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SOLVING RECENT RBC MODELS USING LINEARIZATION: FURTHER RESERVES

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Abstract

Through a simple example, we show that the successive sophistications introduced in the early RBC models in order to improve their internal propagation mechanisms have actually increased their non-linearities, even locally. Accordingly, linearization-based resolution methods become nuch more disputable than they were for early RBC models. Simple comparative studies of impulse-response functions are used to illustrate this point. We conclude by pointing at some alternative resolution techniques that allow the model builder to take non-linearities into account and/or to handle with the presence of large state spaces.

Key Words
RBC Models; Linearization Methods; Impulse-Response Functions.

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1. Introduction

The main *methodological* innovation in macroeconomics over the last twenty five years has probably consisted in building full-fledged stochastic models where fully rational agents maximize well-defined objective functions intertemporally. Along the lines of Frisch (1933), the large majority of these models have kept on presenting macroeconomic fluctuations as the response of the economic system to exogenous shocks to some of its fundamental characteristics (such as technology, money supply or public spendings). Mathematically speaking, this means that for a given value of the exogenous variables and parameters, these models are characterized by the existence of a unique saddle-point trajectory[1] leading progressively the economy to its stationary state equilibrium position.

As these dynamic models become rapidly sufficiently complex to preclude any analytical resolution, a numerical analysis is usually required. In this respect, the methodology initially proposed by the Real Business Cycle literature (particularly Kydland and Prescott (1982) and King, Plosser and Rebelo (1988)) has clearly become a reference. Along these lines, a model should finally be assessed according to its ability to generate[2] pseudo-time series the main statistical properties of which are in accordance with the properties of the observed data. In the large majority of cases, the numerical analysis underlying this methodology is made exclusively on the basis of a linearized version of the structural model. In particular, the method proposed by King, Plosser and Rebelo (1987) (linearization of the first-order equations) is quite often used.

Although the basic principles of the RBC methodology are not in question here, the present note aims at illustrating briefly the limits inherent to a quantitative exercice based on a linearized model. To an important extent, this claim will appear rather trivial and, legitimally enough, the reader could have the feeling that we are simply about to labour an obvious point. By taking this risk, we want to stress that the *new* developments in the real business cycle research program[3] make a quantitative analysis based on a linearization much more disputable than it was for the seminal RBC models. The underlying argument is simple. The recent literature has mainly focused on the improvement of the internal propagation mechanisms of these models. Accordindly, the recent models have integrated additional sources of non linearities, even locally. As we will show on the basis of a very simple example, linearizing such models is very likely to induce an important bias even when the analysed fluctuations remain in a neighbourood (traditionally considered as) close to the stationary state[4]. Therefore, even though a linearization remains certainly a

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[1]... implying a sequence of unique rational expectations equilibria...

[2]...by means of stochastic simulations...

[3]This expression has to be understood in a very broad sense here, including e.g. the numerous developments in the New-Keynesian program using the RBC methodology. See Cooley (1995) for a very thorough introduction to the different directions of the (real) business cycle research program.

[4]The concept of local non-linearities has thus a very concrete meaning here. "Local" refers to the diameter of the neighbooroud of the stationary state in which fluctuations can be observed, given the

quite valuable tool for a purely qualitative analysis, more quantitative exercices relying on a linearization (like exercices of comparison or even discrimation between alternative models) call for an increased caution.

In order to illustrate very simply our argument, we conduct impulse-response functions comparison for two different models. The first one is close the basic King, Plosser and Rebelo's model (1988) with a constant capital depreciation rate. The second one extends the former model by including "a depreciation in use" assumption. In each case, we compute the impulse-response functions to a technological shock as well on the basis of the linearized form as on the basis of the structural nonlinear form using Laffargue's algorithm (1990) (see Boucekkine (1995) for the analysis of the theoretical properties of this algorithm). For a sufficiently small convergence tolerance level, Laffargue's algorithm gives the exact impulse-response functions and so allows the model builder to capture appropriately the effects of nonlinearities that a linearization misses unavoidably. In particular, a direct comparison between the values of the exact and approximated impulse response functions at the time of the shock illustrates the relative size of the linearization bias each time a shock of the same size occurs in a stochastic simulation exercice. Other - and more sophisticated- evaluation procedures could also be used. In particular, a general procedure to test the accuracy of any solution method has been proposed by Den Haan and Marcet (1990)[5]. Since the source of bias inherent to the linearization methods is obvious -namely the local nonlinearities, our comparative analysis of impulse-response is anyway perfectly sufficient to make the bias apparent and our argument clear.

Section 2 first presents the models under consideration in this paper. 3 displays the numerical results of our comparison study. We conclude in section 4.

2. An Illustration

2.1 The two models

In order to remain as short as possible, we only consider a simple framework, which is not -and by far- the most sophisticated one can find. An assumption of "depreciation in use" (as in Greenwood-Hercowitz-Huffman (1988), Burnside-Eichenbaum (1994)) is the only departure with respect to the most standard RBC models. Since the depreciation in use model nests the standard model, it is first described.

We consider a perfectly competitive economy with indivisible labour *à la* Hansen (1986). It is assumed that individuals have identical preferences and are covered by a full un-

assumptions on the distribution of the stochastic shocks in a typical RBC exercice.

[5]If the approximation method is sufficiently accurate, the introduction of the pseudo-time series generated by the approximation into the structural nonlinear Euler equations, must give a series of residuals that satisfy the martingale property (inherent to the rational expectations hypothesis).

employment insurance (so that they all receive the same income irrespectively of their working or not). Let $\mathcal{U}_t$ be the utility function of the representative agent:

$$\mathcal{U}_t = \mathrm{E}_t \sum_{s=0}^{\infty} \beta^s \left[\log(c_{t+s}) + B\,(1 - n_{t+s})\right] \tag{1}$$

where c_t and n_t represent consumption and labour at date t (the total time endowment has been normalized to 1.). β $(0 < \beta < 1)$ is the time preference parameter.

The productive capital stock at date t (k_t) is predetermined but can be used with a variable intensity $u_t > 0$. The production function of the representative firm at each date $t = 0, 1, 2, \ldots$ is Cobb-Douglas:

$$y_t = A_t\,(k_t\,u_t)^{1-\alpha}\; n_t^{\alpha} \qquad \text{with} \quad A_t = A_{t-1}^{\rho}\,\exp(v_t) \tag{2}$$

where y_t is the output level at date t and n_t is the labour input. In the AR(1) process describing the evolution of the total productivity of factors (A_t), $0 < \rho < 1$ and v_t is an i.i.d. variable with zero mean.

To compute the competitive allocation in the above described economy, it is sufficient to analyse the central planner's decision problem. At each date t, he chooses c_t, n_t, u_t and k_{t+1} in order to maximize (1) subject to the macroeconomic ressource constraint:

$$c_t + k_{t+1} - (1 - \bar{\delta}\,u_t^{\phi})\,k_t \leq y_t \tag{3}$$

where y_t is given by (2). The parameter $\bar{\delta}$ $(0 < \bar{\delta} < 1)$ is a depreciation constant. The parameter ϕ $(\phi > 0)$ reflects the sensitivity of the depreciation rate to the capital utilization rate u_t.

The first-order conditions of this maximization program are:

$$B = c_t^{-1}\,\alpha\,\frac{y_t}{n_t} \tag{4}$$

$$c_t^{-1} = \mathrm{E}_t\left[\beta\,c_{t+1}^{-1}\left((1-\alpha)\,\frac{y_{t+1}}{k_{t+1}} + 1 - \bar{\delta}\,u_{t+1}^{\phi}\right)\right] \tag{5}$$

$$\bar{\delta}\,\phi\,u_t^{\phi-1} = (1-\alpha)\,\frac{y_t}{k_t} \tag{6}$$

Moreover, the ressource constraint (3) has to be satisfied with strict equality.

The interpretation of conditions (4) and (5) is obvious. According to condition (6), the optimal capital utilization rate makes the marginal product of a more intensive use of capital equal to the marginal cost of a faster depreciation.

In the sequel, we call M1 the above depreciation in use model (i.e., the system of equations (2) to (6)). In absence of technological shocks, M1 admits a unique non-stochastic stationary state $\{\bar{u}, \bar{c}, \bar{n}, \bar{k}, \bar{y}\}$.

M1 nests the standard model with exogenous depreciation rate ($\phi = 0$): by assuming $u_t = 1$ (for $t = 0, 1, ...$) in equations (2) to (5) and suppressing equation (6), we retrieve the standard model, called M2 hereafter. M2 admits a unique non-stochastic stationary state (resp. $\{\bar{c}', \bar{n}', \bar{k}', \bar{y}'\}$).

Let M1L (resp. M2L) be the linear model obtained from the linearization of M1 (resp. M2) around its non stochastic stationary. At date t, the endogenous variables of the linearized model M1L (resp. M2L) are $\{\hat{u}_t, \hat{c}_t, \hat{n}_t, \hat{k}_{t+1}, \hat{y}_t\}$ (resp. $\{\hat{c}_t, \hat{n}_t, \hat{k}_{t+1}, \hat{y}_t\}$) where the notation $\hat{x}_t$ represents the relative deviation of variable x_t with respect to its stationary state value $\bar{x}$ (i.e., $\hat{x}_t = x_t/\bar{x} - 1$).

2.2 Comparison of the Impulse Response Functions.

From calibrated versions of our models[6], we conduct our impulse response comparisons for a technological shock v occuring at date $t = 1$. As we have already mentionned in the introduction, the impulse response functions are computed by solving the models with Laffargue's algorithm (1990). Given the usual assumptions of the literature on the densition function of the technological shocks and its standard deviation (namely a normal distribution with a standard deviation between 0.008 and 0.015), considering shocks between minus and plus 0.05 is largely sufficient to cover the range of the possible shocks.

For each shock ($v_1 \in [-0.05, +0.05]$), we focus exclusively on the difference between the responses of the structural and linearized models in the first period (i.e., at the time of shock). A comparison limited to the first period is indeed the most meaningful to illustrate the bias induced by the linearized model each time a shock of a given size occurs in a stochastic simulation exercice[7]. For a given shock v_1, let x_1^s be the value of variable x in period 1 obtained from the structural non-linear model (either M1 or M2). $\bar{x}$ is the stationary value of the variable. The difference between $x_1^s/\bar{x} - 1$ and $\hat{x}_1$ (obtained from either M1L or M2L) gives us the relative approximation error induced by the linearized model. In function of the size of the shock, figure 1 hereafter displays this relative error in the first period for output, investment, employment and consumption. In each pannel of

[6]The models are calibrated in order to obtain a stationary equilibrium consistent with a list of stylized facts or available estimations for the US economy. Capital share in production is 0.36 ($\alpha = 0.64$). The discount factor β is set equal to 0.992 (corresponding to an annual interest rate slighty above 3%). The parameters ϕ and $\bar{\delta}$ have next been choosen in order to obtain a quaterly capital output ratio around 13.5 and an investment share in output equal to 0.25 ($\delta = 0.02$ and $\phi = 1.44$). The parameter B has finally been set in order to obtain an average working time close to 1/3 for the representative individual ($B = 2.5$). The calibration of M2 is identical (ϕ is irrelevant).

[7]Even though shocks larger than 5% are never observed, it is worth outlining that in a stochastic simulation exercice, the history of shocks can move temporarily the economy away from its stationary state more than a shock v_1 of plus or minus 5 percent would do. In a stochastic simulation exercice, biases larger than those showed hereafter are thus possible.

Figure 1, the thick (resp. thin) line represents the difference between M1 and M1L (resp. M2 and M2L).

Figure 1: **Linearization bias in $t = 1$ in function of v_1.**

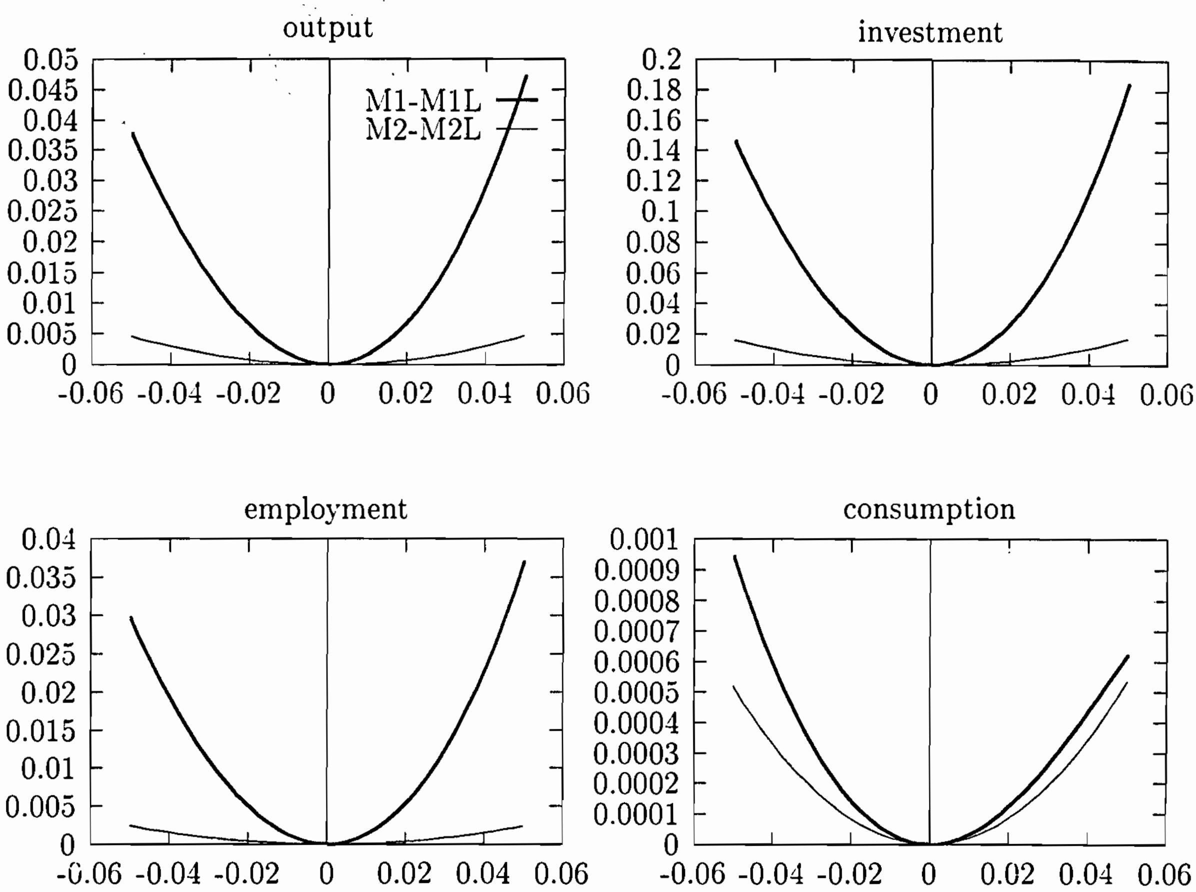

Figure 1 speaks for itself and long comments are probably useless. Quite strikingly, the linearization bias appears much more important in the case of the depreciation in use model. A single departure with respect to the standard model M2 (rather well approximated by M2L) may thus be sufficient to deteriorate markedly the quality of the approximation obtained from the linearized model.

It is also quite clear that all the variables do not suffer from a linearization bias of the same importance. Indeed, the variables that are traditionally the most volatile along the business cycle are the ones for which the linearization bias is the most important. This is true for both the "depreciation in use" and the standard model but in a totally different extent. It is noticable that the behaviour of consumption remains particularly well approximated by the linearized model in both cases. Quite generally, the presence of a variable-specific bias is hardly surprising: a multivariate model may exhibit degrees of non-linearities that differs from one variable to another[8].

Incidentally, Figure 1 also reveals the presence of an assymetric response of the nonlinear model M1 (to positive and negative shock of same size) which is not captured by its linearized version. We do not mean to emphasize this assymetry here because it is only really apparent for shocks above 4%. However, this illustrates that a very simple non-linear model may already produce assymetries even though it does not incorporate any particular economic mechanisms forcing assymetries[9].

By way of conclusion...

On the basis of a model with a single departure from a basic RBC model, we have illustrated the principal argument of this note: the different specifications that are proposed in order to improve the internal propagation mechanisms of these models tend to increase markedly their local-nonlinearities. This suggests that a resolution scheme based on a linearization is much less appropriated to the latest models than to the early RBC models. Given the ultimate objective of a RBC exercice -namely a precise quantitative validation and not a purely qualitative appraisal, this computational aspect cannot be neglected by the current and future related research programs.

In our opinion -and well beyond the computational aspect mentionned above, a linearization-based resolution method puts anyway a brake on this research program in that it disables the study of many important propagation mechanisms relying on strong non-linearities (like pointwise non-differentiabilities and, more generally, any mechanism responsible for assymetric response to positive and negative shocks). Although some very recent contributions have tried to account for these strong non-linearities (see e.g. Christiano and Fisher (1994) and Diaz-Gimenez (1995) for a numerical treatment of boundary constraints), many economists still desist in introducing them in their models because they would require too costly or tedious non-linear resolution techniques. At this point of the article, the reader is thus probably in right to expect that we suggest him alternatives to linearization methods. Since 1990, new methods have indeed been proposed[10], among which the famous expectations parametrization method of Den Haan and Marcet (1990) or the

[8] In a multivariate framework, it may however become very hard to determine (without numerical analysis) the variables that are the most exposed to a linearization bias.

[9] Moreover, our remark in footnote 7 still applies here.

[10] Danthine and Donaldson (1995) propose an introduction to several out of them.

integral equations discretization of Tauchen and Hussey (1991). It is however commonly argued that the numerical precision of the new methods is hard to control. This critique is not unfounded: contrary to the claims of their respective authors, these new numerical methods are not easily implementable for an arbitrarily small level of accuracy. This is especially true when the state space is relatively large[11].

It seems hardly disputable that a valuable and real alternative to the linearization methods should allow the user to handle with large state spaces, nonlinearities and non-differentiabilities.
a) The recent methods mentionned above should satisfy these requirements with massive parallel computing, but this way of proceding is not really familiar to economists (and will probably not be so in the near future).
b) A more obvious numerical method that satisfies the above requirements consists in a stochastic extension of Laffargue's algorithm we used in the numerical section of this note[12]. Unfortunately, this approach approximates rational expectations by perfect foresight so that its stochastic extension generates a bias (for each replication, innovations posterior to the first period are assumed equal to their expected value)[13].
c) In the current state of the art, the only method that has been quite successfully used to solve relatively large state spaces nonlinear models including boundary constraints is the projection technique proposed by Judd (1992). By using this technique, Gilchrist and Williams (1995) developp an algorithm allowing them to solve a general equilibrium model incorporating a putty-clay technology and vintage capital, with eleven state variables. This is clearly a worthwhile advance. However, the algorithm proposed by Gilchrist and Williams relies on a particular polynomial approximation the efficiency of which may vary from one model to another. Moreover, this type of approximation may be useful to compute accurately one particular decision rule but may turn out useless for another[14]. It is thus not obvious that Gilchrist and Williams' algorithm could be so successfully used on other models.

This last observation suggests us that the era of user-friendly algorithms, able to solve any model of a given type without any particular adaptation work, is probably definitively closed. Linearization methods seem to be less and less adapted to the most recent macroeconomic modelling. But there is little doubt that capturing the nonlinearities effects will require a growing computational expertise among economists.

[11]According to the method, "large" means here more than 3 or 4 state variables. On top of that, it should also be said that the convergence properties of some recent methods are not quite clear.

[12]The pseudo-time series are then simply generated by replicating the impulse-response functions obtained from Laffargue's algorithm.

[13]A recent application by Adda and Boucekkine (1995) suggests that obtaining a reasonably accurate solution is however possible with a very strict choice of the experimental parameters of the method.

[14]For example, Gilchrist and Williams show that the cutoff wage variable of their model is accurately approximated by Chebychev polynomial functions but not by direct polynomial functions.

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