Working Paper 94-41
Economics Series 22
October 1994

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A Proof of First-Order Stochastic Dominance for Quantity Constrained Oligopolies

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Abstract

In this paper a proof for First Order Stochastic Dominance for capacity constrained oligopolies exhibiting equilibrium in mixed strategies is derived. The result is an extension of Levitan and Shubik (1972) where they derive the mixed strategy equilibrium for quantity constrained oligopolies. I show that their result has applications in policy issues in Regulation and Trade Theory. The proof of First Order Stochastic Dominance facilitates the comparison of expected prices across different experimental/trade policy designs, enlarging the qualitative implications of the results derived by Levitan et al.

Key Words: Licenses, Voluntary Export Restraints, First Order Stochastic Dominance, Randomisation, Surplus Maximizing Queue, Cumulative Density Functions.

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1. Introduction.

Industrial organization literature has benefitted greatly from the application of game theoretic methods. The richness of the results and the specific characterization of equilibria derived from the application of game theoretic methods has been especially useful in experimental economics and Industrial Oragnization models of trade policy. This has enabled economists to derive and test specific equilibrium predictions.

In experimental economics specific equilibrium predictions have been derived and tested in experimental markets (Kruse et. al., Davis et. al.). The game theoretic models have seen applications in experiments on Bertrandt-Cournot oligopolies (Kruse et. al.), mergers and issues of market power (Davis et. al.). In International Trade policy models of imperfect competition have been developed along the lines of (IO) models of Bertrandt-Cournot competiton (to name a few, Helpman and Krugman (1991), Krishna (1989)). Richness of the results has been enlarged as specific issues have been addressed.

Characterizing all possible equilibrium outcomes in pure/mixed strategies is useful to experimental/policy[1] economists as it enables them to isolate distinct equilibria. Furthermore, specific cases can be studied clearly with the knowledge of the equilibrium predictions. This enables the experimental economist/policy theorist to make specific comments on the nature of the equilibria and the effect on welfare, etc.. If equilibrium outcomes are in pure strategies it is easier to make statements on the implications of the policy tool, or experimental design. However, equilibrium outcomes in mixed strategies need not be barren in information. To quote Krishna;

"..However, the mere existence of a mixed strategy equilibrium does not yield any information about the effects of a VER." p.259.

This need not be the case. For an equilibrium in mixed strategies it is possible, under some conditions, to make a statement on the expected prices. If the cumulative density functions (cdf) for the specific cases, of demand and cost conditions, can be characterized. A specific functional form of the cdf is deriveable (as is in capacity constrained models) from which the upper and lower bounds of the distribution are easily calculated[2]. To enrich the predictive power of the theoretical and experimental models a useful result is the proof of FSD. If FSD can be etsablished then that implies that the cumulative distribution that FSD the other also has a higher expected value.[3]

Derivation of the proof of stochastic dominance can be a non-trivial task[4]. If such a result is established the benefits far outweigh the costs. It is possible to examine certain specific cases and obtain results to this regard. In capacity constrained models, for specific characterisations of demand and cost function, specific forms for the cdf's are easily derived[5] (Levitan et. al. (1972)). Levitan et. al. derive this result for the value queue, or the surplus maximising queue. This makes this proof applicable to a limited class of problems. However, it makes it easier to show FSD for these specific cases.

2. The Framework.

The proof of FSD is derived for oligopolies in two distinct markets[6] under the surplus maximising rationing rule (or, the value queue). In each market the firms have equal capacities, however, across the market the capacities are unequal. This situation is similar to the one where in the first stage the market is unconstrained and then quantity restrictions are imposed on all the firms. As a result each firm has a smaller capacity. Let x_{ij} be the capacity of each firm in each market, i=1,2. Let $x_{1j} > x_{2j}$, such that $\Sigma x_{1i} > \Sigma x_{2j}$. Price is the choice variable in both the markets. Let $G_1(p)$ be the cumulative density function (cdf) for market 1 and $G_2(p)$ the cdf for market 2.

The derivation of cdf's is as in Levitan and Shubik (1972). I assume a downward sloping linear demand, p = a-Q. Demand for the good at a price of zero is Q. Then the derivation of the cdf's for two firms, with equal capacities within a market, is;

(a) $$G_{ij}(p)=[4x_{ij}p-(a-x_{ij})^2]/4p(p+2x_{ij}-a). \quad i = 1,2;\ j=1,2.$$

The case of two firms in each market is considered. Our only interest is to derive the cdf's for the two markets and show FSD.

Let x_1, x_2 ($x_1 > x_2$) be capacities of two firms[7] in market 1 and 2, respectively. The capacities are distinct and can be due to different design treatments in experiments, or due to the different policy tools in international trade policy (Krishna (1989), or issues in regulation economics (Kujal (1993)). Given, $x_{11} = x_{12} > x_{21} = x_{22}$, substituting for x_{1j} and x_{2j}[8] in (a) we get (a1) and (a2).[9]

(a1) $$G_1(p) = [4x_1p-(a-x_1)^2]/4p(p+2x_1-a)].$$

(a2) $$G_2(p) = [4x_2p-(a-x_2)]/4p(p+2x_2-a)].$$

It is easily shown that the lower bound for (a2) is greater than the lower bound for (a1). Set (a1) and (a2) equal to zero and solve for p (the sellers have no incentive to price below the lower bound and hence the cumulative density must be zero). The respective bounds are, (a3) $p_{min1} = [(1/x_1)((a-x_1)/2)^2]$.

(a4) $p_{min2} = [(1/x_2)((a-x_2)/2)^2]$.

It is clear that $p_{min2} > p_{min1}$.

Comparing the right side of (a1) and (a2) we see that the first term on the right hand side of (a1) is less than the first term on the right side of (a2), given $x_1 > x_2$. Also, the second term for (a1) is greater than the second term for (a2). Thus, it is not possible to show that $G_2(p)$ first order stochastically dominates $G_1(p)$. In the following section the proof of stochastic dominance is derived.

3. Proof of Stochastic Dominance.

FSD is defined as in Levy (1987). Let $G_1(x)$ and $G_2(x)$ be the cdf's for two distinct uncertain options. Then G_1 dominates G_2 by First Order Stochastic dominance iff:

(i) $G_2(x) \geq G_1(x)$ for all x FSD

With a strict inequality for at least one value of x. It implies that design-1 (G_1) yields higher expected prices than design-2 (G_2).

<figure-1 here>

Given that the cdf's are continuous and non-decreasing, and that the lower bound of the non-cooperative price distribution for market quotas $p_{min2} > p_{min1}$ and the upper bound $p_{max2} > p_{max1}$. For $G_2(p)$ to first-order stochastically dominate $G_1(p)$, $G_2(p)$ must lie on, or below, $G_{1j}(p)$ at least at one point anywhere in the range$[p_{min2}, p_{max1}]$.

<figure-2 here>

Figure-1

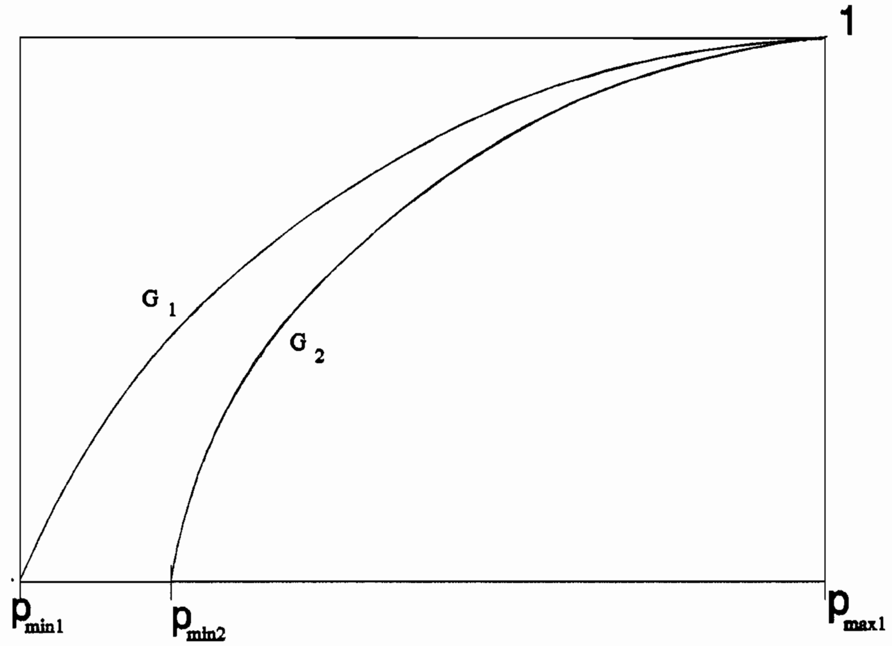
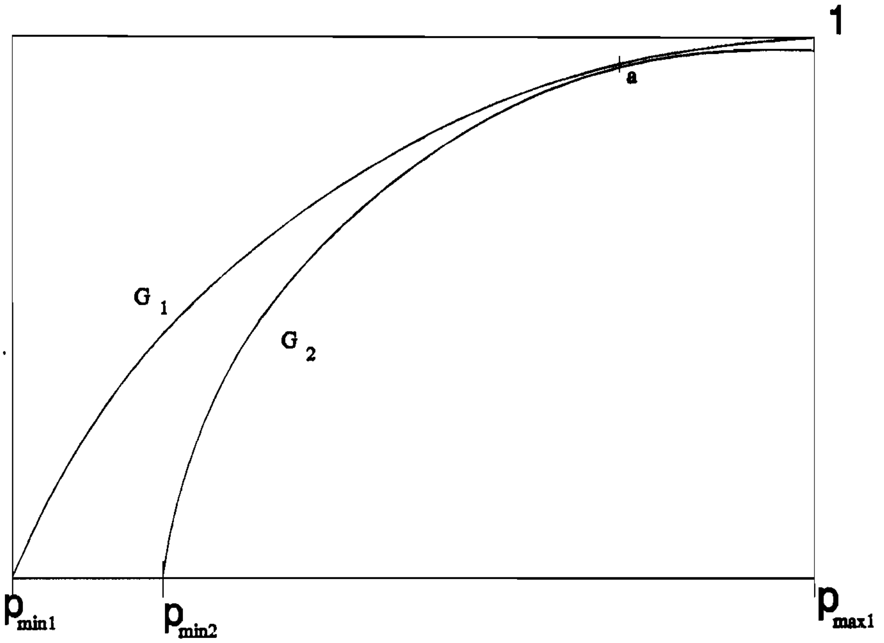


Figure-2



This needs some explanation. If G_2 FSD G_1 then it must lie below G_1 at least at one point for the definition of FSD to be satisfied[10]. This implies that if it can be shown that G_2=G_1 at most at one point in the interval then it must lie below G_1 everywhere else[11]. Which is exactly what we need for the proof.

The proof proceeds as follows. Taking the first difference of $G_1(p)$ and $G_2(p)$ and simplifying we get,

(1) $$2p^2-(x_1+x_2)p+(2x_1x_2+a(x_1+x_2)=0.$$

The solution to the quadratic is,

(2) $$p = \{(x_1+x_2) \pm \sqrt{[(x_1+x_2)^2- 8(2\ x_1\ x_2 + a\ (x_1+x_2)]}\}/4.$$

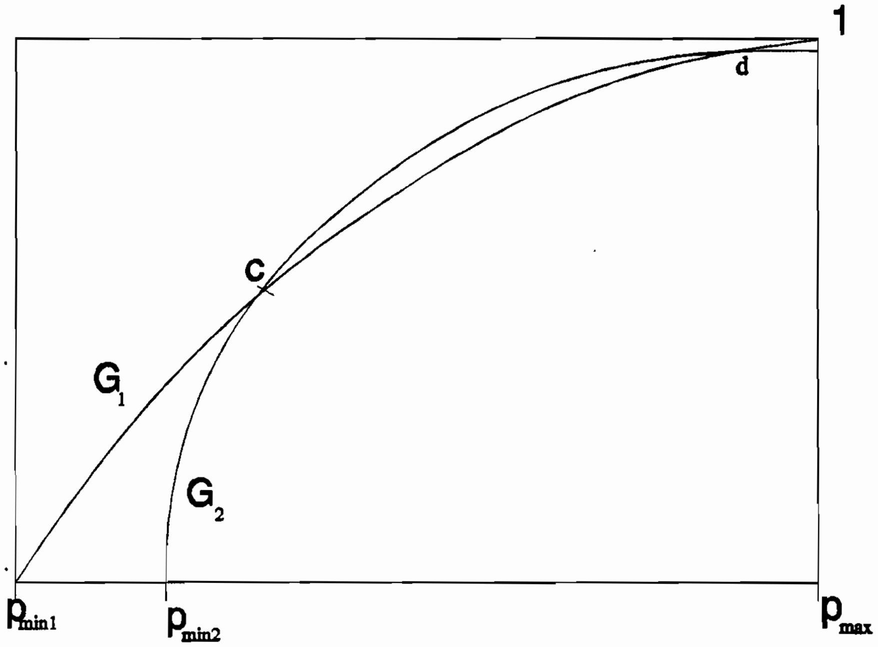
Given that the solution to (1) is non-unique if it is shown that one, or both, of the solutions do not lie in the interval $[p_{min2},p_{max1}]$ then the proof of FSD is established. That is, the cdf lies below at all the points, except one (where it is equal)[12]. The first term on the right hand side of (2), (x_1+x_2), is positive. Two possiblities exist. First, if both the solutions to the quadratic are positive then it needs to be shown that at least one does not lie in the interval $[p_{min2},p_{max1}]$. Second, if the square root is a complex number then the solution can not lie in the set of real numbers $[p_{min2},p_{max1}]$.

It can be easily shown that the term in the square root is negative. Writing the term in the square root,

(3) $$[(x_1+x_2)^2- 8(2\ x_1\ x_2 + a\ (x_1+x_2)]\}.$$

Now, let the capacities be such that $a>x_1>x_2>a/3$[13]. Thus, we can write $x_2 = \alpha\ x_1$, $\alpha \varepsilon\ (0,1)$. Substituting, we get

Figure-3



(4) $$x_1^2(1+\alpha^2) - 14\, x_1^2\, \alpha - 8\, x_1\, a\, (1 + \alpha).$$

The sign for (4) needs to be checked. Now, for the equilibrium in mixed strategies to exist x_1 should be in the interval (a, a/3). That is, the smallest value that **a** can take is x_i itself. Substituting this in (4) we get,

(5) $$x_1^2(1+\alpha^2) - 8\, x_1^2\ (1 + \alpha) - 14\, x_1^2\, \alpha.$$

Now, $(1+\alpha^2) < (1 + \alpha)$ as $\alpha \in (0,1)$. It follows that $x_1^2(1+\alpha^2) < 8\, x_1^2\ (1 + \alpha)$. That is, the expression in (5) is negative. This implies that the solutions to the quadratic is a complex number and can not lie in the set of real numbers (where the complex component is zero). Q.E.D.[14]

This argument can be extended to the case of Krishna (1989). Krishna studies the case of an oligopoly in two countries. Trade restrictions of the form licenses, or Voluntary Export Restrictions, are imposed on the foreign firm. The argument is that trade restrictions work as facilitaing practices in terms of price increases. That is, if sales of the foreign firm are limited by using trade restrictions such as licenses and/or Voluntary Export Restraints then the restrictions facilitate a price increase. A similar, but simpler, argument applies in this case. As the restriction is imposed on the foreign firm it cannot sell more than the amount of the restraint. As a result the domestic firm can increase its price and still sell a positive amount. Both the home and the foreign firm will then have a common lower bound (in the home market) as the home firm prices with zero probability below its lower bound. Thus the relevant cdf is that of the home firm only (both have a common upper bound in the home market). Now, it is easy to show that the resulting lower bound of the cdf for the home firm is higher than before the imposition of the licenses (the same as above). Then all that needs to be done is to compare the two cases, before and after the imposition of the quantity restrictions. Which is exactly the proof developed above. This can be illustrated with a simple example.

Example:

Imagine an equilibrium in pure strategy where both firms have a capacity of **a**. This example is the case of asymmetric capacities in Levitan et al. Now, let a quantity restriction be imposed on the foreign firm such that its capacity is less than **a**. Let the capacity of the foreign firm be β**a** such that $\beta \in (0,1)$. This is sufficient to show that we get an equilibrium in mixed strategies with a lower bound greater than the marginal cost of production. Thus, given that the lower bound is greater than the marginal cost of production the expected prices obtained under the quantity restrictions are greater as well.[15]

4. Conclusion.

In this paper a proof of FSD is derived. It has uses in Regulation and Trade Policy for cases where equilibria in mixed strategies exist. This enables one to make predictions on expected price outcomes across different policy regimes. The result is also useful in Experimental Economics where such scenarios are common[16]. On experiments in market power, mergers, effect of quotas, issues in International Trade Policy, this result extends the richness of the results. Price predictions can be made thereby enabling study of welfare issues. The result is also extended to the environment in Krishna (1989). FSD is easily shown using the proof developed here.

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1. The term policy is used for both regulation and trade policy.

2. The value of the random variable at the upper and the lower bound is obtained by setting the cdf equal to one (at the upper bound), and zero (at the lower bound). Note, the derivation of the upper and the lower bounds is useful. On its own it does not convey any information on expected prices. Without a proof of stochastic dominance it cannot be said that a design yields distince price outcomes over another.

3. Modelling equilibria in mixed strategies price is assumed to be the random variable.

4. In most cases the cumulative density functions cannot be characterized. In this context Krishna's remark is legitimate (see above).

5. Thanks to Levitan and Shubik.

6. It is easy to imagine that we have two regimes. One before the imposition of the quantity restrictions and the other after the imposition of the restrictions. Then we have smaller aggregate capacities after the quantity restrictions have been uniformly imposed on all the firms. I do not consider the case of asymmetric capacities right now.

7. The subscript for the markets is retained.

8. As the firms are homogenous in each market I drop the subscript 'j' for the firms and retain the subscript, '1' and '2', for the two markets.

9. This is a simple scenario. I assume two competing firms in each regime, with and without the licenses.

10. Note, if the cdf's intersect at one point then they have to be equal at at least two points (see Figure-3).

11. Note, if the two cdf's are equal at even two points that may imply that there exist points where one is above the other and vice-versa (figure-1).

12. Of corse, if the cdf's are equal at points outside the lower and the upper bound it is not a problem. Sellers price outside the bounds with a probability zero.

13. This is true if an equlibrium in mixed strategies has to exist.

14. The solution for the case of single step and demand functions is much more straight forward and yields a unique solution to the quadratic. That is, the cdf is below everywhere except at one point.

15. It is curious that Levitan and Shubik solve this case only as *mathematical curiosity and uninteresting* (sic). This proof can be applied to problems in policy where asymmetric capacities are generated if the imposition of the quantity restrictions is discriminative. As is in the case of licenses and VER's.

16. For example, design treatments with different capacities are now comparable.