \frac{\lambda_1+\lambda_3+\lambda_4}{3} - \frac{\lambda_2+\lambda_4}{2} \geq a_{4,3} = \frac{\lambda_1+\lambda_2+\lambda_4}{3} - \frac{\lambda_3+\lambda_4}{2} \end{pmatrix} \tag{3.16}$$

Firstly we prove that the coefficients $\Delta(\beta^2_{m_k},4)\Delta(\beta^3_{m_j},4)$ in (3.12) related to the four diagonals are also ordered. We give the proof only for the first diagonal, the other cases are similar. Again by Lemma 2.4,

$$\begin{array}{ccccc}\Delta(\beta^2_2,4)\Delta(\beta^3_{(1,2)},4) & \geq & \Delta(\beta^2_3,4)\Delta(\beta^3_{(1,3)},4) & \geq & \Delta(\beta^2_4,4)\Delta(\beta^3_{(1,4)},4) \Leftrightarrow \\ \frac{\lambda_2}{s_4}\Delta(\beta^3_{(1,2)},4) & \geq & \frac{\lambda_3}{s_4}\Delta(\beta^3_{(1,3)},4) & \geq & \frac{\lambda_4}{s_4}\Delta(\beta^3_{(1,4)},4)\end{array}$$

where $s_4 = \sum_{k=1}^4 \lambda_k$. Also, we have that

$$\begin{aligned} b_{1,2} &= \Delta(\beta^2_2,4)\Delta(\beta^3_{(1,2)},4)\, e^{-t\left(\frac{\lambda_1+\lambda_3+\lambda_4}{3}+\frac{\lambda_1+\lambda_2}{2}\right)} \\ b_{1,3} &= \Delta(\beta^2_3,4)\Delta(\beta^3_{(1,3)},4)\, e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_4}{3}+\frac{\lambda_1+\lambda_3}{2}\right)} \\ b_{1,4} &= \Delta(\beta^2_4,4)\Delta(\beta^3_{(1,4)},4)\, e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_3}{3}+\frac{\lambda_1+\lambda_4}{2}\right)} \end{aligned}$$

and therefore $b_{1,2} \geq b_{1,3} \geq b_{1,4}$. Hence, by Lemma 2.5,

$$\sum_{\substack{j=1\\ j\neq 1}}^{4} a_{1,j}b_{1,j} \geq \left(\sum_{\substack{j=1\\ j\neq 1}}^{4} a_{1,j}\right)\left(\sum_{\substack{j=1\\ j\neq 1}}^{4} b_{1,j}\right)/3 = a_1 b_1/3$$

Secondly we will verify that $b_1 \geq b_2 \geq b_3 \geq b_4$. To do this, it is sufficient to prove that the corresponding $b_{i,j}$ in the array (3.16) are ordered by rows. We give the proof only for the first two rows, so we will see that $b_{1,2} \geq b_{2,1}, b_{1,3} \geq b_{2,3}$ and $b_{1,4} \geq b_{2,4}$. It is immediate that

$$\begin{aligned} e^{-t\left(\frac{\lambda_1+\lambda_3+\lambda_4}{3}+\frac{\lambda_1+\lambda_2}{2}\right)} &\geq e^{-t\left(\frac{\lambda_2+\lambda_3+\lambda_4}{3}+\frac{\lambda_1+\lambda_2}{2}\right)} \\ e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_4}{3}+\frac{\lambda_1+\lambda_3}{2}\right)} &\geq e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_4}{3}+\frac{\lambda_2+\lambda_3}{2}\right)} \\ e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_3}{3}+\frac{\lambda_1+\lambda_4}{2}\right)} &\geq e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_3}{3}+\frac{\lambda_2+\lambda_4}{2}\right)} \end{aligned}$$

It follows easily that

$$\Delta(\beta^2_2,4)\Delta(\beta^3_{(1,2)},4) \geq \Delta(\beta^2_1,4)\Delta(\beta^3_{(1,2)},4) \Leftrightarrow \Delta(\beta^2_2,4) = \frac{\lambda_2}{s_4} \geq \frac{\lambda_1}{s_4} = \Delta(\beta^2_1,4),$$

then

$$b_{1,2} \geq b_{2,1} = \Delta(\beta^2_1,4)\Delta(\beta^3_{(1,2)},4)e^{-t\left(\frac{\lambda_2+\lambda_3+\lambda_4}{3}+\frac{\lambda_1+\lambda_2}{2}\right)}$$

Now from Lemma 2.4

$$\begin{aligned} \Delta(\beta^2_3,4)\Delta(\beta^3_{(1,3)},4) \geq \Delta(\beta^2_3,4)\Delta(\beta^3_{(2,3)},4) &\Leftrightarrow \Delta(\beta^3_{(1,3)},4) \geq \Delta(\beta^3_{(2,3)},4) \\ \Delta(\beta^2_4,4)\Delta(\beta^3_{(1,4)},4) \geq \Delta(\beta^2_4,4)\Delta(\beta^3_{(2,4)},4) &\Leftrightarrow \Delta(\beta^3_{(1,4)},4) \geq \Delta(\beta^3_{(2,4)},4) \end{aligned}$$

then

$$b_{1,3} \geq b_{2,3} = \Delta(\beta_3^2, 4)\Delta(\beta_{(2,3)}^3, 4)e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_4}{3}+\frac{\lambda_2+\lambda_3}{2}\right)}$$
$$b_{1,4} \geq b_{2,4} = \Delta(\beta_4^2, 4)\Delta(\beta_{(2,4)}^3, 4)e^{-t\left(\frac{\lambda_1+\lambda_2+\lambda_3}{3}+\frac{\lambda_2+\lambda_4}{2}\right)}$$

Consequently

$$b_1 = \sum_{\substack{j=1\\ j\neq 1}}^{4} b_{1,j} \geq \sum_{\substack{j=1\\ j\neq 2}}^{4} b_{2,j} = b_2$$

The same reasoning applies to the other cases. Then $b_1 \geq b_2 \geq b_3 \geq b_4$. Let $a_k = \sum_{\substack{j=1\\ j\neq k}}^{4} a_{k,j}$ be for $k = 1, \ldots, 4$. Clearly, $a_1 \geq a_2 \geq a_3 \geq a_4$. Since $\sum_{k=1}^{4} a_k = 0$, by Lemma 2.5 and (3.14) we conclude that (3.12) holds.

□

4 Generalized spacings

We turn to consider the spacings of the order statistics where now, $\beta_{m_j}^i = \sum_{l=i}^{n} \lambda(r_l)$. From (2.1) one sees immediately that the p.d.f. of $D_{i:n}$ for $1 \leq i \leq n$ is

$$f_i(t) \;=\; \sum_{\mathbf{r}_n} \frac{\prod_{k=1}^{n} \lambda_k}{\prod_{k=1}^{n}\left(\sum_{l=k}^{n} \lambda(r_l)\right)} \sum_{l=i}^{n} \lambda(r_l)\, e^{-t\sum_{l=i}^{n}\lambda(r_l)}$$

which can be written again as (2.3). The probability $\Delta(\beta_{m_j}^i, n)$ in (2.4) is the same in the p.d.f. of $D_{i:n}^*$ and $D_{i:n}$. This condition is essential to the proof of the next result.

Theorem 4.1. *Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i for $i = 1, \ldots, n$, then*

$$D_{i:4} \preccurlyeq_{hr} D_{i+1:4}, \;\; i = 1, \ldots, 4.$$

Proof. We begin by proving $D_{3:4} \preccurlyeq_{hr} D_{4:4}$, then we have to show that 3.6 holds. Here, the matrix of $\beta_{m_k}^3 - \beta_{m_j}^4$ is

$$\begin{pmatrix} \lambda_3+\lambda_4-\lambda_1 & \lambda_3+\lambda_4-\lambda_2 & \lambda_4 & \lambda_3 \\ \lambda_2+\lambda_4-\lambda_1 & \lambda_4 & \lambda_2+\lambda_4-\lambda_3 & \lambda_2 \\ \lambda_2+\lambda_3-\lambda_1 & \lambda_3 & \lambda_2 & \lambda_2+\lambda_3-\lambda_4 \\ \lambda_4 & \lambda_1+\lambda_4-\lambda_2 & \lambda_1+\lambda_4-\lambda_3 & \lambda_1 \\ \lambda_3 & \lambda_1+\lambda_3-\lambda_2 & \lambda_1 & \lambda_1+\lambda_3-\lambda_4 \\ \lambda_2 & \lambda_1 & \lambda_1+\lambda_2-\lambda_3 & \lambda_1+\lambda_2-\lambda_4 \end{pmatrix} \tag{4.17}$$

It is easy to check that there are only four negative coefficients $a_{2,u} = \lambda_j + \lambda_k - \lambda_l$ for $j < k < l$ and $u \notin \{j, k, l\}$. We can consider the term $a_{1,u} = \lambda_j + \lambda_l - \lambda_k \geq 0$ for $u = 1, \ldots, 4$. Notice that $exp\left\{-t(\beta^3_{(j,l)} + \beta^4_k)\right\} = exp\left\{-t(\beta^3_{(j,k)} + \beta^4_l)\right\}$. From equation (3.8) and (3.10) we have

$$b_{u,1} = \Delta(\beta^3_{(j,l)}, 4)\Delta(\beta^4_k, 4) \geq \Delta(\beta^3_{(j,k)}, 4)\Delta(\beta^4_l, 4) = b_{u,2}$$

Hence, by Lemma 2.5

$$\sum_{h=1}^{2} a_{u,h} b_{u,h} \geq \left(\sum_{h=1}^{2} a_{u,h}\right)\left(\sum_{h=1}^{2} b_{u,h}\right)/2 \geq 0$$

This proves the required result. Next, we will see $D_{2:4} \preccurlyeq_{hr} D_{3:4}$. This follows by the same method as in the first part of this proof. The matrix of $\beta^2_{m_k} - \beta^3_{m_j}$ is the transpose of (4.17) and by Lemma 2.4

$$\Delta(\beta^2_j, 4)\Delta(\beta^3_{(u,j)}, 4) \geq \Delta(\beta^2_k, 4)\Delta(\beta^3_{(u,k)}, 4) \Leftrightarrow \frac{\lambda_j}{s_4}\Delta(\beta^3_{(u,j)}, 4) \geq \frac{\lambda_k}{s_4}\Delta(\beta^3_{(u,k)}, 4),$$

for $u = 1, \ldots, 4$. The rest of the proof runs as in the first part of this proof. □

5 Extensions

Following the methodology of the proof of $D^*_{2:4} \preccurlyeq_{hr} D^*_{3:4}$ in the Theorem 3.2, we can see that the second and the third spacing and normalized spacing are ordered according to the hazard rate ordering for any n. But, first we prove the following result.

Theorem 5.1. *Let $\beta^i_{m_k}$ be as defined in (2.2), then*

$$\sum_{k=1}^{M_i} \sum_{j=1}^{M_{i+1}} \beta^i_{m_k} - \beta^{i+1}_{m_j} = 0.$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{M_{i+1}} \sum_{k=1}^{M_i} \beta^i_{m_k} - \beta^{i+1}_{m_j} &= \sum_{k=1}^{M_i} \left[M_{i+1}\beta^i_{m_k} - \sum_{j=1}^{M_{i+1}} \beta^{i+1}_{m_j} \right] \\
&= M_{i+1} \sum_{k=1}^{M_i} \beta^i_{m_k} - M_i \sum_{j=1}^{M_{i+1}} \beta^{i+1}_{m_j} \\
&= \sum_{l=1}^{n} \binom{n}{n-i}\binom{n-1}{n-i} \frac{\lambda_l}{n-i+1} - \sum_{l=1}^{n} \binom{n}{n-i+1}\binom{n-1}{n-i-1} \frac{\lambda_l}{n-i} \\
&= \left[\binom{n}{n-i}\binom{n-1}{n-i} \frac{1}{n-i+1} - \binom{n}{n-i+1}\binom{n-1}{n-i-1} \frac{1}{n-i} \right] \sum_{l=1}^{n} \lambda_l \\
&= 0
\end{aligned}$$

since

$$\begin{aligned}
\binom{n}{n-i}\binom{n-1}{n-i} \frac{1}{n-i+1} &= \left(\frac{n!}{(n-i)!i!} \right) \left(\frac{(n-1)!}{(n-i)!(i-1)!} \right) \frac{1}{n-i+1} \\
&= \frac{n!(n-1)!}{(n-i)!(n-i+1)!(i-1)!i!} \\
&= \left(\frac{n!}{(n-i+1)!(i-1)!} \right) \left(\frac{(n-1)!}{(n-i-1)!i!} \right) \frac{1}{n-i} \\
&= \binom{n}{n-i+1}\binom{n-1}{n-i-1} \frac{1}{n-i}
\end{aligned}$$

□

Theorem 5.2. *Under the same assumptions as in Theorem 4.1*

$$D^*_{2:n} \preccurlyeq_{hr} D^*_{3:n}, \qquad \text{for all} \quad n.$$

Proof. We have to show

$$\sum_{j=1}^{M_3} \sum_{k=1}^{M_2} \Delta(\beta^2_{m_k}, n)\Delta(\beta^3_{m_j}, n)\, e^{-t\left(\beta^2_{m_k}+\beta^3_{m_j}\right)} \left(\beta^2_{m_k} - \beta^3_{m_j} \right) \geq 0 \tag{5.18}$$

To do this, we consider the values of $\beta^2_{m_k} - \beta^3_{m_j}$ which add zero for each $k = 1, \ldots, M_2$ and $j = 1, \ldots, M_3$. To illustrate this idea, see the structure of the representation of the matrix of $\beta^2_{m_k} - \beta^3_{m_j}$ up to $n = 6$ in Figure 1. To simplify the notation,

$$\beta^2_u = \sum_{\substack{h=1 \\ h \neq u}}^{n} \lambda_h/(n-1), \quad \beta^3_{(j,k)} = \sum_{\substack{h=1 \\ h \notin \{j,k\}}}^{n} \lambda_h/(n-2), \quad u = 1, \ldots, M_2.$$

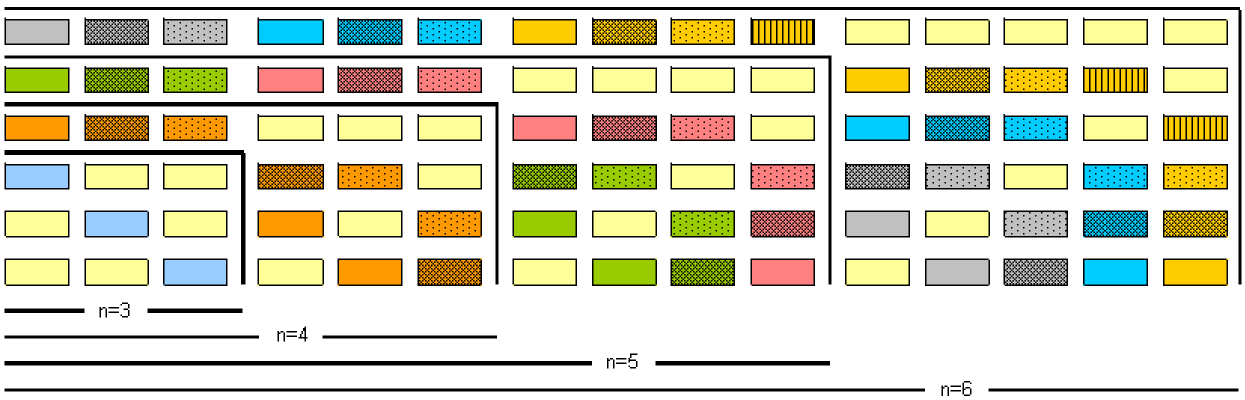

Figure 1: Representation of the matrix of $\beta^2_{m_k} - \beta^3_{m_j}$ up to $n = 6$

There are a number of elements in each row of this matrix that sum to zero, that is:

$$\sum_{\substack{k=1\\k\neq u}}^{n} \beta^2_u - \beta^3_{(u,k)} = (n-1)\beta^2_u - \sum_{\substack{k=1\\k\neq u}}^{n} \beta^3_{(u,k)} = 0$$

These values correspond to the yellow squares in the Figure 1. We are interested in proving

$$\Delta(\beta^2_u, n)\, e^{-t\beta^2_u} \sum_{\substack{k=1\\k\neq u}}^{n} \Delta(\beta^3_{(u,k)}, n)\, e^{-t\beta^3_{(u,k)}} \left(\beta^2_u - \beta^3_{(u,k)}\right) \geq 0 \tag{5.19}$$

for $u = 1, \ldots, M_2$. Notice that $a_{u,k} = \beta^2_u - \beta^3_{(u,k)}$ and $exp\left\{\beta^2_u + \beta^3_{(u,k)}\right\}$ are two sequence increasing in $k \in \{1, \ldots, n\} - u$. It follows from Lemma 2.4

$$\Delta(\beta^3_{(u,k)}, n) \geq \Delta(\beta^3_{(u,k')}, n) \quad \text{for} \quad k > k'$$

Then $b_{u,k} = \Delta(\beta^3_{(u,k)}, n)\, e^{-t\beta^3_{(u,k)}}$ are increasing in k. Finally, by Lemma 2.5, we conclude that (5.19) holds.

We group the remaining values in $\binom{n}{3}$ diagonals, each of them has a different color and aspect in the Figure 1. We fix a combination of three elements $j < k < l$ so that

$$a_{u,1} = \beta^2_j - \beta^3_{(k,l)} \geq a_{u,2} = \beta^2_k - \beta^3_{(j,l)} \geq a_{u,3} = \beta^2_l - \beta^3_{(j,k)}.$$

Since $\Delta(\beta^2_j, n) = \lambda_j / s_n$ where $s_n = \sum_{h=1}^{n} \lambda_h$, it is immediate from Lemma 2.4 that

$$\Delta(\beta^2_j, n)\Delta(\beta^3_{(k,l)}, n) \geq \Delta(\beta^2_k, n)\Delta(\beta^3_{(j,l)}, n) \geq \Delta(\beta^2_l, n)\Delta(\beta^3_{(j,k)}, n). \tag{5.20}$$

Then, again from Lemma 2.5

$$\sum_{h=1}^{3} a_{u,h} b_{u,h} \geq \left(\sum_{h=1}^{3} a_{u,h}\right)\left(\sum_{h=1}^{3} b_{u,h}\right)/3, \quad \text{for} \quad u = 1, \ldots, \binom{n}{3}$$

where

$$\begin{aligned} b_{u,1} &= \Delta(\beta_j^2, n)\Delta(\beta_{(k,l)}^3, n)e^{-t\left(\beta_j^2+\beta_{(k,l)}^3\right)} \\ b_{u,2} &= \Delta(\beta_k^2, n)\Delta(\beta_{(j,l)}^3, n)e^{-t\left(\beta_k^2+\beta_{(j,l)}^3\right)} \\ b_{u,3} &= \Delta(\beta_l^2, n)\Delta(\beta_{(j,k)}^3, n)e^{-t\left(\beta_l^2+\beta_{(j,k)}^3\right)}. \end{aligned}$$

Let $A_u = \sum_{h=1}^{3} a_{u,h}$ and $B_u = \sum_{h=1}^{3} b_{u,h}$ be. Next, we group diagonals such that,

$$\sum_{u \in \text{group } 1} A_u \geq \sum_{u' \in \text{group } 2} A_{u'}.$$

Then, it is necessary to prove that the respective B_u are also ordered. The approach is as in the last part of Theorem 3.2. Each group is formed by the (three or more) diagonals in Figure (1) which have the same color. In this way, we can apply Lemma 2.5 as many times as necessary until we can obtain the sums of the differences of the betas, as, by Theorem 5.1, this is equal to zero. □

Theorem 5.3. *Under the same assumptions that in Theorem 4.1*

$$D_{2:n} \preccurlyeq_{hr} D_{3:n}.$$

Proof. We have to show that (5.18) holds. We can use the same approach as in the proof of Theorem 4.1. It is easy to see that for each negative element of the matrix $\beta_{m_k}^2 - \beta_{m_j}^3$ there exists another positive element of form

$$a_{u,1} = \beta_k^2 - \beta_{(j,l)}^3 = (\lambda_j + \lambda_l) - \lambda_k \geq (\lambda_j + \lambda_k) - \lambda_l = \beta_l^2 - \beta_{(j,k)}^3 = a_{u,2}$$

since $\beta_k^2 = \sum_{\substack{h=1 \\ h \neq k}}^{n} \lambda_h$, $\beta_{(j,l)}^2 = \sum_{\substack{h=1 \\ h \notin \{j,l\}}}^{n} \lambda_h$ and $j < k < l$. Notice

$$\beta_k^2 + \beta_{(j,l)}^3 = (\lambda_j + \lambda_k + \lambda_l) + 2 \sum_{\substack{h=1 \\ h \notin \{j,k,l\}}}^{n} \lambda_h = \beta_l^2 + \beta_{(j,k)}^3,$$

then $e^{-t\left(\beta_k^2+\beta_{(j,l)}^3\right)} = e^{-t\left(\beta_l^2+\beta_{(j,k)}^3\right)}$. From equation (5.20) and by Lemma 2.5

$$\sum_{h=1}^{2} a_{u,h} b_{u,h} \geq \left(\sum_{h=1}^{2} a_{u,h}\right)\left(\sum_{h=1}^{2} b_{u,h}\right)/2 \geq 0$$

where $b_{u,1} = \Delta(\beta_k^2, n)\Delta(\beta_{(j,l)}^3, n)$ and $b_{u,2} = \Delta(\beta_l^2, n)\Delta(\beta_{(j,k)}^3, n)$. This proves the required result. □

6 Conclusions and conjectures

This paper is devoted to establishing the proof of the conjecture of Kochar and Korwar [7]. They obtain some new results on normalized spacings of independent exponential random variables with possibly different scale parameters, but proposed open problems. We show the conjecture is true for $n = 4$ for both normalized spacings and spacings. We establish hazard rate between the second and the third spacings and normalized spacings for any n. We conclude that in each row of the matrix of differences between the betas there exist a number of elements with sum zero. Furthermore, we believe that by studying the structure of these matrices, an adequate form of applying Chebyshev's inequality can be found. To end this paper, we make the following conjectures.

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having failure rate λ_i for each i. Let $\Delta(\beta^{n-1}_{m_j}, n)$ be as defined in (2.4). Let $\beta^{n-1}_{m_k} = \frac{\lambda_j+\lambda_l}{2}$ and $\beta^{n}_{m_j} = \lambda_j$ be, if $m_k = (j,l)$ and $m_j = j$, respectively. Then,

(1) $\Delta(\beta^{n-1}_{(k,l)}, n)\Delta(\beta^{n}_{j}, n) \geq \Delta(\beta^{n-1}_{(j,k)}, n)\Delta(\beta^{n}_{l}, n)$

(2) $\Delta(\beta^{n-1}_{(j,l)}, n)\Delta(\beta^{n}_{k}, n) \geq \Delta(\beta^{n-1}_{(j,k)}, n)\Delta(\beta^{n}_{l}, n)$

(3) $\Delta(\beta^{n-1}_{(k,l)}, n)\Delta(\beta^{n}_{j}, n) \geq \Delta(\beta^{n-1}_{(j,l)}, n)\Delta(\beta^{n}_{k}, n)$ for $j = 1$ and $k = 2$ if $\beta^{n-1}_{(j,l)} - \beta^{n}_{k} < 0$

(4) $\Delta(\beta^{n-1}_{(k,l)}, n)\Delta(\beta^{n}_{j}, n) \geq \Delta(\beta^{n-1}_{(j,l)}, n)\Delta(\beta^{n}_{k}, n)$ for $j \neq 1$ or $k \neq 2$.

Assuming that our conjectures hold, it would then be possible to prove that

$$D^*_{n-1:n} \preccurlyeq_{hr} D^*_{n:n} \quad \text{and} \quad D_{n-1:n} \preccurlyeq_{hr} D_{n:n} \qquad \text{for all} \quad n.$$

A Appendix

In this appendix we show that equations (3.8)-(3.11) hold.

Lemma A.1. *Let $\Delta(\beta^{i}_{m_j}, n)$ be as defined in (2.4), and $m_j = j$ and $m_k = (j,l)$. Then*

$$\Delta(\beta^{3}_{(3,4)}, 4)\Delta(\beta^{4}_{j}, 4) \geq \Delta(\beta^{3}_{(j,4)}, 4)\Delta(\beta^{4}_{3}, 4) \geq \Delta(\beta^{3}_{(j,3)}, 4)\Delta(\beta^{4}_{4}, 4).$$

Proof. We divide the proof into two parts

(a) $\Delta(\beta^{3}_{(3,4)}, 4)\Delta(\beta^{j}_{1}, 4) \geq \Delta(\beta^{3}_{(j,4)}, 4)\Delta(\beta^{4}_{3}, 4)$

(b) $\Delta(\beta^{3}_{(j,4)}, 4)\Delta(\beta^{4}_{3}, 4) \geq \Delta(\beta^{3}_{(j,3)}, 4)\Delta(\beta^{4}_{4}, 4)$

We give the proof only for the case $j = 1$; the other case is similar with 1 replaced by 2. After a few manipulations, we have

$$
\begin{aligned}
\Delta(\beta^3_{(3,4)}, 4)\Delta(\beta^4_1, 4) &= \left(\frac{\lambda_1\lambda_2^2\lambda_3\lambda_4}{Ss_4s_3s_2^2s_1}\right)\frac{s_1 + s_2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)}Z_1 \\
\Delta(\beta^3_{(1,4)}, 4)\Delta(\beta^4_3, 4) &= \left(\frac{\lambda_1\lambda_2^2\lambda_3\lambda_4}{Ss_4s_3s_2^2s_1}\right)\frac{s_2 + s_3}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_4)}Z_2 \\
\Delta(\beta^3_{(1,3)}, 4)\Delta(\beta^4_4, 4) &= \left(\frac{\lambda_1\lambda_2^2\lambda_3\lambda_4}{Ss_4s_3s_2^2s_1}\right)\frac{s_2 + s_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)}Z_3
\end{aligned}
$$

where

$$
\begin{aligned}
Z_1 &= s_2s_3(\lambda_1 + \lambda_4)(s_4 + \lambda_1) + s_2s_4(\lambda_1 + \lambda_3)(s_3 + \lambda_1) + s_3s_4(\lambda_1 + \lambda_2)(s_2 + \lambda_1) \\
Z_2 &= s_1s_2(\lambda_3 + \lambda_4)(s_4 + \lambda_3) + s_1s_4(\lambda_2 + \lambda_3)(s_2 + \lambda_3) + s_2s_4(\lambda_1 + \lambda_3)(s_1 + \lambda_3) \\
Z_3 &= s_1s_2(\lambda_3 + \lambda_4)(s_3 + \lambda_4) + s_1s_3(\lambda_2 + \lambda_4)(s_2 + \lambda_4) + s_2s_3(\lambda_1 + \lambda_4)(s_1 + \lambda_4)
\end{aligned}
$$

and

$$
\begin{aligned}
S &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\
s_i &= \sum_{\substack{j=1 \\ j \neq i}}^{4} \lambda_j
\end{aligned}
$$

Then:

(a) $\Delta(\beta^3_{(3,4)},4)\Delta(\beta^4_1,4) \geq \Delta(\beta^3_{(1,4)},4)\Delta(\beta^4_3,4) \Leftrightarrow$

$$
\begin{aligned}
&(s_1+s_2)(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)Z_1 - (s_2+s_3)(\lambda_1+\lambda_2)(\lambda_1+\lambda_4)Z_2 = \\
&(\lambda_3-\lambda_1)\big(2\lambda_1^5\lambda_2 + 4\lambda_1^5\lambda_3 + 2\lambda_1^5\lambda_4 + 7\lambda_1^4\lambda_2^2 + 19\lambda_1^4\lambda_2\lambda_3 + 9\lambda_1^4\lambda_2\lambda_4 + 16\lambda_1^4\lambda_3^2 + \\
&\quad 21\lambda_1^4\lambda_3\lambda_4 + 8\lambda_1^4\lambda_4^2 + 9\lambda_1^3\lambda_2^3 + 34\lambda_1^3\lambda_2^2\lambda_3 + 16\lambda_1^3\lambda_2^2\lambda_4 + 47\lambda_1^3\lambda_2\lambda_3^2 + \\
&\quad 51\lambda_1^3\lambda_2\lambda_3\lambda_4 + 13\lambda_1^3\lambda_2\lambda_4^2 + 24\lambda_1^3\lambda_3^3 + 51\lambda_1^3\lambda_3^2\lambda_4 + 39\lambda_1^3\lambda_3\lambda_4^2 + 12\lambda_1^3\lambda_4^3 + \\
&\quad 5\lambda_1^2\lambda_2^4 + 28\lambda_1^2\lambda_2^3\lambda_3 + 13\lambda_1^2\lambda_2^3\lambda_4 + 54\lambda_1^2\lambda_2^2\lambda_3^2 + 51\lambda_1^2\lambda_2^2\lambda_3\lambda_4 + \\
&\quad 7\lambda_1^2\lambda_2^2\lambda_4^2 + 47\lambda_1^2\lambda_2\lambda_3^3 + 83\lambda_1^2\lambda_2\lambda_3^2\lambda_4 + 44\lambda_1^2\lambda_2\lambda_3\lambda_4^2 + \\
&\quad 9\lambda_1^2\lambda_2\lambda_4^3 + 16\lambda_1^2\lambda_3^4 + 51\lambda_1^2\lambda_3^3\lambda_4 + 61\lambda_1^2\lambda_3^2\lambda_4^2 + 35\lambda_1^2\lambda_3\lambda_4^3 + 8\lambda_1^2\lambda_4^4 + \\
&\quad \lambda_1\lambda_2^5 + 10\lambda_1\lambda_2^4\lambda_3 + 4\lambda_1\lambda_2^4\lambda_4 + 28\lambda_1\lambda_2^3\lambda_3^2 + 25\lambda_1\lambda_2^3\lambda_3\lambda_4 + \\
&\quad 2\lambda_1\lambda_2^3\lambda_4^2 + 34\lambda_1\lambda_2^2\lambda_3^3 + 51\lambda_1\lambda_2^2\lambda_3^2\lambda_4 + 13\lambda_1\lambda_2^2\lambda_3\lambda_4^2 + 19\lambda_1\lambda_2\lambda_3^4 + \\
&\quad 51\lambda_1\lambda_2\lambda_3^3\lambda_4 + 12\lambda_3^3\lambda_4^3 + 44\lambda_1\lambda_2\lambda_3^2\lambda_4^2 + 17\lambda_1\lambda_2\lambda_3\lambda_4^3 + 4\lambda_1\lambda_3^5 + \\
&\quad 21\lambda_1\lambda_3^4\lambda_4 + 39\lambda_1\lambda_3^3\lambda_4^2 + 35\lambda_1\lambda_3^2\lambda_4^3 + 15\lambda_1\lambda_3\lambda_4^4 + 2\lambda_1\lambda_4^5 + 8\lambda_3^4\lambda_4^2 + \\
&\quad \lambda_2^5\lambda_3 + 5\lambda_2^4\lambda_3^2 + 4\lambda_2^4\lambda_3\lambda_4 + 9\lambda_2^3\lambda_3^3 + 13\lambda_2^3\lambda_3^2\lambda_4 + 2\lambda_2^3\lambda_3\lambda_4^2 + 7\lambda_2^2\lambda_3^4 + \\
&\quad 16\lambda_2^2\lambda_3^3\lambda_4 + 7\lambda_2^2\lambda_3^2\lambda_4^2 + 2\lambda_3^5\lambda_4 + 2\lambda_2\lambda_3^5 + 9\lambda_2\lambda_3^4\lambda_4 + 13\lambda_2\lambda_3^3\lambda_4^2 + \\
&\quad 9\lambda_2\lambda_3^2\lambda_4^3 + \lambda_1\lambda_2\lambda_4^4 + \lambda_2\lambda_3\lambda_4^4 + 7\lambda_3^2\lambda_4^4 + 2\lambda_1\lambda_2\lambda_4^3(\lambda_4-\lambda_1) + \\
&\quad 2\lambda_2\lambda_3\lambda_4^3(\lambda_4-\lambda_2) + \lambda_4^3\left(\lambda_3^2\lambda_4-\lambda_2^3\right) + \lambda_3\lambda_4^5 + \lambda_4^4\left(\lambda_3\lambda_4-\lambda_2^2\right)\big) \geq 0
\end{aligned}
$$

Notice that the last four terms are positive because λ_k's are increasing, even if 1 is replacing by 2.

(b) $\Delta(\beta^3_{(1,4)},4)\Delta(\beta^4_3,4) \geq \Delta(\beta^3_{(1,3)},4)\Delta(\beta^4_4,4) \Leftrightarrow$

$$
\begin{aligned}
&(s_2+s_3)(\lambda_1+\lambda_4)(\lambda_2+\lambda_4)Z_2 - (s_2+s_4)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)Z_3 = \\
&(\lambda_4-\lambda_3)\big(2\lambda_1^5\lambda_3 + 2\lambda_1^5\lambda_4 + 3\lambda_1^4\lambda_2\lambda_3 + 3\lambda_1^4\lambda_2\lambda_4 + 8\lambda_1^4\lambda_3^2 + \\
&\quad 15\lambda_1^4\lambda_3\lambda_4 + 8\lambda_1^4\lambda_4^2 + 9\lambda_1^3\lambda_2\lambda_3^2 + 17\lambda_1^3\lambda_2\lambda_3\lambda_4 + 9\lambda_1^3\lambda_2\lambda_4^2 + \\
&\quad 12\lambda_1^3\lambda_3^3 + 35\lambda_1^3\lambda_3^2\lambda_4 + 35\lambda_1^3\lambda_3\lambda_4^2 + 12\lambda_1^3\lambda_4^3 + \\
&\quad 2\lambda_1^2\lambda_2^3\lambda_3 + 7\lambda_1^2\lambda_2^2\lambda_3^2 + 13\lambda_1^2\lambda_2^2\lambda_3\lambda_4 + 7\lambda_1^2\lambda_2^2\lambda_4^2 + 4\lambda_3\lambda_4^5 + \\
&\quad 13\lambda_1^2\lambda_2\lambda_3^3 + 44\lambda_1^2\lambda_2\lambda_3^2\lambda_4 + 44\lambda_1^2\lambda_2\lambda_3\lambda_4^2 + 13\lambda_1^2\lambda_2\lambda_4^3 + 8\lambda_1^2\lambda_3^4 + \\
&\quad 4\lambda_3^5\lambda_4 + 16\lambda_3^4\lambda_4^2 + 24\lambda_3^3\lambda_4^3 + 16\lambda_3^2\lambda_4^4 + 39\lambda_1^2\lambda_3^3\lambda_4 + \\
&\quad 61\lambda_1^2\lambda_3^2\lambda_4^2 + 39\lambda_1^2\lambda_3\lambda_4^3 + 8\lambda_1^2\lambda_4^4 + 4\lambda_1\lambda_2^4\lambda_3 + 4\lambda_1\lambda_2^4\lambda_4 + \\
&\quad 13\lambda_1\lambda_2^3\lambda_3^2 + 25\lambda_1\lambda_2^3\lambda_3\lambda_4 + 13\lambda_1\lambda_2^3\lambda_4^2 + 16\lambda_1\lambda_2^2\lambda_3^3 + 51\lambda_1\lambda_2^2\lambda_3^2\lambda_4 + \\
&\quad 51\lambda_1\lambda_2^2\lambda_3\lambda_4^2 + 16\lambda_1\lambda_2^2\lambda_4^3 + 9\lambda_1\lambda_2\lambda_3^4 + 51\lambda_1\lambda_2\lambda_3^3\lambda_4 + 83\lambda_1\lambda_2\lambda_3^2\lambda_4^2 + \\
&\quad 51\lambda_1\lambda_2\lambda_3\lambda_4^3 + 9\lambda_1\lambda_2\lambda_4^4 + \lambda_2^5\lambda_4 + 21\lambda_1\lambda_3^4\lambda_4 + 51\lambda_1\lambda_3^3\lambda_4^2 + \\
&\quad 51\lambda_1\lambda_3^2\lambda_4^3 + 21\lambda_1\lambda_3\lambda_4^4 + \lambda_2^5\lambda_3 + 5\lambda_2^4\lambda_3^2 + 10\lambda_2^4\lambda_3\lambda_4 + \\
&\quad 5\lambda_2^4\lambda_4^2 + 9\lambda_2^3\lambda_3^3 + 28\lambda_2^3\lambda_3^2\lambda_4 + 28\lambda_2^3\lambda_3\lambda_4^2 + 9\lambda_2^3\lambda_4^3 + 7\lambda_2^2\lambda_3^4 + \\
&\quad 34\lambda_2^2\lambda_3^3\lambda_4 + 54\lambda_2^2\lambda_3^2\lambda_4^2 + 34\lambda_2^2\lambda_3\lambda_4^3 + 7\lambda_2^2\lambda_4^4 + \\
&\quad 19\lambda_2\lambda_3^4\lambda_4 + 47\lambda_2\lambda_3^3\lambda_4^2 + 47\lambda_2\lambda_3^2\lambda_4^3 + 19\lambda_2\lambda_3\lambda_4^4 + \\
&\quad 2\lambda_1^2\lambda_2^3\lambda_4 + 2\lambda_2\lambda_4^5 + 2\lambda_2\lambda_3(\lambda_3^4 - \lambda_1^3\lambda_2) + 2\lambda_1\lambda_4(\lambda_4^4 - \lambda_1^2\lambda_2^2) + \\
&\quad \lambda_1(\lambda_3^5 - \lambda_1^3\lambda_2^2) + \lambda_1(\lambda_3^5 - \lambda_1^2\lambda_2^3)\big) \geq 0
\end{aligned}
$$

Using the ordering between λ_k's, we can see that the last four terms are positive as before. □

Lemma A.2. *Under the same assumptions that in Lemma A.1*

(a) $\Delta(\beta^3_{(2,l)},4)\Delta(\beta^4_1,4) \geq \Delta(\beta^3_{(1,2)},4)\Delta(\beta^4_l,4)$

(b) $\Delta(\beta^3_{(1,l)},4)\Delta(\beta^4_2,4) \geq \Delta(\beta^3_{(1,2)},4)\Delta(\beta^4_l,4)$

(c) if $\beta^3_{1,l} - \beta^4_2 < 0$*, then* $\Delta(\beta^3_{(2,l)},4)\Delta(\beta^4_1,4) \geq \Delta(\beta^3_{(1,l)},4)\Delta(\beta^4_2,4)$

Proof. We give the proof only for the case $l = 3$; the other case is similar with 3 replaced by 4. After a few manipulations, we have

$$
\begin{aligned}
\Delta(\beta^3_{2,3},4)\Delta(\beta^4_1,4) &= \left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4^2}{Ss_4^2s_3s_2s_1}\right)\frac{s_1+s_4}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)}Y_1 \\
\Delta(\beta^3_{1,3},4)\Delta(\beta^4_2,4) &= \left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4^2}{Ss_4^2s_3s_2s_1}\right)\frac{s_2+s_4}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)}Y_2 \\
\Delta(\beta^3_{1,2},4)\Delta(\beta^4_3,4) &= \left(\frac{\lambda_1\lambda_2\lambda_3\lambda_4^2}{Ss_4^2s_3s_2s_1}\right)\frac{s_3+s_4}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)}Y_3
\end{aligned}
$$

where

$$Y_1 = s_2s_3(\lambda_1+\lambda_4)(s_4+\lambda_1)+s_2s_4(\lambda_1+\lambda_3)(s_3+\lambda_1)+s_3s_4(\lambda_1+\lambda_2)(s_2+\lambda_1)$$
$$Y_2 = s_1s_3(\lambda_2+\lambda_4)(s_4+\lambda_2)+s_1s_4(\lambda_2+\lambda_3)(s_3+\lambda_2)+s_3s_4(\lambda_1+\lambda_2)(s_1+\lambda_2)$$
$$Y_3 = s_1s_2(\lambda_3+\lambda_4)(s_4+\lambda_3)+s_1s_4(\lambda_2+\lambda_3)(s_2+\lambda_3)+s_2s_4(\lambda_1+\lambda_3)(s_1+\lambda_3)$$

(a) $\Delta(\beta^3_{2,3},4)\Delta(\beta^4_1,4) \geq \Delta(\beta^3_{1,2},4)\Delta(\beta^4_3,4) \Leftrightarrow$

$$\begin{aligned}
&(s_1+s_4)(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)Y_1-(s_3+s_4)(\lambda_1+\lambda_2)(\lambda_1+\lambda_4)Y_3 = \\
&(\lambda_3-\lambda_1)\big(2\lambda_1^5\lambda_2+4\lambda_1^5\lambda_3+2\lambda_1^5\lambda_4+8\lambda_1^4\lambda_2^2+21\lambda_1^4\lambda_2\lambda_3+9\lambda_1^4\lambda_2\lambda_4+ \\
&\quad 16\lambda_1^4\lambda_3^2+19\lambda_1^4\lambda_3\lambda_4+7\lambda_1^4\lambda_4^2+12\lambda_1^3\lambda_2^3+39\lambda_1^3\lambda_2^2\lambda_3+ \\
&\quad 13\lambda_1^3\lambda_2^2\lambda_4+51\lambda_1^3\lambda_2\lambda_3^2+51\lambda_1^3\lambda_2\lambda_3\lambda_4+16\lambda_1^3\lambda_2\lambda_4^2+24\lambda_1^3\lambda_3^3+ \\
&\quad 47\lambda_1^3\lambda_3^2\lambda_4+34\lambda_1^3\lambda_3\lambda_4^2+9\lambda_1^3\lambda_4^3+8\lambda_1^2\lambda_2^4+35\lambda_1^2\lambda_2^3\lambda_3+ \\
&\quad 9\lambda_1^2\lambda_2^3\lambda_4+61\lambda_1^2\lambda_2^2\lambda_3^2+44\lambda_1^2\lambda_2^2\lambda_3\lambda_4+7\lambda_1^2\lambda_2^2\lambda_4^2+51\lambda_1^2\lambda_2\lambda_3^3+ \\
&\quad 83\lambda_1^2\lambda_2\lambda_3^2\lambda_4+51\lambda_1^2\lambda_2\lambda_3\lambda_4^2+13\lambda_1^2\lambda_2\lambda_4^3+16\lambda_1^2\lambda_3^4+47\lambda_1^2\lambda_3^3\lambda_4+ \\
&\quad 54\lambda_1^2\lambda_3^2\lambda_4^2+28\lambda_1^2\lambda_3\lambda_4^3+5\lambda_1^2\lambda_4^4+2\lambda_1\lambda_2^5+15\lambda_1\lambda_2^4\lambda_3+9\lambda_3^3\lambda_4^3+ \\
&\quad 3\lambda_1\lambda_2^4\lambda_4+35\lambda_1\lambda_2^3\lambda_3^2+17\lambda_1\lambda_2^3\lambda_3\lambda_4+39\lambda_1\lambda_2^2\lambda_3^3+44\lambda_1\lambda_2^2\lambda_3^2\lambda_4+ \\
&\quad 13\lambda_1\lambda_2^2\lambda_3\lambda_4^2+21\lambda_1\lambda_2\lambda_3^4+51\lambda_1\lambda_2\lambda_3^3\lambda_4+51\lambda_1\lambda_2\lambda_3^2\lambda_4^2+25\lambda_1\lambda_2\lambda_3\lambda_4^3+ \\
&\quad 4\lambda_1\lambda_2\lambda_4^4+4\lambda_1\lambda_3^5+19\lambda_1\lambda_3^4\lambda_4+34\lambda_1\lambda_3^3\lambda_4^2+28\lambda_1\lambda_3^2\lambda_4^3+ \\
&\quad 10\lambda_1\lambda_3\lambda_4^4+\lambda_1\lambda_4^5+2\lambda_2^5\lambda_3+8\lambda_2^4\lambda_3^2+3\lambda_2^4\lambda_3\lambda_4+12\lambda_2^3\lambda_3^3+ \\
&\quad 9\lambda_2^3\lambda_3^2\lambda_4+8\lambda_2^2\lambda_3^4+13\lambda_2^2\lambda_3^3\lambda_4+7\lambda_2^2\lambda_3^2\lambda_4^2+2\lambda_2\lambda_3^5+9\lambda_2\lambda_3^4\lambda_4+ \\
&\quad 16\lambda_2\lambda_3^3\lambda_4^2+13\lambda_2\lambda_3^2\lambda_4^3+4\lambda_2\lambda_3\lambda_4^4+2\lambda_3^5\lambda_4+7\lambda_3^4\lambda_4^2+4\lambda_3^2\lambda_4^4+ \\
&\quad 2\lambda_2^2\lambda_4^2\left(\lambda_4-\lambda_2\right)\left(\lambda_3+\lambda_1\right)+\lambda_4^2\left(\lambda_3^2\lambda_4^2-\lambda_2^4\right)+\lambda_4^3\left(\lambda_3\lambda_4^2-\lambda_2^3\right)\big) \geq 0
\end{aligned}$$

Notice that the last three terms are positive because λ_k's are increasing, even if 3 is replacing by 4.

(b) $\Delta(\beta^3_{1,3},4)\Delta(\beta^4_2,4) \geq \Delta(\beta^3_{1,2},4)\Delta(\beta^4_3,4) \Leftrightarrow$

$$\begin{aligned}
&(s_2+s_4)(\lambda_1+\lambda_3)(\lambda_3+\lambda_4)Y_2-(s_3+s_4)(\lambda_1+\lambda_2)(\lambda_2+\lambda_4)Y_3 = \\
&(\lambda_3-\lambda_2)\big(2\lambda_1^5\lambda_2+2\lambda_1^5\lambda_3+8\lambda_1^4\lambda_2^2+15\lambda_1^4\lambda_2\lambda_3+3\lambda_1^4\lambda_2\lambda_4+8\lambda_1^4\lambda_3^2+3\lambda_1^4\lambda_3\lambda_4+ \\
&\quad 12\lambda_1^3\lambda_2^3+35\lambda_1^3\lambda_2^2\lambda_3+9\lambda_1^3\lambda_2^2\lambda_4+35\lambda_1^3\lambda_2\lambda_3^2+17\lambda_1^3\lambda_2\lambda_3\lambda_4+ \\
&\quad 12\lambda_1^3\lambda_3^3+9\lambda_1^3\lambda_3^2\lambda_4+8\lambda_1^2\lambda_2^4+39\lambda_1^2\lambda_2^3\lambda_3+9\lambda_3^3\lambda_4^3+5\lambda_3^2\lambda_4^4+ \\
&\quad 2\lambda_3^5\lambda_4+7\lambda_3^4\lambda_4^2 61\lambda_1^2\lambda_2^2\lambda_3^2+44\lambda_1^2\lambda_2^2\lambda_3\lambda_4+7\lambda_1^2\lambda_2^2\lambda_4^2+39\lambda_1^2\lambda_2\lambda_3^3+ \\
&\quad 44\lambda_1^2\lambda_2\lambda_3^2\lambda_4+13\lambda_1^2\lambda_2\lambda_3\lambda_4^2+8\lambda_1^2\lambda_3^4+13\lambda_1^2\lambda_3^3\lambda_4+19\lambda_2\lambda_3^4\lambda_4+ \\
&\quad 7\lambda_1^2\lambda_3^2\lambda_4^2+2\lambda_1\lambda_2^5+21\lambda_1\lambda_2^4\lambda_3+9\lambda_1\lambda_2^4\lambda_4+51\lambda_1\lambda_2^3\lambda_3^2+ \\
&\quad 51\lambda_1\lambda_2^3\lambda_3\lambda_4+16\lambda_1\lambda_2^3\lambda_4^2+51\lambda_1\lambda_2^2\lambda_3^3+13\lambda_1^2\lambda_2^3\lambda_4+83\lambda_1\lambda_2^2\lambda_3^2\lambda_4+ \\
&\quad 51\lambda_1\lambda_2^2\lambda_3\lambda_4^2+13\lambda_1\lambda_2^2\lambda_4^3+21\lambda_1\lambda_2\lambda_3^4+51\lambda_1\lambda_2\lambda_3^3\lambda_4+51\lambda_1\lambda_2\lambda_3^2\lambda_4^2+ \\
&\quad 25\lambda_1\lambda_2\lambda_3\lambda_4^3+4\lambda_1\lambda_2\lambda_4^4+2\lambda_1\lambda_3^5+9\lambda_1\lambda_3^4\lambda_4+16\lambda_1\lambda_3^3\lambda_4^2+ \\
&\quad 13\lambda_1\lambda_3^2\lambda_4^3+4\lambda_1\lambda_3\lambda_4^4+4\lambda_2^5\lambda_3+2\lambda_2^5\lambda_4+16\lambda_2^4\lambda_3^2+19\lambda_2^4\lambda_3\lambda_4+ \\
&\quad 6\lambda_2^4\lambda_4^2+24\lambda_2^3\lambda_3^3+47\lambda_2^3\lambda_3^2\lambda_4+34\lambda_2^3\lambda_3\lambda_4^2+8\lambda_2^3\lambda_4^3+16\lambda_2^2\lambda_3^4+ \\
&\quad 47\lambda_2^2\lambda_3^3\lambda_4+54\lambda_2^2\lambda_3^2\lambda_4^2+28\lambda_2^2\lambda_3\lambda_4^3+5\lambda_2^2\lambda_4^4+4\lambda_2\lambda_3^5+34\lambda_2\lambda_3^3\lambda_4^2+ \\
&\quad 28\lambda_2\lambda_3^2\lambda_4^3+10\lambda_2\lambda_3\lambda_4^4+\lambda_4^3(\lambda_3\lambda_4^2-\lambda_1^3)+\lambda_4^2(\lambda_2\lambda_4^3-\lambda_1^4)+ \\
&\quad 2\lambda_1^2\lambda_4^2\left(\lambda_4-\lambda_1\right)\left(\lambda_2+\lambda_3\right)+\lambda_4^2(\lambda_2^4-\lambda_1^4)+\lambda_4^3(\lambda_2^3-\lambda_1^3)\big) \geq 0
\end{aligned}$$

In the same manner than before we can see that the last five terms are positive too.

(c) $\Delta(\beta^3_{2,3},4)\Delta(\beta^4_1,4) \geq \Delta(\beta^3_{1,3},4)\Delta(\beta^4_2,4) \Leftrightarrow$

$$
\begin{aligned}
&(s_1+s_4)(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)Y_1 - (s_2+s_4)(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)Y_2 = \\
&(\lambda_2-\lambda_1)\big(4\lambda_1\lambda_2^5 + 4\lambda_1^5\lambda_2 + 2\lambda_1\lambda_3^5 + 2\lambda_1^5\lambda_3 + \lambda_1\lambda_4^5 + 2\lambda_2\lambda_3^5 + 2\lambda_1^5\lambda_4 + \\
&\quad 2\lambda_2^5\lambda_3 + \lambda_2\lambda_4^5 + 2\lambda_2^5\lambda_4 + 15\lambda_1\lambda_2\lambda_3^4 + 21\lambda_1\lambda_2^4\lambda_3 + \\
&\quad 21\lambda_1^4\lambda_2\lambda_3 + 10\lambda_1\lambda_2\lambda_4^4 + 19\lambda_1\lambda_2^4\lambda_4 + 19\lambda_1^4\lambda_2\lambda_4 + 4\lambda_1\lambda_3\lambda_4^4 + \\
&\quad 3\lambda_1\lambda_3^4\lambda_4 + 9\lambda_1^4\lambda_3\lambda_4 + 4\lambda_2\lambda_3\lambda_4^4 + 3\lambda_2\lambda_3^4\lambda_4 + 9\lambda_2^4\lambda_3\lambda_4 + \\
&\quad 25\lambda_1\lambda_2\lambda_3\lambda_4^3 + 17\lambda_1\lambda_2\lambda_3^3\lambda_4 + 51\lambda_1\lambda_2^3\lambda_3\lambda_4 + 51\lambda_1^3\lambda_2\lambda_3\lambda_4 + 16\lambda_1^2\lambda_2^4 + \\
&\quad 24\lambda_1^3\lambda_2^3 + 16\lambda_1^4\lambda_2^2 + 8\lambda_1^2\lambda_3^4 + 12\lambda_1^3\lambda_3^3 + 8\lambda_1^4\lambda_3^2 + \\
&\quad 5\lambda_1^2\lambda_4^4 + 8\lambda_2^2\lambda_3^4 + 12\lambda_2^3\lambda_3^3 + 8\lambda_2^4\lambda_3^2 + 5\lambda_2^2\lambda_4^4 + 7\lambda_1^2\lambda_3^2\lambda_4^2 + \\
&\quad 7\lambda_2^2\lambda_3^2\lambda_4^2 + 16\lambda_1^3\lambda_3\lambda_4^2 35\lambda_1\lambda_2^2\lambda_3^3 + 39\lambda_1\lambda_2^3\lambda_3^2 + 35\lambda_1^2\lambda_2\lambda_3^3 + \\
&\quad 51\lambda_1^2\lambda_2^3\lambda_3 + 39\lambda_1^3\lambda_2\lambda_3^2 + 51\lambda_1^3\lambda_2^2\lambda_3 + 47\lambda_1^2\lambda_2^3\lambda_4 + 47\lambda_1^3\lambda_2^2\lambda_4 + \\
&\quad 2\lambda_1\lambda_3^2\lambda_4^3 + 13\lambda_1^2\lambda_3\lambda_4^3 + 9\lambda_1^2\lambda_3^3\lambda_4 + 13\lambda_1^3\lambda_3^2\lambda_4 + 2\lambda_2\lambda_3^2\lambda_4^3 + \\
&\quad 9\lambda_2^2\lambda_3^3\lambda_4 + 13\lambda_1\lambda_2\lambda_3^2\lambda_4^2 + 51\lambda_1\lambda_2^2\lambda_3\lambda_4^2 + 44\lambda_1\lambda_2^2\lambda_3^2\lambda_4 + 51\lambda_1^2\lambda_2\lambda_3\lambda_4^2 + \\
&\quad 44\lambda_1^2\lambda_2\lambda_3^2\lambda_4 + 83\lambda_1^2\lambda_2^2\lambda_3\lambda_4 + 61\lambda_1^2\lambda_2^2\lambda_3^2 + 13\lambda_2^3\lambda_3^2\lambda_4 + \\
&\quad 32\lambda_1^3\lambda_2\lambda_4^2 + 42\lambda_1^2\lambda_2^2\lambda_4^2 + 22\lambda_1^2\lambda_2\lambda_4^3 + 74\lambda_1\lambda_2^3\lambda_4^2 + 40\lambda_1\lambda_2^2\lambda_4^3 + 8\lambda_1^4\lambda_4^2 + \\
&\quad 10\lambda_1^3\lambda_4^3 + \lambda_2^3\lambda_4^3 + 4\lambda_2^3\lambda_3\lambda_4^2 + 12\lambda_4^2\lambda_2^3(\lambda_3-\lambda_2) + 13\lambda_4^2\lambda_2^2(\lambda_3\lambda_4-\lambda_2^2) + \\
&\quad \lambda_4^2((2\lambda_2-\lambda_1)^3-\lambda_3^3)(2\lambda_1+2\lambda_2+\lambda_4) + \lambda_4^2((2\lambda_2-\lambda_1)^4-\lambda_3^4)\big) \geq 0
\end{aligned}
$$

Since λ_k's are increasing, the last two terms in the penultimate row are positive, and the last two terms are positive if $2\lambda_2 - \lambda_1 > \lambda_3$, i.e., if $\beta^3_{(1,l)} - \beta^4_2 < 0$ for $l = 3$ or 4, which is assumed.

□

References

[1] N. Balakrishnan, A.P. Basu, The Exponential Distribution: Theory, Methods and Applications, Gordon and Breach, London, 1995.

[2] N. Balakrishnan, C.R. Rao, Handbook os Statistics 16 - Order Statistics: Theory and Methods, Elsevier, New York, 1998.

[3] N. Balakrishnan, C.R. Rao, Handbook os Statistics 17 - Order Statistics: Applications, Elsevier, New York, 1998.

[4] R.E. Barlow and F. Proschan, Inequalities for linear combinations of order statistics from restricted families, Ann. Math. Statis. 37(1966)1574-1592.

[5] B. Khaledi, S.C. Kochar, Stochastic Orderings among Order Statistics and Sample Spacings, in: J.C. Misra (Ed.), Uncertainty and Optimality: probability, statistics and operations research, Singapore, 2002, p.p. 167-203.

[6] S.C. Kochar, S.N.U.A. Kirmani, Some results on normalized spacings from restrictited families of distributions , J. Statis. Plann. 46 (1995) 47-57.

[7] S.C. Kochar, R. Korwar, Stochastic orders for spacings of heterogeneous exponential random variables, J. Multivariate Anal. 57(1996)69-83.

[8] D.S. Mitrinovic, Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[9] G. Pledger, F. Proschan, Comparisons of order satistics from heterogeneous populations, with applications in reliability, in: J.S. Rustagi Ed., Optimizing Methods in Statistics, Academic Press, New York, 1971, p.p. 89-113.

[10] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer, Berlin, 2006.

[11] P. V. Sukhatme, Tests of significance for samples of the χ^2 population with two degrees of freedom, Annals of Eugenics, 8(1937)52-56.

[12] S. Wen, Q. Lu, and T. Hu, Likelihood ratio orderings of spacings heterogeneous exponential random variables, J. Multivariate Anal. 98(2007)743-756.