PRECISION PERFORMANCES OF TERMINAL
CONDITIONS FOR SHORT TIME HORIZONS
FORWARD-LOOKING SYSTEMS

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Abstract

In this paper, we investigate both theoretically and empirically the numerical bias due to the truncation of structurally infinite time forward-looking models, by the means of various terminal conditions. We shed light on the difficulties of numerical control using the latter instruments, and recommend a prior investigation of the individual dynamics generated by each variable of the models under consideration.

Key Words

Expectations, Large scale models, Solution time horizons, Terminal conditions.
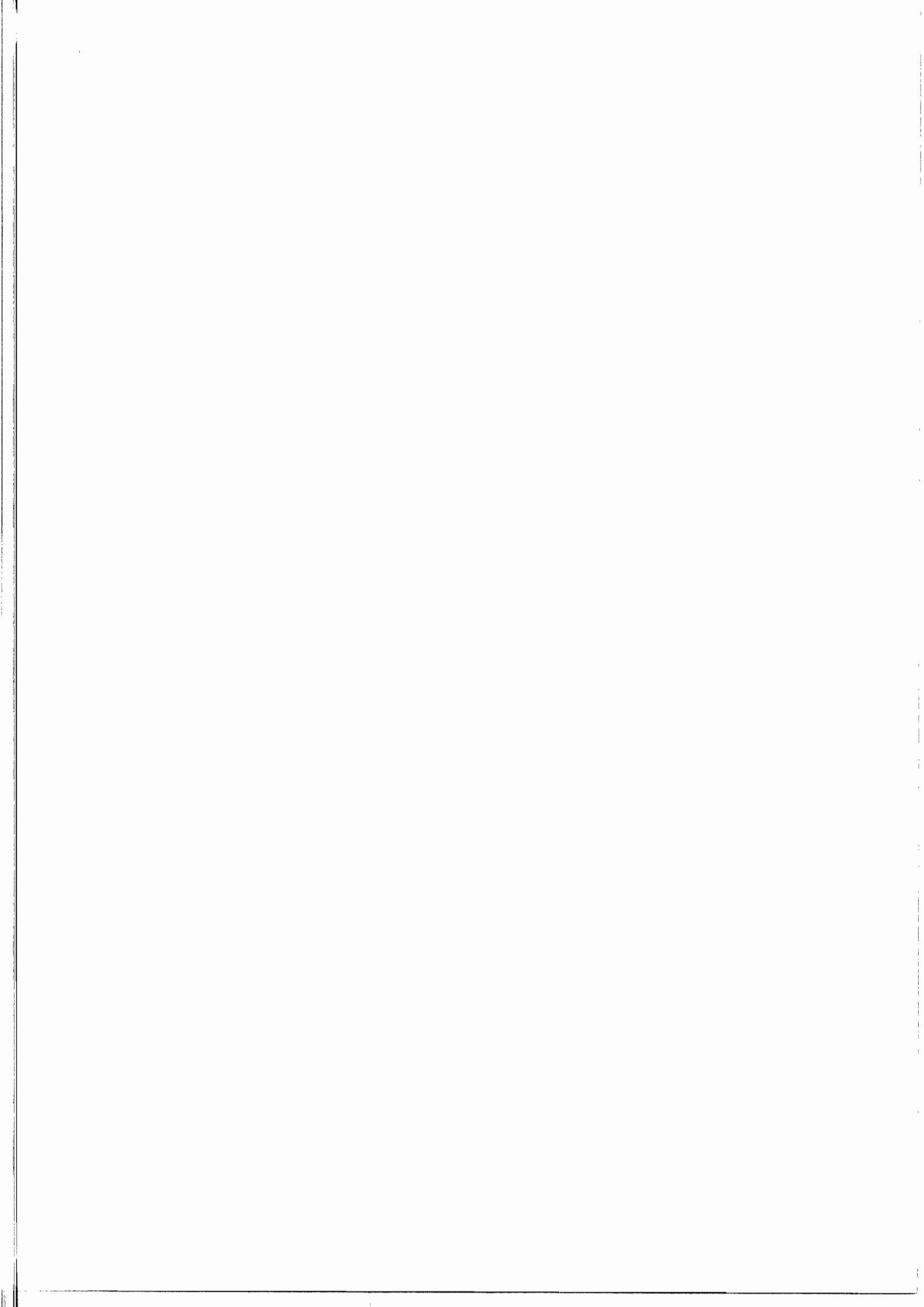
1. Introduction

Beginning with the last decade, the construction of forward looking models raised a number of important computational issues, which motivated the initiation of some research programs, most of them in progress. These computational problems are twofold: part of them derives from the specific mathematical and numerical implications of forward expectations schemes, and the other consists in the feasibility limitations arising in the simulation of large scale models (say models of some hundreds of equations).

Forward expectations based models are dynamically non-recursive, in the sense that initial boundary values (initializing lagged variables) are no longer sufficient to compute the solution paths. To solve such models, second boundary values are needed. However, as most macroeconometric models have an infinite time support, the latter values are actually chosen by the practitioners according to some precise criteria.

If we take the following elementary forward-looking model:

$$f(y_{t-1}, y_t, y_{t+1}, z_t) = 0, \quad t \geq 1$$
$$\text{and } y_0 \text{ given,}$$

with y (resp. z) the vector of endogenous (resp. exogenous) variables, $f(\cdot)$ a well-dimensioned vector function and t the time index, the second boundary value or terminal condition can be represented by the constraint $g(y_{T+1}, y_T, y_{T-1}) = 0$, where $g(\cdot)$ is a chosen vector function with the same dimension as function $f(\cdot)$ and T the selected solution time horizon. The finite-time approximation to be solved is:

$$S_T \qquad \begin{cases} y_0 \quad \text{given,} \\ f(y_{t-1}, y_t, y_{t+1}, z_t) = 0, \quad 1 \leq t \leq T \\ g(y_{T+1}, y_T, y_{T-1}) = 0. \end{cases}$$

There is an abundant literature concerned with the specification of the terminal constraint (see for example Wallis et alii (1986) or Fisher (1992)). In this literature, the choice of the terminal conditions depends upon the existence of computed long run equilibria for the models under consideration. For a given model, if a long run equilibrium

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exists, say y^s, then the fixed value condition, $y_{T+1} = y^s$, is legitimated. Otherwise, one could use for example the alternative constant level condition $y_{T+1} = y_T$. The latter restriction is of course more general and in certain cases, it provides better performances for the precision issue addressed in this paper, as we will show in the next section. In our setting we assume that both terminal conditions can be used, or equivalently that we can characterize a kind of long run equilibria of the models under consideration. We take this approach for two reasons at least:

(i) First, we just consider the "nonexistence" of long run equilibria as a non-property, rather than a valuable unavoidable characteristic of the models. It is known (see for example, Deleau et alii (1990)) that slight modifications in the specification or the parameterization of the models allow generally to find out such equilibria, including large scale models (see Loufir and Malgrange (1994)) in the case of the multicountry model MULTIMOD developed by Masson et alii (1990)).

(ii) Although long run equilibra do not "exist" on the structural forms of the models, adding residuals to the models' equations in a convenient way permits the identification of baseline solutions paths, playing the role of long run equilibria for the residuals-augmented forms (see also Masson et alii (1990) in the case of MULTIMOD)..

This paper investigates the role of the terminal condition specification in the goodness of the finite time approximation, represented by the system S_T. We do not address the difficult problem of the optimal terminal condition regarding to the approximation quality, which seems quite intractable. Indeed, we study a precise computational issue related to the simulation of large scale forward-looking models. In the latter case, the use of high solution time horizons is impossible: in practice, the two boundary values systems, represented by S_T, are solved within solution horizons of some tens of periods. Given this feasibility constraint, some authors put forward a number of experimental instruments to improve the approximation quality of systems S_T, for such short solution time horizons. A natural instrument turns out to be the terminal condition specification: in our setting, different admissible analytical forms for function $g(\cdot)$ will not provide the

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same solution paths if the solution time horizon is not sufficiently high. The key issue consists consequently to find out the terminal constraints which guarantee an acceptable approximation quality.

A first contribution on this topic is due to Fisher (1992). The author considers the following model (page 85):

$$p_t = \alpha p_{t+1} + u_t$$
$$u_t = \gamma_0 + \gamma_1 u_{t-1} \quad \text{with } u_0 \text{ given.}$$

Then, he evaluates the approximation quality of the corresponding systems S_T, for different terminal conditions and different spectra of the considered model, by comparison with the explicit solutions.

In this paper, we present the corresponding results on a general multivariate optimization based model. The model is the principal example considered in Stokey and Lucas (1989), chapter 6, to formalize economic central planner problems (ie: optimization under constraints of a unique objective function). On this model, we study the robustness of Fisher's results increasing the scope of some of them and challenging the others.

Almost all the simulation exercises are performed using Laffargue's algorithm (1990). The algorithm uses Newton-Raphson relaxation steps, with a specific triangulation procedure allowing to compute per-relaxation step Newton-Raphson improvements. A theoretical analysis of this technique is given by Boucekkine (1995). The algorithm has been adapted to Gauss language, coded and extended by Juillard (1994a, 1994b). Of course, given the issue addressed here, our results are quite insensitive to the simulation method.

The paper is organized as follows: section 2 provides the theoretical results on a linearized central planner model. Section 3 is devoted to numerical corroboration. We conclude by some methodological recommendations on the simulation of short time horizon forward-looking systems.

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2. Theoretical analysis

As announced in the introductive section, we begin by finding out some theoretical results on the precision performances of different terminal conditions. The analysis is conducted on the linearized form of the central planner model considered by Stokey and Lucas (1989), chapter 6.

Beginning with the canonical optimization problem

$$\sup_{x_t \in \mathbb{R}^n} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t. } x_{t+1} \in \Gamma(x_t)$$

where x_0 given and $\Gamma(x_t)$ being the set of feasible states depending on x_t, they show that the linear approximation around the steady state $\{x_t^*\}$ of the Euler equations, can be written as:

$$y_{t+1} = Ay_t + By_{t-1}, \quad t \geq 1, \quad y_0 \text{ given}$$

with $y = x - x^*$ an $(n \times 1)$ vector and A, B two $(n \times n)$ square matrices of constants.

Relatively to the model considered in section 1, we suppress the exogenous variables vector z_t for ease of exposition. We assume that the model has no unit root, to deal with a unique stationary equilibrium, $y^* = 0$. The initial value y_0 is obviously assumed non zero, otherwise the solutions y_t, would remain at $y^* = 0$ for every t.

The model can be rewritten as:

$$(M) \qquad \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix} \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix},$$

where $I(n)$ (resp. $0(n)$) is the identity matrix (resp. null matrix) of dimension n. We set

$$F = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix},$$

and we assume that

(H_1) matrix F is diagonalizable.

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So, it exists an invertible matrix P and a diagonal matrix Λ such that $F = P\Lambda P^{-1}$. We can always assume that elements of Λ are in increasing order. P, Λ and P^{-1} are partitioned as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with P_{11} (resp. P_{22}), the submatrix $(n \times n)$ corresponding to the first (resp. last) n rows and columns of P, and P_{12} (resp. P_{21}), the submatrix $(n \times n)$ corresponding to the first (resp. last) n rows of P and to the last (resp. first) columns of P,

$$P^{-1} = \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix}$$

with the same conventions as before, and

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}$$

with Λ_1 and Λ_2 of dimension n, the eigenvalues being ordered in an increasing order.

We assume that the saddlepoint conditions are fulfilled to avoid multiplicity (see Boucekkine (1993) for a general discussion on the same model).

(H_2)   All the elements of Λ_1 (resp. Λ_2) are less than one (resp. greater than one) in modulus.

We specify now the terminal conditions under consideration. As we are dealing with a linear model, we set the two following linear terminal constraints:

(i) The fixed-value terminal condition: $y_{T+1} = y^*(= 0)$, and

(ii) the constant-level terminal condition: $y_{T+1} = y_T$.

The two conditions involve two different finite time approximations:

$$S_T(FV) \qquad \begin{cases} y_0 \neq 0 \text{ given,} \\ \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = F \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} & \text{for } 1 \leq t \leq T, \\ y_{T+1} \neq 0 \text{ given,} \end{cases}$$

and

$$S_T(CL) \qquad \begin{cases} y_0 \neq 0 \text{ given,} \\ \begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = F \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} & \text{for } 1 \leq t \leq T, \\ y_{T+1} = y_T \text{ given,} \end{cases}$$

Under assumption H_2, the two systems are obviously equivalent when T goes to infinity, but not in the short run. The true value of y_1 can be for example computed by solving the system $S_T(FV)$ for T going to infinity.

Our general approach consists first in computing this true value, denoted $y_1(\infty)$ hereafter, and then computing the solution value of y_1 obtained for both systems for a fixed "short" horizon T. We denote by $y_1^{FV}(T)$ (resp. $y_1^{CL}(T)$) this solution value for the system $S_T(FV)$ (resp. $S_T(CL)$).

To obtain explicit solutions, we set the following regularity condition:

(H_3) Submatrices P_{11} and P_{12} are invertible.

The results are summarized in the following proposition:

Proposition *Under assumption H_3:*

(i) *All the submatrices P_{ij} and P^{ij}, for i and $j = 1, 2$, are invertible.*
Let us define matrices $X_1(T)$, $Z_1(T)$, $X_2(T)$, and $Z_2(T)$ as follows, assuming that H_3 holds:

$$X_1(T) = (P^{21})^{-1} \Lambda_2^{-1-T} (P_{22})^{-1} P_{21} \Lambda_1^{1+T} P^{11},$$

$$X_2(T) = (P^{21})^{-1} \Lambda_2^{-T} (\Lambda_2 - I(n))^{-1} (P_{22})^{-1} P_{21} (\Lambda_1 - I(n)) \Lambda_1^T P^{11},$$

$$Z_1(T) = \Lambda_2^{-1-T} (P_{22})^{-1} P_{21} \Lambda_1^{1+T} P^{12} (P_{22})^{-1}, \quad and$$

$$Z_2(T) = \Lambda_2^{-T} (\Lambda_2 - I(n))^{-1} (P_{22})^{-1} P_{21} (\Lambda_1 - I(n)) \Lambda_1^T P^{12} (P_{22})^{-1},$$

Then

(ii) *For a fixed T, the system $S_T(FV)$ (resp. $S_T(CL)$) admits a unique solution if and only if matrix $I(n) + X_1(T)$ (resp. $I(n) + X_2(T)$) is invertible*

If the conditions stated in (ii) *hold, then:*

(iii) *The absolute error of both terminal conditions is given by*

(a) $y_1^{FV}(T) - y_1(\infty) = \{(P^{21})^{-1} P^{22} - [\, I(n) + X_1(T)\,]^{-1} (P^{21})^{-1} [\, I(n) + Z_1(T)\,] P^{22}\} y_0$

(b) $y_1^{CL}(T) - y_1(\infty) = \{(P^{21})^{-1} P^{22} - [\, I(n) + X_2(T)\,]^{-1} (P^{21})^{-1} [\, I(n) + Z_2(T)\,] P^{22}\} y_0$

Proof of the proposition: Property (i) is derived in the appendix 1, using traditional linear algebra arguments. To establish property (ii), we need to solve both systems $S_T(FV)$ and $S_T(CL)$. For the former, observe that the terminal condition $y_{T+1} = 0$ could be written as $Ay_T + By_{T-1} = 0$, or $[A\ B]\begin{bmatrix} y_T \\ y_{T-1} \end{bmatrix} = 0$. Thus, using the stacked equation form (M), it yields

$$[A\ B]F^{T-1}\begin{bmatrix} y_1 \\ y_0 \end{bmatrix} = 0.$$

Setting C, D the matrices defined by the relation

$$[C'D] = [A\ B]P,$$

and given the partitioned forms of matrices P, P^{-1} and Λ, it follows

$$(EFV) \qquad (C\Lambda_1^{T-1}P^{11} + D\Lambda_2^{T-1}P^{21})y_1^{FV} = -(C\Lambda_1^{T-1}P^{12} + D\Lambda_2^{T-1}P^{22})y_0.$$

Similarly, observing that the terminal condition $y_{T+1} = y_T$ could be rewritten as $(A - I(n))y_T + By_{T-1} = 0$, it is easy to prove that $y_1^{CL}(T)$ is determined by

$$(ECL) \qquad (C'\Lambda_1^{T-1}P^{11} + D'\Lambda_2^{T-1}P^{21})y_1^{CL} = -(C'\Lambda_1^{T-1}P^{12} + D'\Lambda_2^{T-1}P^{22})y_0$$

with C' and D' the matrices defined by

$$[C'\ D'] = [A - I(n)\ B]P.$$

To achieve the proof, we need the following intermediate result:

Lemma *Under assumption H_1, and by definition of the matrices C, D, C' and D', the following equalities hold:*

$$\begin{aligned} D &= P_{22}(\Lambda_2)^2, & C &= P_{21}(\Lambda_1)^2, \\ D' &= P_{22}\Lambda_2(\Lambda_2 - I(n)), \quad and & C' &= P_{21}\Lambda_1(\Lambda_1 - I(n)). \end{aligned}$$

The proof of the lemma is given in the appendix 1.

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Observe that the four matrices are therefore invertible under assumptions H_2 and H_3. Multiplying equation (EFV) by $(D\Lambda_2^{T-1}P^{21})^{-1}$ and then replacing C and D by their expressions given in the lemma, it follows:

$$[\ I(n) + X_1(T)\ ]y_1^{FV}(T) = Ly_0,$$

L being the altered right side of equation (EFV). It follows that for a fixed T, (EFV) admits a unique solution if and only if the matrix $I(n) + X_1(T)$ is invertible. The same device could be used to show that equation (ECL) admits a unique solution if and only if matrix $I(n) + X_2(T)$ is invertible.

Actually, given that the saddlepoint conditions are assumed to hold, $I(n)+X_1(T)$ and $I(n) + X_2(T)$ must be invertible, for sufficiently high solution horizons T. We assume that this property holds either for short T. Let us prove now the most important property (iii).

It is straightforward to show that the term L of equation (EFV) is given by

$$L = (P^{21})^{-1}[\ I(n) + Z_1(T)\ ]P^{22}.$$

Thus

$$y_1^{FV}(T) = [\ I(n) + X_1(T)\ ]^{-1}(P^{21})^{-1}[\ I(n) + Z_1(T)\ ]P^{22}y_0.$$

The true value $y_1(\infty)$ is obtained, for example, as the limit of $y_1^{FV}(T)$ when T goes to infinity. Under assumption H_2, both $X_1(T)$ and $Z_1(T)$ go to zero, such that

$$y_1(\infty) = (P^{21})^{-1}P^{22}y_0.$$

We can check that the difference $y_1^{FV}(T) - y_1(\infty)$ is given by the expression (a) of the proposition. Similarly the difference $y_1^{CL}(T) - y_1(\infty)$ is given by the proposition— expression (b).   Q.E.D.

The previous theoretical analysis allows to bring out various interesting conclusions on the small-sample precision properties of both terminal conditions. More concretely, this analysis provides some theoretical foundations to some heuristic results put forward

by certain practitioners (as Fisher (1992)), as well as it rejects some others. We focus on two heuristic findings emphasized by Fisher (1992), page 84:

(P_1) If the lowest unstable root is close to one, then if the largest stable root is not close to one, the fixed value terminal condition dominates in terms of precision.

(P_2) For intermediate spectral situations, the constant level terminal condition is more efficient in terms of precision.

Property P_1 can be checked in our framework. As one can see, the difference between the absolute precision indicators of the two terminal conditions consists in the terms $(I(n) - \Lambda_2)^{-1}$ and $(I(n) - \Lambda_1)$ affecting the constant level terminal condition precision indicator. When the lowest unstable root goes to unity whereas the greatest stable root is not close to unity, the precision of the constant level condition worsens considerably because of the term $(I(n) - \Lambda_2)^{-1}$, while the solution value $y_1^{FV}(T)$ adjusts quickly to the true value $y_1(\infty)$.

On the contrary, it is not clear at all why the constant level condition should dominate for intermediate spectral configurations. The only conclusion allowed by our theoretical treatment is that the precision difference between the two terminal constraints should be relatively small in such cases. In the following subsection, we present a numerical counterexample against property P_2. Our theoretical analysis allows to conclude for another case:

(P_3) If the largest stable root is close to one, then if the lowest unstable root is not close to one, the constant level condition dominates.

However, we cannot conclude for the case where both the lowest unstable root and the largest stable root go to one. The cumulative effect of the terms $(I(n) - \Lambda_2)^{-1}$ and $(I(n) - \Lambda_1)$ in the expression of the constant level condition precision indicator is definitively unclear.

Indeed, the general forms of the error expressions given in the proposition are so complicated that it is impossible to bring out simple conclusions, except in the two cases

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mentioned above (statements P_1 and P_3). One would like to get out general compar-
ison results of the type: "if the lowest unstable root is closer to one than the largest
stable root, then the fixed value terminal condition dominates in terms of precision."
Unfortunately, despite the simplified structure of the considered formal model, this is
not possible: a quick look at the error expressions is sufficient to account for the model
dependence of the results, through the terms in matrices P and P^{-1}. As acknowledged
by Fisher (1992), chapter 4, this feature (model dependence) is inherent to the use of
terminal conditions as numerical control instruments.

We propose now a numerical evaluation of the latter theoretical outcomes, first on a
central planner model, then on a general equilibrium model.

3. Numerical study

3.1. A corroboration

We consider the following model à la Ramsey:

$$c_t + k_t + (1 - \delta)k_{t-1} = Az_t k_{t-1}^{\alpha} \tag{1}$$

$$c_t^{-\gamma} = (1 + \beta)^{-1} c_{t+1}^{-\gamma} (A\alpha z_{t+1} k_t^{\alpha - 1} + 1 - \delta) \tag{2}$$

All variables are per capita. Equation (1) is the supply-demand constraint of the econ-
omy; at each date t the consumption level c_t plus the investment level $k_t - (1 - \delta)k_{t-1}$,
where k_t is the capital stock, are equal to production, z being the productivity exogenous
shock. Equation (2) is the Euler equation associated to the intertemporal optimization
behaviour of the consumer with an isoelastic utility function. The model obviously fits
the formalization of the previous section. Replacing c_t and c_{t+1} in equation (2) by their
corresponding expressions in k_t, k_{t-1} and k_{t+1}, using equation (1), we can see that the
model is reducible to a single equation involving only the capital variable in its three
temporal forms, k_t, k_{t-1} and k_{t+1}. Linearizing this equation, we obtain the linear form
adopted in our theoretical section (with $y_t = k_t$ and $n = 1$). However, we solve here the
structural nonlinear form of the model using Laffargue's Newton Raphson algorithm.
Indeed, as the convergence of the algorithm (for the considered tolerance level) requires

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at most three linearizations, the model is quasi-linear and so, the numerical results presented in this subsection are rather an illustration of the theoretical outcomes seen above.

We assume, as in Taylor and Uhlig (1990), that the exogenous shock evolves according to:

$$\ln(z_t) = \rho \ln(z_{t-1}) + u_t.$$

We assume that u_t is always zero except at $t = 1$ and $z_0 = 1$. To obtain explicit solutions for the model, we assume that the capital stock has a unitary depreciation rate $\delta = 1$; the underlying utility function is set logarithmic $\gamma = 1$. The exact solutions are consequently:

$$k_t^* = (1 - \beta)^{-1} \alpha A z_t k_{t-1}^\alpha, \quad k_0 \text{ given,}$$
$$\text{and} \quad c_t^* = A z_t k_{t-1}^\alpha - k_t^*.$$

We set $z_1 = 0.97$ and $\rho = 0.25$. We solve numerically the model with Laffargue's algorithm for both terminal constraints, initializing the capital variable by its stationary value. To generate the different local spectra configurations, we vary α, β and A.

With the values assigned to z_1 and ρ, the exogenous variable returns to its long run value (ie. returns to one) at $t = 12$. We set $T = 12$. We compare the numerical solutions for consumption and capital with the exact ones, over the time interval $[0\ 10]$. Denoting by $\{k_t^{FV}, c_t^{FV},\ t = 1, \ldots, 10\}$ (resp. $\{k_t^{CL}, c_t^{CL},\ t = 1, \ldots, 10\}$) the numerical solutions obtained with the fixed value (resp. constant level) condition, the finite time approximation error indicator is defined as

$$E_m = \sum_{t=0}^{10} \left\{ \frac{|k_t^m - k_t^*|}{k_s} + \frac{|c_t^m - c_t^*|}{c_s} \right\}$$

with $m \in \{FV, CL\}$ and k_s, c_s the stationary values for capital and consuption respectively.

The table 1 below displays the precision results obtained with a convergence tolerance level (of the resolution algorithm) set to 10^{-4}:

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As announced in the theoretical subsection, the fixed value condition is shown to dominate for certain intermediate spectral configurations (in the sense of Fisher). For both parameterizations 1 and 3, the results contradict property P_2. For parameterization 2, the comparison is favorable to the constant level condition, as the unstable root presents a larger deviation with respect to unity than for the parameterizations 1 and 3. The outcome fits the statement P_3 of the theoretical subsection. Actually, as emphasized before, it is worth pointing out that the precision difference between the two conditions is quite negligible for intermediate spectral situations. This is definitely not the case when both roots are close to unity—parameterizations 4 and 5. Especially for the last parameterization, the fixed value condition is twice more precise than the constant level one. Nonetheless, parameterization 4 shows that no clear comparison outcome can be got out for such spectra. On the other hand, observe that in both parameterizations 4 and 5, the stable root is closer to one than the unstable eigenvalue, but the precision results are extremely different. This confirms our remark ending the theoretical section: it is really impossible to conclude in a simple way just regarding to the eigenvalues magnitudes, for such important spectral configurations. Only a posteriori qualitative reasonings are allowed. We will follow this approach in the general equilibrium model example studied below.

Before, let us point at a final methodological detail. As we use numerical solutions to evaluate the precision outcomes, we must separate the numerical bias due to the finite time approximation and the one generated by the numerical resolution. Here the second bias is indeed negligible (see Boucekkine (1994), for the numerical precision performances of Laffargue's algorithm on the same model).

3.2. A general equilibrium model example

We consider now a general equilibrium model example, contrasting with the simple central planner setting considered so far. Formally, the former models are obviously more sparse because they involve different optimization blocks (ie: household's be-

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haviour, firm's behaviour, etc...), each of them with its own control variables. In general, this involves that some of the matrices, assumed invertible in our theoretical section, will be singular.

Nonetheless, following the empirical work conducted by the practitioners of the ESRC Macroeconomic Modelling Bureau on various models (see again, Fisher (1992)), it is very likely that properties P_1 and P_3, stated in section 2, still hold on such sparse models. In fact, they should hold for intuitive reasons. For example, we can justify property P_1 as follows, introducing the usual dichotomy-implicte in our theoretical setting- between forward-looking variables driven by the unstable roots of the models and the predetermined variables driven by their stable roots. Given that the constant level terminal condition consists in a general level stability rule on forward-looking variables, the convergence speed of the latter (to the corresponding long run values) is the major determinant of the approximation quality of this terminal condition. As unstable roots drive the dynamics of forward-looking variables, the approximation error induced by the constant level condition should be important in the short run if the lowest unstable root is very close to one. On the other hand, the stickiness of forward-looking variables plays by construction a less important role in the approximation quality of the fixed-value terminal condition. In the latter case, the stickiness of predetermined and non-predertermined variables are equally important, such that in the case of sticky forward-looking variables and quickly converging predertermined variable (ie. when the stable roots depart significantly from unity), the approximation quality of the fixed-value terminal condition should be better, which corresponds exactly to the statement of property P_1. To justify property P_3, one can use a symmetrical reasoning.

Here, we present a number of numerical experiments much more relevant in practice. Indeed, a basic characteristic of the existing macroeconometric models is the closeness to unity of both the unstable and the stable roots of these models. By considering two parameterizations of an example model, involving two representative local spectra, we study the respective finite-time approximation bias. In particular, we illustrate a major feature: more importantly than the terminal condition specification, the practitioner

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needs to primarily isolate high convergence speed economic variables (to their corresponding long run values) from the "slow" variables. For short time solution horizons, the per variable bias looks indeed very heterogeneous, and the practitioner should take into account this feature for conveniently solving the models.

In this subsection, we consider the general equilibrium model called PLM, developed by Laffargue et alii (1992). The model includes about twenty equations and is numerically calibrated on quarterly French data. It describes an economy working under imperfect competition on both labor and goods markets. The closure of the model is obtained by household consumption (permanent income), by public expenditures and by foreign trade.

The considered first parameterization is the one used by the authors. It generates the following local spectrum (around the steady state): 1.2085, 1.0676, 1.0094, 0.9937, 0.9418 and 0.8439. Both the largest stable root and the lowest unstable root are very close to one. We denote this model M_1.

To obtain eigenvalues less close to one, we modify a parameter of the models equation relating interest rates to foreing debt, denoted ϕ_1 by the authors. Increasing this parameter from 0.1 to 1.4, we get a more "stable" model with the following local spectrum: 1.2094, 1.0688, 1.0300, 0.9732, 0.9414 and 0.8431. We denote this model M_2 hereafter.

The simulation exercise consists in computing the model response, initially on rest, to a transitory increase in public expenditures. Precisely, we assume that the expenditures G increase from 0.7972 by $\Delta G = 0.05$ during the first ten periods.

Solving the models using Laffargue's algorithm, for short time horizons, $T = 10, 20, 30$ and 40, and for both fixed value and constant level terminal conditions, we measure the average percentage error occuring in the computation of the multipliers of six variables: production, consumption, prices, imports, sales and inventories. Denoting by $X_t(T)$ the solution value of a variable X for a solution horizon T, at the date t, and by $X_t(\infty)$ the corresponding "exact" solution obtained for very high solution time horizons, we use

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the following error measure:

$$E_X(T) = \frac{1}{T_0} \left\{ \sum_{t=1}^{T_0} \left| \frac{m_t^X(T) - m_t^X(\infty)}{m_t^X(\infty)} \right| \right\} \times 100$$

$$\text{with } m_t^X(T) = \frac{X_t(T) - X_s}{\Delta G},$$

$$m_t^X(\infty) = \frac{X_t(\infty) - X_s}{\Delta G},$$

where X_s is the long run value of variable X and $T_0 > 1$ a chosen integer. Given that the models are initially on rest, $m_t^X(T)$ measures variable X's multiplier obtained for the solution horizon T whereas $m_t^X(\infty)$ gives the corresponding "exact" value. Tables 2 and 3 provide the error values obtained for each parameterization, when $T_0 = 5$; with a tolerance convergence level equal to 10^{-4} for the resolution algorithm:

[Tables 2 and 3 around here]

Let us comment the previous results in the following few points:

(i) First, observe that the finite time approximation is better in the case of parameterization M_2. This is indeed a trivial outcome deriving from the fact that the roots are less close to one in the latter case. The apparent erratic convergence of the production variable for parameterization M_1 (ie: an average error equal to 15% at $T = 10$, to 19% at $T = 30$ and 17% at $T = 40$) is purely anecdotic. For this typical parameterization where the largest stable root and the lowest unstable root are very close to one, the required solution time horizon to reach the steady state (for an absolute precision of 10^{-4}) is around 500 periods. As the model is nonlinear, such a phenomenon is not impossible when convergence is far from being achieved. We check that the bias decreases continuously beginning with $T = 30$ for all the variables.

(ii) In both parameterizations, the fixed value terminal condition dominates in terms of precision, although the largest stable root is closer to one than the lowest unstable root, in both cases. But the most interesting result follows from the second parame-

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terization M_2: for $T = 40$, the precision performances of the two terminal conditions are analogous. In fact, for such local spectra, the precision differential between the two terminal conditions considerably decreases beginning with $T = 30$. We can check this heuristic finding on the comparable parameterizations 4 and 5 of Ramsey's model given in the previous subsection. For macroeconomic models like MULTIMOD (the roots closest to one, for example of its German block being 0.9836 and 1.0212), it is very likely that the choice of the terminal condition is not crucial if the solution horizon is set to 40 or 50. But the bias registred at the solution time horizon values is not negligible and depends strongly upon the variables.

(iii) Indeed, in the case of parameterization M_2, for example, the per variable bias is very heterogeneous. For $T = 40$ and for the fixed value terminal condition, production and price variables exhibit a bias around 7% whereas inventories and sales variables produce an average error around 1%. This differenciated bias depends of course upon the convergence speed of each variable to its corresponding equilibrium value. This feature is more "dramatic" for parameterization M_1. More than the terminal condition specification, the heterogenous per variable bias, and so the convergence speed of each variable, seems to be the most important aspect to investigate when solving short time horizons forward-looking systems. Even if such prior investigations will not allow to surmount the feasibility limitations, they are likely to provide the "touch of rigor" that lack most of the experiments usually conducted on large scale models, simply because they permit the discrimination between the acceptable variables solution paths and the others.

4. Conclusion

In this paper, we provide a critical evaluation on the use of terminal conditions specifications to improve the numerical precision of short time horizons forward-looking models solution paths. Using the basic economic optimization model, we show how it is theoretically impossible to bring out simple devices on terminal conditions specification selection. We point at the model-dependence inconvenient of such instruments.

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Finally, after providing a number of heuristic results, we emphasize the heterogenous per variable bias outcome of such experiments. A prior investigation of the convergence speed of each variable seems to us the unique way to conveniently utilize the solution paths of short time horizons forward-looking systems. This suggests the extension of some of the existing methods (see, for example, the technique developed for backward-looking models by Le Van and Malgrange (1988)) taking into account the large scale characteristic of the models under consideration.

Technical appendix

Proof of the property (i) To establish this property, we use the particular form of matrix

$$F = \begin{bmatrix} A & B \\ I(n) & 0(n) \end{bmatrix}.$$

If λ_i is an eigenvalue of F, then the associated eigenvector takes the form $E_i = \begin{bmatrix} \lambda_i e_i \\ e_i \end{bmatrix}$ with e_i, a normalized element of the kernel of the matrix with e_i, a normalized element of the kernel of the matrix with e_i, a normalized element of the kernel of the matrix $\lambda_i^2 I(n) - \lambda_i A - B$.

Consequently, if F is assumed diagonalizable (assumption H_1), the submatrices of the transition matrix P, defined in the text, must satisfy the relations

$$P_{11} = P_{21}\Lambda_1 \quad \text{and} \quad P_{12} = P_{22}\Lambda_2.$$

By assumption H_3, P_{11} and P_{12} are invertible. It follows that P_{21} and P_{22} are also invertible. By symmetry, the corresponding submatrices of P^{-1} are necessarily regular. Q.E.D.

Proof of the lemma Let us prove the results concerning matrices D and D', the proof for matrices C and C' being very similar.

By definition we have $D = AP_{12} + BP_{22}$. As

$$P_{12} = [\; \lambda_{n+1}e_{n+1}, \lambda_{n+2}e_{n+2}, \ldots, \lambda_{2n}e_{2n} \;] \quad \text{and} \quad P_{22} = [\; e_{n+1}, e_{n+2}, \ldots, e_{2n} \;]$$

it follows

$$D = [\; \lambda_{n+1}Ae_{n+1} + Be_{n+1}, \lambda_{n+2}Ae_{n+2} + Be_{n+2}, \ldots, \lambda_{2n}Ae_{2n} + Be_{2n} \;].$$

As the eigenvectors e_k, $n+1 \le k \le 2n$, are in the kernel of the matrix $\lambda_k^2 I(n) - \lambda_k A - B$, it yields

$$D = [\; \lambda_{n+1}^2 e_{n+1}, \lambda_{n+2}^2 e_{n+2}, \ldots, \lambda_{2n}^2 e_{2n} \;],$$

so $D = P_{22}\Lambda_2^2$.

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On the other hand, $D' = (A - I(n))P_{12} + BP_{22}$. Thus, using the expressions of P_{12} and P_{22} given above:

$$D = [\ \lambda_{n+1}Ae_{n+1} + Be_{n+1} - \lambda_{n+1}e_{n+1}, \ldots, \lambda_{2n}Ae_{2n} + Be_{2n} - \lambda_{2n}e_{2n}\ ].$$

As e_k, $n + 1 \leq k \leq 2n$, are in the kernel of $\lambda_k^2 I(n) - \lambda_k A - B$, then

$$D' = [\ (\lambda_{n+1}^2 - \lambda_{n+1})e_{n+1}, (\lambda_{n+2}^2 - \lambda_{n+2})e_{n+2}, \ldots, (\lambda_{2n}^2 - \lambda_{2n})e_{2n}\ ],$$

so $D' = P_{22}(\Lambda_2^2 - \Lambda_2) = P_{22}\Lambda_2(\Lambda_2 - I(n)) = P_{22}(\Lambda_2 - I(n))\Lambda_2$. $\qquad$ Q.E.D.

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Table 1. Ramsey's model parameterizations and precision results

Model	α	β	A	Eigenvalues	E_{FV}	E_{CL}
1	0.8	0.05	1.5	1.328; 0.814	0.01332	0.01412
2	0.8	0.25	1.75	1.575; 0.817	0.00825	0.00576
3	0.8	0.001	1.25	1.268; 0.813	0.01533	0.01933
4	0.96	0.05	1.1	1.105; 0.979	0.30289	0.26902
5	0.96	0.001	1.1	1.061; 0.972	0.27503	0.61052

* E_{FV}: Absolute error due to the fixed value terminal condition for a solution time horizon $T = 12$.

* E_{CL}: Absolute error due to the constant level terminal condition for a solution time horizon $T = 12$.

Table 2. The error measures in % obtained for the parameterization M_1 of PLM

T	10	20	30	40
Production	$15; 100$	$15; 78$	$19; 55$	$17; 40$
Consumption	$44; 252$	$18; 90$	$14; 50$	$12; 33$
Prices	$19; 100$	$16; 82$	$20; 58$	$18; 42$
Sales	$12; 100$	$8; 40$	$8; 25$	$7; 17$
Imports	$2; 100$	$11; 55$	$12; 36$	$11; 25$
Inventories	$31; 100$	$3; 14$	$0.5; 4$	$0.4; 2$

Note: For a given variable X and a selected time horizon value T, the first (resp. second) figure provides the error measure $E_X(T)$ obtained with the fixed value terminal condition (resp. constant level terminal condition).

Table 3. The error measures in % obtained for the

parameterization M_2 of PLM

T	10	20	30	40
Production	45; 101	33; 59	16; 23	7; 9
Consumption	54; 137	20; 32	8; 10	3; 4
Prices	46; 101	37; 67	18; 27	8; 10
Sales	42; 100	13; 21	5; 7	2; 2.5
Imports	43; 100	17; 28	7; 10	3; 4
Inventories	42; 100	10; 14	3; 4	1; 1

Note: For a given variable X and a selected time horizon value T, the first (resp. second) figure provides the error measure $E_X(T)$ obtained with the fixed value terminal condition (resp. constant level terminal condition).