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Markov Perfect Nash Equilibrium in Stochastic Differential Games as Solution of a Generalized Euler Equations System[1]

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Abstract

This paper gives a new method to characterize Markov Perfect Nash Equilibrium in stochastic differential games by means of a set of Generalized Euler Equations. Necessary and sufficient conditions are given.

Keywords: Stochastic differential games, dynamic programming, Hamilton–Jacobi–Bellman equation, semilinear parabolic equation, stochastic productive assets.

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1 Introduction

The paper develops a new approach that characterizes directly Markov Perfect Nash Equilibrium (MPNE) in stochastic differential games as solution of a system of semilinear partial differential equations (PDEs), that can be seen as a set of Generalized Euler Equations (GEEs). That is, we obtain the counterpart for the continuous time case of the Euler equations for discrete dynamic programming. This system is obtained from the optimality conditions of the stochastic maximum principle.

The main antecedents of this aproach date back to the paper Bourdache–Siguerdidjane and Fliess (1987) for deterministic control problems, Rincón–Zapatero, Martínez and Martín–Herrán (1998) and Rincón–Zapatero (2004), in deterministic differential games, and Josa–Fombellida and Rincón–Zapatero (2007), in stochastic control. The aim of this paper is to extend this methodology to an important class of stochastic differential games, which presents the following characteristics: (i) the diffusion coefficient in the state variable process is independent of the strategies of the players, (ii) the MPNE is interior and smooth, and (iii) the number of strategies of each player is greater than or equal to the number of state variables. These characteristics appear in some important models arising in economics.

The paper is organized as follows. In Section 2 we present the differential game and the first definitions, hypotheses and notations. In Section 3 the GEE system characterizing MPNE is presented as a new necessary condition, and some interesting consequences are deduced, as the possibility of jumps in the first order derivatives of the MPNE. In Section 4 we establish sufficient conditions for MPNE in terms of the GEE system and a concavity property of the Hamiltonians of the players. Some applications come in Section 5: (i) determination of robust equilibria, (ii) identification of games with constant MPNE, and (iii) some remarks on a game of competition for consumption of a productive asset. Concluding remarks are stated in Section 6.

2 The game formulation

In this section we establish the formulation for the stochastic differential games considered along the paper. We shall use the following notation. The partial derivatives are indicated by subscripts and ∂_x stands for *total derivation*; the partial derivative of a scalar function with respect to a vector is a column vector; given a real vector function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a vector $z \in \mathbb{R}^n$, g_z is defined as the matrix $(\partial g^i/\partial z^j)_{i,j}$; for a matrix A, $A^{(i)}$ denotes the ith column and A^{ij} denotes the (i,j) element; vectors $v \in \mathbb{R}^n$ are column vectors and v^i is the ith component; $^\top$ denotes transposition; finally, for $z \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $j = 1, 2, \ldots, n$, we denote $(z|x_{-j}) = (x^1, \ldots, x^{j-1}, z, x^{j+1}, \ldots, x^n)$.

We consider an N–person differential game over a fixed and bounded time interval $[0, T]$ with $0 < T \leq \infty$. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space. Assume that on this space a d–dimensional Brownian motion $\{w(t), \mathcal{F}_t\}_{t\in[0,T]}$ is defined with $\{\mathcal{F}_t\}_{t\in[0,T]}$ being the Brownian filtration. Let E denote expectation under the probability measure P. We also consider the function space $L^2_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ of all processes $X(\cdot)$ with values in $\mathbb{R}^n$ adapted to filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathrm{E} \int_0^T \|X(t)\|^2 \, dt < \infty$.

The state space is $\mathbb{R}^n$ and the set of admissible profiles of the players is some subset $U = U^1 \times U^2 \times \cdots \times U^N$, where $U^i \subseteq \mathbb{R}^{m_i}$, with[4] $m_i = n$, for all $i = 1, \ldots, N$. *A U–valued process of strategic profiles* $\{(u(s); \mathcal{F}_s) = ((u^1(s), u^2(s), \ldots, u^N(s)); \mathcal{F}_s)\}$ defined on $[t, T] \times \Omega$ is an $\mathcal{F}_s$–progressively measurable map $(r, \omega) \to u(r, \omega)$ from $[t, s] \times \Omega$ into U, that is, $u(t, \omega)$ is $\mathcal{B}_s \times \mathcal{F}_s$–measurable for each $s \in [t, T]$, where $\mathcal{B}_s$ denotes the Borel σ–field in $[t, s]$. For simplicity, we will denote $u(t)$ to $u(t, \omega)$.

The state process $\xi \in \mathbb{R}^n$ satisfies the system of controlled stochastic differential equations (SDEs)

$$d\xi(s) = f(s, \xi(s), u(s))\, ds + \sigma(s, \xi(s))\, dw(s), \quad t \leq s \leq T, \tag{1}$$

with initial condition $\xi(t) = x$, $t \in [0, T]$, $x \in \mathbb{R}^n$. Observe that the diffusion coefficient, σ, is independent of the control variable, u. The functions $f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ are both assumed to be of class $\mathcal{C}^2$ with respect to (x, u) and of class $\mathcal{C}^1$ with respect to t. Since our aim is to work with the MPNE concept, we will consider the game for every initial condition (t, x).

Definition 2.1 (Admissible strategies) *A strategic profile* $\{(u(t); \mathcal{F}_t)\}_{t\in[0,T]} = \{((u^1(t), u^2(t), \ldots, u^N(t)); \mathcal{F}_t)\}_{t\in[0,T]}$ *is called admissible if*

(i) *for every (t, x) the system of SDEs* (1) *with initial condition $\xi(t) = x$ admits a pathwise unique strong solution;*

(ii) *for each $i = 1, \ldots, N$, there exists some function $\phi^i : [0, T] \times \mathbb{R}^n \longrightarrow U^i$ of class $\mathcal{C}^{1,2}$ such that u^i is in relative feedback to ϕ^i, i.e. $u^i(s) = \phi^i(s, \xi(s))$ for every $s \in [0, T]$.*

Let $\mathcal{U}^i(t, x)$ denote the set of admissible strategies of player i and $\mathcal{U} = \mathcal{U}^1 \times \cdots \times \mathcal{U}^N$ the set of admissible strategies profiles, corresponding to the initial condition $(t, x) \in [0, T] \times \mathbb{R}^n$.

[4]The case $m_i > n$ could also be analyzed, by means of a reduction to the case $m_i = n$ as in Josa–Fombellida and Rincón–Zapatero (2007).

Part (ii) in Definition 2.1 means that players use Markov strategies. When ϕ^i is time independent, we will say that the strategy is a stationary Markovian strategy.

The instantaneous utility function of player i is denoted by L^i and his or her bequest function by S^i. Given initial conditions $(t, x) \in [0, T] \times \mathbb{R}^n$ and an admissible strategic profile u, the *payoff function* of each player (to be maximized) is given by

$$J^i(t, x; u) = \mathrm{E}_{tx} \left\{ \int_t^T e^{-\rho^i(s-t)} L^i(s, \xi(s), u(s))\, ds + e^{-\rho^i(T-t)} S^i(T, \xi(T)) \right\}, \qquad (2)$$

where E_{tx} denotes conditional expectation with respect to the initial condition (t, x). In the following, the subscript will be eliminated if there is no confusion. The functions $L^i : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}$ and $S^i : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, \ldots, N$, are both of class $\mathcal{C}^2$ with respect to (x, u) and of class $\mathcal{C}^1$ with respect to t. The constant $\rho^i \geq 0$ is the rate of discount. $J^i(t, x; u)$ denotes the utility obtained by player i when the game starts at (t, x) and the profile of strategies is u. Given that our aim is to solve the problem *for every* $(t, x) \in [0, T] \times \mathbb{R}^n$, $\mathcal{U}$ will often be written instead of $\mathcal{U}(t, x)$.

In the infinite horizon case the bequest functions S^i are null. In this case, if the problem is autonomous and the strategies are Markov stationary, the value function is independent of time, and the initial condition is simply x.

In a non-cooperative setting the aim of the players is to maximize their individual payoff J^i. Since this aspiration depends on the strategies selected by the other players also, it is generally impossible to attain[5]. An adequate concept of solution is Nash equilibrium, which prevents unilateral deviations of the players from its recommendation of play.

Definition 2.2 (MPNE) *An N–tuple of strategies $\widehat{\phi} \in \mathcal{U}$ is called a Markov perfect Nash equilibrium if for every $(t, x) \in [0, T] \times \mathbb{R}^n$, for every $\phi^i \in \mathcal{U}^i$*

$$J^i(t, x; (\phi^i | \widehat{\phi}_{-i})) \leq J^i(t, x; \widehat{\phi}),$$

for all $i = 1, \ldots, N$.

Remember that $(\phi^i | \widehat{\phi}_{-i})$ denotes $(\widehat{\phi}^1, \ldots, \widehat{\phi}^{i-1}, \phi^i, \widehat{\phi}^{i-1}, \ldots, \widehat{\phi}^N)$. Note that with a MPNE no player has incentives to deviate unilaterally from the equilibrium, whatever the initial condition (t, x) is.

The standard approach adopted in the literature to determine a MPNE is to solve the

[5]But in some models the MPNE is also Pareto optimal; see Martín–Herrán and Rincón–Zapatero (2005).

HJB system of PDEs

$$\rho^i V^i(s,x) = V^i_t(s,x) + \max_{u^i \in U^i} \left\{ H^i(s,x,(u^i|\phi_{-i}), V^i_x(s,x)) \right\} + \frac{1}{2} \operatorname{Tr} \left\{ (\sigma\sigma^\top V^i_{xx})(s,x) \right\}, \quad (3)$$

$$V^i(T,x) = S^i(T,x), \quad t \le s \le T, \quad (4)$$

where H^i is the (current–value) *deterministic Hamiltonian* function of the ith player, corresponding to the associated deterministic problem with $\sigma \equiv 0$,

$$H^i(s,x,u,p^i) = L^i(s,x,u) + (p^i)^\top f(s,x,u),$$

for all $i = 1,\dots,N$, for every initial condition $(t,x) \in [0,T] \times \mathbb{R}^n$. Denote by $\widehat{\phi} = (\widehat{\phi}^1,\dots,\widehat{\phi}^N)$, with $\widehat{\phi}^i(t,x) = \phi^i_0(t,x,V^1_x(t,x),\dots,V^N_x(t,x))$ an argument maximizing the right–hand side in (3), for all i. A classical result establishes that if $V = (V^1,\dots,V^N)$ is a solution to (3)–(4) of class $\mathcal{C}^{1,2}$, then $\widehat{\phi}$ is an MPNE of class $\mathcal{C}^{1,2}$; see Dockner *et al* (2000), Theorem 8.5.

Definition 2.3 (Value functions) *Let $\widehat{\phi}$ be a MPNE of the game. The value function V^i of the ith player is*

$$V^i(t,x) = \max_{\phi^i \in \mathcal{U}^i} \Big\{ J^i(t,x;(\phi^i|\widehat{\phi}_{-i})) : d\xi(s) = f(s,\xi(s),(\phi^i|\widehat{\phi}_{-i})(s,\xi(s)))\,ds + \sigma(s,\xi(s))\,dw(s),$$

$$\xi(t) = x, \quad \forall s \in [t,T] \Big\},$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, for all $i = 1,\dots,N$.

It is known that under suitable smoothness conditions, $V = (V^1,\dots,V^N)$ satisfies problem (3)–(4).

Now we consider the associated deterministic game to the initial stochastic game, that is, making $\sigma = 0$, so that the state equation is

$$\dot{\xi}(s) = f(s,\xi(s),u(s)), \quad t \le s \le T, \quad (5)$$

with initial condition $\xi(t) = x$, $t \in [0,T]$, $x \in \mathbb{R}^n$. The objective of the ith player is to maximize

$$J^i(t,x;u) = \int_t^T e^{-\rho^i(s-t)} L^i(s,\xi(s),u(s))\,ds + e^{-\rho^i(T-t)} S^i(T,\xi(T)), \quad i = 1,\dots,N, \quad (6)$$

once the remainder players have fixed their strategies u_{-i}.

Definition 2.4 (Robust MPNE) *A MPNE of the associated deterministic game* (5), (6) *is robust if it is a MPNE of the stochastic game* (1), (2).

It is not easy to obtain a condition to robust equilibrium directly from HJB system since the value functions of the respective problems, deterministic and stochastic are, in general, different.

3 Necessary conditions for MPNE

We deduce in this section a set of GEEs that a smooth interior MPNE must satisfy. This new system consists of PDEs of semilinear type, and constitutes an alternative to the classical HJB system.

Given the initial condition $(t, x) \in [0, T] \times \mathbb{R}^n$, let us suppose $\widehat{\phi} \in \mathcal{U}$ is a MPNE of the game. More specifically consider $(\widehat{\xi}, \widehat{\phi}) \in L^2_{\mathcal{F}}([0, T]; \mathbb{R}^n) \times \mathcal{U}[t, x]$, where $\widehat{\xi}$ is the optimal state process corresponding to $\widehat{\phi}$. Then for every $i = 1, \ldots, N$, $\widehat{\phi}^i \in \mathcal{U}^i$ is an optimal Markov control of problem

$$\max_{\phi^i \in \mathcal{U}^i} J^i(t, x; (\phi^i | \widehat{\phi}_{-i})), \tag{7}$$

subject to (1). We suppose that the data of the game are regular enough to apply the stochastic maximum principle; see Assumption A1 in Josa–Fombellida and Rincón–Zapatero (2007) or Yong and Zhou (1999), p. 114. Thus, there exist square integrable processes $p^i \in L^2_{\mathcal{F}}([0, T]; \mathbb{R}^n)$, $q^i \in \left(L^2_{\mathcal{F}}([0, T]; \mathbb{R}^n)\right)^d$ satisfying for $s \in [t, T]$ the first order adjoint equations

$$dp^i(s) = \left(\rho^i p^i(s) - \partial_x H^i(s, \widehat{\xi}(s), \widehat{\phi}(s, \widehat{\xi}(s)), p^i(s)) - \sum_{j=1}^{d} (\sigma^{(j)})_x(s, \widehat{\xi}(s))^\top (q^{(j)})^i(s)\right) ds + q^i(s)\, dw(s), \tag{8}$$

$$p^i(T) = S^i_x(T, \widehat{\xi}(T)), \tag{9}$$

for each $i = 1, \ldots, N$. Notice that we are considering current–value adjoint variables, and that they depend on (t, x): $p^i(s; t, x)$, $q^i(s; t, x)$ with $s \in [t, T]$. Most often we will suppress this dependence in the notation to facilitate the exposition. On the other hand, in (8) we have used the total derivation symbol, $\partial_x H^i$, instead of partial derivation. This is correct in our framework, since we are supposing that $\widehat{\phi}$ is smooth and the MPNE is interior. In these circumstances,

$$\partial_x H^i = H^i_x + \sum_{j \neq i}^{N} H^i_{u^j} \widehat{\phi}^j_x + H^i_{u^i} \widehat{\phi}^i_x$$

coincides with the usual term H^i_x of the one–player case, because $H^i_{u^i} = 0$ for interior MPNE by the maximum principle; see (11) below. In the game with N players, the term

$\sum_{j\neq i}^N H^i_{u^j}\widehat{\phi}^j_x$ must be added to H^i_x. Both possibilities are handled using $\partial_x H^i$ into the expression defining dp^i.

Definition 3.1 (Adjoint feedback) *A function* $\gamma^i : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *is called an adjoint feedback of the ith player if it express the adjoint process* p^i *in terms of the state variable* ξ, $p^i(s) = \gamma^i(s,\xi(s))$.

The following maximization condition holds for every[6] $s\in[t,T]$, P a.s.

$$\forall i = 1,\dots,N, \quad H^i(s,\widehat{\xi}(s),\widehat{\phi}(s,\widehat{\xi}(s)),p^i(s)) = \max_{u^i\in U^i} H^i(s,\widehat{\xi}(s),(u^i|\widehat{\phi}_{-i})(s,\widehat{\xi}(s)),p^i(s)). \tag{10}$$

Since we have imposed that the maximizing argument $\widehat{\phi}^i$ is interior to U^i, (10) implies

$$\forall i = 1,\dots,N, \quad H^i_{u^i}(s,\widehat{\xi}(s),\widehat{\phi}(s,\xi(s)),p^i(s)) = 0, \quad \forall s\in[t,T], \quad \text{P a.s.}. \tag{11}$$

Assuming that f_{u^i} is invertible for all $(t,x,u)\in[0,T]\times\mathbb{R}^n\times U$, it is possible to obtain the adjoint variable p^i as the unique solution of the above algebraic linear system, $L^i_{u^i}+f^\top_{u^i}p^i = 0$

$$\forall i = 1,\dots,N, \quad p^i = -f^{-\top}_{u^i}L^i_{u^i}.$$

Define for each player the *adjoint function* Γ^i on $[0,T]\times\mathbb{R}^n\times U$ by

$$\Gamma^i(t,x,u) := -f^{-\top}_{u^i}L^i_{u^i}(t,x,u). \tag{12}$$

Note that if γ^i is an adjoint feedback and $\widehat{\phi}$ is an optimal control then $\gamma^i(t,x) = \Gamma^i(t,x,\widehat{\phi}(t,x))$, for all $(t,x)\in[0,T]\times\mathbb{R}^n$.

Next result establishes a necessary condition for MPNE in terms of a new system of PDEs. As stated at the beginning of this section, the system can be seen as composed of GEEs which are the counterpart to the continuous case of those appearing in discrete–time dynamic programming. We state the theorem only for the finite horizon case, and afterwards we make some remarks concerning the infinite horizon case.

Theorem 3.1 (Necessary condition) *Suppose, for all* $i = 1,\dots,N$, *that* γ^i *is an adjoint feedback and* $\widehat{\phi}\in\mathcal{U}$ *is an interior MPNE, both continuous on* $[0,T]\times\mathbb{R}^n$ *and of class* $\mathcal{C}^{1,2}$ *on* $[0,T)\times\mathbb{R}^n$. *Then* $\widehat{\phi}$ *satisfies the semilinear system of PDEs of second order*

$$\begin{aligned}&\rho^i\Gamma^i(t,x,\phi(t,x))\\ &\quad = \partial_t\Gamma^i(t,x,\phi(t,x)) + \partial_x\left(\mathcal{H}^i(t,x,\phi(t,x)) + \frac{1}{2}\operatorname{Tr}\left\{\sigma(t,x)\sigma(t,x)^\top\partial_x\Gamma^i(t,x,\phi(t,x))\right\}\right),\end{aligned} \tag{13}$$

[6]That is, fixed $(s,\xi(s))$, the MPNE is a Nash equilibrium of the static game with payoff H^i for each player, $i = 1,\dots,N$.

where $\mathcal{H}^i(t,x,u) := H^i(t,x,u,\Gamma^i(t,x,u))$, *with terminal condition*

$$L^i_{u^i}(T,x,\phi(T,x)) + S^i_x(T,x)^\top f_{u^i}(T,x,\phi(T,x)) = 0, \tag{14}$$

for all $i = 1,\dots,N$.

Proof. As $\widehat{\phi} \in \mathcal{U}$ is a MPNE of the game, then $\widehat{\phi}^i \in \mathcal{U}^i$ is an interior optimal Markov control of problem (1), (7), for all $i = 1,\dots,N$, when the equilibrium strategies of the other players, $\widehat{\phi}_{-i}$, are fixed.

Consider player ith; by hypothesis γ^i is a function of class $\mathcal{C}^{1,2}$. Applying Itô's rule to $p^i(s) = \gamma^i(s,\widehat{\xi}(s))$, $t \le s \le T$, we obtain

$$dp^i(s) = \left(\gamma^i_s + \gamma^i_x f + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top \gamma^i_{xx}\}\right) ds + \gamma^i_x \sigma dw(s), \quad s \in [t,T], \tag{15}$$

where the arguments have been eliminated to simplify the notation. Thus, equating the diffusion parameters of (8) and (15) we obtain

$$q^i = \gamma^i_x \sigma; \tag{16}$$

and equating now the drift terms of (8) and (15),

$$\rho^i\gamma^i = \gamma^i_s + \gamma^i_x f + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top\gamma^i_{xx}\} + \partial_x L^i + \gamma^i\partial_x f + \sum_{j=1}^d (\sigma^{(j)})^\top_x (q^{(j)})^i. \tag{17}$$

Inserting (16) in (17), for all $(t,x) \in [0,T]\times\mathbb{R}^n$ we obtain

$$\rho^i\gamma^i = \gamma^i_s + \partial_x\left(L^i + (\gamma^i)^\top f + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top\gamma^i_x\}\right).$$

Now, taking limits $s \to t$ we obtain (13). Finally, the terminal condition (14) is obtained from the stochastic maximum principle, concretely using (9) and (12). □

Observe that the dependent variables (unknowns) in system (13) are $\phi^1,\dots,\phi^N$ and t,x are the independent variables. The system is formed by $n \times N$ PDEs of semilinear type, meaning that the second order derivatives appear linearly. Notice that the nonlinearity affect the first derivative in a very specific way, because they appear at the square, $(\phi^i_x)^2$. This can be seen taking the total derivatives; see Section 5.3 below for specific examples. In comparison, the HJB system (3) consists of N equations and it is also of semilinear type, but the non linearity with respect to ϕ^i_x can be much more general and not only of quadratic type.

Remark 3.1 (Infinite horizon). System (13) is also valid for the infinite horizon case, $T = \infty$, but the final condition (14) obviously does not apply now. As it is well known in the literature, this makes possible the existence of multiple smooth solutions and a tranversality condition must be used in order to isolate the correct solution. For autonomous problems and stationary Markov strategies, $\partial_t \Gamma(x, \phi(x)) = 0$, thus the GEE system reduces to: $\forall i = 1, \dots, N$,

$$\rho^i \Gamma^i(x, \phi(t,x)) = \partial_x \Big(\mathcal{H}^i(x, \phi(t,x)) + \frac{1}{2} \operatorname{Tr} \left\{ \sigma(x)\sigma(x)^\top \partial_x \Gamma^i(x, \phi(t,x)) \right\} \Big), \tag{18}$$

which in the scalar case, $n = 1$, is in fact a coupled system of second order ordinary differential equations.

The special structure of the system (13) allows us to consider solutions–in a generalized sense– with jumps in the first derivatives. Let us show the method in the scalar case, with infinite horizon, $T = \infty$, and on an autonomous problem. Suppose that $\phi = (\phi^1, \dots, \phi^N)$ is a classical solution in a punctured open interval around some $x = z$, that ϕ^i is continuous at z, and that both side limits $\phi^i_x(z_-) = \lim_{x \to z_-} \phi^i_x(x)$, $\phi^i_x(z_+) = \lim_{x \to z_+} \phi^i_x(x)$ exist for every $i = 1, \dots, N$, although possibly $\phi^j_x(z_-) \neq \phi^j_x(z_+)$ for some $j = 1, \dots, N$. That is, the strategy of some player may experience a jump in the first derivative at $x = z$. If (18) is integrated with respect to x in an interval $[z - \varepsilon, z - \varepsilon]$ with $\varepsilon > 0$, then using Barrow's rule and taking limits as $\varepsilon \to 0$, we get

$$\forall i = 1, \dots, N, \quad \sigma(z)\sigma(z)^\top \sum_{j=1}^{N} \Gamma^i_{u^j}(z, \phi(z)) \Big(\phi^j_x(z_+) - \phi^j_x(z_-) \Big) = 0. \tag{19}$$

In the degenerated case $\sigma(z)\sigma(z)^\top = 0$, this does not impose any condition in the jump, and makes clear why non–smooth solutions may appear when $\sigma(z)\sigma(z)^\top = 0$ at some z. Letting aside this situation, (19) imposes a necessary condition for jumps in the derivatives of a MPNE. Thus, we have proved the following result

Proposition 3.1 (Jumps). In the autonomous scalar game with infinite horizon, suppose that ϕ is a continuous MPNE profile, which is smooth except at some z, and such that $\phi^i_x(z_\pm)$ exist. Suppose also that $\sigma(z)\sigma(z)^\top \neq 0$. Then, ϕ_x satisfies the jump condition

$$\forall i = 1, \dots, N, \quad \sum_{j=1}^{N} \Gamma^i_{u^j}(z, \phi^1(z), \dots, \phi^N(z)) \Big(\phi^j_x(z_+) - \phi^j_x(z_-) \Big) = 0.$$

Consider now the frequent case in applications where $\Gamma^i_{u^j} = 0$ for $j \neq i$ and $\Gamma^i_{u^i} \neq 0$, see Section 5.3 below. Then, it is obvious that the MPNE cannot have jumps in the first order derivatives.

4 Sufficient conditions and connection with the HJB system

In this section we show that a solution $\widehat{\phi}$ of class $\mathcal{C}^{1,2}$ of the semilinear system (13), (14), maximizing the deterministic hamiltonian for all (t,x), is a MPNE of the differential game.

For each player i and for an admissible strategy $\widehat{\phi}$ solving the GEE system we define the adjoint feedback of the ith player $\gamma^i(t,x) = \Gamma^i(t,x,\widehat{\phi}(t,x)) = -f_{u^i}^{-\top} L^i_{u^i}(t,x,\widehat{\phi}(t,x))$, for all $(t,x) \in [0,T] \times \mathbb{R}^n$. We also establish the connection existing between the GEE system (13), (14), and the HJB system (3), (4).

The following result establishes that, for each ith player, the adjoint process $p^i(s) = \gamma^i(s,\xi(s))$ is the gradient with respect to x of the objective functional, $J^i_x(s,\xi(s);\widehat{\phi})$. This result, of independent interest, is a previous step in the formulation of the sufficiency theorem that will be stated later. For the result it is needed to impose a technical condition, consisting in that the processes $(B^i)^{jk}$, for $i = 1,\dots,N$, $j,k = 1,\dots,n$, defined in the proof below, is square integrable. We give the proof for the finite horizon case, and then a remark concerning the infinite horizon case.

Proposition 4.1 (Shadow prices) Let $\widehat{\phi} \in \mathcal{U}$ be a solution of class $\mathcal{C}^{1,2}$ of (13), (14). Then J^i is twice differentiable with respect to x and the derivatives are

$$J^i_x(t,x;\widehat{\phi}) = \gamma^i(t,x) = p^i(t),$$

$$J^i_{xx}(t,x;\widehat{\phi})\sigma(t,x) = q^i(t),$$

for every $(t,x) \in [0,T] \times \mathbb{R}^n$ and for every $i = 1,\dots,N$.

Proof. Consider player ith, and a smooth solution $\widehat{\phi}$ of (13), (14). By the hypotheses γ^i is a function of class $\mathcal{C}^{1,2}$, thus we can use the Itô's formula with process $p^i(s) = \gamma^i(s,\xi(s))$, $t \le s \le T$, and obtain (15). From (16), $(q^i)^{jk} = \sum_{l=1}^n (\gamma^i)^j_{x^l} \sigma^{lk}$.

Denote by ξ_x the $n \times n$ matrix of partial derivatives, $\xi^j_{x^k}$, of ξ^j with respect to x^k. This process ξ_x satisfies the linear system of stochastic differential equations

$$d\xi_x(s) = f_y \xi_x \, ds + \sigma_y \xi_x dw(s), \quad t \le s \le T,$$

with $\xi_x(t) = I_{n\times n}$, see Fleming and Rishel (1975), p. 174. Thus every $\xi^j_{x^r}$ satisfies the linear system of stochastic differential equations

$$d\xi^j_{x^r} = \sum_{l=1}^n f^j_{x^l} \xi^l_{x^r} \, ds + \sum_{k=1}^d \sum_{l=1}^n \sigma^{jk}_{x^l} \xi^l_{x^r} dw^k(s), \tag{20}$$

with $\xi^j_{x^r}(t) = \delta_{rj}$, with δ_{rj} denoting Kronecker's delta.

Applying the Itô's formula, the product $\xi^j_{x^r}(p^i)^j$ satisfies the following SDE

$$d(\xi^j_{x^r}(p^i)^j) = (p^i)^j d\xi^j_{x^r} + \xi^j_{x^r} d(p^i)^j + (A^i)^{rj}\, ds, \tag{21}$$

where $(A^i)^{rj} := \sum_{k=1}^d \sum_{l=1}^n \sigma^{jk}_{x^l} \xi^l_{x^r}(q^i)^{jk}$. Analogously the product $e^{-\rho^i(s-t)}\xi^j_{x^r}(p^i)^j$ satisfies the SDE

$$d(e^{-\rho^i(s-t)}\xi^j_{x^r}(p^i)^j) = e^{-\rho^i(s-t)}\left(-\rho^i \xi^j_{x^r}(p^i)^j + d(\xi^j_{x^r}(p^i)^j)\right). \tag{22}$$

Now by means of a simple calculation using (15), (20), (21), (22) and (13) the following equality holds

$$\sum_{j=1}^n d(e^{-\rho^i(s-t)}\xi^j_{x^r}(p^i)^j) = e^{-\rho^i(s-t)}\Big(- \sum_{j=1}^n L_{x^j}\xi^j_{x^r}\, ds + \sum_{k=1}^d (B^i)^{rk}\, dw^k(s)\Big), \tag{23}$$

with $(B^i)^{rk} := \sum_{j=1}^n \sum_{l=1}^n (\sigma^{jk}_{x^l}\xi^j_{x^r}(p^i)^j + (q^i)^{jk}\xi^j_{x^r})$. Taking conditional expectation with respect to the initial condition (t, x) in (23) we obtain

$$\begin{aligned}
&\sum_{j=1}^n \mathrm{E}_{tx}\left\{e^{-\rho^i(T-t)}\xi^j_{x^r}(T)(p^i)^j(T)\right\} \\
&\quad = \sum_{j=1}^n \mathrm{E}_{tx}\left\{\xi^j_{x^r}(t)(p^i)^j(t)\right\} \\
&\quad\quad - \mathrm{E}_{tx}\left\{\int_t^T \sum_{j=1}^n e^{-\rho^i(s-t)} L^i_{x^j}\big(s, \xi(s), \widehat{\phi}(s, \xi(s))\big)\xi^j_{x^r}(s)\, ds\right\},
\end{aligned} \tag{24}$$

since we are supposing that $(B^i)^{rk}$ are all square integrable.

Obviously, $\sum_{j=1}^n \mathrm{E}_{tx}\left\{\xi^j_{x^r}(t)(p^i)^j(t)\right\} = (p^i)^r(t)$ and because $\widehat{\phi}$ satisfies the final condition (14), the equality

$$\sum_{j=1}^n \mathrm{E}_{tx}\left\{e^{-\rho^i(T-t)}\xi^j_{x^r}(T)(p^i)^j(T)\right\} = \sum_{j=1}^n \mathrm{E}_{tx}\left\{e^{-\rho^i(T-t)} S^i_{x^j}(T, \xi(T))\xi^j_{x^r}(T)\right\}$$

holds. The following step is to interchange the order of integration and derivation and also the expectation operator in (24) to obtain

$$\begin{aligned}
(\gamma^i)^r(t, x) &= (\gamma^i)^r(t, \xi(t)) \\
&= (p^i)^r(t) \\
&= \frac{\partial}{\partial x^r}\mathrm{E}_{tx}\left\{\int_t^T e^{-\rho^i(s-t)} L^i\big(s, \xi(s), \widehat{\phi}(s, \xi(s))\big)\, ds + e^{-\rho^i(T-t)} S^i(T, \xi(T))\right\} \\
&= J^i_{x^r}(t, x; \widehat{\phi}),
\end{aligned}$$

for all $r = 1, \dots, n$.

Finally, note that $J^i_{xx}(t,x;\widehat{\phi})\sigma(t,x) = \gamma^i_x(t,x)\,\sigma(t,x) = q^i(t)$. $\square$

Remark 4.1 (Infinite horizon) In the case $T = \infty$, further conditions are needed in order to assure the identities in Proposition 4.1. These conditions can be obtained by the same method as in the proof, letting $T \to \infty$, and having into account that no final condition exists. We get now from (24)

$$(p^i)^r(t) = \frac{\partial}{\partial x^r}\mathrm{E}_{tx}\left\{\int_t^T e^{-\rho^i(s-t)}L^i\big(s,\xi(s),\widehat{\phi}(s,\xi(s))\big)\,ds\right\}+\sum_{j=1}^n \mathrm{E}_{tx}\left\{e^{-\rho^i(T-t)}\xi^j_{x^r}(T)(p^i)^j(T)\right\}.$$

Thus, to show $(p^i)^r(t) = J^i_{x^r}(t,x;\widehat{\phi})$ in the infinite horizon case, we need to impose convergence of the first summand as $T \to \infty$, as well as the transversality condition

$$\forall i = 1,\dots,N, \forall j = 1,\dots,n, \quad \lim_{T\to\infty} \mathrm{E}_{tx}\big\{e^{-\rho^i(T-t)}(p^i)^j(T)\,\xi^j_{x^r}(T)\big\} = 0.$$

Once we have the identification of the adjoint variables with the gradient of the objective functional, we have that γ^i is the gradient with respect to the variable x of the function $J^i(t,x;\widehat{\phi})$, which is of class $\mathcal{C}^3$. In consequence, $(\gamma^i)^k_{x^l} = (\gamma^i)^l_{x^k}$ is satisfied for every $k,l = 1,\dots,n$, because the crossed second order partial derivatives of the function J^i coincide.

Now we are in position to establish the following sufficient condition for optimality. For $u \in U$ and $i \in \{1,\dots,N\}$, we denote $h^{i,u}(t,x) = \mathcal{H}^i(t,x,u)$, for $(t,x) \in [0,T]\times\mathbb{R}^n$. As in previous results, we state the result only for the finite horizon case.

Theorem 4.1 (Value functions and sufficient conditions for MPNE) *Let $\widehat{\phi}$ be an admissible control satisfying* (13), (14), *and such that*

$$\forall i = 1,\dots,N, \forall (t,x) \in [0,T]\times\mathbb{R}^n, \forall \phi^i \in \mathcal{U}^i, \qquad h^{i,\widehat{\phi}}(t,x) \geq h^{i,(\phi^i|\widehat{\phi}_{-i}))}(t,x). \tag{25}$$

Then, for each $i = 1,\dots,N$, and for any arbitrary constant α, the functions W^i given by

$$W^i(t,x) = \int_\alpha^{x^j}(\gamma^i)^j(t,(z|x_{-j}))dz + (g^i)^j(t,(\alpha|x_{-j})), \quad (t,x)\in[0,T]\times\mathbb{R}^n, \tag{26}$$

where, for $j = 1,2,\dots,n$, the functions $(g^i)^j$ satisfy

$$\begin{aligned}(g^i)^j(t,(\alpha|x_{-j})) &= e^{-\rho^i(T-t)}S^i(T,(\alpha|x_{-j}))\\ &\quad + \int_t^T e^{-\rho^i(s-t)}\left(h^i(s,(\alpha|x_{-j})) + \frac{1}{2}\,\mathrm{Tr}\left\{(\sigma\sigma^\top\gamma^i_x)(s,(\alpha|x_{-j}))\right\}\right)ds,\end{aligned} \tag{27}$$

are solutions of class $\mathcal{C}^{1,3}$ of the HJB system and satisfies the final condition $W^i(T,x) = S(T,x)$. Moreover, if γ^i is polynomially bounded, then $W^i = V^i$ is the value function of the ith player, and $\widehat{\phi}$ is a MPNE.

Proof. It is obvious that W^i, defined in (26), is a function of class $\mathcal{C}^{1,3}$ and it is unambiguously defined, because the integrability condition $(\gamma^i)^k_{x^l} = (\gamma^i)^l_{x^k}$ holds for all $k, l = 1, \ldots, n$, as was stated immediately after Proposition 4.1. Our purpose is to prove that W^i satisfies the HJB equation (3) and final condition (4). Let us check that $W^i_{x^k}(t,x) = (\gamma^i)^k(t,x)$ for $k = 1, \ldots, n$ whichever index j is taken in (26). In the case where $k = j$ this is obvious. For $k \neq j$

$$
\begin{aligned}
W^i_{x^k}(t,x) &= \int_\alpha^{x^j} (\gamma^i)^j_{x^k}(t,(z|x_{-j}))\,dz + (g^i)^j_{x^k}(t,(\alpha|x_{-j})) \\
&= \int_\alpha^{x^j} (\gamma^i)^k_{x^j}(t,(z|x_{-j}))\,dz + (g^i)^j_{x^k}(t,(\alpha|x_{-j})) \\
&= (\gamma^i)^k(t,x) - (\gamma^i)^k(t,(\alpha|x_{-j})) + (g^i)^j_{x^k}(t,(\alpha|x_{-j})) = (\gamma^i)^k(t,x).
\end{aligned}
$$

The latter equality comes from $(g^i)^j_{x^k}(t,(\alpha|x_{-j})) = (\gamma^i)^k(t,(\alpha|x_{-j}))$ for $k \neq j$. Let us prove this claim. Considering the expression given in (27) and deriving partially with respect to x^k we have

$$
\begin{aligned}
(g^i)^j_{x^k}(t,(\alpha|x_{-j})) = {} & e^{-\rho^i(T-t)} S^i_{x^k}(T,(\alpha|x_{-j})) \\
& + \int_t^T e^{-\rho^i(s-t)} \Big(h^i(s,(\alpha|x_{-j})) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top \gamma^i_x)(s,(\alpha|x_{-j}))\} \Big)_{x^k} ds \\
= {} & e^{-\rho^i(T-t)} S^i_{x^k}(T,(\alpha|x_{-j})) + \int_t^T e^{-\rho^i(s-t)} \Big(-(\gamma^i)^k_t + \rho^i(\gamma^i)^k \Big)(s,(\alpha|x_{-j}))\,ds \\
= {} & e^{-\rho^i(T-t)} S^i_{x^k}(T,(\alpha|x_{-j})) - e^{-\rho^i(s-t)}(\gamma^i)^k(s,(\alpha|x_{-j}))\Big]_t^T \\
= {} & (\gamma^i)^k(t,(\alpha|x_{-j})).
\end{aligned}
$$

The second equality is due to (13) and the last because $S^i_{x^k}$ coincides with $(\gamma^i)^k$ at the final time T, as it is established by the maximum principle. Therefore, the function W^i defined in (26) satisfies $W^i_x = \gamma^i$ and then $W^i_{xx} = \gamma^i_x$.

Integrating with respect to x^j in (13) and exchanging the order of integration and derivation, we have (in terms of $\gamma^i = \Gamma^i(\cdot,\cdot,\widehat{\phi})$):

$$-\rho^i \int_\alpha^{x^j} (\gamma^i)^j(t,(z|x_{-j}))\,dz + \frac{\partial}{\partial t}\int_\alpha^{x^j} (\gamma^i)^j(t,(z|x_{-j}))\,dz + \mathcal{H}^i(t,x,\widehat{\phi}(t,x))$$
$$- \; h^i(t,(\alpha|x_{-j})) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top \gamma^i_x)(t,x)\} - \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top\gamma^i_x)(t,(\alpha|x_{-j}) = 0$$

for $j = 1,\ldots,n$. Therefore,

$$-\rho^i \int_\alpha^{x^j} (\gamma^i)^j(t,(z|x_{-j}))\,dz + W^i_t(t,x) - (g^i)^j_t(t,(\alpha|x_{-j})) + \mathcal{H}^i(t,x,\widehat{\phi}(t,x))$$
$$- \; h^i(t,(\alpha|x_{-j})) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top W^i_{xx})(t,x)\} - \frac{1}{2}\operatorname{Tr}\left\{(\sigma\sigma^\top\gamma^i_x)(s,(\alpha|x_{-j})\right\} = 0, \qquad (28)$$

because (26) implies $\frac{\partial}{\partial t}\int_\alpha^{x^j}(\gamma^i)^j\,dz = W^i_t - (g^i)^j_t$. It is immediate then that (28) reduces to:

$$-\rho^i W^i(t,x) + W^i_t(t,x) + L^i(t,x,\widehat{\phi}(t,x))$$
$$+ W^i_x(t,x)^\top f(t,x,\widehat{\phi}(t,x)) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top W^i_{xx})\}(t,x) = 0, \qquad (29)$$

because $(g^i)^j_t = \rho^i (g^i)^j - (h^i + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top\gamma^i_x\})$. On the other hand, (25) allows us to deduce

$$L^i(t,x,\widehat{\phi}(t,x)) + W^i_x(t,x)^\top f(t,x,\widehat{\phi}(t,x)) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top W^i_{xx})(t,x)\}$$
$$\geq L^i(t,x,(u^i|\widehat{\phi}_{-i})) + W^i_x(t,x)^\top f(t,x,(u^i|\widehat{\phi}_{-i})) + \frac{1}{2}\operatorname{Tr}\{(\sigma\sigma^\top W^i_{xx})(t,x)\}, \qquad (30)$$

for all $u^i \in U^i$. Thus, (29) and (30) imply (3). The final condition, $W^i(T,x) = S^i(T,x)$, is satisfied by definition of the function $(g^i)^j$.

Finally, the independence of W^i with respect to the constant α is deduced by satisfying that the derivative of W^i with respect to α is zero,

$$W^i_\alpha(t,x) = -\,(\gamma^i)^j(t,(\alpha|x_{-j})) + e^{-\rho^i(T-t)} S^i_{x^j}(T,(\alpha|x_{-j})) + \int_t^T e^{-\rho^i(s-t)} h^i_{x^j}(s,(\alpha|x_{-j}))\,ds$$
$$+ \frac{1}{2}\int_t^T e^{-\rho^i(s-t)}\left(\operatorname{Tr}\{(\sigma\sigma^\top\gamma^i_x)(s,(\alpha|x_{-j})\}\right)_{x^j} ds$$
$$= e^{-\rho^i(s-t)}(\gamma^i)^j(s,(\alpha|x_{-j}))\Big]_t^T + \int_t^T e^{-\rho^i(s-t)}\partial_{x^j}\left(h^i + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top\gamma^i_x\}\right)(s,(\alpha|x_{-j}))\,ds$$
$$= \int_t^T e^{-\rho^i(s-t)}\left(-\rho^i(\gamma^i)^j + (\gamma^i)^j_s + \partial_{x^j}\left(h^i + \frac{1}{2}\operatorname{Tr}\{\sigma\sigma^\top\gamma^i_x\}\right)\right)(s,(\alpha|x_{-j}))\,ds = 0,$$

where the second equality holds because $e^{-\rho^i(s-t)}(\gamma^i)^j\Big]_t^T = \int_t^T \partial_s\big(e^{-\rho^i(s-t)}(\gamma^i)^j\big)ds$ and the last equality is implied by (13).

To continue with the proof, if γ^i is polynomially bounded, then W^i is also, hence to get that W^i is the value function of player i and $\widehat{\phi}$ is indeed a MPNE; applying a verification theorem of Başar and Olsder (1999), Theorem 6.27, or Dockner *et al* (2000), Theorem 8.5. □

Remark 4.2 (Concavity condition) Condition (25) is satisfied when $\widehat{\phi}$ is interior to the control set U and the Hamiltonian function of the ith player is concave with respect to u^i, for every t, x, p^i. To see this, note that $H^i_{u^i}(t, x, \widehat{\phi}, \Gamma^i(t, x, \widehat{\phi})) = 0$ is trivially fulfilled by the definition of Γ^i, hence $\widehat{\phi}^i$ is a critical point of the concave function $u^i \mapsto H^i(\cdot, \cdot, (u^i|\widehat{\phi}_{-i}), \cdot)$, so $\widehat{\phi}^i$ is a global maximum of H^i.

Remark 4.3 (Infinite horizon) Theorem 4.1 can be extended to the infinite–horizon case, $T = \infty$, adding the transversality condition

$$\forall i = 1, \dots, N, \quad \limsup_{T\to\infty} \mathrm{E}_{tx} \left\{ e^{-\rho^i(T-t)} V^i(T, \xi^{(\phi^i|\widehat{\phi}_{-i})}(T)) \right\} = 0,$$

for all $\phi^i \in \mathcal{U}^i$. This follows from e.g. Dockner et al. (2000), Theorem 8.5.

In the autonomous case with stationary Markov strategies, the value functions are given by

$$\forall i = 1, \dots, N, \quad V^i(x) = \int_\alpha^{x^j} (\gamma^i)^j(z|x_{-j})\, dz + (g^i)^j(\alpha|x_{-j}), \tag{31}$$

for $j = 1, \dots, n$, where the functions $(g^i)^j$ satisfy (with $\rho^i > 0$):

$$(g^i)^j(\alpha|x_{-j}) = \frac{1}{\rho^i} \left(h^i + \frac{1}{2} \mathrm{Tr}\{\sigma\sigma^\top \gamma^i_x\} \right) (\alpha|x_{-j}). \tag{32}$$

As in the finite horizon case, it is easy to check that the expression for V^i is well defined and is independent on the constant α.

Remark 4.4 (Particular cases) In the deterministic case, $\sigma = 0$, the system of partial differential equations (13) is of first order. Clearly, the results remain valid now for $\mathcal{C}^1$ solutions. The system was obtained in Rincón–Zapatero, Martínez, and Martín–Herrán (1998) and further explored in Rincón–Zapatero (2004) for deterministic differential games. Previously, it was found in deterministic optimal control in Bourdache–Siguerdidjane and Fliess (1987). The stochastic control case, where $N = 1$, is studied in Josa–Fombellida and Rincón–Zapatero (2007).

5 Applications

5.1 Robust MPNE

In the following proposition we establish a necessary condition for a robust MPNE to exist. It can be seen as a certainty equivalence principle for the class of stochastic differential games studied in this paper. Notice that we do not confine ourselves to the familiar linear–quadratic case (see Başar and Olsder (1999) Corollary 6.12).

Proposition 5.1 (Robust equilibria) *Suppose $\widehat{\phi}$ is a robust MPNE of the deterministic game. Then there exist functions $A^i(t)$ such that*

$$\forall i = 1, \ldots, N, \quad \forall t \in [0, T), \quad \mathrm{Tr}\left\{\sigma\sigma^\top(t, x)\partial_x \Gamma^i(t, x, \widehat{\phi}(t, x))\right\} = A^i(t). \tag{33}$$

Moreover, when $T < \infty$ there exist constants $A^i(T)$ such that

$$\forall i = 1, \ldots, N, \quad \mathrm{Tr}\left\{\sigma\sigma^\top(T, x) S^i_{xx}(T, x)\right\} = A^i(T). \tag{34}$$

Proof. The proof is trivial based in the GEEs found for the stochastic and the deterministic game associated. Denote by $\Gamma^{i,\det}$ the adjoint variable of the player i of the associated deterministic game ($\sigma = 0$), for $i = 1, \ldots, N$. In both games $\Gamma^i = \Gamma^{i,\det} = -f_{u^i}^{-\top} L^i_{u^i}$, for all $i = 1, \ldots, N$; see (12). On the other hand, as $\widehat{\phi}$ is a MPNE of both the deterministic and the stochastic game, (13) implies (33). To finish the proof, notice that (34) is obtained from (33) by continuity of the involved functions and derivatives, and the final condition (14). □

The set of conditions (34), which apply when T is finite, place a strong link[7] between the bequest function S^i of each player and the diffusion matrix, σ, and may serve as a criterium for disregard robust equilibrium in games where (34) is not fulfilled. Obviously, this has no effect in the infinite horizon case.

Remark 5.1 In the one dimensional case, $n = 1$, (33) can be made more operative: If $\widehat{\phi}$ is a robust MPNE of the deterministic game, then there exist functions $A^i(t)$ such that

$$\forall i = 1, \ldots, N, \qquad L^i_{u^i}(t, x, \widehat{\phi}(t, x)) = -A^i(t)\Theta(t, x) f_{u^i}(t, x, \widehat{\phi}(t, x)), \tag{35}$$

where $\Theta(t, x)$ a primitive of $1/\sigma\sigma^\top(t, x)$ with respect to x. This fact will be used in Section 5.3.2 where we determine utility functions L^i leading to robust MPNE in a game non–cooperative game of productive assets.

[7] A notable exception occurs when S^i is linear in x for every i; see Sorger (1989) for an interesting application to advertising.

5.2 Games with constant MPNE

Suppose $N = 2$, $n = 1$, $U^i = \mathbb{R}$, constant discount $\rho^i > 0$ for $i = 1, 2$, infinite horizon ($T = \infty$) and that the following structure on the data of the game

$$\forall i = 1, 2, \quad L^i(t, x, u^i) = h_i(x)\ell_i(u^i),$$
$$f(x, u^1, u^2) = g_1(x)f_1(u^1) + g_2(x)f_2(u^2),$$

where all the functions involved are of class $\mathcal{C}^2$. Assume that f_1', f_2', g_1 and g_2 are different from zero and that the Hamiltonian of each player is concave with respect to own strategy u^i. Furthermore, suppose:

(i) there exists a constant A satisfying $g_2(x)/g_1(x) = A$;

(ii) there exists a unique constant profile (λ^1, λ^2) such that

$$\ell_1(\lambda^1)f_1'(\lambda^1) - \ell_1'(\lambda^1)(f_1(\lambda^1) + Af_2(\lambda^2)) = 0,$$
$$\ell_2(\lambda^2)f_2'(\lambda^2) - \ell_2'(\lambda^2)((1/A)f_1(\lambda^1) + f_2(\lambda^2)) = 0;$$

(iii) the functions $k_i = h_i/g_i$, $i = 1, 2$, satisfy the linear second order differential equations

$$\forall i = 1, 2, \quad -\rho^i k_i(x) + \frac{1}{2}(\sigma^2(x)k_i'(x))' = 0.$$

We have a lot of information about the problem and the question is whether this is enough to obtain a solution to the HJB equation for the value function of each player i, $i = 1, 2$, which is given by

$$-\rho^i V^i(x) + \max_{u^i \in \mathbb{R}} \Big\{ h_i(x)\ell_i(u^i) + (V^i)'(x)(g_1(x)f_1(u^1) + g_2(x)f_2(u^2)) \Big\} + \frac{\sigma^2(x)}{2}(V^i)''(x) = 0,$$

since we are considering stationary Markov strategies. At first sight is not apparent what the solution is; it is even difficult to get an idea of the explicit form of this non linear equation, given that the maximization cannot be done explicitly. Let us turn our attention to the system (13) for the MPNE which, in contradistinction, is always *explicit*. Recalling the notation introduced in Theorem 3.1 we have

$$\Gamma^i(x, u^1, u^2) = -\frac{\ell_i'(u^i)}{f_i'(u^i)}k_i(x), \quad i = 1, 2,$$
$$\mathcal{H}^1(x, u^1, u^2) = \frac{h_1(x)}{f_1'(u^1)}\Big(\ell_1(u^1)f_1'(u^1) - \ell_1'(u^1)\big(f_1(u^1) + Af_2(u^2)\big)\Big),$$
$$\mathcal{H}^2(x, u^1, u^2) = \frac{h_2(x)}{f_2'(u^2)}\Big(\ell_2(u^2)f_2'(u^2) - \ell_2'(u^2)\big((1/A)f_1(u^1) + f_2(u^2)\big)\Big)$$

and then $\mathcal{H}^i(x, \lambda^1, \lambda^2) = 0$, $i = 1, 2$, for all $x \in \mathbb{R}$, by (ii). If we look at equations (13), we see that the *constant profile* of strategies (λ^1, λ^2) is a solution if

$$\forall i = 1, 2, \quad \forall x \in \mathbb{R}, \quad \frac{\ell_i'(\lambda^i)}{f_i'(\lambda^i)}\Big(- \rho^i k_i(x) + \frac{1}{2}(\sigma^2(x) k_i'(x))'\Big) = 0,$$

and this is asserted in (iii). The solution of the HJB system is thus, according to (31) and (32),

$$V^i(x) = -\frac{1}{2\rho^i}\frac{\ell_i'(\lambda^i)}{f_i'(\lambda^i)}\sigma^2(x) k_i'(x),$$

$i = 1, 2$, by (iii). Further assumptions on the coefficient functions would imply that the constant profile (λ^1, λ^2) is a MPNE of the stochastic differential game and that V^i, $i = 1, 2$, are the value functions. Once this solution is known, it is obvious that $(u^1, u^2) = (\lambda^1, \lambda^2)$ is the maximizing argument in the HJB system, but without this knowledge, it is difficult to guess a tentative form for the solution. Thus the system of PDEs (13) has allowed to solve the game because it directly characterizes the MPNE.

A particular case appears when the game is symmetric, $\rho^1 = \rho^2 = \rho$, $h_1 = h_2 = h$, $\ell_1 = \ell_2 = \ell$, $f_1 = f_2 = f_0$, $g_1 = g_2 = g$. In this case we look for symmetric Nash equilibria, $u^1 = u^2 = \lambda$. The dimension of problem is reduced and the value functions are identical, $V^1 = V^2 = V$. Thus conditions (ii), (iii) are simply

$$\ell f_0' - 2\ell' f_0 = 0, \qquad -\rho k + \frac{1}{2}(\sigma^2 k')' = 0,$$

respectively, where $k_1 = k_2 = k$; and the value function is

$$V(x) = -\frac{1}{2\rho}\frac{\ell'(\lambda)}{f_0'(\lambda)}\sigma^2(x) k'(x).$$

These conditions can be extended for the symmetric MPNE in N–person games. In particular, (ii) becomes $\ell f_0' - N\ell' f_0 = 0$ and (iii) remains the same.

Example 5.1 As an specific example, consider the N–player symmetric game with $h(x) = x^\delta$, $\ell(u) = (au + b)^{1-1/a}$, $g(x) = x$, $f_0(u) = (\mu/N - u)$ and $\sigma(x) = \sigma x$, with $\delta > 1$, $a > 1$, $\mu \geq 0$ and $\sigma > 0$. The game is defined

$$\max_{u^i} E \int_0^\infty e^{-\rho t}\xi^\delta (au^i + b)^{1-1/a}\, dt$$

subject to $d\xi = \xi(s)\big(\mu - \sum_{i=1}^N u^i(s)\big)\, ds + \sigma\xi(s)\, dw(s)$. This formulation models the non–cooperative exploitation of a renewable resource x in free access by N identical agents that derive utility both from the consumption rate and the stock level. The dynamics shows a technology that makes costly to obtain the resource as it becomes scarce. We are supposing that the recruitment function is linear.

Notice that (i) defines $A = 1$. If the constants ρ, σ and δ are linked by the relation $\rho = (1/2)(\delta - 1)\delta\sigma^2$, then (iii) holds. Imposing (ii), we find

$$\lambda = \frac{b - (1-a)\mu}{a(N-1) - N},$$

which is valid for $a(N-1)-N \neq 0$, and is non–negative for suitable values of the parameters involved. The value function is $V(x) = (1/\delta)(a\lambda + b)^{1-1/a}x^{\delta}$.

5.3 Competition for consumption of a productive asset

Consider N agents in an economy that choose consumption in an optimal way, knowing that the productive asset is stochastic. Given the non–stationary Markov strategy c_{-i} of the remainder players, the ith agent chooses consumption c^i to maximize the expected total utility of consumption, discounted at rate $\rho^i > 0$ and over a fixed time horizon $[0, T]$,

$$J^i(t, x; (c^1, \dots, c^N) = E_{tx} \left\{ \int_t^T e^{-\rho^i(s-t)} L^i(c^i(s))\, ds + e^{-\rho^i(T-t)} S^i(\xi(T)) \right\}, \qquad (36)$$

subject to

$$d\xi(s) = \left(F(\xi(s)) - \sum_{i=1}^N c^i(s) \right) ds + \sigma(\xi(s))dw(s), \qquad \xi(t) = x. \qquad (37)$$

That is, the stock $\xi(s)$ of the asset is consumed at rate $c^1(s)+\cdots+c^N(s)$. The drift includes also the production/recruitment function $F(x)$ and the diffusion is given by function $\sigma(x)$. We assume that there is only one source of uncertainty, thus $d = 1$. Given an initial stock x, the asset process obeys the SDE above. The class of admissible strategies is given as in Definition 2.1, but incorporating the obvious condition $c^i \geq 0$ for each player i. We suppose that both the instantaneous utility L^i and bequest function S^i are smooth enough for the computations below; moreover, L^i is supposed to be strictly concave.

It is easy to obtain the N GEEs for MPNE in this game (36), (37) from (13). All what is needed is Hamiltonian and the adjoint function of each player.

$$\text{Hamiltonian: } H^i(x, c^1, \dots, c^N, p^i) = L^i(c^i) + \left(F(x) - \sum_{i=1}^N c^i(s) \right) p^i,$$

$$\text{Adjoint function: } \Gamma^i(x, c^1, \dots, c^N) = (L^i)'(c^i).$$

Then the GEE system (13) for this game is

$$\forall i=1,\dots,N,\quad -\rho^i (L^i)' + \partial_t (L^i)' + \partial_x \left(L^i + \left(F(x) - \sum_{j=1}^{N} c^j \right) (L^i)' + \frac{1}{2}\sigma^2(x)(L^i)'' c_x^i \right) = 0, \tag{38}$$

with final condition

$$c^i(T,x) \overset{\text{not.}}{=} \varphi^i(x) = \begin{cases} ((L^i)')^{-1}((S^i)'(x)), & \text{if } x>0; \\ 0, & \text{if } x=0. \end{cases} \tag{39}$$

In the infinite horizon case with stationary Markov strategies the system is also valid, and the terms $\partial_t (L^i)'$ can be deleted. It is obvious for this example that the comments placed just below Proposition 3.1 apply, since $(L^i)'' < 0$ and L^i is independent of the consumption of the remaining players. Thus, in this game there is no possibility of jumps in the first order derivatives of the equilibrium strategies.

After taking total derivatives, (38) takes the following form: $\forall i=1,\dots,N$

$$\begin{aligned} c_t^i + \left(F(x) - \sum_{j=1}^{N} c^j + \sigma'(x)\sigma(x) \right) c_x^i + \frac{1}{2}\sigma(x)^2 \mathcal{E}_2^i(c^i)(c_x^i)^2 + \frac{1}{2}\sigma(x)^2 c_{xx}^i \\ + \left(\rho^i - F'(x) + \sum_{j=1, j\neq i}^{N} c_x^j \right) \mathcal{E}_1^i(c^i) = 0, \end{aligned} \tag{40}$$

where $\mathcal{E}_1^i(c^i) = -(L^i)'/(L^i)''$ and $\mathcal{E}_2^i(c^i) = (L^i)'''/(L^i)''$. The system obtained is coupled, since in equation i appears also the strategies of the remainder players, c_{-i}, but the second order derivatives, which appear linearly, only are affected by player i. In the symmetric case, $L^1 = \dots = L^N = L$, $S^1 = \dots = S^N = S$, $\rho^1 = \dots = \rho^N = \rho$, $\mathcal{E}_j^1 = \dots = \mathcal{E}_j^N = \mathcal{E}_j$, $j=1,2$, and $\mathcal{U}^1 = \dots = \mathcal{U}^N = \mathcal{U}$, the symmetric MPNE leads, after rearrangement, to the single GEE

$$\begin{aligned} c_t + \Big(F(x) - Nc + (N-1)\mathcal{E}_1(c) + \sigma'(x)\sigma(x) \Big) c_x + \frac{1}{2}\sigma(x)^2 \mathcal{E}_2(c) c_x^2 + \frac{1}{2}\sigma(x)^2 c_{xx} \\ + \Big(\rho - F'(x) \Big) \mathcal{E}_1(c) = 0. \end{aligned} \tag{41}$$

In Section 5.3.1 below we will obtain analytical optimal solutions in the symmetric case and, in Section 5.3.2, solve an inverse problem related with the certainty equivalence principle. Besides theoretical work related with existence, uniqueness and sensitivity analysis the set of GEEs (40) allows for implementation of numerical algorithms. Several suitable methods can be found in e.g. Judd (1998).

Smooth solutions of the GEE system are automatically MPNE (if they are admissible in the sense of Definition (2.1)) when the utility functions L^i are all polynomially bounded (see Theorem 4.1), since the Hamiltonian of each player is concave in its own variable.

5.3.1 Analytical solution

Suppose $L(c) = Nc^{1/N}$, where N is the number of players. This case has been considered in Dockner and Sorger (1996) in a deterministic framework and infinite horizon. These authors show existence of (non–smooth) infinitely many MPNE; see also Sorger (1998) for an extension to more general isoelastic utility functions.

We prove here that in the stochastic case it is still possible to give a closed form solution to the problem in the finite horizon case. We left aside the infinite horizon case for further research. With the above specifications, $\mathcal{E}_1(c) = (N/(N-1))c$, and $\mathcal{E}_2(c) = (1/N-2)c^{-1}$, thereby the GEE (41) becomes

$$c_t + \Big(F(x) + \sigma'(x)\sigma(x)\Big)c_x + \frac{1}{2}\Big(\frac{1}{N} - 2\Big)\sigma(x)^2\frac{c_x^2}{c} + \frac{1}{2}\sigma(x)^2 c_{xx} + \frac{N}{N-1}\Big(\rho - F'(x)\Big)c = 0.$$

with the final condition (39), $c(T,x) = \varphi(x) = S'(x)^{N/(1-N)}$. Let us consider the change of variable $\Psi = c^{1/N-1}$. A simple computation shows that Ψ satisfies the linear PDE

$$\Psi_t + \Big(F(x) + \sigma'(x)\sigma(x)\Big)\Psi_x + \frac{1}{2}\sigma(x)^2\Psi_{xx} - \Big(\rho - F'(x)\Big)\Psi = 0. \tag{42}$$

Proposition 5.2 *If a smooth symmetric MPNE exists, then it must be given by the expression*

$$c(t,x) = \left(E_{tx}\left\{e^{\int_t^T(\rho - F'(X(s))\,ds}S'(X(T))\right\}\right)^{N/(1-N)}, \tag{43}$$

where X satisfies the SDE

$$dX(s) = \Big(F(X(s)) + (\sigma'\sigma)(X(s))\Big)\,ds + \sigma(X(s))\,dw_0(s), \qquad X(t) = x,$$

where w_0 is a one dimensional standard Brownian motion.

Proof. It is a simple consequence of the Feynman–Kac formula applied to the PDE (42) with final condition $\Psi(T,x) = S'(x)$; see e.g. Yong and Zhou (1999), p. 373. □

We can provide a more explicit expression than (43) in some specific cases. For instance, suppose that $F(x) = \mu x$ with $\mu \geq 0$ and that $\sigma(x) = \sigma x$ with $\sigma > 0$. Then, since $X(s) = xe^{(\mu+\frac{1}{2}\sigma^2)(s-t)+\sigma w_0(s)}$ for $s \geq t$, and $w_0(s)/\sqrt{s}$ has a standard normal distribution, (43) yields (with $\eta = N/(1-N)$)

$$c(t,x) = \left(\int_{-\infty}^{\infty} S'\left(xe^{(\mu+\frac{1}{2}\sigma^2)(T-t)+\sigma z}\right)\frac{e^{(\rho-\mu)(T-t)+\frac{-z^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}\,dz\right)^{\eta},$$

that is,

$$c(t,x) = e^{\eta(\rho-\mu)(T-t)}(2\pi(T-t))^{-\eta/2}\left(\int_{-\infty}^{\infty} S'\left(xe^{(\mu+\frac{1}{2}\sigma^2)(T-t)+\sigma z}\right)e^{\frac{-z^2}{2(T-t)}}dz\right)^{\eta}.$$

5.3.2 Inverse problem for robust MPNE

Our purpose is to apply our results for solving a certain class of inverse problems, which are related with the certainty equivalence principle. In our specific setting, the inverse problem can be described as follows: given the productive asset evolution (37), find strictly concave and strictly increasing utility functions L, S, such that the game admits a robust equilibrium. These type of problems are significant in economics, since that when a policy function can be rationalized by a "well behaved" utility function, it means that the prescribed behavior is consistent with an optimizing behavior. Relevant papers in the topic are Kurz (1969) and Chang (1988), but in the single–agent case. Inverse problems are easily handled with the GEE, since it characterizes directly the MPNE. Note that the HJB equation is not well suited for this class of problems, since they require knowledge both of the value function and the prescribed control law.

By concision of the paper we consider only the infinite horizon case, since similar results are obtained for the finite horizon case. The GEE (38) in this case for stationary ($c_t = 0$), symmetric MPNE is

$$-\rho L'(c) + \partial_x\left(L(c) + (F(x) - Nc)L'(c) + \frac{1}{2}\sigma(x)^2\partial_x L'(c)\right) = 0. \tag{44}$$

By Proposition 5.1, the necessary condition (33) for a robust MPNE is $\sigma(x)^2\partial_x L'(c(x))$ constant. Since we want to find explicit expressions, let us suppose that $\sigma(x) = \sigma x$, with $\sigma > 0$. Then, integrating with respect to x, we get $L'(c(x)) = A/x + B$, where A, B are constants (see (35)). Thus

$$L(c) = A\int^{c}\frac{dy}{\zeta(y)} + Bc, \tag{45}$$

where ζ is the inverse of the consumption function with respect to x, i.e. $x = \zeta(c(x))$, that we suppose that exists globally. For this to be true, it suffices that the consumption function be strictly increasing in (positive) wealth, which is a plausible economic assumption. From $L'(c) = A/x + B$ we can obtain the second derivative $L''(c) = -A/(x^2 c_x(t,x))$, thus L is strictly concave if and only if $A > 0$, because, as pointed out above, we are supposing that c is strictly increasing in wealth. On the other hand, $A > 0$ and $B \geq 0$ makes L strictly increasing in the relevant region $x > 0$, thus we impose these conditions on A and B.

To determine L explicitly from (45), it is necessary to find ζ. Here we suppose that $F(x) = \mu x$, with $\mu \geq 0$. We begin finding c, observing that it is also a solution to the deterministic problem, thus it must be a solution of (44) with $\sigma = 0$. This GEE can be explicitly solved once $L'(c) = A/x + B$ is substituted into, obtaining a linear ODE for c with non–constant coefficients: $c'(x) + P(x)c(x) + Q(x) = 0$, where

$$P(x) = \frac{-NA}{(N-1)(Ax + Bx^2)}, \qquad Q(x) = \frac{1}{N-1}\left(\rho - \frac{\mu Bx}{A + Bx}\right).$$

It remain the task of determining further conditions on A and B such that the solution remains positive and strictly increasing in the region $x > 0$. Once this is assured the problem is finished, with L given in (45). We do not pursue here to attain this generality, but showing instead some explicit examples. Consequently, suppose $B = 0$. Then it is easy to find the solution $c(x) = \rho x + Dx^{N/(N-1)}$, where D is another (non–negative) constant. Let us distinguish two cases.

- $D = 0$ and N general. The consumption is linear, $c(x) = \rho x$, so that $\zeta(c) = c/\rho$. Substituting into (45) we find a logarithmic utility, $L(c) = A\rho \ln c$, where A is an arbitrary, positive constant. The asset evolves according to the SDE

  $$d\xi(s) = (\mu - N\rho)\xi(s)ds + \sigma\xi(s)dw(s), \quad \xi(0) = x,$$

  that is a geometric Brownian motion

  $$\xi(t) = xe^{(\mu - N\rho - \sigma^2/2)t + \sigma w(t)}, \quad t \geq 0.$$

  The value functions of the deterministic ($V^{\det}$) and stochastic problems are easily found from Remark 4.3 to be

  $$V^{\det}(x) = A\ln x + A\left(\ln\rho + \frac{\mu}{\rho} - N\right),$$
  $$V(x) = V^{\det}(x) - \frac{A\sigma^2}{2\rho},$$

  respectively.

- $N = 2$ and $D > 0$. The robust MPNE is a quadratic function of x, $\widehat{c}(t, x) = \rho x + Dx^2$, so that ζ is given by $\zeta(c) = \frac{-\rho + \sqrt{\rho^2 + 4Dc}}{2D}$ and the bi–parametric family of increasing and strictly concave utility functions that rationalize the quadratic consumption function for both the stochastic and the deterministic problem is

  $$L(c) = A\rho \ln\left(\sqrt{\rho^2 + 4Dc} - \rho\right) + A\sqrt{\rho^2 + 4Dc},$$

  with $A > 0$ and $D > 0$ arbitrary constants.

Because the game is symmetric, the asset evolves as

$$d\xi(s) = \big((\mu - N\rho)\xi(s) - ND\xi^2(s)\big)\, ds + \xi(s)\sigma dw(s), \quad \xi(0) = x.$$

The solution to this nonlinear SDE is (see e.g. Øksendal (2003), p. 78)

$$\xi(t) = \frac{e^{(\mu - N\rho - \frac{1}{2}\sigma^2)t + \sigma w(t)}}{\frac{1}{x} + ND\int_0^t e^{(\mu - N\rho - \frac{1}{2}\sigma^2)s + \sigma w(s)}\, ds}, \quad t \geq 0.$$

The explicit expressions for the value functions are

$$V^{\det}(x) = A\ln x + A\bigg(\ln(2D) + \frac{\mu}{\rho} - 1\bigg),$$
$$V(x) = V^{\det}(x) - \frac{A\sigma^2}{2\rho}.$$

6 Conclusions

This paper provides a new perspective for the analysis of stochastic differential games. Instead of the classical method based on HJB equation, we propose the system of GEEs which directly characterizes the MPNE. The main contributions of the paper are to provide a systematic way to find GEE in general stochastic differential games and to establish a sufficient condition in terms of this new system of PDEs. Further research should be directed to extend the methodology to games where the diffusion coefficient of the state process depends on the strategies of players, and to apply the method to the study of models in economics and operations research.

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