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# EQUITABLE OPPORTUNITIES IN ECONOMIC ENVIRONMENTS

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Abstract

In this paper, we extend the axiomatic analysis of equitable opportunities developed in Kranich [6] from finite to continuous opportunity sets. This extended framework is amenable to economic applications. The main results establish conditions under which an ordinal ranking of profiles of opportunity sets can be represented by a cardinal *advantage function* which describes both the extent of inequality and the distribution of advantage among the agents.

Keywords: Equity, fairness, opportunities, advantage.

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## 1. Introduction

In Kranich [6], we proposed an axiomatic approach to ranking distributions of opportunities on the basis of fairness. Specifically, we considered the case of n agents, each of whom faces a finite set of opportunities, and we demonstrated that it is possible to construct a complete, reflexive, and transitive relation defined over profiles of such sets that satisfies several intuitively appealing criteria for fairness. However, the restriction to finite opportunity sets is quite limiting and precludes most economic applications. In this paper, we extend the framework to general topological spaces. In particular, this allows for connected subsets of a finite Euclidean space.

As in the previous paper, our approach is axiomatic. But here, rather than concern ourselves with characterizing a particular ordinal relation, we establish conditions under which such a relation admits of a cardinal representation of a particularly intuitive form, which we call an *advantage function*. Such a function indicates both the extent of inequality and the distribution of advantage among the agents.

We begin with the two-agent case in Sections 2 and 3, where we develop the basic framework. Then, in Sections 4 and 5, we build upon the two-agent results to extend the analysis to include additional agents. Section 6 contains a brief conclusion in which we discuss the significance of the results and their relationship to our earlier work.

## 2. The Two-Agent Case

Let X be a universal set of *opportunities*. For reasons of generality, we do not specify the nature of the elements in X. Let $\mathbb{P}(X)$ denote the set of nonempty subsets of X. We consider a topological space $\mathcal{X}\subseteq\mathbb{P}(X)$.[1] An *opportunity set* for agent i is an element $O^i\in\mathcal{X}$.

In this section, we consider the case of only two agents. Hence, a *profile* of opportunity sets is a pair $\mathbf{O}=(O^1,O^2)\in\mathcal{X}^2\equiv\mathcal{X}\times\mathcal{X}$, and an *equality relation* is a complete, reflexive, and transitive relation (i.e., a weak order) $\succsim^2$ defined on $\mathcal{X}^2$.[2] For $\mathbf{O},\mathbf{O}'\in\mathcal{X}^2$, we write $\mathbf{O} \succsim^2 \mathbf{O}'$, meaning the opportunity sets in $\mathbf{O}$ are at least as equitable as those in $\mathbf{O}'$. $\succ^2$ (more equitable than) and $\sim^2$ (equally equitable) are the asymmetric and symmetric components of $\succsim^2$, respectively, and are defined in the usual way.

Some examples of equality relations are the following:[3]

EXAMPLE 1. Let X be a (Lebesque) measurable subset of $\mathbb{R}^\ell$, and suppose one's opportunities can be described by a measurable subset of X. Then let $\mathcal{X}$ be the $\sigma$-algebra of all such subsets. And define the *Lebesgue difference* in $\mathbf{O}=(O^1,O^2)$ by $LD(\mathbf{O})\equiv|\mu(O^1)-\mu(O^2)|$, where $\mu$ is the Lebesgue measure restricted to $\mathcal{X}$. The *Lebesgue difference relation* $\succsim^2_{LD}$ is defined by

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[1] We formulate the problem in general terms since different topologies may be appropriate for different domains. For discussions of topologies defined on spaces of subsets, see Kuratowski [7] and Bourbaki [2]. For additional references, see Klein and Thompson [5].

[2] In "$\succsim^2$" and elsewhere, we use the superscript "2" to distinguish the two-agent case.

[3] $\mathbb{R}^\ell$ denotes the $\ell$-dimensional Euclidean space, and $\mathbb{R}^\ell_+$ its nonnegative cone. For $x,y\in\mathbb{R}^\ell$, $x\cdot y$ denotes the Euclidean inner product, and $|\cdot|$ denotes absolute value.

$$\mathbf{O} \succsim^2_{LD} \mathbf{O}' \iff LD(\mathbf{O}) \le LD(\mathbf{O}').$$

EXAMPLE 2. Consider the case of a large market economy in which each individual has a private production set. For two agents, neither of whom can influence prices, we can compare consumption (trading) opportunities on the basis of the maximum values of their production sets as follows.

Let $X=\mathbb{R}^\ell$ and let $\mathcal{X}$ be the set of *neoclassical production sets.*[4] Also, let $p\in\Delta^{\ell-1}$ be given, where $\Delta^{\ell-1}$ denotes the $(\ell-1)$-dimensional unit simplex. For $O\in\mathcal{X}$, define $v(O;p)\equiv\max\{p\cdot x\,|\,x\in O\}$. Then the *v-difference*, or *value difference*, in $\mathbf{O}=(O^1,O^2)$ is given by $VD(\mathbf{O})\equiv|v(O^1;p)-v(O^2;p)|$. The *value difference relation* $\succsim^2_{VD}$ is defined by

$$\mathbf{O} \succsim^2_{VD} \mathbf{O}' \iff VD(\mathbf{O}) \le VD(\mathbf{O}').^5$$

EXAMPLE 3. Since individuals may be willing to trade off potential earnings for other job amenities, the *actual* distribution of earnings among equally abled individuals might be quite skewed. Therefore, from society's perspective, an appropriate comparison of employment opportunities might be based, first, on *potential* earnings and, second, on other job characteristics. For simplicity, consider the case of a single nonwage characteristic, called "quality." Let $x_1$ measure earnings and $x_2$ measure quality. We take X to be $\mathbb{R}^2_+$ and $\mathcal{X}$ to be the set of all combinations of $x_1$ and $x_2$ that an individual might face. Then from society's point of view, comparisons can be made lexicographically as follows:

For $\mathbf{O}\in\mathcal{X}^2$, let $\hat{x}^1_1=\max\{x^1_1\,|\,(x^1_1,x^1_2)\in O^1\}$ and $\hat{x}^1_2=\max\{x^1_2\,|\,(\hat{x}^1_1,x^1_2)\in O^1\}$. Define the

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[4] I.e., each $Y\in\mathcal{X}$ is closed, convex, bounded above, and satisfies $-\mathbb{R}^\ell_+\subseteq Y$ and $Y\cap\mathbb{R}^\ell_+=\{0\}$.

[5] Note that this relation is parametric on the price vector p.

*lexicographic difference* in $\mathbf{O}=(O^1,O^2)$, by

$$XD(\mathbf{O}) \equiv \begin{cases} |\hat{x}^1_1 - \hat{x}^2_1| & \text{if } \hat{x}^1_1 \neq \hat{x}^2_1, \\ |\hat{x}^1_2 - \hat{x}^2_2| & \text{otherwise.} \end{cases}$$

The *lexicographic difference relation* $\succsim^2_{XD}$ is defined by

$$\mathbf{O} \succsim^2_{XD} \mathbf{O}' \iff XD(\mathbf{O}) \leq XD(\mathbf{O}').$$

In each of the above examples, the comparison between $\mathbf{O}$ and $\mathbf{O}'$ is based on the magnitude of a real-valued function. Generally, we will say an equality relation $\succsim^2$ is *represented* by a function $f:\mathcal{X}^2 \to \mathbb{R}$ if

$$\mathbf{O} \succsim^2 \mathbf{O}' \text{ if and only if } f(\mathbf{O}) \geq f(\mathbf{O}').$$

(Clearly, if f represents $\succsim^2$, then $-f$ also provides an appropriate ranking. That is, for all $\mathbf{O},\mathbf{O}' \in \mathcal{X}^2$, $\mathbf{O} \succsim^2 \mathbf{O}'$ if and only if $-f(\mathbf{O}) \leq -f(\mathbf{O}')$. Although an abuse of terminology, we will say $-f$ represents $\succsim^2$ as well. Intuitively, while f provides an index of fairness, $-f$ measures unfairness.)

In the sequel, we will investigate conditions under which an equality relation admits of a representation of a particularly intuitive form. But first, given $\succsim^2$, we will say *i is poor relative to j at* $\mathbf{O}$, denoted $i \in P(\mathbf{O};j)$, if there exists $O'^i \in \mathcal{X}$ such that $O'^i \supset O^i$ and $(O'^i,O^j) \succ^2 (O^i,O^j)$.[6] We will write $P(\mathbf{O}) \neq \emptyset$ if $i \in P(\mathbf{O};j)$, for some i,j. For future reference, if $i \notin P(\mathbf{O};j)$ and

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[6]Intuitively, if an expansion of 1's opportunity set were to increase equality, that would identify her as "poor" and thus agent 2 as "rich." Similarly, if a contraction of 2's set were to increase equality, that too would establish the same relative ranking, and the analysis could be carried out using this alternative identification procedure. Note, however, that the opposite is not true due to rank reversals. That is, if an expansion of 1's set were to decrease fairness, that would not identify her as rich. It may be the case that initially agent 1 is poor relative to 2, and yet 1's opportunity set expands to such an extent that the relative ranking is reversed and the overall skewness increased.

$j \notin P(\mathbf{O};i)$, then we will say *i is in the same class as j* and write $i \in C(\mathbf{O};j)$. And if $i \in P(\mathbf{O};j)$ or $i \in C(\mathbf{O};j)$, we will say *i is weakly poor relative to j* (or *no richer than j*) and write $i \in WP(\mathbf{O};j)$.

DEFINITION 2.1. A (2-agent) *advantage function* is a mapping $a^2:\mathcal{X}^2 \rightarrow \mathbb{R}^2$ that associates with each $\mathbf{O} \in \mathcal{X}^2$ a pair of real numbers $(a^2_1(\mathbf{O}), a^2_2(\mathbf{O}))$ such that (1) $a^2_1(\mathbf{O}) + a^2_2(\mathbf{O}) = 0$, and (2) $a^2_i(\mathbf{O}) < a^2_j(\mathbf{O})$ if and only if $i \in P(\mathbf{O};j)$.[7]

An advantage function indicates the direction and magnitude of the skewness of $\mathbf{O}$. If $a^2_i(\mathbf{O}) < a^2_j(\mathbf{O})$, then j enjoys an advantage relative to i, or i suffers a disadvantage. The magnitude of the advantage can be measured by $\sum_i |a^2_i(\mathbf{O})|$.[8] Consistent with the above terminology (although again a slight abuse), we will say the advantage function $a^2:\mathcal{X}^2 \rightarrow \mathbb{R}^2$ represents $\succsim^2$ if

$$\mathbf{O} \succsim^2 \mathbf{O}' \quad \text{if and only if} \quad \sum_i |a^2_i(\mathbf{O})| \leq \sum_i |a^2_i(\mathbf{O}')|.$$

## 3. A Representation Theorem for $\succsim^2$

Let $\mathcal{E}^2$ denote the subdomain of $\mathcal{X}^2$ in which the agents have identical opportunity sets, i.e., $\mathcal{E}^2 \equiv \{\mathbf{O} \in \mathcal{X}^2 \mid O^1 = O^2\}$. We refer to $\mathcal{E}^2$ as the *egalitarian domain*, and we denote a generic element of $\mathcal{E}^2$ by $\mathbf{O}^e$.

We define the following properties of an equality relation $\succsim^2$:

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[7] With only two agents, conditions (1) and (2) imply $a^2_i(\mathbf{O}) < 0 < a^2_j(\mathbf{O})$, when $i \in P(\mathbf{O};j)$.

[8] Alternatively, since $|a^2_i(\mathbf{O})| = \frac{1}{2}\sum_i |a^2_i(\mathbf{O})|$, $|a^2_i(\mathbf{O})|$ would yield the same ranking of profiles as defined below.

Acyclicity of the Strict Relative Poverty Relation ($ACYCL^2$): For all $\mathbf{O}\in\mathcal{X}^2$, for $i,j=1,2$, $i\neq j$, if $i\in P(\mathbf{O};j)$, then $j\notin P(\mathbf{O};i)$.

Uniformity ($UNIF^2$): For all $\mathbf{O}^e\in\mathcal{E}^2$ and for all $\mathbf{O}\in\mathcal{X}^2$, $\mathbf{O}^e \succsim^2 \mathbf{O}$.

Continuity ($CONT^2$): For all $\mathbf{O}\in\mathcal{X}^2$, $\{\mathbf{O}'\in\mathcal{X}^2 \mid \mathbf{O}' \succsim^2 \mathbf{O}\}$ and $\{\mathbf{O}'\in\mathcal{X}^2 \mid \mathbf{O} \succsim^2 \mathbf{O}'\}$ are closed (in the product topology on $\mathcal{X}^2$).

According to $ACYCL^2$, if an expansion of agent 1's opportunity set is equality enhancing, then expansions of 2's should be (weakly) equality detracting. Thus, it requires that $\succsim^2$ be logically consistent: having identified i as poor relative to j, it cannot identify j as poor relative to i. $UNIF^2$ means the most equitable distributions are those in which the agents have identical, or uniform, opportunities. $CONT^2$ is a standard technical condition.

THEOREM 3.1. Let $\mathcal{X}$ be a connected, separable, topological space[9,10]; and let $\succsim^2$ satisfy $ACYCL^2$, $UNIF^2$, and $CONT^2$. Then there exists a continuous advantage function $a^2:\mathcal{X}^2\to\mathbb{R}^2$ that represents $\succsim^2$.

*Proof.* First, note that since $\mathcal{X}$ is connected and separable, the product $\mathcal{X}^2$ is

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[9] A topological space is <u>connected</u> if it cannot be partitioned into two disjoint, nonempty, closed sets. It is <u>separable</u> if it contains a countable dense subset.

[10] For example, if X is a (Lebesque) measurable subset of $\mathbb{R}^2$ and $\mathcal{X}$ is the $\sigma$-algebra of all measurable subsets of X, then $\mathcal{X}$ is separable and connected in the topology constructed in Berliant [1] to describe preferences defined over parcels of land.

as well.[11] And by definition, $\succsim^2$ is a complete preorder on $\mathcal{X}^2$. Since $\succsim^2$ satisfies CONT$^2$, it follows from Eilenberg [4][12] that there is a continuous function $w:\mathcal{X}^2 \to \mathbb{R}$ such that

$$\mathbf{O} \succsim^2 \mathbf{O}' \text{ if and only if } w(\mathbf{O}) \geq w(\mathbf{O}'). \tag{1}$$

Clearly, (1) is preserved when translating $w$ by a constant. Therefore, by UNIF$^2$, we may assume $0=w(\mathbf{O}^e)\geq w(\mathbf{O})$, for all $\mathbf{O}^e \in \mathcal{E}^2$ and for all $\mathbf{O} \in \mathcal{X}^2$.

We must show that whenever $P(\mathbf{O})\neq\emptyset$, $w(\mathbf{O})<0$. However, this follows easily from the definition of relative poverty. Suppose, for instance, that $1\in P(\mathbf{O};2)$. Then, there exists $O'^1 \in \mathcal{X}$, $O'^1 \supset O^1$, such that $(O'^1,O^2) \succ^2 \mathbf{O}$. Therefore, $0\geq w(O'^1,O^2) > w(\mathbf{O})$.

Next, we derive an advantage function $a^2$ from $w$ as follows. Let

$$a_1^2(\mathbf{O}) = \begin{cases} -w(\mathbf{O}) & 2\in P(\mathbf{O};1) \\ w(\mathbf{O}) \quad \text{if} & 1\in P(\mathbf{O};2), \\ 0 & \text{otherwise} \end{cases}$$

and let $a_2^2(\mathbf{O})=-a_1^2(\mathbf{O})$. Note that by ACYCL$^2$, $a^2$ is well-defined. That is, if $i\in P(\mathbf{O};j)$, then it cannot be the case that $j\in P(\mathbf{O};i)$, and vice versa. By construction, $a_i^2(\mathbf{O})<0<a_j^2(\mathbf{O})$ if and only if $i\in P(\mathbf{O};j)$, and also by construction, $a_1^2(\mathbf{O})+a_2^2(\mathbf{O})=0$. Finally, since $w$ is continuous, $a^2$ is as well. ∎

For Examples 1 and 2 in Section 2, we can easily construct a continuous advantage function. First, for $\mathbf{O}\in\mathcal{X}^2$, define $\bar{\mu}(\mathbf{O})\equiv \frac{1}{2}(\mu(O^1)+\mu(O^2))$ and $\bar{v}(\mathbf{O})\equiv \frac{1}{2}(v(O^1;p)+v(O^2;p))$. Then $a_{LD}^2(\mathbf{O})\equiv(\mu(O^1)-\bar{\mu}(\mathbf{O}),\mu(O^2)-\bar{\mu}(\mathbf{O}))$ represents $\succsim_{LD}^2$, and $a_{VD}^2(\mathbf{O})\equiv(v(O^1;p)-\bar{v}(\mathbf{O}),v(O^2;p)-\bar{v}(\mathbf{O}))$ represents $\succsim_{VD}^2$.

The lexicographic difference relation in Example 3, however, violates the

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[11] See Munkres [8, p.190-192].

[12] See also Debreu [3, Proposition 4].

assumption of $CONT^2$. Thus, while it is clearly representable as described above, the representation is not continuous.

## 4. An n-Agent Generalization

In this section, we take $\succsim^2$ satisfying the conditions in Theorem 1 as given. Consequently, there exists a continuous representation $a^2$ of $\succsim^2$ which we take as given as well. Then based on the results of the previous section, we use $\succsim^2$ to extend the analysis to include additional agents. We begin by generalizing the notation.

Let $N=\{1,\dots,n\}$ be a finite set of agents, with $n>2$. We now consider profiles of opportunity sets of the form $\mathbf{O}=(O^1,\dots,O^n)\in\mathcal{X}^n$, where $\mathcal{X}^n \equiv \bigtimes_{i\in N} \mathcal{X}$. Although the domain of an equality relation is now $\mathcal{X}^n$, the interpretation remains the same; we write $\mathbf{O} \succsim^n \mathbf{O}'$.

To be consistent with our earlier reasoning, whatever principles of fairness we apply to the n-agent case should generalize the axioms in Section 3. Indeed, one might be tempted to consider n-agent relations $\succsim^n \subseteq \mathcal{X}^n x \mathcal{X}^n$ for which the projections onto the two-agent subspaces are consistent with the previous axioms. This is inappropriate, however; with additional agents, new considerations arise. Suppose, for example, there are three agents and agent 1 is poor relative to agent 2. In the subspace pertaining to agents 1 and 2, expanding 2's opportunity set should decrease fairness. However, in the three agent problem, expanding 2's set might increase fairness (say, by making 2's and 3's sets more equitable). In other words, it is unreasonable to suppose that the evaluation of fairness of the projection of $(O^1,O^2,O^3)$ onto $(O^1,O^2)$ should be independent of $O^3$. Instead, we will directly construct an appropriate n-agent relation from $\succsim^2$.

First, based on $\succsim^2$, we can establish a ranking of the agents by applying $\succsim^2$ to all pairs. Given $\mathbf{O}\in\mathcal{X}^n$, we will say *i is poor relative to j at* **O** if there exists $O'^i\in\mathcal{X}$ such that $O'^i \supset O^i$ and $(O'^i,O^j) \succ^2 (O^i,O^j)$, and again we denote by $P(\mathbf{O};j)$ the set of agents who are poor relative to j. ($C(\mathbf{O};j)$ and $WP(\mathbf{O};j)$ are defined analogously.)

Definition 4.1. An *n-agent advantage function* is a mapping $a^n:\mathcal{X}^n\to\mathbb{R}^n$ that associates with each $\mathbf{O}\in\mathcal{X}^n$ a list of real numbers $(a^n_1(\mathbf{O}),\ldots,a^n_n(\mathbf{O}))$ such that (1) $\sum_{i\in N} a^n_i(\mathbf{O})=0$, and (2) $a^n_i(\mathbf{O})<a^n_j(\mathbf{O})$ if and only if $i\in P(\mathbf{O};j)$.

Again an advantage function indicates the direction and magnitude of the skewness of **O**. Bilaterally, if $a^n_i(\mathbf{O})<a^n_j(\mathbf{O})$, then j enjoys an advantage relative to i, and globally, those agents for whom $a^n_i(\mathbf{O})>0$ enjoy an advantage relative to those for whom $a^n_i(\mathbf{O})<0$. The magnitude of the (aggregate) advantage can be measured by $\sum_i|a^n_i(\mathbf{O})|$.[13] We will say $a^n:\mathcal{X}^n\to\mathbb{R}^n$ represents $\succsim^n$ if

$$\mathbf{O} \succsim^n \mathbf{O}' \quad \text{if and only if} \quad \sum_i|a^n_i(\mathbf{O})| \le \sum_i|a^n_i(\mathbf{O}')|.$$

## 5. A Representation Theorem for $\succsim^n$

The following properties generalize those of Section 3:

Strong Transitivity of the Weak Relative Poverty Relation (STRANS$^n$): For all $\mathbf{O}\in\mathcal{X}^n$, and for all $i,j,k\in N$, if $i\in WP(\mathbf{O};j)$ and $j\in WP(\mathbf{O};k)$, then $i\in WP(\mathbf{O};k)$; and if $i\in P(\mathbf{O};j)$ and $j\in P(\mathbf{O};k)$, then $a^2_k(O^i,O^k) > a^2_k(O^j,O^k)$.

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[13] Here an alternative measure would be $\sum_{i\in A(\mathbf{O})} a^n_i(\mathbf{O})$, where $A(\mathbf{O})\equiv\{i\in N \mid a^n_i(\mathbf{O})>0\}$, since this is simply $\frac{1}{2}\sum_i|a^n_i(\mathbf{O})|$.

Uniformity ($\text{UNIF}^n$): For all $\mathbf{O}^e \in \mathcal{E}^n$ and $\mathbf{O}' \in \mathcal{X}^n$, $\mathbf{O}^e \succsim^n \mathbf{O}'$.[14]

Continuity ($\text{CONT}^n$): For all $\mathbf{O} \in \mathcal{X}^n$, $\{\mathbf{O}' \in \mathcal{X}^n \mid \mathbf{O}' \succsim^n \mathbf{O}\}$ and $\{\mathbf{O}' \in \mathcal{X}^n \mid \mathbf{O} \succsim^n \mathbf{O}'\}$ are closed (in the product topology on $\mathcal{X}^n$).

$\text{UNIF}^n$ and $\text{CONT}^n$ are straightforward generalizations of the axioms in Section 3. $\text{STRANS}^n$ is somewhat more restrictive than transitivity of the weak relative poverty relation. In addition to requiring the latter, it also imposes the quite reasonable condition that in the event $i \in P(\mathbf{O};j)$ and $j \in P(\mathbf{O};k)$, the extent of the advantage enjoyed by k relative to i should exceed that enjoyed by k relative to j. Note that with only two agents, $\text{STRANS}^n$ is equivalent to $\text{ACYCL}^2$.

THEOREM 5.1. Let $\mathcal{X}$ be a connected, separable, topological space; and let $\succsim^n$ satisfy $\text{STRANS}^n$, $\text{UNIF}^n$, and $\text{CONT}^n$. Then there exists a continuous advantage function $a^n : \mathcal{X}^n \to R^n$ that represents $\succsim^n$.

*Proof*. Let $\succsim^n$ satisfy the above axioms. First, we apply $a^2$ to each pair of agents separately. Then, for $i \in N$, we define $a_i^n(\mathbf{O}) \equiv \sum_{j \neq i} a_i^2(O^i, O^j)$. We will show that $a^n = (a_1^n, \ldots, a_n^n)$ is a continuous advantage function that represents $\succsim^n$.

Notice that $\sum_{i \in N} a_i^n(\mathbf{O}) = \sum_{i \in N} \sum_{j \neq i} a_i^2(O^i, O^j) = \sum_{i,j \in N} \left( a_i^2(O^i, O^j) + a_j^2(O^i, O^j) \right) = 0.$
Moreover, since each $a_i^2$ is continuous, $a_i^n$ is as well. We need only establish that $a_i^n(\mathbf{O}) < a_j^n(\mathbf{O})$ if and only if $i \in P(\mathbf{O};j)$.

First, suppose $i \in P(\mathbf{O};j)$ for some $i,j \in N$, $i \neq j$. We must show that

[14] The definition of $\mathcal{E}^n$ is obvious.

$\sum_{k\neq i} a_i^2(O^i,O^k) < \sum_{k\neq j} a_j^2(O^j,O^k)$. For this, it is sufficient to show that for each $k\neq i,j$, $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$. We consider several mutually exclusive and exhaustive cases, each of which follows from $\text{STRANS}^n$:

(1) $j\in P(\mathbf{O};k)$.

Then $i\in P(\mathbf{O};k)$. Hence, both $a_i^2(O^i,O^k)<0$ and $a_j^2(O^j,O^k)<0$. Also, $|a_k^2(O^i,O^k)| > |a_k^2(O^j,O^k)|$. Therefore, since $a_i^2(O^i,O^k)+a_k^2(O^i,O^k)=a_j^2(O^j,O^k)+a_k^2(O^j,O^k)=0$, $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$.

(2) $k\in C(\mathbf{O};j)$.

Then by definition, $k\notin P(\mathbf{O};j)$ and $j\notin P(\mathbf{O};k)$. Hence, $a_j^2(O^j,O^k)=0$. Also, $i\in P(\mathbf{O};k)$ and so $a_i^2(O^i,O^k)<0$. Therefore, $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$.

(3) $i\in P(\mathbf{O};k)$ and $k\in P(\mathbf{O};j)$.

If $i\in P(\mathbf{O};k)$ and $k\in P(\mathbf{O};j)$, then $a_i^2(O^i,O^k)<0$ and $a_j^2(O^j,O^k)>0$. Clearly, therefore, $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$.

(4) $k\in C(\mathbf{O};i)$.

Then by definition, $k\notin P(\mathbf{O};i)$ and $i\notin P(\mathbf{O};k)$. Hence, $a_i^2(O^i,O^k)=0$. Also, $k\in P(\mathbf{O};j)$ and so $a_j^2(O^j,O^k)>0$. Therefore, $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$.

(5) $k\in P(\mathbf{O};i)$.

Then $k\in P(\mathbf{O};j)$. Hence, both $a_i^2(O^i,O^k)>0$ and $a_j^2(O^j,O^k)>0$. It then follows immediately that $a_i^2(O^i,O^k)<a_j^2(O^j,O^k)$.

Finally, since $WP(\mathbf{O};\cdot)$ is complete (i.e., for all $i,j\in N$, either $i\in WP(\mathbf{O};j)$ or $j\in WP(\mathbf{O};i)$), it is straightforward to show $a_i^n(\mathbf{O})<a_j^n(\mathbf{O})$ implies $i\in P(\mathbf{O};j)$. ∎

Examples 1 and 2 in Section 2 can easily be generalized to include additional agents.

First, define the *mean (pairwise) Lebesgue difference* in $\mathbf{O}\in\mathcal{X}^n$ by

$$\mathrm{MLD}(\mathbf{O}) \equiv \sum_i \sum_j \frac{|\mu(O^i)-\mu(O^j)|}{2n(n-1)} .$$

Then we can define an ordinal generalization of $\succsim^2_{LD}$ by

$$\mathbf{O} \succsim^n_{MLD} \mathbf{O}' \iff \mathrm{MLD}(\mathbf{O}) \leq \mathrm{MLD}(\mathbf{O}').$$

An advantage function representing $\succsim^n_{MLD}$ is

$$a^2_{MLD}(\mathbf{O}) \equiv (\mu(O^1)-\bar{\mu}(\mathbf{O}),\ldots,\mu(O^n)-\bar{\mu}(\mathbf{O})),$$

where $\bar{\mu}(\mathbf{O}) \equiv \frac{1}{n} \sum_i \mu(O^i)$.

Similarly, define the *mean (pairwise) value difference* by

$$\mathrm{MVD}(\mathbf{O}) \equiv \sum_i \sum_j \frac{|v(O^1;p)-v(O^2;p)|}{2n(n-1)} .$$

We can define an ordinal generalization of $\succsim^2_{VD}$ by

$$\mathbf{O} \succsim^n_{MVD} \mathbf{O}' \iff \mathrm{MVD}(\mathbf{O}) \leq \mathrm{MVD}(\mathbf{O}').$$

And an advantage function representing $\succsim^n_{MVD}$ is

$$a^2_{MVD}(\mathbf{O}) \equiv (v(O^1;p)-\bar{v}(\mathbf{O}),,\ldots,v(O^n;p)-\bar{v}(\mathbf{O})),$$

where $\bar{v}(\mathbf{O}) \equiv \frac{1}{n} \sum_i v(O^i;p)$.

## 6. Conclusion

In this paper, we have established conditions under which an ordinal equality relation defined on profiles of opportunity sets admits a particularly intuitive cardinal representation in the form of an advantage function. The primary difference between this paper and [6] is that the latter considered only finite opportunity sets, whereas the present analysis pertains to general topological spaces. Consequently, the present analysis is amenable to economic applications.

A second distinction concerns the representations themselves. For the n-agent case, [6] identified conditions under which profiles of finite sets can be compared on the basis of a linear function of the cardinalities of the sets. Clearly, the present result affords less structure, but it applies to a much broader class of environments. The principle advantage of this result is that it reduces the complexity of comparing sets to a single dimension, and, under the conditions of Theorem 5.1, it establishes that there is no loss of generality in assuming that profiles are ranked *as if* according to an advantage function. Consequently, an advantage function may provide a useful tool in evaluating equitable opportunities.

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