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GARCH MODELS WITH LEVERAGE EFFECT: DIFFERENCES AND SIMILARITIES.

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Abstract

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Keywords: EGARCH, GJR, QGARCH, TGARCH, APARCH.

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GARCH models with leverage effect: Differences and similarities.

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1. INTRODUCTION

It is widely accepted that the volatility of financial returns evolves over time. Furthermore, there is also empirical evidence that the increase in volatility is larger when the returns are negative than when they are positive. This characteristic, originally put forward by Black (1976), is known as *leverage effect.* Consider the following model for returns

$$y_t = \varepsilon_t \sigma_t \tag{1}$$

where ε_t is a serially independent sequence with zero mean, variance one and symmetric density and σ_t is the volatility. To represent the asymmetric evolution of σ_t, we consider five of the most popular asymmetric GARCH models namely the QGARCH of Sentana (1995), the TGARCH of Zaköian (1994), the GJR of Glosten et al. (1993), the EGARCH of Nelson (1991) and the APARCH of Ding et al. (1993). We only consider the simplest formulations of these models which specify the conditional variances as non-linear functions of one-lagged conditional variances and returns. Each of these models can potentially represent the excess kurtosis, positive and persistent autocorrelations of squares and the negative cross-correlations between returns and future squared returns often observed in real time series. Unfortunately, faced with the problem of choosing a particular specification among the available alternatives, empirical researchers do not have prior statistical tools at their disposal. These five models have been successfully implemented to represent the volatility of the real series of financial returns; see, for example, Franses and Van Dijk (1996), Loudon et al. (2000) and Awartani and Corradi (2004), among many others, for empirical applications.

In this paper, we compare the statistical properties of these five models for asymmetric volatilities when they are restricted to satisfy the positivity, stationarity and finite fourth order moment restrictions[1]. Furthermore, we also analyze empirically whether the estimates of the conditional standard deviations generated by the alternative models differ among them.

We show that it is not unusual that the parameters of the GJR do not satisfy the restrictions

[1]Several authors have tried before to compare some of these models in terms of their predictive power; see, for example, Loudon et al. (2000), Awartani and Corradi (2004), Balaban (2004) and Hansen and Lunde (2005). Note that one should be careful with the use of acronyms as they have not been fully consistent in the existing literature. For example, A-GARCH has been used to represent four different specifications.

for finite fourth order moment to represent the leverage effect present in the data. The restrictions imposed on the TGARCH model to have finite fourth order moment also restrict the dynamics that this model can represent but to a lesser degree. On the other hand, the asymmetry that the QGARCH model is able to represent can be restricted when the parameters satisfy the positivity restrictions. The APARCH model is apparently more flexible although there are not general necessary conditions for the stationarity and finite fourth order moment. Finally, the EGARCH model is the most flexible within the models considered in this paper. However, when the models are implemented in practice to estimate the conditional standard deviations of the two series of return considered in this paper, the TGARCH, EGARCH and APARCH estimates are rather similar.

The rest of this paper is organized as follows. Section 2 analyses the flexibility of the five models with leverage effect considered to represent the empirical properties of time series returns when the parameters are restricted to satisfy the positivity, stationarity and finite fourth order conditions. The results are illustrated in Section 3 with simulated data by fitting all the models considered to series generated by each of the other models. We show the implications of the restrictions on the estimated parameters of each of the models. Section 4 contains an empirical application to two series of financial returns. Finally, Section 5 summarizes the main conclusions and gives some guidelines for future research.

2. GARCH-TYPE MODELS WITH LEVERAGE EFFECT

In this section, we analyze the flexibility of each of the five models considered to represent the combinations of kurtosis, acf of squares and cross-correlations between returns and future squared returns often observed in real time series. The models are restricted to satisfy the positivity, stationary and finite fourth order moment restrictions.

2.1 The QGARCH model

The QGARCH volatility is given by

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 + \delta_Q y_{t-1}. \tag{2}$$

The main properties of the QGARCH model have been derived by Sentana (1995) who shows that it is stationary if $p = \alpha + \beta < 1$. In this case, the marginal variance is $\sigma_y^2 = \frac{\omega}{1-p}$. Note that the stationarity of the model does not depend on the asymmetry parameter, δ_Q. However, if σ_y^2 is finite, the asymmetry parameter has to be restricted to guarantee the positivity of σ_t^2. In particular $\delta_Q^2 \leq 4\alpha\sigma_y^2(1-p)$. To avoid the dependence of the asymmetry parameter on the marginal variance, we analyze the positivity restrictions in terms of the parameter $\delta_Q^* = \frac{\delta_Q}{\sigma_y}$. Consequently, if the model is stationary, the positivity restriction is given by

$$\delta_Q^{*2} \leq 4\alpha(1-\alpha-\beta). \tag{3}$$

This restriction implies that, for fixed α, the maximum absolute asymmetry parameter decreases as β and, consequently, the persistence, increases. This result is illustrated in Figure 1 that plots the maximum absolute value of δ_Q^* that satisfy the positivity restriction when $\alpha + \beta < 1$. On the other hand, for fixed β, the maximum absolute value of δ_Q^* increases with α when $\alpha < 0.5(1-\beta)$ and decreases otherwise. Finally, for fix $\alpha + \beta$, the maximum value of $\left|\delta_Q^*\right|$ increases with α. Note that, for the parameter values usually encountered in practice, i.e. small α and large β, the maximum absolute value of δ_Q^* and, consequently, the leverage effect that the QGARCH model is able to represent, is very small. Therefore, regardless of the error distribution, the positivity restriction of the QGARCH model can unduly restrict the dynamics of the conditional variance.

Consider now the restriction for the existence of the fourth order moment which is given by

$$(k_\varepsilon - 1)\alpha^2 + p^2 < 1, \tag{4}$$

where k_ε is the kurtosis of ε_t; see He and Teräsvirta (1999a). This restriction does not depend on the asymmetry parameter. However, the restrictions imposed on α and β by the existence of the fourth order moment of y_t are stronger as the kurtosis of ε_t increases. When restriction (4) is satisfied, the kurtosis of y_t is given by $k_y = k_\varepsilon \frac{1-p^2+\delta_Q^{*2}}{1-[(\kappa_\varepsilon-1)\alpha^2+p^2]}$.

The dynamic properties of QGARCH models appear in the acf of squared observations and in the cross-correlations between squares and original observations. In particular, using the results in He and

Teräsvirta (1999a), the acf of y_t^2 can be derived as follows

$$\rho_2(\tau) = \begin{cases} \frac{2\alpha(1-p+\alpha p)+\delta_Q^{*2}(k_\varepsilon \alpha+\beta)}{2(1-p^2+\alpha^2)+k_\varepsilon \delta_Q^{*2}}, & \tau = 1 \\ \\ p^{\tau-1}\rho_2(1), & \tau > 1. \end{cases} \tag{5}$$

The rate of decay of the acf of y_t^2 does not depend on the asymmetry parameter, δ_Q^*. The autocorrelations decay exponentially with parameter p as in the symmetric GARCH model; see Sentana (1995). The presence of the leverage effect only affects the first order autocorrelation which is larger than in the corresponding symmetric model. However, remember that if the persistence, p, is close to one and α close to zero, the maximum absolute value of δ_Q^* is very small if one wants to guarantee the positivity of σ_t^2. Consequently, the effect of the leverage effect on the autocorrelations of squares is negligible. As an illustration, Figure 2 plots the acf of squares of four QGARCH models with parameters $\{\omega, \alpha, \beta, \delta_Q^*\}$ given by $\{0.05, 0.15, 0.8, -0.17\}$, $\{0.03, 0.1, 0.87, -0.109\}$, $\{0.02, 0.1, 0.88, -0.089\}$ and $\{0.01, 0.09, 0.9, -0.06\}$ and Gaussian errors. The parameter values of ω, α, β have been chosen to resemble the values usually estimated when GARCH models are fitted to real time series of financial returns. The marginal variance is always one and the fourth order moment is finite. On the other hand, δ_Q^* has been chosen at its maximum value to guarantee the positivity of σ_t^2. The persistences of the models are 0.95, 0.97, 0.98 and 0.99, with kurtosis 7.22, 5.44, 7.27 and 19.05, respectively. Note that, the four models chosen for this illustration have increasing persistence and, consequently, the maximum value of the asymmetry parameter is decreasing. Figure 2 also plots the acf corresponding to the models with $\delta_Q = 0$. This figure shows that, for the parameter values considered, the autocorrelations of squares are nearly indistinguishable in the QGARCH models with leverage effect with respect to the corresponding symmetric models.

The cross-correlations between y_t^2 and $y_{t-\tau}$, derived by Sentana (1995), are given by

$$\rho_{21}(\tau) = \begin{cases} \frac{\delta_Q^*}{(\kappa_y - 1)^{1/2}}, & \tau = 1 \\ \\ (\alpha+\beta)\rho_{21}(\tau-1), & \tau > 1. \end{cases} \tag{6}$$

The second column of Figure 2 plots the cross-correlations of the same models considered above. This figure shows that, in the cases of interest from the empirical point of view, the QGARCH model generates very small cross-correlations between returns and future squared returns. Furthermore, we can also observe that the correlations decrease as the persistence of the volatility increases because δ_Q^* has to decrease to guarantee the positivity of σ_t^2.

2.2 The TGARCH model

The TGARCH model is given by

$$\sigma_t = \omega + \alpha \left| y_{t-1} \right| + \beta \sigma_{t-1} + \delta_T y_{t-1}. \tag{7}$$

Note that there are not restrictions to guarantee the positivity of σ_t^2. However, the parameters of the TGARCH model have to be restricted to guarantee stationarity and the existence of the fourth order moment. The stationarity condition is given by $p < 1$, which implies

$$\delta_T^2 < 1 - \alpha^2 - \beta^2 - 2\alpha\beta\nu_1 \tag{8}$$

where $\nu_1 = E\left(|\varepsilon_t|\right)$ which is given by $\sqrt{\frac{2}{\pi}}$ when ε_t is Gaussian and by $\sqrt{\frac{(\nu-2)}{\pi}\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}}$ when ε_t has a Student-ν distribution; see He and Teräsvirta (1999a).

Note that the stationarity of the TGARCH model depends on the distribution of ε_t. However, the values of ν_1 are not very different when ε_t is assumed to be Gaussian or Student-ν. For example, when $\nu = 5$, $\nu_1 = 0.735$ and $\nu_1 = 0.798$ when ε_t is Gaussian. Consequently, the restrictions imposed on δ_T for the TGARCH model to be stationary are rather similar for different error distributions. When the stationary restriction in (8) is satisfied, the marginal variance is given by

$$\sigma_y^2 = \omega^2 \frac{1+q}{(1-q)\,(1-p)}$$

where $q = \alpha\nu_1 + \beta$, $p = q^2 + \alpha^2(1-\nu_1^2)$. Figure 3 represents the admissible values of δ_T that guarantee the stationarity of the TGARCH model as a function of positive α and β parameters, when the errors are Normal. Observe that when β is close to one, the maximum asymmetry allowed is rather small. On the other hand, the asymmetry of the TGARCH model decreases when α increases.

The parameters of the TGARCH model have to be further restricted to guarantee finite fourth order moment. The kurtosis of y_t, derived by He and Teräsvirta (1999a), is given by

$$k_y = k_\varepsilon \frac{[(1-p)(1-q)(3d+5p+3q+3dp+5dq+3pq+dpq+1]}{(1+q)^2(1-d)(1-f)} \tag{9}$$

where $d = (\alpha+\beta)^3 + \alpha^3(\nu_3 - 1) + 3\alpha\beta^2(\nu_1 - 1) + 3\delta_T^2(\alpha\nu_3 + \beta)$ and $f = (\alpha+\beta)^4 + \alpha^4(k_\varepsilon - 1) + 4\alpha\beta\left[\beta^2(\nu_1 - 1) + \alpha^2(\nu_3 - 1)\right] + 6\delta_T^2\left[2\alpha\beta\nu_3 + \alpha^2 k_\varepsilon + \beta^2\right]$ with $\nu_3 = \mathbb{E}(|\varepsilon_t|^3)$. Given that when α and β are positive, as it is the case in the empirically relevant models, $f < 1$ implies that $d < 1$, the kurtosis is finite if

$$f < 1. \tag{10}$$

Given that in expression (10) it is not obvious which is the relationship between the asymmetry, the parameters α and β and the error distribution, Figure 3 also represents the values of the asymmetry parameter that guarantee finite kurtosis as a function of positive α and β. This figure shows that the restrictions imposed on δ_T become stronger as the degrees of freedom decrease. Therefore, as the errors are allowed to have more kurtosis, the leverage effect represented by the TGARCH model should be smaller and it loses part of its flexibility.

On the other hand, (10) implies that for fixed δ_T, the parameters α and β and, consequently, the persistence should be strongly restricted. Consequently, the TGARCH model may have difficulties to represent simultaneously leverage effect with finite kurtosis and large persistence.

The following expression of the acf of y_t^2 is given by He and Teräsvirta (1999a)

$$\rho_2(\tau) = \begin{cases} \frac{(1-q)(1-p)\{2\bar{q}(1-f)\Delta_3^0 + \bar{p}\Delta_4^0\} - (1+q)(1-d)(1-f)\{2q + p(1-q)\}}{\Delta^0}, & \tau = 1 \\ \\ p\rho_2(\tau-1) + \theta^0 q^{\tau-1}, & \tau > 1, \end{cases} \tag{11}$$

where

$$\begin{aligned} \Delta^0 &= k_\varepsilon \Delta_4^0 (1-q)(1-p) - (1+q)^2(1-d)(1-f), \\ \theta^0 &= (1/\Delta^0)\{2(1-p)(1-f)[\Delta_3^0\bar{q}(1-q) - q(1+q)(1-d)]\} \\ \Delta_3^0 &= (1+p)(1+q) + 2(p+q) \\ \Delta_4^0 &= (1+p)(1+q)(1+d) + 2(1+p)(q+d) + 4(qd+p) \\ \bar{q} &= \alpha\nu_3 + \beta \\ \bar{p} &= \beta^2 + 2\alpha\beta\nu_3 + k_\varepsilon(\alpha^2 + \delta_T^2). \end{aligned}$$

Figure 4 plots the acf of squares of four TGARCH models with Gaussian errors and parameters $\{\omega, \alpha, \beta, \delta_T\}$ given by $\{0.057, 0.14, 0.825, -0.12\}$, $\{0.07, 0.15, 0.8, -0.16\}$, $\{0.049, 0.1, 0.865, -0.14\}$ and

$\{0.025, 0.09, 0.9, -0.1\}$, respectively. Once more, the parameters have been chosen in such a way that the marginal variance is one and the asymmetry parameter has its maximum value to guarantee the existence of the kurtosis of y_t. The persistence of these four models are $0.88, 0.9, 0.92$ and 0.96 with kurtosis $10.10, 19.02, 15.88, 20.04$, respectively. For comparison shake, Figure 4 also plots the autocorrelations of squares of the corresponding symmetric models. Observe that in the TGARCH model, the presence of asymmetries could generate large differences in the autocorrelations.

The cross-correlations between y_t^2 and $y_{t-\tau}$ can be computed using the results in He et al. (2008) and are given by

$$\rho_{21}(\tau) = \left\{ \frac{2\sqrt{(1-q)(1-p)}\delta_T(\sum_{j=0}^{\tau-1} q^{\tau-1-j}p^j + p^{\tau-1}\bar{q}(1+pq)(1-d)^{-1}(1+q)^{-1})}{\sqrt{(1+q)(k_y-1)}}, \quad \tau \geq 1. \right. \tag{12}$$

As an illustration, Figure 4 plots the cross-correlations for the same four models considered above. These cross-correlations are rather large when compared with those of the QGARCH model. They decay exponentially. Therefore, it seems that although the TGARCH model has difficulties to represent series with finite kurtosis and persistent shocks to volatility, the restrictions imposed on its asymmetry parameter to guarantee finite kurtosis are milder than those imposed on the QGARCH model to guarantee positive conditional variances.

2.3 The GJR model

The GJR model specifies the conditional variance as

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta\sigma_{t-1}^2 + \delta_G I(\varepsilon_{t-1} < 0)y_{t-1}^2 \tag{13}$$

where $I(.)$ is the indicator function that takes value 1 when the argument is true. Note that the GJR model is similar to the TGARCH model but the volatility is specified in terms of σ_t^2 instead of σ_t. Consequently, it is necessary to restrict its parameters in order to ensure positivity. In particular, Hentschel (1995) shows that σ_t^2 is positive if

$$\omega > 0, \ \alpha, \beta, \delta_G \geq 0. \tag{14}$$

On the other hand, the model is stationary if

$$\delta_G < 2(1 - \alpha - \beta). \tag{15}$$

Note that this restriction is the same regardless of the conditional distribution of ε_t. When the stationarity condition is satisfied, the marginal variance of y_t is given by $\sigma_y^2 = \frac{\omega}{1-p}$ where $p = \alpha + \beta + 0.5\delta_G$. Figure 5, that plots the parameter space of the GJR model that guarantees stationarity, illustrates that, as in the TGARCH model, the maximum value of δ_G decreases with α and β. For small values of α and values of β close to one, the allowed values of δ_G are rather small.

He and Teräsvirta (1999a) derive the condition for the existence of the fourth order moment which is given by

$$p^2 + \alpha(\kappa_\varepsilon - 1)(\alpha + \delta_G) + 0.25\delta_G^2(2\kappa_\varepsilon - 1) < 1. \tag{16}$$

When condition (16) is satisfied, the kurtosis of y_t is given by

$$k_y = k_\varepsilon \frac{1 - p^2}{1 - p^2 - \alpha(k_\varepsilon - 1)(\alpha + \delta_G) - 0.25\delta_G^2(2k_\varepsilon - 1)}.$$

The existence of the fourth order moment depends on the distribution of ε_t. For fixed values of α and β, larger kurtosis of ε_t imply more restrictive conditions on δ_G. Figure 5 also plots the maximum allowed values of δ_G that guarantee finite kurtosis of y_t as a function of the parameters α and β when ε_t has Normal, Student-5 and Student-7 distributions. Note that as expected, given the close relationship between the TGARCH and GJR models, the shapes of these surfaces are very similar to those plotted in Figure 3 for the TGARCH model. Finally note that (16) implies that the GJR model has also difficulties to represent high persistence together with finite kurtosis.

The acf of y_t^2, derived by He and Teräsvirta (1999a), is given by

$$\rho_2(\tau) = \begin{cases} \frac{(\beta + k_\varepsilon(\alpha+\delta_G))(1-p^2) - p(1-p^2-\alpha(k_\varepsilon-1)(\alpha+\delta_G)-0.25\delta_G^2(2k_\varepsilon-1))}{k_\varepsilon(1-p^2)-(1-p^2-\alpha(k_\varepsilon-1)(\alpha+\delta_G)-0.25\delta_G^2(2k_\varepsilon-1))} & \tau = 1 \\ \\ \rho_2(1)p^{\tau-1} & \tau > 1. \end{cases} \tag{17}$$

Figure 6 plots the acf of y_t^2 corresponding to four GJR models with Gaussian errors and parameters $\{\omega, \alpha, \beta, \delta_G\}$, given by $\{0.045, 0.1, 0.8, 0.11\}$, $\{0.035, 0.1, 0.83, 0.07\}$, $\{0.025, 0.10, 0.845, 0.06\}$, $\{0.015, 0.07, 0.885, 0.06\}$. Once more, the parameters have been chosen such that the marginal variance is one and the asymmetry parameter has the maximum allowed value to guarantee finite kurtosis. The persistences of the models are 0.955, 0.965, 0.975 and 0.985, with kurtosis 8.55, 7.20, 11.50, 12.62, respectively. Note that in the four models chosen, the differences between the acf of squares in the

GJR model and in the corresponding symmetric model are even larger than those observed in the TGARCH model.

Using the results in He et al. (2008), we have derived the cross-correlations between y_t^2 and $y_{t-\tau}$ which are given by

$$\rho_{21}(\tau) = \begin{cases} \frac{\delta_G \nu_3 \gamma_3}{(\sqrt{\sigma_y^2})^3 \sqrt{k_y - 1}}, & \tau = 1 \\ \\ p^{\tau-1} \rho_{21}(1), & \tau > 1 \end{cases} \tag{18}$$

where $\gamma_3 = E(\sigma_t^3)$. This quantity has to be computed by simulation as suggested by He et al. (2008)[2]. The corresponding cross-correlations, plotted in the second column of Figure 6 for the same four GJR models considered above, have an exponential decay similar to this observed in the QGARCH and TGARCH models. The magnitudes of the cross-correlations are very small, in line with the values shown by the QGARCH model and far from the TGARCH case. However, note that the cross-correlations of the GJR model are less reliable as they are based on simulated moments.

2.4 The EGARCH model

The EGARCH model specifies the conditional variance as follows

$$\log \sigma_t = \omega + \beta \log \sigma_{t-1} + \alpha^* \left(|\varepsilon_{t-1}| - E\left|\varepsilon_{t-1}\right| \right) + \delta_E \varepsilon_{t-1}. \tag{19}$$

The specification of the volatility in terms of its logarithmic transformation implies that there are not restrictions on the parameters to guarantee the positivity of the variance. In addition, Nelson (1991) show that the EGARCH model is stationary and has finite kurtosis if $|\beta| < 1$. Therefore, there are not restrictions on the leverage effect that the model can represent imposed by the positivity, stationarity or the finite fourth order moment restrictions. The unconditional variance, kurtosis and acf of squares can be derived using the results in Nelson (1991). They have respectively the following expressions

$$\sigma_y^2 = \exp\left(\frac{\omega}{1-\beta}\right) \prod_{i=1}^{\infty} E(\exp(\beta^{i-1} g(\varepsilon_{t-i}))) \tag{20}$$

[2] For each model, 1500 series of size 5000 are generated. For each series the median of σ_t^3 is computed. Then, the median of $MED(\sigma_t^3)$ is computed through replicates.

$$k_y = k_\varepsilon \prod_{i=1}^{\infty} \frac{E(\exp(2\beta^{i-1} g(\varepsilon_{t-i})))}{\left[E(\exp(\beta^{i-1} g(\varepsilon_{t-i})))\right]^2} \tag{21}$$

$$\rho_2(\tau) = \frac{E(\varepsilon_t^2 \exp(\beta^{\tau-1} g(\varepsilon_t)))P_1 P_2 - P_3}{k_\varepsilon P_4 - P_3} \tag{22}$$

where $g(\varepsilon_t) = \alpha^* \left(|\varepsilon_t| - E\,|\varepsilon_t|\right) + \delta_E \varepsilon_t$, $P_1 = \prod_{i=1}^{\tau-1} E(\exp(\beta^{i-1} g(\varepsilon_{t-i})))$, $P_2 = \prod_{i=1}^{\infty} E(\exp((1+\beta^\tau)\beta^{i-1} g(\varepsilon_{t-i})))$, $P_3 = \prod_{i=1}^{\infty} \left[E(\exp(\beta^{i-1} g(\varepsilon_{t-i}))\right]^2$, $P_4 = \prod_{i=1}^{\infty} E(\exp(2\beta^{i-1} g(\varepsilon_{t-i})))$. Nelson (1991) derived closed form expressions for some the expectations involved in equations (20) to (22) for several distributions of ${\varepsilon_t}$[3]. It is interesting to remark that the rate of decay of the autocorrelations of squares is not constant; see Carnero et al. (2004) who show that this rate tends to β for large lags whereas for small ones it depends on α^* and $\delta_{E.}$

Figure 7 plots the acf of squares of four Gaussian EGARCH models with parameters $\{\omega, \alpha, \beta, \delta_E\}$, given by $\{-0.002, 0.09, 0.98, -0.06\}$, $\{-0.003, 0.06, 0.985, -0.1\}$, $\{-0.003, 0.12, 0.99, -0.08\}$, $\{-0.001$, 0.05, -0.995, $-0.07\}$, respectively. In this case, the asymmetry parameter can be freely chosen. All the models have marginal variance equal to one and finite kurtosis. The persistences are 0.98, 0.985, 0.99 and 0.995 with kurtosis 3.60, 4.54, 5.84 and 5.58, respectively. Figure 7 also plots the acf of squares of the corresponding symmetric models. This figure illustrates that the EGARCH model can generate first order autocorrelations clearly larger than in the corresponding symmetric models.

Ruiz and Veiga (2008) derived the following expression of the cross-correlations between y_t^2 and $y_{t-\tau}$

$$\rho_{21}(\tau) = \frac{E(\varepsilon_t \exp(\beta^{\tau-1} g(\varepsilon_t)))P_1 P_5}{P_3^{1/4}[k_\varepsilon P_4 - P_3]^{1/2}} \tag{23}$$

where $P_5 = \prod_{i=1}^{\infty} E(\exp(\beta^{i+\tau-1} + \frac{1}{2}\beta^{i-1}) g(\varepsilon_{t-i}))$; see also Demos (2002) who derives the cross-correlation function under Gaussian errors and Karanasos and Kim (2003) who obtain a general expression of $\rho_{21}(\tau)$ in ARMA(r,s)-EGARCH(p,q) models from which expression (23) can be obtained as a particular case.

[3] He et al. (2002) derived the acf of $|y_t|^\theta$ for the EGARCH(1,1) model giving closed form expressions for some of the expectations involved in (20) to (22) when the errors are Normal or Generalized Errors Distribution (GED). Karanasos and Kim (2003) derive the acf of $|y_t|^\theta$ of a general EGARCH(p,q) model with Gaussian, GED and Double Exponencial errors.

The second column of Figure 7 plots the cross-correlation functions for the four EGARCH models described before. The exponential decay is similar to this observed in QGARCH, TGARCH or GJR models.

2.5 The APARCH model

The APARCH model is given by

$$\sigma_t^{\lambda} = \omega + \alpha_A \left(|y_{t-1}| - \delta_A y_{t-1}\right)^{\lambda} + \beta\sigma_{t-1}^{\lambda}. \tag{24}$$

This model nests some of the GARCH models with leverage effect described before. For example, when $\lambda = 1$ and $\delta_T = \alpha_A \delta_A$, we obtain the TGARCH model while when $\lambda = 2$, the GJR model in (13) is obtained with $\delta_G = 4\alpha_A\delta_A$ and $\alpha = \alpha_A(1 - \delta_A)$.

The restrictions for the positivity of σ_t^{λ} are given by Ding et al. (1993) as follows:

$$\omega > 0, \lambda \geq 0, -1 < \delta_A < 1, \alpha_A \geq 0 \text{ and } \beta \geq 0. \tag{25}$$

However, it is not clear whether σ_t^{λ} should be positive for all λ. Consider for example that $\lambda = 1$, as in the TGARCH model. In this case, σ_t^2 is always positive regardless of whether σ_t is positive or negative. Therefore (25) is a sufficient but not necessary condition for the positivity of the conditional variance.

The condition for the existence of $E\left(\sigma_t\right)^{\lambda}$ is given by

$$\alpha_A E\left(|\varepsilon_t| - \delta_A \varepsilon_t\right)^{\lambda} + \beta < 1. \tag{26}$$

Expression (26) depends on the density of the errors. Ding et al. (1993) derive the expression of $E\left(|\varepsilon_t| - \delta_A\varepsilon_t\right)^{\lambda}$ for Gaussian errors and Karanasos and Kim (2006) extended it for Student, GED and Double exponential distributions. Note that this condition is sufficient for stationary when $\lambda \geq 2$, otherwise it guarantees the existence of a moment that not necessarily implies the existence of the marginal variance. Consider, once more, $\lambda = 1$, then expression (26) guarantees the existence of $E\left(\sigma_t\right)$ which is necessary but not sufficient for the existence of σ_y^2.

He and Teräsvirta (1999b) derived the following condition for the existence of the moment $E(\sigma_t^{2\lambda})$

$$\frac{\alpha_A^2}{2}[(1+\delta_A)^{2\lambda} + (1-\delta_A)^{2\lambda}]E\left(|\varepsilon_t|^{2\lambda}\right) + \alpha_A\beta[(1+\delta_A)^{\lambda} + (1-\delta_A)^{\lambda}]E\left(|\varepsilon_t|^{\lambda}\right) + \beta^2 < 1. \tag{27}$$

Again, condition (27) does not imply finite kurtosis for all λ. For example, if $\lambda = 1$, it only guarantees the existence of the variance as in the TGARCH model. However, when $\lambda = 2$, it reduces to $\beta^2 + \alpha_A^2(1 + 6\delta_A^2 + \delta_A^4) + 2\alpha_A\beta(1 + \delta_A^2) < 1$ which is the condition for finite kurtosis in the GJR model. Therefore, condition (27) is sufficient for stationarity when $\lambda \geq 1$ while it is sufficient for finite kurtosis when $\lambda \geq 2$. Consequently, it is not possible to carry out a comparative analysis of the maximum allowed values of δ_A when the model is stationary, has finite kurtosis and the conditional variances are positive for general values of λ. In practice, λ uses to be estimated between 1 and 2. In this case, the restriction in (27) guarantees the existence of the variance but not of the kurtosis. In any case, this restriction will be sufficient but not necessary. Therefore, we do not pursue the same kind of analysis carried out for the other four models considered in this paper.

It is important to note that there are not close form expressions of the variance and kurtosis of y_t. These expressions are only available when $\lambda = 1, 2$ and, in these cases, they coincide with the expressions given above for the TGARCH and GJR models. Moreover, the autocorrelations of squares and cross-correlations are also available only for these two particular cases.

3. THE RESTRICTIONS WITH ESTIMATED PARAMETERS.

In the previous section, we have analyzed the restrictions imposed in the parameter space of each model to guarantee positivity and finite variance and kurtosis. However, in practice, the parameters are estimated and the restrictions are checked using the corresponding estimates. In this section, we analyze whether the conclusions obtained by checking the restrictions on estimated parameters are in concordance with the true existence of moments. For this purpose, we generate $R = 1000$ series of size $T = 2000$ by each of the models[4]. Then, all five models are fitted to each series and the corresponding plug-in moments are obtained by substituting the estimated parameters in the analytical expressions of the moments[5]. In particular, we have generated series by a QGARCH(1,1) model with parameters $\{0.03, 0.1, 0.87, -0.109\}$, a TGARCH model with parameters $\{0.057, 0.14, 0.825, -0.12\}$, a GJR model with parameters $\{0.015, 0.07, 0.885, 0.06\}$, an EGARCH model with $\{-0.003, 0.12, 0.99, -0.08\}$ and, finally, an APARCH model with $\lambda = 1.2$ and $\{0.03, 0.1, 0.8, 0.05\}$. All the models are stationary, have

[4]Results for $T = 5000$ are available upon request. They are similar to those reported in this paper.

[5]The parameters have been estimated by QML using software developed in Matlab by the first author.

unit variance and finite kurtosis and satisfy the restrictions for positivity of the conditional variances.

Table 1 reports the percentage of estimated models that satisfy the conditions for positivity, finite variance and kurtosis. Consider first the results obtained when the QGARCH model is fitted. The results in Table 1 show that the QGARCH model may have problems to satisfy the positivity restriction when the series are generated by the QGARCH and TGARCH models and to satisfy the condition for finite kurtosis when they are generated by the GJR and EGARCH models. Remember that all the models used to generate the series are such that the positivity restrictions are satisfied. However, when fitting the QGARCH model to the series generated by itself only 50.7% of the estimated parameters satisfy this restriction. The situation is even worse when the series are generated by the TGARCH model as, in this case, only 32% of the estimated QGARCH models satisfy the restriction. The stationarity condition is almost always satisfied. However, when looking at the existence of the kurtosis, there are 13.8% and 9% of the estimated QGARCH that do not satisfy this condition when the series are generated by GJR and EGARCH models, respectively.

Looking now at the results obtained when the TGARCH model is fitted, we can observe that both the stationarity and finite fourth order conditions are satisfied in nearly all cases. Only when the series are generated by the EGARCH model, there are 7.4% of the TGARCH models that do not satisfy the condition (10) for finite kurtosis.

The next model fitted is the GJR model. In this case, there is a large percentage of series when the estimated parameters do not satisfy the positivity condition mainly when the series are generated by the TGARCH, EGARCH and APARCH models. The stationarity condition is satisfied in nearly all cases. Only when the series are generated by the EGARCH model, there are 13.5% of the estimated GJR models that do not satisfy this condition. However, the percentages of estimated models that satisfy the condition for the existence of the kurtosis are very small. Note, for example, that only 10% of the GJR models fitted when the series are generated by the TGARCH model satisfy the condition for the existence of the kurtosis. Even when the series are generated by a GJR model with finite kurtosis, only 85.3% of the estimated GJR models satisfy the condition for the existence of the kurtosis.

On the other hand, the EGARCH estimates always satisfy the conditions for the existence of the kurtosis.

Finally, in the APARCH model, we have imposed a priori the positivity restriction[6]. On the other hand, remember that conditions (26) and (27) are the stationarity conditions when $\lambda \geq 2$ and $\lambda \geq 1$, respectively. Therefore, we have computed the percentage of series that satisfy (26) when $\hat{\lambda} \geq 2$ and that satisfy (27) when $\hat{\lambda} \geq 1$. This is the quantity reported as the percentage of series that satisfy the stationarity condition in Table 1. On the other hand, (27) is the condition for the existence of kurtosis when $\lambda \geq 2$. Therefore, the percentage reported in Table 1 correspond to the percentage of series that satisfy (27) among those in which $\hat{\lambda} \geq 2$. When the series are generated by the GJR a the EGARCH models and the APARCH model is fitted, there is a large percentage of series in which the stationarity condition is not satisfied. This percentage is even larger when looking at the condition for finite kurtosis. In this case, with the exception of series generated by the own APARCH model, the APARCH estimates only satisfy the finite kurtosis restriction in very small percentage of series.

Summarizing, it seems that the QGARCH model may seem to estimate non-positive variances when fitted to series in which the conditional variance is always positive while the GJR and APARCH estimates may lead to think that the kurtosis is not finite when the series has finite kurtosis.

4. EMPIRICAL APPLICATION

In this section, the five GARCH models with leverage effect previously described are fitted to represent the evolution of the volatility of two series of daily returns. For each model and series, we check whether the estimated parameters satisfy the positivity, finite variance and kurtosis restrictions and analyze whether we obtain contradictory conclusions. Furthermore, and given that the final goal when fitting a conditionally heteroscedastic model is to obtain estimates of the underlying volatility, we compare the estimated volatilities obtained with the alternative models. The series analyzed are daily returns of the S&P500 index observed from January 5th 1999 to May 9th 2006 and of the exchange rates of US Dollar against the Australian Dollar observed from January 2nd 1990 to May 9th 2006. To avoid the misleading effects of outliers on the estimation of the volatility, the series have been filtered by equalling all observations larger than $5\widehat{\sigma}_t$ to $\widehat{\sigma}_t sign(y_t)$ where $\widehat{\sigma}_t$ is an estimate of the conditional standard deviation. The filtered series have been plotted in Figure 9 together with their

[6] Without this restriction, the estimator did not converge in may replicates.

correlogram of squares and their sample cross-correlations. The volatility clustering observed in the series of returns is reflected in the positive and significant autocorrelations of squares. Moreover, note that, as usual, these autocorrelations are not very large and highly persistent. On the other hand, the cross-correlations are also significant and negative, suggesting the presence of leverage effect.

Table 2 reports the estimation results obtained when fitting the QGARCH, TGARCH, GJR, EGARCH and APARCH[7] models with Student-ν errors to S&P500 returns. For all the models, the estimated degrees of freedom are over 20. Therefore, the conditional distribution of the errors seems to be well approximated by the Normal distribution. Furthermore, the estimated power parameter of the APARCH model is 1.188 which is closer to the TGARCH model than to the GJR formulation. In fact, the parameters estimated when the TGARCH model is fitted are very similar to those estimated for the APARCH model. In any case, the estimates of β are always very close to 1 while the estimates of α are small although significant. Finally, note that the estimates of the asymmetry parameter are also significant in all the models.

After estimating the conditional deviations for each of the models, $\widehat{\sigma}_t$, the residuals are computed as $\hat{\varepsilon}_t = y_t/\widehat{\sigma}_t$. Table 2 also reports several diagnostics. For all the models fitted, the kurtosis have been clearly reduced with respect to the sample kurtosis of S&P500 returns which is 4.41 and they are closer to 3, the kurtosis of the Normal distribution, although still significantly different from it. Furthermore, the autocorrelations of squared residuals and the cross-correlations between returns and future squared returns are not significant. Therefore, it seems that all the fitted models have been successful in representing the dynamic evolution of the squares and part of the kurtosis observed in the S&P500 returns.

Table 2 also reports whether the positivity and finite variance and kurtosis restrictions are satisfied. All models fitted satisfy the positivity restrictions. However, the estimates of the TGARCH and GJR models are such that the stationarity condition is not satisfied. Consequently, the variance, kurtosis, autocorrelations of squares and cross-correlations are apparently not defined for S&P500 returns. In the other two models, the QGARCH and the EGARCH, the estimated parameters satisfy the three conditions. Finally, in the APARCH model, we can only check whether the stationarity condition is

[7] All models have been estimated with E-views.

satisfied.

As mentioned above, the final goal when fitting conditionally heterocedastic models is to obtain estimates of the underlying volatilities. The main diagonal of Figure 10 plots these estimates obtained after fitting each of the five models considered to the S&P500 returns. The lower triangle of this figure plots the differences between the volatilities estimates considered two by two. Finally, the upper triangle of Figure 10 plots scatter plots of the estimated volatilities taken two by two. The general shape of the estimated volatilities is similar. However, it is remarkable the similarity between the variances estimated by the TGARCH, EGARCH and APARCH models. The similarity between TGARCH and APARCH models could be expected given that, as we mentioned before, the parameter estimates in both models are very similar. On the other hand, the variances estimated by the QGARCH and GJR models are different of the variances estimated by any of the other alternative specifications and different between them. Note that, in general, when the volatility is large, the QGARCH and GJR estimated variances are smaller than the variances estimated by any of the other models.

The estimation results corresponding to the Australian Dollar/US Dollar exchange rates are reported in Table 3. In this case, the estimated degrees of freedom are approximately 7 implying a leptokurtic distribution of errors. Also, it is interesting to note that the asymmetry parameters are not significant in the QGARCH and GJR models which is expected given that exchange rates show smaller leverage effects than equity indexes[8]. Again, the APARCH model suggest an specification closer to the TGARCH model, with the estimated power parameter being $\hat{\lambda} = 1.32$. Its parameters are also close to those estimated by the TGARCH model. As usual, all the estimates of α are small and those of β are close to 1. With respect to the diagnostics, the residuals have slightly smaller kurtosis than the returns which is 4.3. Note, that if the errors have a Student-7 distribution, their kurtosis is 5. The autocorrelation of squared residuals and the cross-correlations of residuals are not significant.

The last part of the Table 3 reports whether the restrictions for positivity and finite variance and kurtosis are satisfied. All the models satisfy the positivity restrictions. However, the estimated GJR model is not stationary and the QGARCH estimates do not satisfy the conditions for the existence of

[8]When the errors are assumed to be Gaussian, the estimated leverage effect parameters were significant. It seems that although the assumption on the error distribution does not affect the other parameters of the volatility, it has an effect on the estimated asymmetry parameter; this is in concordance with the results of Zhang and King (2008).

the kurtosis. The EGARCH and the TGARCH models are the only ones with defined kurtosis and, consequently, autocorrelations of squares and cross-correlations.

Figure 11 plots the same quantities plotted in Figure 10 for the exchange rates volatilities. Once more, we observe a great similarity between the conditional variances estimated by the TGARCH, EGARCH and APARCH models. In this case, there is also similarity between the QGARCH and GJR estimates.

Summarizing, when the GJR model is fitted, the estimated parameters imply that the variance is not defined. Something similar happens in the QGARCH model which does not satisfy the conditions for the existence of the kurtosis of the exchange rate returns and on the TGARCH model which is not stationary in the case of S&P500 returns. The estimates of the GJR model do not satisfy the restrictions for the existence of the marginal variance in any of the two series of returns analyzed in this paper. This could be an artifact due to their lack of flexibility to represent the moments observed in real data. Finally, in the two examples considered, the conditional standard deviations estimated by the TGARCH, EGARCH and APARCH models are very similar. Therefore, if the objective is to estimate the underlying volatilities of a series of returns, choosing any of these three models seems to give the same answer. However, the conclusions on the existence of moments implied by the estimated parameters are not reliable in some of the models because they are not flexible to represent simultaneously high persistence and kurtosis together with leverage effect.

5. CONCLUSIONS

There is a large number of alternative GARCH models proposed in the financial econometrics literature to represent the dynamic evolution of volatilities with leverage effects. In this paper, we compare the properties of five popular asymmetric GARCH models when they are restricted to guarantee positivity of conditional standard deviations, stationarity and existence of fourth order moments. In particular, we consider the QGARCH, TGARCH, GJR, EGARCH and APARCH models. We show that the leverage effect that the QGARCH, TGARCH and GJR models can represent is heavily restricted when these models guarantee positive conditional standard deviations and finite fourth order moments. The EGARCH model is more flexible. Finally, for the APARCH model, the results of interest correspond to particular cases that coincide with the TGARCH and GJR models. In the empirical application, we show that the estimates of the underlying volatilities of the two series of financial returns analyzed are surprisingly close when the TGARCH, EGARCH and APARCH models are fitted. However, the estimates of the parameters of the TGARCH and GJR models are such that they do not satisfy the conditions for the existence of the kurtosis of returns. Note that the estimates of the volatility obtained by the EGARCH model with finite fourth order moment are nearly the same as those of the TGARCH model without finite order moment. This result could be attributed to the fact that when these restrictions are satisfied, the level of asymmetry that the TGARCH model can represent is very limited. Therefore, in order to represent the leverage effect truly present in the data, the estimates of the parameters break the restriction for the existence of the kurtosis.

Summarizing, among the models considered in this paper, the EGARCH model seems to be more flexible to represent leverage effect and simultaneously satisfy the restrictions for positivity and existence of the kurtosis. However, the TGARCH model generates estimates of the underlying volatilities very close to the ones generated by the EGARCH model although its parameter does not satisfy the restrictions.

There are many other alternative models proposed in the literature to represent the asymmetric response of volatilities to positive and negative returns. For example Brännäs and Gooijer (2004) propose an extension of the QGARCH model that allows more flexibility in the asymmetric response of volatility. However, it seems that in this model even the low order moments lack explicit analytical expressions. Also, Babsiri and Zaköian (2001) proposed a model that introduces contemporaneous asymmetry on the returns and Wu and Xiao (2002) conclude that an extension of the EGARCH model with separate coefficients for large and small negative shocks is better able to capture the asymmetry effect than the standard EGARCH model. It could be also of interest to extend the analysis to models with effects in the mean as those proposed by He et al. (2008) or Arvanitis and Demos (2004).

There are also several proposals to model the leverage effect in the context of Stochastic Volatility (SV) models. Harvey and Shephard (1996) proposed to represent the asymmetric response of volatilities by introducing correlation between the level and volatility noises. So et al. (2002) proposed a Threshold Stochastic Volatility model to represent simultaneously the mean and variance asymmetries. Recently, Demos (2002) has proposed a model that encompasses both the Asymmetric SV (A-SV) and the EGARCH models as particular cases[9]. Very recently, Kawakatsu (2007) has proposed a new A-SV model that generalizes the one proposed by Harvey and Shephard (1996). In this model, the log-volatility is a quadratic function of a latent variable. The comparison between the GARCH and SV models with leverage effect is left for further research.

[9] Carnero et al. (2004) show that the A-SV model is more flexible than the EGARCH model.

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Figure 1.- Parameter space for the stationary QGARCH(1,1) model when the positivity restriction is satisfied.

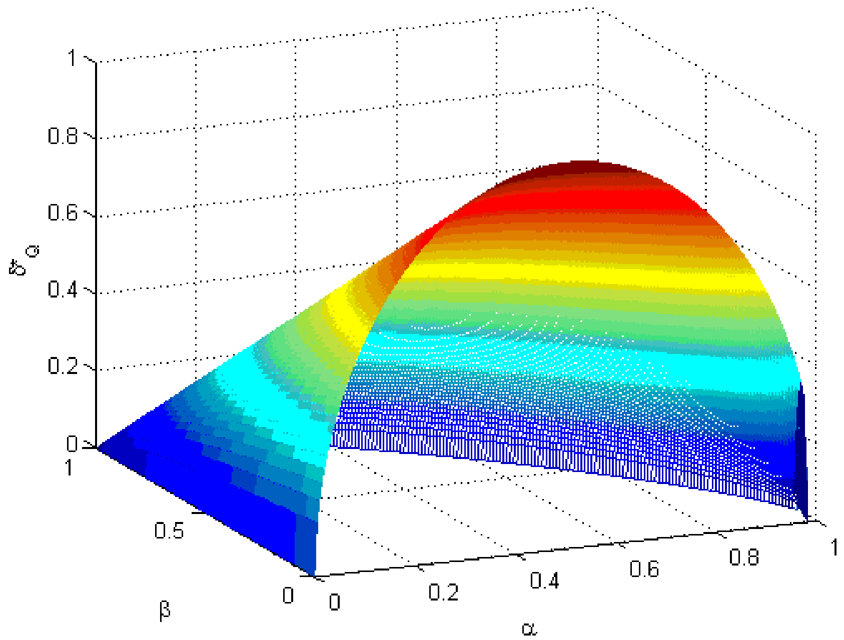
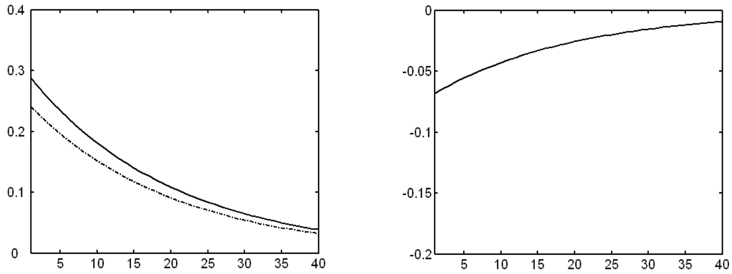
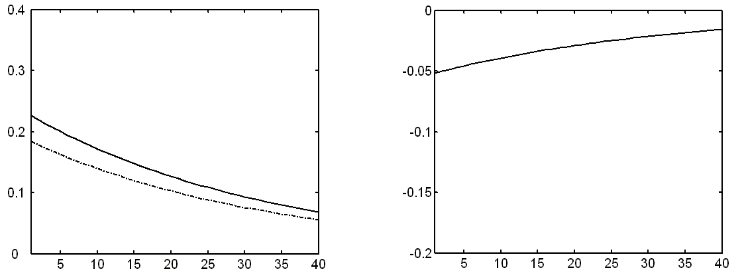


Figure 2.- Autocorrelations of squares (left column) and cross-correlations (right column) of different QGARCH models with Gaussian noises. The solid lines correspond to the autocorrelations when $\delta_Q \neq 0$ while the dashed lines are the corresponding autocorrelations when $\delta_Q = 0$.

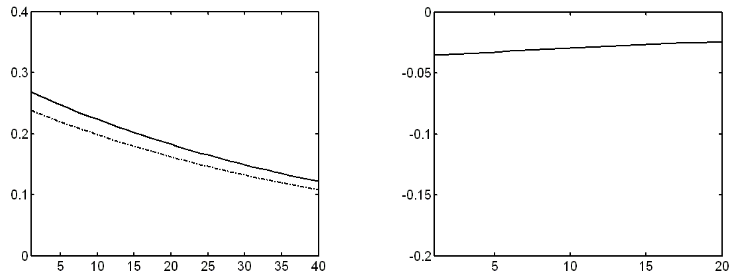
$\omega = 0.05, \alpha = 0.15, \beta = 0.80, \delta_Q = -0.170$



$\omega = 0.03, \alpha = 0.1, \beta = 0.87, \delta_Q = -0.109$



$\omega = 0.02, \alpha = 0.1, \beta = 0.88, \delta_Q = -0.089$



$\omega = 0.01, \alpha = 0.09, \beta = 0.9, \delta_Q = -0.06$

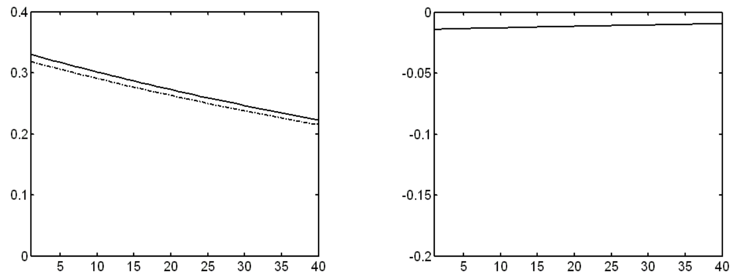


Figure 3.- Parameter space for stationarity TGARCH(1,1) models with finite fourth order moments and different error distributions.

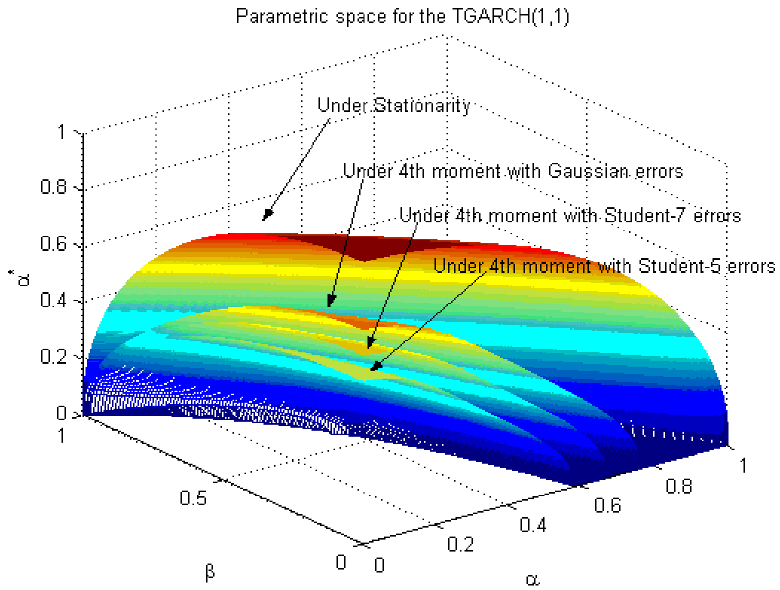
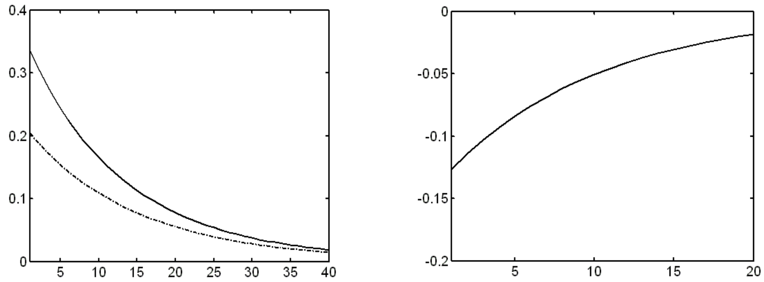
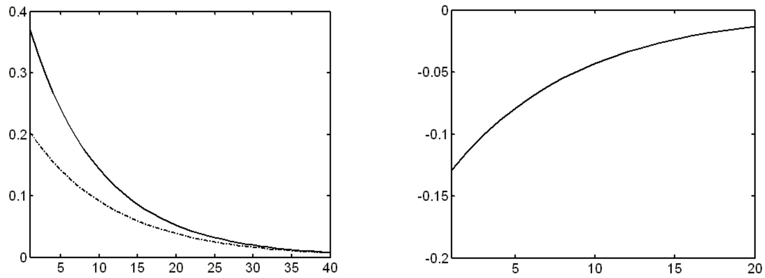


Figure 4.- Autocorrelations of squares (left column) and cross-correlations (right column) of different TGARCH models with Gaussian noises. The solid lines correspond to the autocorrelations when $\delta_T \neq 0$ while the dashed lines are the corresponding autocorrelations when $\delta_T = 0$.

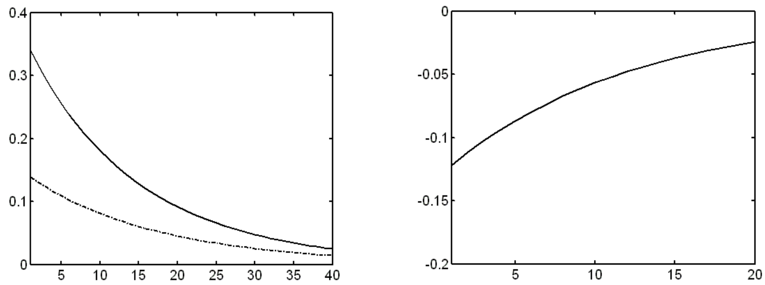
$$\omega = 0.057, \alpha = 0.14, \beta = 0.825, \delta_T = -0.12$$



$$\omega = 0.07, \alpha = 0.15, \beta = 0.8, \delta_T = -0.16$$



$$\omega = 0.049, \alpha = 0.1, \beta = 0.865, \delta_T = -0.14$$



$$\omega = 0.025, \alpha = 0.09, \beta = 0.9, \delta_T = -0.1$$

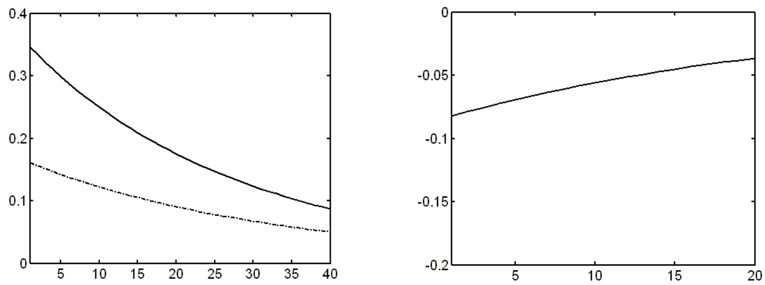


Figure 5.- Parameter space for stationarity GJR models with finite fourth order moments and different error distributions.

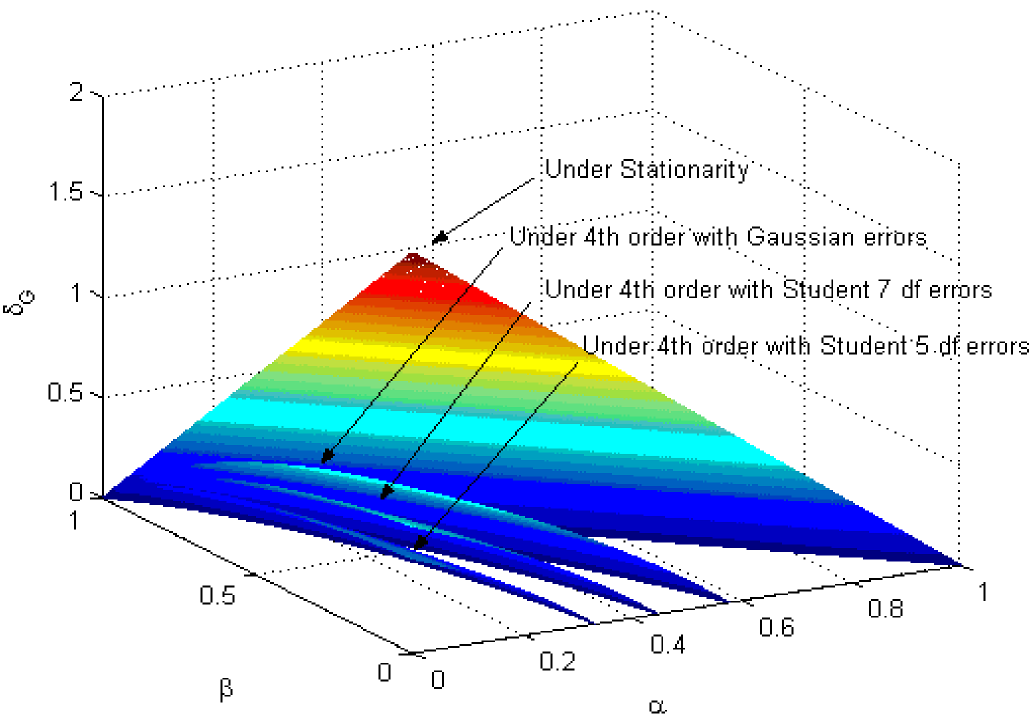
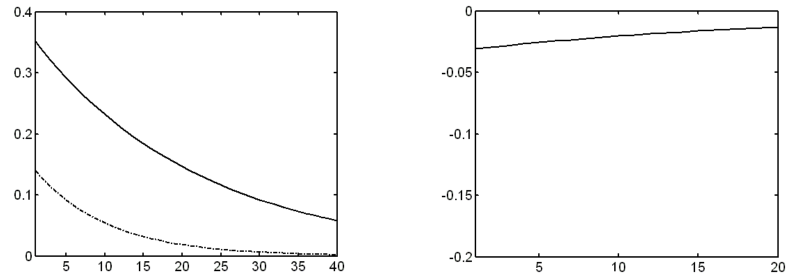
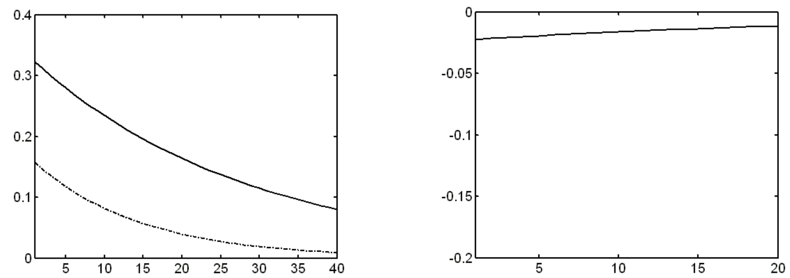


Figure 6.- Autocorrelations of squares (left column) and cross-correlations (right column) of different GJR models with Gaussian noises. The solid lines correspond to the autocorrelations when $\delta_G \neq 0$ while the dashed lines are the corresponding autocorrelations when $\delta_G = 0$.

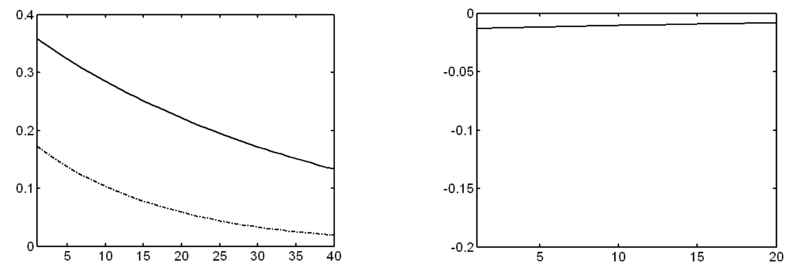
$$\omega = 0.045, \alpha = 0.1, \beta = 0.8, \delta_G = 0.11$$



$$\omega = 0.035, \alpha = 0.1, \beta = 0.83, \delta_G = 0.07$$



$$\omega = 0.025, \alpha = 0.1, \beta = 0.845, \delta_G = 0.06$$



$$\omega = 0.015, \alpha = 0.07, \beta = 0.885, \delta_G = 0.06$$

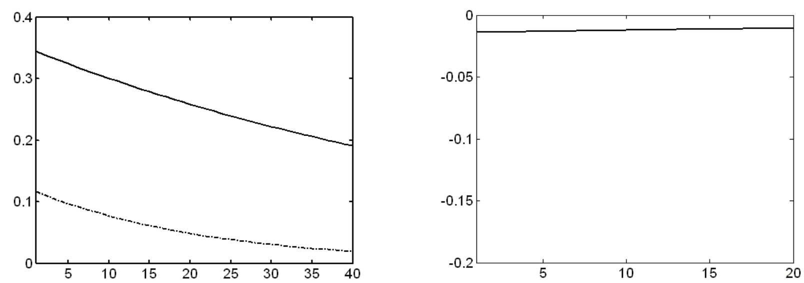


Figure 7.- Autocorrelations of squares (left column) and cross-correlations (right column) of different EGARCH models with Gaussian noises. The solid lines correspond to the autocorrelations when $\delta_E \neq 0$ while the dashed lines are the corresponding autocorrelations when $\delta_E = 0$.

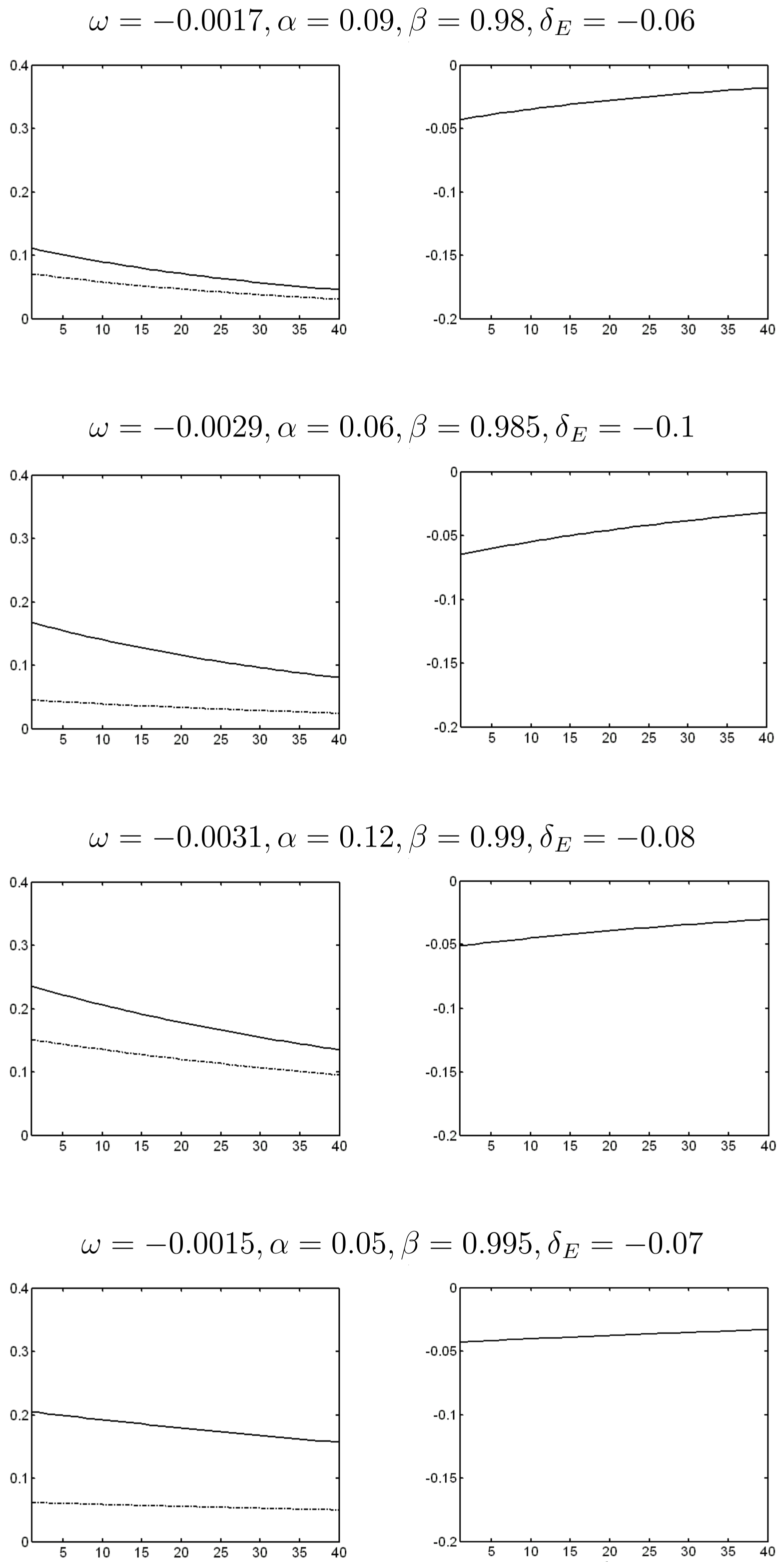
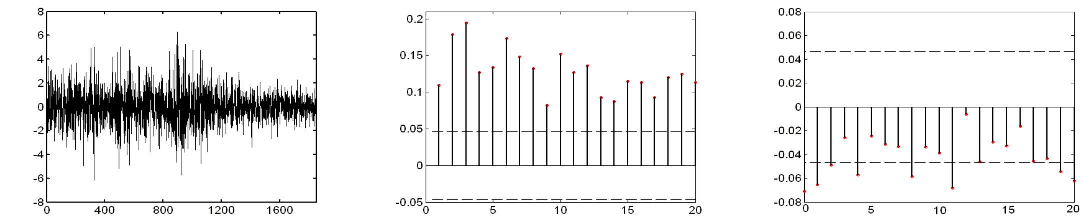


Table 1.-Percentages of fitted models that satisfy the restrictions for positivity and finite variance and kurtosis when T=2000.

Fitted	DGP	QGARCH	TGARCH	GJR	EGARCH	APARCH
QGARCH	*positivity*	50.7	32.0	99.6	96.9	100.0
	σ^2	100.0	100.0	100.0	99.8	100.0
	κ	99.5	100.0	86.2	91.0	100.0
TGARCH	*positivity*	*Always*	*Always*	*Always*	*Always*	*Always*
	σ^2	99.8	100.0	99.8	98.6	100.0
	κ	99.7	100.0	99.5	92.6	100.0
GJR	*positivity*	92.0	60.1	99.7	87.1	78.0
	σ^2	100.0	100.0	100.0	86.5	100.0
	κ	58.2	10.0	85.3	24.5	100.0
EGARCH	*positivity*	*Always*	*Always*	*Always*	*Always*	*Always*
	σ^2	100.0	100.0	100.0	100.0	100.0
	κ	100.0	100.0	100.0	100.0	100.0
APARCH	*positivity*	*Imposed*	*Imposed*	*Imposed*	*Imposed*	*Imposed*
	σ^2	91.96	99.20	77.12	77.22	100
	κ	24.67	66.67	21.78	22.27	100

Figure 9.- Daily returns y_t (first column), sample autocorrelation of y_t^2 (second column), and cross-correlations between y_t and y_{t-1}^2 (third column) for S&P500 and AUD/USD observed returns.

S&P500



AUD/USD

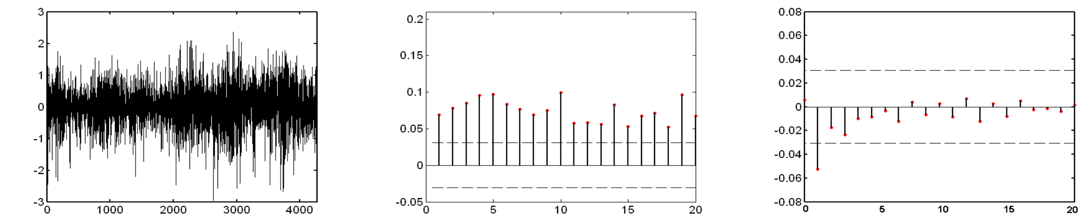


Table 2.- Estimated models for daily S&P500 returns.

	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.013* (0.004)	0.008* (0.004)	0.006* (0.004)	0.002* (0.000)	0.008* (0.004)
α	0.031* (0.007)	0.039* (0.008)	−0.007 (0.009)	0.063* (0.014)	0.036 (0.101)
β	0.96* (0.007)	1* (0.24)	0.961* (0.008)	0.994* (0.003)	0.96* (0.008)
δ	−0.069* (0.014)	−0.038* (0.009)	0.087 (0.014)	−0.078* (0.011)	−1 (4.512)
λ					1.188* (0.228)
ν	20.58	20.64	21.81	21	21.57
Residuals					
Mean	−0.009	−0.009	−0.009	−0.009	−0.009
S.D.	0.999	0.999	0.999	1.1	0.999
Skewness	−0.122*	−0.149*	−0.131*	−0.150*	−0.146*
Kurtosis	3.428*	3.477*	3.373*	3.467*	3.428*
$Jarque - Bera$	18.71*	24.39*	16.02*	23.74*	20.66*
$Q(20)$	27.30	28.42	26.26	28.69	27.74
$Q_2(20)$	13.32	15.46	12.24	15.71	13.91
$Q_{21}(20)$	23.92	22.83	24.19	22.31	23.25
Restrictions					
Positivity	*Yes*	*Always*	*Yes*	*Always*	*Yes*
σ_y^2	1.682	–	–	1.814	*Yes*
k_y	5.054	–	–	6.811	*Unknown*

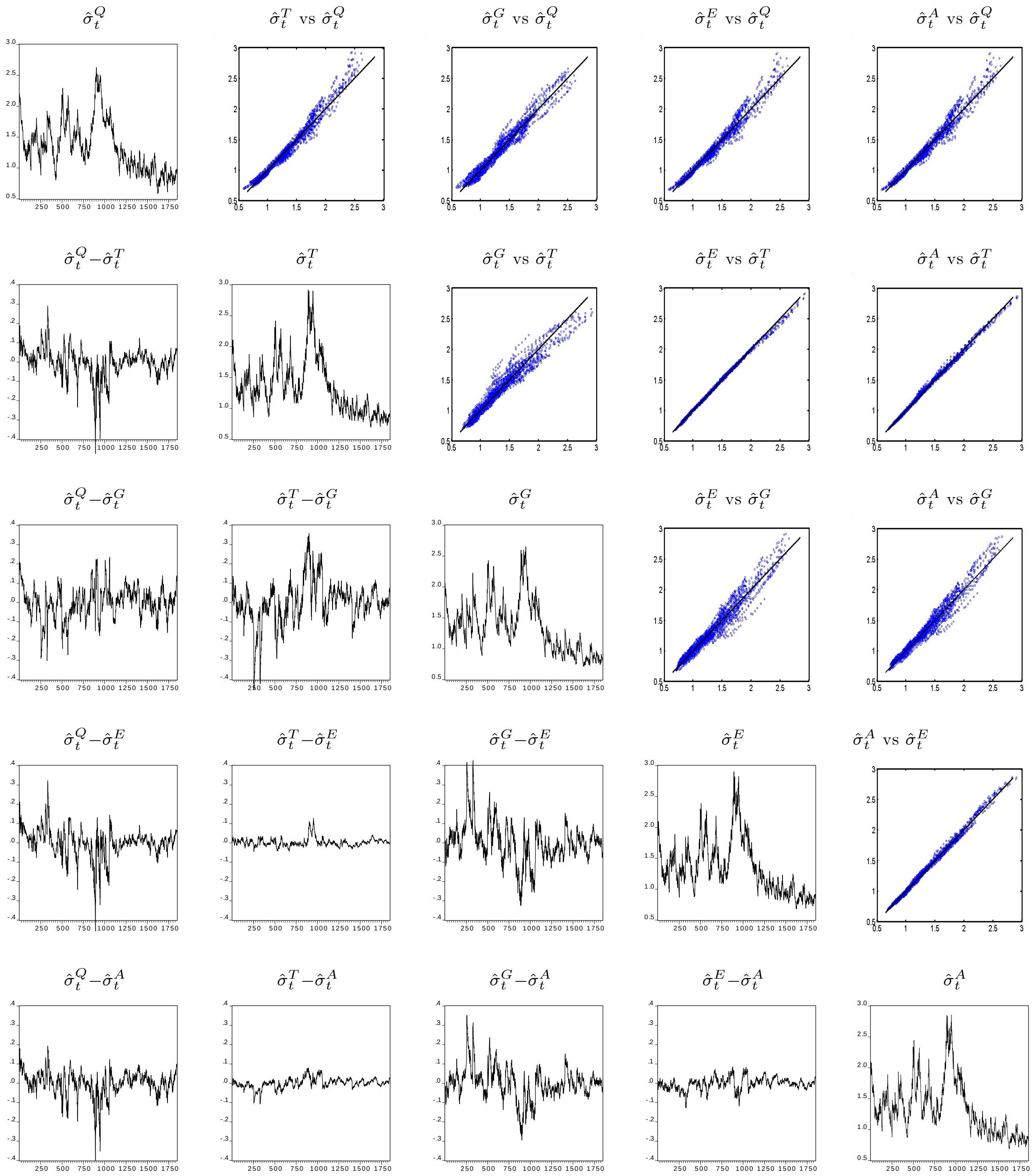
- Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Table 3.- Estimated models for daily AUD/USD exchange returns.

	QGARCH	TGARCH	GJR	EGARCH	APARCH
ω	0.001*	0.005*	0.001*	−0.008*	0.003*
	(0.000)	(0.001)	(0.000)	(0.002)	(0.002)
α	0.035*	0.044*	0.039*	0.085*	0.042*
	(0.005)	(0.006)	(0.006)	(0.012)	(0.006)
β	0.963*	0.959*	0.963*	0.992*	0.961*
	(0.005)	(0.006)	(0.006)	(0.002)	(0.006)
δ	−0.006	−0.007*	0.010	−0.013*	−0.132*
	(0.005)	(0.001)	(0.007)	(0.007)	(0.085)
λ					1.321*
					(0.310)
ν	7.18	7.28	7.17	7.25	7.26
Residuals					
Mean	0.003	0.002	0.003	0.002	0.002
S.D.	0.997	0.997	0.997	0.997	0.998
Skewness	0.299*	0.309*	0.318*	0.311*	0.314*
Kurtosis	4.256*	4.241*	4.300*	4.249*	4.237*
$Jarque-Bera$	403.96*	392.98*	437.49*	400.27*	412.85*
$Q(20)$	16.34	18.66	18.74	18.76	18.63
$Q_2(20)$	19.89	18.58	16.18	18.73	17.90
$Q_{21}(20)$	29.57	30.53	29.51	30.49	30.15
Restrictions					
Positivity	*Yes*	*Always*	*Yes*	*Always*	*Yes*
σ_y^2	0.584	0.471	–	0.525	*Yes*
k_y	–	6.237	–	6.331	*Unknown*

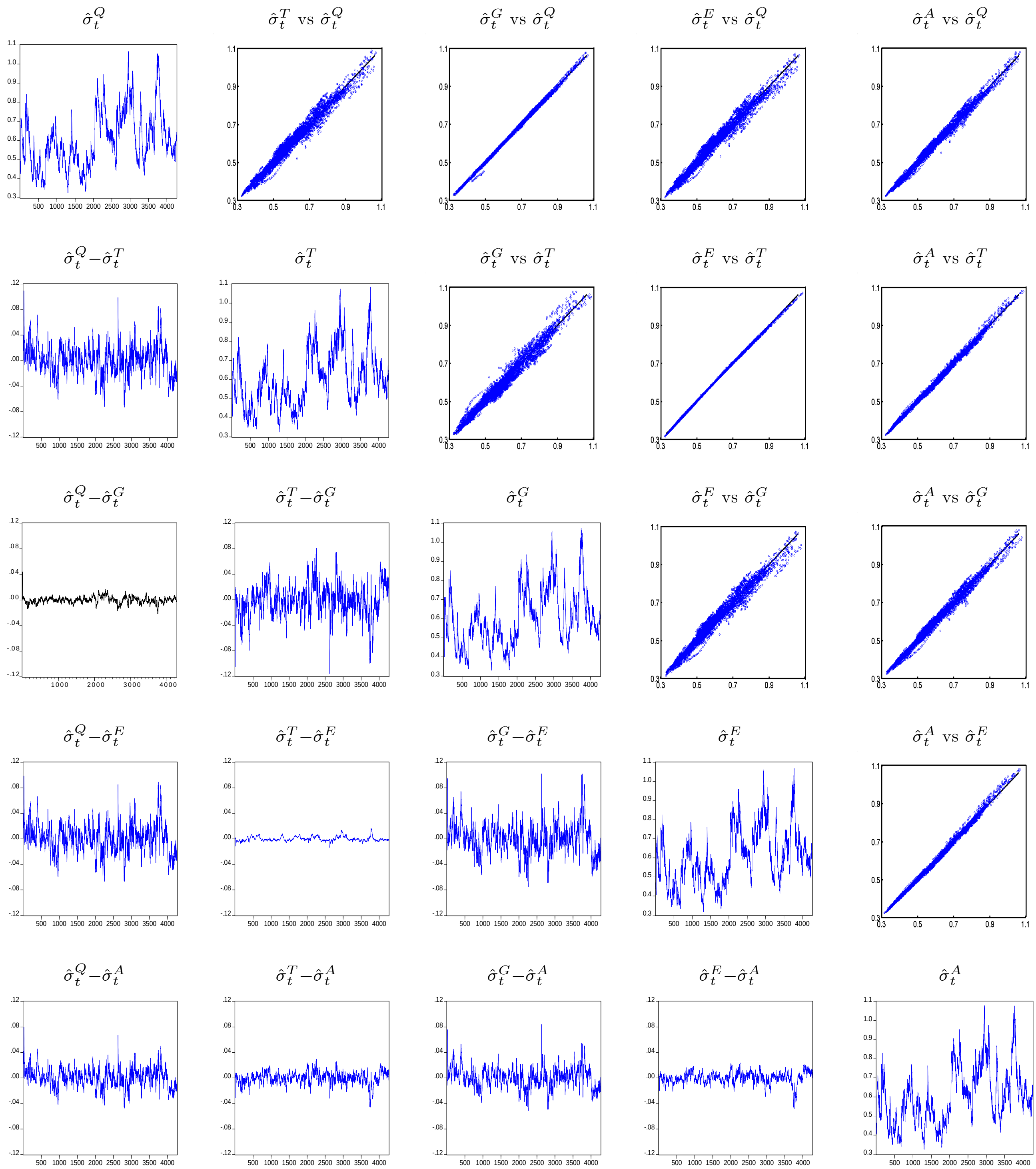
- Means that the moment is not defined.
* Significant at 5% level.
Asymptotic standard deviations in parenthesis.

Figure 10.- Estimated conditional standard deviations, differences between them and scatter plots between conditional standard deviations of the S&P500 returns.



$\hat{\sigma}_t^Q, \hat{\sigma}_t^T, \hat{\sigma}_t^G, \hat{\sigma}_t^E \hat{\sigma}_t^A$ are the conditional standard deviations estimated by the QGARCH, TGARCH GJR, EGARCH and APARCH, respectively.

Figure 11.- Estimated conditional standard deviations, differences between them and scatter plots between conditional standard deviations of the AUD/USD exchange.



$\hat{\sigma}_t^Q, \hat{\sigma}_t^T, \hat{\sigma}_t^G, \hat{\sigma}_t^E \hat{\sigma}_t^A$ are the conditional standard deviations estimated by the QGARCH, TGARCH GJR, EGARCH and APARCH, respectively.