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ON EXPENDITURE FUNCTIONS[1]

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Abstract ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

In this paper we present complete characterizations of the expenditure function for both utility representations and preference structures. Building upon these results, we also establish under minimal assumptions duality theorems for expenditure functions and utility representations, and for expenditure functions and preference structures. These results generalize previous work in this area; moreover, in the case of preferences structures they apply to non-complete preorders.

Key words:
Expenditure functions, utility representations, duality, non-complete preorders.

ON EXPENDITURE FUNCTIONS

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1. INTRODUCTION

In his survey on demand analysis, Deaton (1986) has placed the expenditure function as the centerpiece of consumption theory. Indeed, the expenditure function is a very convenient tool to address questions of welfare and efficiency. Moreover, it is generally possible to obtain good estimates of expenditure functions from real data sets, and then derive the underlying (regularized) preference structures.

It is well known that under some assumptions on preferences, an expenditure function must satisfy certain properties [e.g., see Diewert (1982)]. It seems to be unknown, however, what are the defining properties of this function: A set of necessary and sufficient conditions to establish that a function must be the expenditure function of some specified preference structure. This question is of some relevance since it is usually easier to observe or estimate the expenditure function of a consumer (or the cost function of a producer) than the underlying preferences. [See Blundell (1988) and Deaton (1986) for excellent surveys on the field.]

In this paper we present complete characterizations of the expenditure function for both utility representations and preference structures. Moreover, in the case of preference structures such characterization applies to non-complete preference orderings. Building upon these results, we also establish under minimal assumptions duality theorems for expenditure functions and utility representations, and for expenditure functions and preference structures. These duality theorems generalize previous work in this area [e.g., Diewert (1982), Jacobsen (1970), McFadden (1978), Shephard (1970) and Uzawa (1962)]. Furthermore, our duality theorems hold under conditions other than those required for the duality of direct and indirect preferences [cf., Martínez-Legaz and Santos (1993)].

Previous research on this topic [see Diewert (1982) for an extensive survey] has been essentially concerned with sufficient, rather than necessary, conditions for the existence of expenditure functions and duality. As already suggested, necessary conditions are a useful line of inquiry, since duality arguments entail that expenditure functions are primitive objects of consumption theory.

In contrast to previous work, our results extend to infinite-dimensional spaces and to non-complete preference orderings, and are not based upon particular topological and boundary conditions. Although continuity and boundary conditions are essential properties in a wide range of economic applications, they become artificial for the analytical issues under consideration.

The paper is structured as follows. Section 2 is concerned with expenditure functions and utility representations, and Section 3 is concerned with the more general setting of

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expenditure functions and preferences. Both sections present under minimal assumptions a characterization of the expenditure function, and duality results for expenditure functions and preferences. For utility functions the conditions characteristic of an expenditure function are weaker than those required for the duality of expenditure functions and utilities. Finally, Section 4 reports on additional properties of expenditure functions. With obvious changes in notation and terminology, our analysis can be recast in an environment of technologies and production. For convenience, however, we shall focus on the standard consumer framework.

2. EXPENDITURE AND UTILITY FUNCTIONS

2.1. Notation and Preliminary Definitions

Assume that X is a locally convex, topological vector space, with topological dual X^*. Let K be a closed convex cone in X such that K is always different from the zero element, i.e. $K \neq \{0\}$. Let $\geq$ be the canonical ordering on $K : x \geq x'$ if and only if $x - x' \in K$, for all vectors x, x' in K. Let $K^* = \{p \in X^* \mid p \cdot x \geq 0 \text{ for all } x \text{ in } K\}$. Say that $p \geq p'$ if and only if $p - p' \in K^*$, for all vectors p, p' in K^*. Observe that $K = \{x \mid p \cdot x \geq 0 \text{ for all } p \in K^*\}$.

Let $\succeq$ be a preference ordering on K. Assume that $\succeq$ can be represented by a utility function, $u : K \to R$ [i.e., a real valued function such that for all vectors x, x' in K, $x \succeq x'$ if and only if $u(x) \geq u(x')$]. Define the expenditure function $e_u : K^* \times R \to R_+ \cup\{+\infty\}$, by

$$e_u(p, \lambda) = \inf \{p \cdot x \mid u(x) \geq \lambda\}$$

Sometimes, for convenience, subscript u in this definition will be dropped. Also, for a given function, u, let $S^\lambda(u)$ stand for the upper-contour set, $S^\lambda(u) = \{x \mid u(x) \geq \lambda\}$.

For a function, $e : K^* \times R \to R_+ \cup\{+\infty\}$, where, for fixed λ, $e(\cdot, \lambda) : K^* \to R_+ \cup\{+\infty\}$ is a concave mapping with $e(0, \lambda) = 0$, define the "regularized" utility function, $u_e : K \to R \cup\{+\infty\}$, by

$$u_e(x) = \sup \{\lambda \in R \mid x \in \partial e(\cdot, \lambda)(0)\}$$

Here $\partial e(\cdot, \lambda)(0)$ connotes the superdifferential of the mapping $e(\cdot, \lambda)$ at $p = 0$, that is, $\partial e(\cdot, \lambda)(0) = \{x \in X \mid p \cdot x \geq e(p, \lambda) \text{ for all } p \in K^*\}$.

Finally, for a set A, let $\bar{co}(A)$ stand for the closed convex hull, and for two sets, A, B, let $B \setminus A = \{x \in B \mid x \notin A\}$.

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2.2. A Characterization of the Expenditure Function

In our proof of the set of necessary and sufficient conditions that characterize an expenditure function, we shall make use of the following technical result.

LEMMA 2.1: *Let A and B be two closed convex subsets of K such that $A \subset B$, $A \neq B$, and B has at least two (hence, infinitely many) points. Let $\lambda \in R$. Then there exists a function $f_\lambda : B \setminus A \to (-\infty, \lambda)$ such that*

$$B \setminus A \subset \bigcap_{\mu < \lambda} \bar{co} S^\mu(f_\lambda)$$

PROOF: Assume first that A is non-empty. Pick $\bar{x} \in A$. Since each point in $B \setminus A$ belongs exactly to a ray r emanating from $\bar{x}$, it suffices to define f_λ on each non-empty intersection $r \cap (B \setminus A)$. Such an intersection must be either a right-closed segment $(x_A, x_B]$ with $x_A \in A$ and $x_B \in B$, or an open ray in a given direction, $d \neq 0$, with some end-point, $x_A \in A$. In the first case, we define $f_\lambda(\alpha x_A + (1-\alpha)x_B) = \lambda - \alpha(1-\alpha)$ for $\alpha \in (0,1)$, and $f_\lambda(x_B) = \lambda - 1$; in the second case, we define $f_\lambda(x_A + \alpha d) = \lambda - \alpha e^{-\alpha}$ for $\alpha > 0$. One easily checks that the function f_λ constructed in this way satisfies the required condition.

Assume now that A is the empty set. Pick $\bar{x} \in B$. Let f_λ be the function obtained from the above construction with the set A replaced by $\{\bar{x}\}$. Since B is closed, convex and has at least two elements, we have that $B \setminus \{\bar{x}\} \subset \bigcap_{\mu < \lambda} \bar{co} S^\mu(f_\lambda)$. Also, $B \subset \bigcap_{\mu < \lambda} \bar{co} S^\mu(f_\lambda)$. The lemma is thus established.

REMARK: Observe that in the preceding proof, we have

$$\lim_{\alpha \to 0^+} f_\lambda(\alpha x_A + (1-\alpha)x_B) = \lim_{\alpha \to 1^-} f_\lambda(\alpha x_A + (1-\alpha)x_B) =$$
$$\lim_{\alpha \to 0^+} f_\lambda(x_A + \alpha d) = \lim_{\alpha \to +\infty} f_\lambda(x_A + \alpha d) = \lambda$$

These are the only properties required for our purposes —besides the fact that f_λ takes on values in $(-\infty, \lambda)$. The particular expressions, $\lambda - \alpha(1-\alpha)$ and $\lambda - \alpha e^{-\alpha}$, are just instances of functions satisfying these conditions.

THEOREM 2.2: *Let $e : K^* \times R \to R_+ \cup \{+\infty\}$. Then e is the expenditure function for some utility function, $u : K \to R$, if and only if the following conditions hold*

(a) *For each $\lambda \in R$, either $e(\cdot, \lambda)$ is finite-valued, concave, linearly homogeneous and weak* upper-semicontinuous, or it is identically equal to $+\infty$.*

(b) *$e(p, \cdot)$ is non-decreasing for each fixed $p \in K^*$.*

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(c) $\bigcup_{\lambda \in R} \partial e(\cdot, \lambda)(0) = K$.

PROOF: Assume first that $e : K^* \times R \to R_+ \cup \{+\infty\}$ is the expenditure function of $u : K \to R$. Observe that each $x \in K$ can be regarded as a linear function on X^*, continuous is the weak* topology. Since $e(\cdot, \lambda)$ is the pointwise infimum of the family of linear functions $\{x \mid u(x) \geq \lambda\}$, condition (a) must necessarily hold. Also, condition (b) must always be satisfied. Let us prove (c). If $\lambda \in R$ and $x \in \partial e(\cdot, \lambda)(0)$, then $p \cdot x \geq e(p, \lambda) \geq 0$ for all $p \in K^*$. Hence $x \in K$, and so we have proved one direction of the inclusion in (c). Moreover, the simple fact that $x \in \partial e(\cdot, u(x))(0)$, for all $x \in K$, establishes the equality in (c).

Conversely, suppose that $e : K^* \times R \to R_+ \cup \{+\infty\}$ satisfies $(a) - (c)$. Define $\tilde{\lambda} : K \to R \cup \{+\infty\}$ by

$$\tilde{\lambda}(x) = \sup \{\lambda \in R \mid x \in \partial e(\cdot, \lambda)(0)\}$$

Condition (c) implies that $\tilde{\lambda}(x) > -\infty$. For convenience of notation, let $\partial e(\cdot, +\infty)(0) = \phi$. By condition (b), the sets $\partial e(\cdot, \lambda)(0)$ are non-increasing in λ. For each $\lambda \in R \cup \{+\infty\}$, let $M_\lambda = (\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)) \setminus \partial e(\cdot, \lambda)(0)$. Observe that, for every x in K, either $x \in \partial e(\cdot, \tilde{\lambda}(x))$ or $x \in M_{\tilde{\lambda}(x)}$. If M_λ is non-empty, define a function $f_\lambda : M_\lambda \to (-\infty, \lambda)$ such that $M_\lambda \subset \bigcap_{\mu < \lambda} \bar{c}oS^\mu(f_\lambda)$. The existence of such a function follows from Lemma 2.1, since $M_\lambda \neq \phi$ implies that $\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)$ is never a singleton. Indeed, for every such λ we have

$$\left[\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)\right] + K = \bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)$$

Let us now define $u : K \to R$ by

$$u(x) = \begin{cases} \tilde{\lambda}(x) & \text{if } x \in \partial e(\cdot, \tilde{\lambda}(x))(0) \\ f_{\tilde{\lambda}(x)}(x) & \text{if } x \in M_{\tilde{\lambda}(x)} \end{cases}$$

One readily checks from this definition that $S^\lambda(u) \subset \partial e(\cdot, \lambda)(0)$, for all λ in R. Hence $\bar{c}oS^\lambda(u) \subset \partial e(\cdot, \lambda)(0)$. Let us prove that the converse inclusion also holds. Let $x \in \partial e(\cdot, \lambda)(0)$. Then $\tilde{\lambda}(x) \geq \lambda$. If $x \in \partial e(\cdot, \tilde{\lambda}(x))(0)$, then $u(x) = \tilde{\lambda}(x) \geq \lambda$. Hence, $x \in S^\lambda(u)$. If $x \notin \partial e(\cdot, \tilde{\lambda}(x))(0)$, then $x \in M_{\tilde{\lambda}(x)}$. Moreover, given that $x \in \partial e(\cdot, \lambda)(0)$, we must have $\lambda < \tilde{\lambda}(x)$. In this case, we get from our previous lemma that $x \in M_{\tilde{\lambda}(x)} \subset \bar{c}oS^\lambda(f_{\tilde{\lambda}(x)}) = \bar{c}oS^\lambda(u \mid_{M_{\tilde{\lambda}(x)}})$, where $u \mid_{M_{\tilde{\lambda}(x)}}$ means u restricted to $M_{\tilde{\lambda}(x)}$. We have thus proved that $\partial e(\cdot, \lambda)(0) \subset \bar{c}oS^\lambda(u)$, and so $\bar{c}oS^\lambda(u) = \partial e(\cdot, \lambda)(0)$ for all $\lambda \in R$. Furthermore, it is a consequence of condition (a) that

$$e(p, \lambda) = \inf \{p \cdot x \mid x \in \partial e(\cdot, \lambda)(0)\}$$

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Therefore,

$$
\begin{aligned}
e(p,\lambda) &= \inf\{p \cdot x \mid x \in \bar{co}S^\lambda(u)\} = \\
&= \inf\{p \cdot x \mid x \in S^\lambda(u)\} = \\
&= \inf\{p \cdot x \mid u(x) \geq \lambda\}
\end{aligned}
$$

where the first equality comes from our previous result, and the second equality from the fact that the infimum of a linear function over a set remains unchanged if this set is replaced by its closed convex hull. This shows that e is the expenditure function of the function u. The proof is complete.

REMARKS: (1) Condition (a) implies that, for each $\lambda \in R$, the mapping $e(\cdot,\lambda) :$ $K^* \to R_+ \cup\{+\infty\}$ is non-decreasing. Indeed, as in the preceding proof, one easily shows that $\partial e(\cdot,\lambda)(0) \subset K$. Moreover, if $p_1, p_2 \in K^*$, with $p_1 \geq p_2$, we have

$$
\begin{aligned}
e(p_1,\lambda) &= \inf\{p_1 \cdot x \mid x \in \partial e(\cdot,\lambda)(0)\} \\
&\geq \inf\{p_2 \cdot x \mid x \in \partial e(\cdot,\lambda)(0)\} = e(p_2,\lambda)
\end{aligned}
$$

(2) If X is a finite-dimensional space and the cone K^* is locally simplicial (e.g., $K = R^n_+$, the non-negative orthant of Euclidean space R^n), then every concave function is lower-semicontinuous [Rockafellar (1970, Th. 10.2)]. Hence, in this case the upper-semicontinuity referred to in property (a) amounts to the usual continuity.

(3) In the (pathological) case, $K = \{0\}$, ruled out in Sect. 2.1, conditions $(a) - (c)$ are still necessary for e to be an expenditure function. These conditions, however, are no longer sufficient. Indeed, in this case an expenditure function takes the form

$$
e(p,\lambda) = \begin{cases} 0 & \text{if } \lambda \leq \bar{\lambda} \\ +\infty & \text{if } \lambda > \bar{\lambda} \end{cases}
$$

with $\bar{\lambda} \in R$. But any other function obtained from this expression by replacing the first weak inequality, $\leq$, by the strict inequality, $<$, and the second strict inequality, $>$, by the weak one, $\geq$, also satisfies $(a) - (c)$, even though such function is not an expenditure function.

2.3. Duality between Expenditure Functions and Utility Representations

In this section, we present under minimal assumptions a duality theorem for expenditure and utility functions in an infinite-dimensional framework. This theorem is a generalization of previous results in duality theory [see Shephard (1970), and especially Diewert

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(1982) for an account of this theory]. Unlike previous results on finite-dimensional spaces, lower-semicontinuity of the mapping $e(p, \lambda)$ on λ is no longer a necessary condition to establish duality.

Our duality theorem is a consequence of two basic results which are proved independently. We first explore in Proposition 2.3 the conditions under which the regularized utility function, u_{e_u}, is equal to the original utility function, u, on K. Then we explore in Proposition 2.4 the conditions under which the derived expenditure function, e_{u_e}, is equal to the original expenditure function e.

In order to guarantee that the regularized utility function, u_e, is always finite-valued, we shall introduce the following condition

(d) $\bigcap_{\lambda \in R} \partial e(\cdot, \lambda)(0) = \phi$

PROPOSITION 2.3: *Let* $u : K \to R$. *Then* $u_{e_u} = u$ *if and only if* u *is quasiconcave, non-decreasing, and upper-semicontinuous. Besides, under these properties* e_u *satisfies condition* (d).

PROOF: Let $e : K^* \times R \to R_+ \cup \{+\infty\}$ be an expenditure function. By Theorem 2.2, the function e satisfies $(a) - (c)$. Moreover, by the definition of u_e one can easily check that $S^\lambda(u_e) = \bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)$ for all $\lambda \in R$. Since these level sets are convex, closed, and satisfy $S^\lambda(u_e) + K = S^\lambda(u_e)$ for every non-empty $S^\lambda(u_e)$, the function u_e is quasiconcave, non-decreasing, and upper-semicontinuous.

Conversely, assume that $u : K \to R$ satisfies these properties. Then one readily shows that $S^\lambda(u) = \partial e_u(\cdot, \lambda)(0)$ for all $\lambda \in R$. Hence, for every $x \in K$,

$$
\begin{aligned}
u(x) &= \sup\{\lambda \in R \mid x \in S^\lambda(u)\} = \\
&= \sup\{\lambda \in R \mid x \in \partial e_u(\cdot, \lambda)(0)\} = u_{e_u}(x)
\end{aligned}
$$

Moreover, as $S^\lambda(u) = \partial e_u(\cdot, \lambda)(0)$, and $\bigcap_{\lambda \in R} S^\lambda(u) = \phi$, the expenditure function e_u satisfies condition (d). The proposition is proved.

REMARK: According to the preceding proof, for any expenditure function e satisfying (d), u_e is finite-valued, quasiconcave, non-decreasing, and upper-semicontinuous. Hence, $u_{e_{u_e}}$ is equal to u_e.

PROPOSITION 2.4: *Let* $e : K^* \times R \to R_+ \cup \{+\infty\}$ *satisfy conditions* $(a) - (d)$. *Then* $e_{u_e} = e$ *if and only if* e *satisfies condition*

(e) $\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0) = \partial e(\cdot, \lambda)(0)$ *for all* $\lambda \in R$.

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In this case, u_e is the greatest function $u : K \to R$ such that $e_u = e$, and the unique one that is quasiconcave, non-decreasing, and upper-semicontinuous.

PROOF: Let $e : K^* \times R \to R_+ \cup \{+\infty\}$ satisfy conditions $(a) - (d)$. If $e_{u_e} = e$, then from the proof of the preceding proposition it follows that for every $\lambda \in R$,

$$\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0) = S^\lambda(u_e) = \partial e_{u_e}(\cdot, \lambda)(0) = \partial e(\cdot, \lambda)(0).$$

Hence, condition (e) must be satisfied.

Conversely, if the function e satisfies conditions $(a)-(e)$, then for every $(p, \lambda) \in K^* \times R$ we have

$$
\begin{aligned}
e(p, \lambda) &= \inf \{p \cdot x \mid x \in \partial e(\cdot, \lambda)(0)\} = \\
&= \inf \{p \cdot x \mid x \in \bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0)\} = \\
&= \inf \{p \cdot x \mid x \in S^\lambda(u_e)\} = \\
&= \inf \{p \cdot x \mid u_e(x) \geq \lambda\} = e_{u_e}(p, \lambda)
\end{aligned}
$$

To prove the remaining part of the proposition, suppose now that $u : K \to R$ is such that $e_u = e$. Since $S^\lambda(u) \subset \partial e_u(\cdot, \lambda)(0)$ for all $\lambda \in R$, and $S^\lambda(u_e) = \bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0) = \partial e(\cdot, \lambda)(0)$, we have that u_e is the greatest function whose associated expenditure function is e. Moreover, from the proof of the preceding proposition, it follows that u_e is quasiconcave, non-decreasing, and upper-semicontinuous. Further, if $u : K \to R$ is quasiconcave, non-decreasing, and upper-semicontinuous, then it again follows from the preceding proposition that $u = u_{e_u}$, and so it is the unique function with these properties.

REMARK: It is an easy exercise to check that in the case $K = \{0\}$, conditions $(a) - (e)$ are necessary and sufficient to characterize an expenditure function.

THEOREM 2.5 (Duality between Expenditure and Utility Functions): *The mapping $u \to e_u$ is a bijection from the set of quasiconcave, non-decreasing and upper-semicontinuous utility functions $u : K \to R$ onto the set of expenditure functions $e : K^* \times R \to R_+ \cup \{+\infty\}$ that satisfy $(a) - (e)$. Furthermore, the inverse mapping is $e \to u_e$.*

PROOF: This is a straightforward consequence of the preceding propositions.

REMARKS: (1) For functions $e : K^* \times R \to R_+ \cup \{+\infty\}$ satisfying condition (a), condition (e) is stronger than conditon (b), since (b) can be restated as

$$\partial e(\cdot, \lambda)(0) \subset \partial e(\cdot, \mu)(0) \text{ for } \lambda > \mu$$

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(2) If X is finite dimensional, it can be shown that if (a) is satisfied, then condition (e) implies lower-semicontinuity of $e(p, \lambda)$ in λ for $p \in int(K^*)$.

(3) Contrary to what it is commonly believed, the conditions that characterize an expenditure function (Theorem 2.2) are not the same as those that guarantee duality (Theorem 2.5). As one could infer from the construction in the proof of Theorem 2.2, there are certain expenditure functions that cannot be generated by the class of quasiconcave, non-decreasing and upper-semicontinuous utility functions. The following is an illustrative example.

Assume that $u : R_+ \to R_+$ is given by

$$
u(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}
$$

The corresponding expenditure function is

$$
e(p, \lambda) = \begin{cases} \lambda p & \text{if } 0 \le \lambda < 1 \\ 2p & \text{if } \lambda = 1 \\ +\infty & \text{otherwise} \end{cases}
$$

This expenditure function cannot be generated by a non-decreasing utility function, since for any such representation if $e(p, \lambda) = \lambda p$ for $0 \le \lambda < 1$, then necessarily $e(p, \lambda) = p$ for $\lambda = 1$. The example also shows that condition (e) is stronger than (b).

3. EXPENDITURE FUNCTIONS AND PREFERENCES

3.1. Notation and Basic Definitions

We assume that $\succeq$ is a preorder on a convex cone $K \subset X$, where X is again a locally convex topological vector space, and K^* is the dual cone of K, $K \ne \{0\}$.

Given a preorder $\succeq$ on K, define the expenditure function $e_\succeq : K^* \times K \to R_+$ by

$$
e_\succeq(p, x) = \inf \{p \cdot x' \mid x' \succeq x\}
$$

Let $S^x_\succeq$ represent the upper-contour set $\{x' \mid x' \succeq x\}$. We observe that even if preorder $\succeq$ admits a (real-valued) utility representation, function $e_\succeq$ is still useful to address questions

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of welfare and efficiency. This function is variously known as the "money metric utility function," the "minimum income function," and by other forms [e.g., Varian (1992)]. We shall simply refer to $e_\succeq$ as the expenditure function.

Given an expenditure function, $e : K^* \times K \to R_+$, let $\succeq_e$ be the preorder on K defined by $x_1 \succeq_e x_2$ if and only if $e(\cdot, x_1) \geq e(\cdot, x_2)$, where $\geq$ here denotes the pointwise ordering of functions $[e(\cdot, x_1) \geq e(\cdot, x_2)$ if and only if $e(p, x_1) \geq e(p, x_2)$ for all p in $K^*]$. Finally, for $x \in K$, let $\partial e(\cdot, x)(0)$ denote the superdifferential of the mapping $e(\cdot, x)$ at the point 0; that is, $\partial e(\cdot, x)(0) = \{x' \in X \mid p \cdot x' \geq e(p, x)$ for all $p \in K^*\}$.

3.2. A Characterization of the Expenditure Function

As in the preceding section, we now set forth a set of necessary and sufficient conditions that single out the family of expenditure functions.

THEOREM 3.1: *Let $e : K^* \times K \to R_+$. Then e is an expenditure function for some preorder $\succeq$ on K if and only if it satisfies the following conditions*

(a) *For each $x \in K$, the mapping $e(\cdot, x)$ is concave, linearly homogeneous, and weak* upper-semicontinuous.*

(b) *For each $x \in K$,*

$$\bar{co}(\{x' \in K \mid e(\cdot, x') \geq e(\cdot, x)\} + K) = \partial e(\cdot, x)(0)$$

Moreover, under these conditions e is the expenditure function of $\succeq_e$, that is, $e_{\succeq_e} = e$.

PROOF: Pick $x \in K$. Observe that each $x' \in K$ can be regarded as a linear function on X^*, continuous in the weak* topology. Since $e(\cdot, x)$ is defined as the pointwise infimum of the family of linear functions $\{x'\}_{x' \succeq x}$, condition (a) must necessarily hold.

Let $x \in K$. Assume that $x', x'' \in K$ with $e(\cdot, x') \geq e(\cdot, x)$. Then $p \cdot (x' + x'') = p \cdot x' + p \cdot x'' \geq p \cdot x' \geq e(p, x') \geq e(p, x)$, for all $p \in K^*$. Thus, $x' + x'' \in \partial e(\cdot, x)(0)$. Since $\partial e(\cdot, x)(0)$ is a closed convex set, we have proved one direction of the inclusion in part (b). However, if this inclusion is strict then there exists $x_0 \in \partial e(\cdot, x)(0)$ with $x_0 \notin \bar{co}(\{x' \in K \mid e(\cdot, x') \geq e(\cdot, x)\} + K)$. By a classical separation theorem, there are $p \in X^*$ and $t \in R$ such that $p \cdot x_0 < t \leq p \cdot (x' + x'')$ for all $x', x'' \in K$ with $e(\cdot, x') \geq e(\cdot, x)$. Hence, $p \cdot x'' \geq 0$ for all $x'' \in K$, i.e., $p \in K^*$. Now, assume that $x' \succeq x$. Then we get $e(\cdot, x') \geq e(\cdot, x)$. Setting $x'' = 0$, we therefore obtain $p \cdot x' \geq t$. Hence, $e(p, x) \geq t > p \cdot x_0$. Also, $p \cdot x_0 \geq e(p, x)$ as $x_0 \in \partial e(\cdot, x)(0)$. This contradiction shows that (b) must hold with equality.

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Conversely, assume that e satisfies $(a) - (b)$. It follows from condition (a) that for every p in K^* and x in X,

$$e(p, x) = \inf \{ p \cdot x' \mid x' \in \partial e(\cdot, x)(0) \}$$

Furthermore, from property (b) and the fact that the infimum of a continuous linear function over a set does not change when the set is replaced by its closed convex hull, we must have

$$\begin{aligned}
e(p, x) &= \inf \{ p \cdot (x' + x'') \mid e(\cdot, x') \geq e(\cdot, x), \, x'' \in K \} = \\
&= \inf \{ p \cdot x' \mid e(\cdot, x') \geq e(\cdot, x) \} = \\
&= \inf \{ p \cdot x' \mid x' \succeq_e x \}
\end{aligned}$$

where the last equality comes from the definition of $\succeq_e$. The theorem is thus established.

REMARKS: (1) If $\succeq$ is a complete (total) preorder on K, then $e = e_{\succeq}$ satisfies condition

(c) The pointwise ordering of functions, $\geq$, on the set $\{ e(\cdot, x) \}_{x \in K}$ is a complete order.

Moreover, if (c) holds then $\succeq_e$ is a complete preorder.

(2) By the same arguments of Theorem 2.2 the mapping $e(\cdot, x)$ is non-decreasing on K^* [Remark (1), Th. 2.2]. Also, in the finite-dimensional case if K^* is a locally simplicial cone, then $e(\cdot, x)$ is also continuous [Remark (2), Th. 2.2].

3.3. Duality between Expenditure Functions and Preferences

We now prove two duality theorems between expenditure functions and preferences. In contrast to Theorem 2.5, our first duality result exploits specific properties of the function $e(p, x)$ and includes a class of preorders with upper-contour sets that are non-necessarily closed and convex. Our second duality result applies to the regular class of preorders which are non-decreasing, and with closed and convex upper-contour sets. In all these results it is not presumed that such preorders, $\succeq$, satisfy the completeness assumption: For any two vectors x_1, x_2 in K, either $x_1 \succeq x_2$ or $x_2 \succeq x_1$.

THEOREM 3.2: *Let $\succeq$ be a preorder on K. Then $\succeq_{e_{\succeq}} = \succeq$ if and only if all the upper-contour sets $S^x_{\succeq}$ satisfy the following "Hull Cancellation Property":*

(HCP) For all $x_1, x_2 \in K$, $\bar{co}(S^{x_1}_{\succeq} + K) \subset \bar{co}(S^{x_2}_{\succeq} + K)$ only if $S^{x_1}_{\succeq} \subset S^{x_2}_{\succeq}$.

PROOF: Observe that, for every $x \in K$, one has $e_{\succeq}(p, x) = \inf\{ p \cdot x' \mid x' \in \bar{co}(S^x_{\succeq} + K) \}$ and $\bar{co}(S^x_{\succeq} + K) = \partial e_{\succeq}(\cdot, x)(0)$. Also, by the transitivity of $\succeq$, we have that $x_1 \succeq x_2$ if

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and only if $S_{\succeq}^{x_1} \subset S_{\succeq}^{x_2}$. Hence, by (HCP) and the definition of $e_{\succeq}$, $x_1 \succeq x_2$ if and only if $e_{\succeq}(p, x_1) \geq e_{\succeq}(p, x_2)$ for all $p \in K^*$. That is, if and only if $x_1 \succeq_{e_{\succeq}} x_2$. The theorem is proved.

REMARKS: (1) If $\succeq$ is a complete preorder, then (HCP) can be equivalently written as

(HCP') For all $x_1, x_2 \in K$, if $\bar{co}(S_{\succeq}^{x_1} + K) = \bar{co}(S_{\succeq}^{x_2} + K)$, then $S_{\succeq}^{x_1} = S_{\succeq}^{x_2}$.

However, for non-complete preorders, (HCP') is generally weaker than (HCP).

(2) For any preorder $\succeq$ on K, $\succeq_{e_{\succeq}}$ is an extension of $\succeq$, that is, $x_1 \succeq x_2$ implies $x_1 \succeq_{e_{\succeq}} x_2$.

THEOREM 3.3 *(Duality between Expenditure Functions and Preferences): The mapping* $\succeq \to e_{\succeq}$ *is a bijection from the set of all preorders* $\succeq$ *on* K *whose upper-contour sets* $S_{\succeq}^x$ *satisfy (HCP) onto the set of functions* $e : K^* \times K \to R_+$ *satisfying (a) and (b). Furthermore, the inverse is* $e \to \succeq_e$.

PROOF: The proof follows directly from Theorems 3.1 and 3.2.

REMARK: It is worth observing the asymmetry of the results in the utility case (Ths. 2.2 and 2.5) and those of preferences (Ths. 3.1 and 3.3). In the utility case, there are expenditure functions that cannot be obtained from the duality mapping. In the case of preferences, however, duality has been established for every expenditure function, as (HCP) includes a more general class of preorders. Indeed, (HCP) includes the regular class of preorders, $\succeq$, that are non-decreasing and whose upper-level sets, $S_{\succeq}^x$, are closed and convex. [$\succeq$ non-decreasing simply means that $S_{\succeq}^x + K = S_{\succeq}^x$ for each x.] If $\succeq$ is non-decreasing, closed and convex, then $\bar{co}(S_{\succeq}^x + K) = S_{\succeq}^x$, and so (HCP) holds for such a regular class. Moreover, in this case the following stronger characterization is available.

PROPOSITION 3.4: *Assume that* $\succeq$ *is a non-decreasing preorder on* K *such that for every* x *the set* $S_{\succeq}^x$ *is closed and convex. Then for all* x_1, x_2 *in* K *the following conditions are equivalent*

(1) $x_1 \succeq x_2$

(2) $e_{\succeq}^{y^*}(x_1) \geq e_{\succeq}^{y^*}(x_2)$ *for all* $y \in K$, *where, for every* x *and* y, $e_{\succeq}^{y^*}(x) = \inf_{p \in K^*} \{p \cdot x - e_{\succeq}(p, y)\}$

(3) $x_1 \in \partial e_{\succeq}(\cdot, x_2)(0)$

PROOF: The proof follows from the following steps.
(1) is equivalent to (3). This is a consequence of the fact that $S_{\succeq}^x = \bar{co}(S_{\succeq}^x + K) =$

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$\partial e_{\succeq}(\cdot, x)(0)$, for every x in K.

(1) implies (2). Observe first that since $e_{\succeq}(\cdot, y)$ is linearly homogeneous and K^* is a cone, $e_{\succeq}^{y*}(x)$ is either 0 or $-\infty$. Suppose that $e_{\succeq}^{y*}(x_2) = 0$ [i.e., $p \cdot x_2 \geq e_{\succeq}(p, y)$ for all $p \in K^*$]. Then $x_2 \in \partial e_{\succeq}(\cdot, y)(0)$. By virtue of the equivalence of (1) and (3), we have $x_2 \succeq y$. Thus, $x_1 \succeq y$. Invoking again the equivalence of (1) and (3) we have $x_1 \in \partial e_{\succeq}(\cdot, y)(0)$. Therefore, $p \cdot x_1 \geq e_{\succeq}(p, y)$ for all $p \in K^*$. Consequently, $e^{y*}(x_1) = 0$. It follows then that $e_{\succeq}^{y*}(x_1) \geq e_{\succeq}^{y*}(x_2)$ for all $y \in K$.

(2) implies (3). By the equivalence of (1) and (3), we first note that, for all $x \in K$, one has $x \in \partial e_{\succeq}(\cdot, x)(0)$, or equivalently $e_{\succeq}^{x*}(x) = 0$. Thus, if (2) holds, then $e_{\succeq}^{x_2*}(x_1) \geq e_{\succeq}^{x_2*}(x_2) = 0$. We have therefore that $x_1 \in \partial e_{\succeq}(\cdot, x_2)(0)$. This completes the proof of the proposition.

REMARK: Observe that if either (1) and (2), or (1) and (3), are equivalent, then $\succeq$ is non-decreasing with closed, convex upper-contour sets. Indeed, under the equivalence of (1) and (2), one has for every $x \in K$ that $S_{\succeq}^x = \{x' \in K \mid e_{\succeq}^{y*}(x') \geq e_{\succeq}^{y*}(x) \text{ for all } y \in K\}$. Since the functions e^{y*} are concave, non-decreasing and weak* upper-semicontinuous, these sets are closed, convex and satisfy $S_{\succeq}^x + K \subset S_{\succeq}^x$. (As already remarked, this latter condition means that $\succeq$ is non-decreasing.) Also, under the equivalence of (1) and (3), one has $S_{\succeq}^x = \partial e_{\succeq}(\cdot, x)(0)$, and the superdifferential $\partial e_{\succeq}(\cdot, x)(0)$ is closed, convex and $\partial e_{\succeq}(\cdot, x)(0) + K \subset \partial e_{\succeq}(\cdot, x)(0)$, for each x in K.

THEOREM 3.5 (Duality between Expenditure Functions and Regular Preferences): The mapping $\succeq \rightarrow e_{\succeq}$ is a bijection from the set of non-decreasing preorders, $\succeq$, on K with closed and convex upper-contour sets, $S_{\succeq}^x$, for each x, onto the set of expenditure functions, $e : K^* \times K \rightarrow R_+$, satisfying condition (a) and

(b') For every $x \in K$, $\{x' \in K \mid e(\cdot, x') \geq e(\cdot, x)\} = \partial e(\cdot, x)(0)$.

Moreover, the inverse mapping $e \rightarrow \succeq_e$ is given by $x' \succeq_e x$ if and only if $x' \in \partial e(\cdot, x)(0)$.

PROOF: By Theorem 3.1, the function $e_{\succeq}$ satisfies condition (a). Moreover, if $\succeq$ is a non-decreasing preorder such that $S_{\succeq}^x$ is closed and convex for every $x \in K$, then it follows from Theorem 3.2 that for every x in K

$$\{x' \in K \mid e_{\succeq}(\cdot, x') \geq e_{\succeq}(\cdot, x)\} = S_{\succeq_{e_{\succeq}}}^x = S_{\succeq}^x = \bar{co}(S_{\succeq}^x + K)$$

Hence, condition (b') also holds as $\partial e(\cdot, x)(0) = \bar{co}(S_{\succeq}^x + K)$.

Conversely, assume that $e : K^* \times K \rightarrow R_+$ satisfies (a) and (b'). From condition (b') we have that $S_{\succeq_e}^x = \partial e(\cdot, x)(0)$, for every x in K. Moreover, in view of property (a), $S_{\succeq_e}^x$ is closed and convex, and $S_{\succeq_e}^x + K \subset S_{\succeq_e}^x$. Thus, $\succeq_e$ is a non-decreasing preorder with closed and convex upper-contour sets.

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Finally, it follows then from Corollary 3.3 that the mapping $\succeq \to e_{\succeq}$ is a bijection. Furthermore, by condition (b') the inverse $e \to \succeq_e$ can be defined by $x' \succeq_e x$ if and only if $x' \in \partial e(\cdot, x)(0)$.

4. FURTHER PROPERTIES OF EXPENDITURE FUNCTIONS

Theorem 2.2 illustrates that for fixed p in K^* the mapping $e_u(p, \cdot) : R \to R_+ \cup \{\infty\}$ is non-decreasing, but in general this mapping has no further properties. Nonetheless, it is well known that certain important economic assumptions, such as continuity, separability, concavity and homogeneity of the utility function u, impose further restrictions on $e_u(p, \cdot)$ [see, for instance, Blackbory, Primont and Russell (1978)]. However, relatively little is known on properties satisfied by the mapping $e_{\succeq}(p, \cdot) : K \to R_+$ [see Jacobsen (1970) for some results].

Our goal now is to analyze further properties of the functions e_u and $e_{\succeq}$. Section 4.1 focuses on preorders, $\succeq$, that admit a utility representation, u, and presents certain related properties of the functions e_u and $e_{\succeq}$ in order to gain further insights into the representation of preferences via an expenditure function. Section 4.2 is devoted to the more general class of preorders, $\succeq$, which may not admit a utility representation and establishes further properties of the mapping $e_{\succeq}(p, \cdot)$ on K for fixed p in K^*. Finally, Section 4.3 extends some classical results on expenditure functions and Hicksian demands.

4.1. Utility Representations and Expenditure Functions

We now focus on those preorders $\succeq$ on K that can be represented by a real-valued function u on K with $K \neq \{0\}$. We first show certain regular properties of expenditure functions e_u and $e_{\succeq}$, where such functions are defined as in Sections 2.1 and 3.1, respectively. Then we present a characterization of the function $e_{\succeq}$ for this class of preorders.

PROPOSITION 4.1: *For all (p, x) in $K^* \times K$ and all λ in R, we have the following*

(1) $e_u(p, u(x)) = e_{\succeq}(p, x)$

(2) $e_u(p, \lambda) = \inf \{e_{\succeq}(p, x) \mid u(x) \geq \lambda\}$

(3) *If the upper-contour set $S^\lambda(u)$ has a minimal (least preferred) element, x_λ, then*
$e_u(p, \lambda) = e_{\succeq}(p, u(x_\lambda))$

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PROOF: (1) follows from the following equalities involving the definitions of u, e_u and $e_\succeq$. That is,

$$
\begin{aligned}
e_u(p, u(x)) &= \inf\{p \cdot x' \mid u(x') \geq u(x)\} = \\
&= \inf\{p \cdot x' \mid x' \succeq x\} = e_\succeq(p, x)
\end{aligned}
$$

(2) follows similarly from the previous definitions and part (1). More precisely,

$$
\begin{aligned}
e_u(p, \lambda) &= \inf\{p \cdot x \mid u(x) \geq \lambda\} = \\
&= \inf\{\inf\{p \cdot x' \mid u(x') \geq u(x)\} \mid u(x) \geq \lambda\} = \\
&= \inf\{e_u(p, u(x)) \mid u(x) \geq \lambda\} = \\
&= \inf\{e_\succeq(p, x) \mid u(x) \geq \lambda\}
\end{aligned}
$$

(3) is again a simple consequence of the previous definitions and part (1), since

$$
\begin{aligned}
e_u(p, \lambda) &= \inf\{p \cdot x \mid u(x) \geq \lambda\} = \\
&= \inf\{p \cdot x \mid u(x) \geq u(x_\lambda)\} = \\
&= e_u(p, u(x_\lambda)) = e_\succeq(p, x_\lambda)
\end{aligned}
$$

The proof is complete.

As in Section 3 we define the order $\geq$ on $\{e(\cdot, x)\}_{x \in K}$ such that for any two elements $e(\cdot, x)$ and $e(\cdot, x')$, we say that $e(\cdot, x) \geq e(\cdot, x')$ if and only if $e(p, x) \geq e(p, x')$ for all p in K^*. Also, we shall say that the ordered set $\{e(\cdot, x)\}_{x \in K}$ admits a utility representation, if there exists a real-valued function $\tilde{u}$ on $\{e(\cdot, x)\}_{x \in K}$ such that $\tilde{u}(e(\cdot, x)) \geq \tilde{u}(e(\cdot, x'))$ if and only if $e(\cdot, x) \geq e(\cdot, x')$ for all x and x' in K.

PROPOSITION 4.2: *Let $e : K^* \times K \to R_+$. Then e is an expenditure function for some preorder $\succeq$ on K that admits a real-valued utility representation if and only if e satisfies (a) and (b) of Theorem 3.1 and the ordered set $\{e(\cdot, x)\}_{x \in K}$ admits a real-valued utility representation. Moreover, under these properties $\succeq_e$ is representable by a utility function.*

PROOF: Assume that $\succeq$ can be represented by some utility function $u : K \to R$. By Theorem 3.1, e satisfies conditions (a) and (b). Moreover, under these properties e is the expenditure function of $\succeq_e$. Now, let $\sim_e$ be the equivalence relation defined on K by $x' \sim_e x$ if $e(\cdot, x') = e(\cdot, x)$. For every equivalence class Γ of $\sim_e$, choose a point $x_\Gamma \in \Gamma$. Given x in K, denote by $\Gamma(x)$ the equivalence class containing x. Then define $\tilde{u} : \{e(\cdot, x)\}_{x \in K} \to R$ by $\tilde{u}(e(\cdot, x)) = u(x_{\Gamma(x)})$. One can see that $\tilde{u}$ is well defined, since $e(\cdot, x') = e(\cdot, x)$ implies $\Gamma(x') = \Gamma(x)$. Moreover, we claim that $\tilde{u}$ is a utility function

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on the ordered set $\{e(\cdot,x)\}_{x\in K}$. To see this, let us pick two arbitrary points x,x' in K and suppose that $\tilde{u}(e(\cdot,x')) \geq \tilde{u}(e(\cdot,x))$, that is, $u(x_{\Gamma(x')}) \geq u(x_{\Gamma(x)})$. As $e = e_{\succeq}$ and $x_{\Gamma(x')} \succeq x_{\Gamma(x)}$, we then have $e(\cdot,x') = e(\cdot,x_{\Gamma(x')}) \geq e(\cdot,x_{\Gamma(x)}) = e(\cdot,x)$. In view of the definition of $\tilde{u}$ over equivalence classes, this is sufficient to establish our claim.

Conversely, let e satisfy (a) and (b) and let $\tilde{u}$ be a utility function of the totally ordered set $\{e(\cdot,x)\}_{x\in K}$. By Theorem 3.1, $e = e_{\succeq_e}$. Hence, to conclude the proof we only need to observe that $u : K \to R$, defined by $u(x) = \tilde{u}(e(\cdot,x))$, is a utility representation for $\succeq_e$.

COROLLARY 4.3: *Let $\succeq$ be a preorder on K such that all the upper-contour sets $S_{\succeq}^x$ satisfy (HCP) of Theorem 3.2. Then $\succeq$ is representable by a utility function if and only if the totally ordered set $\{e_{\succeq}(\cdot,x)\}_{x\in K}$ is representable by a utility function.*

PROOF: This is a straightforward consequence of the previous proposition and Theorem 3.3.

4.2. Some Properties of $e_{\succeq}(p,\cdot)$ on K

We shall now be concerned with complete preorders $\succeq$ on K and present further properties of the mapping $e(p,\cdot)$ on K, where p is a fixed vector in K^*. We shall assume that $int(K) \neq \phi$. Also, we shall make use of the following hypotheses.

(A) $\succeq$ is a non-decreasing preorder.

(B) For all x in K, $S_{\succeq}^x$ is convex and closed.

(C) For all x in K, $int(S_{\succeq}^x) = \{x' \in K \mid x' \succ x\}$.

Here, $\succ$ denotes the strict preorder, i.e., $x' \succ x$ if and only if $x' \succeq x$ and not $x \succeq x'$. Observe that these assumptions are not sufficient for preorder $\succeq$ to admit a continuous utility representation. [See Mas-Colell (1986) for some results on utility representations in infinite-dimensional spaces.]

PROPOSITION 4.4: *Under Assumptions (A) to (C), the function $e(p,\cdot)$ is non-decreasing and evenly quasiconcave[1] for all $p \in K^*$.*

---

[1] A function $f : K \to R$ is evenly quasiconcave if all upper-contour sets $S^\lambda(f)$ are evenly convex sets. A set is evenly convex if it can be expressed as an intersection of open half-spaces. In other words, an evenly convex set is a convex set such that every outside point lies in a closed hyperplane disjoint from the given set. Examples (and further references) of evenly convex sets are given in Martínez-Legaz and Santos (1993).

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PROOF: Let $p \in K^*$. Assume that $x_2 \geq x_1$ for $x_1, x_2 \in K$. Then $x_2 \succeq x_1$. As $x' \succeq x_2$ implies $x' \succeq x_1$, we obtain that $e(p, x_2) \geq e(p, x_1)$. Thus, $e(p, \cdot)$ is non-decreasing. Let $\lambda \geq 0$ and define $S^\lambda(e(p, \cdot)) = \{x \in K \mid e(p, x) \geq \lambda\}$. Then $S^\lambda(e(p, \cdot)) = \bigcap_{x' \, s.t. \, p \cdot x' < \lambda} \{x \in K \mid x \succ x'\}$, an intersection of open convex sets. Hence, $S^\lambda(e(p, \cdot))$ is evenly quasiconcave. This completes the proof.

REMARK: If (C) is replaced by the weaker assumption

(C') For all x in K, the set $\{x' \in K \mid x' \succ x\}$ is convex,

then it is readily shown that $e(p, \cdot)$ is non-decreasing and quasiconcave.

PROPOSITION 4.5: *Assume that for given p in K^* the set $\{x \in K \mid p \cdot x \leq \lambda\} \cap K$ is compact, for all $\lambda \geq 0$. Then under the conditions of Proposition 4.4, the function $e(p, \cdot)$ is non-decreasing, evenly quasiconcave, and continuous on K.*

PROOF: It only remains to prove continuity. Let $\lambda > 0$ be such that $S^\lambda(e(p, \cdot)) \neq \phi$ and $x_0 \notin S^\lambda(e(p, \cdot))$. From the compactness assumption there is $\hat{x}_0 \succeq x_0$ with $e(p, x_0) = p \cdot \hat{x}_0$. The vector $\hat{x}_0$ is not maximal for $\succeq$ since, otherwise, for all $x \in K$, we would have $\hat{x}_0 \succeq x$ and hence $\lambda > e(p, x_0) = p \cdot \hat{x}_0 \geq e(p, x)$, a contradiction to $S^\lambda(e(p, \cdot)) \neq \phi$. Thus, $\{x \in K \mid x \succ \hat{x}_0\} \neq \phi$. Therefore, by Assumption (C) we can choose $\epsilon > 0$ small and a vector $\hat{x}_\epsilon \succ x_0$ with $p \cdot \hat{x}_\epsilon < \lambda - \epsilon$. Given that $x \in S^\lambda(e(p, \cdot))$, we have $e(p, x) \geq \lambda > \lambda - \epsilon > p \cdot \hat{x}_\epsilon$, and so $x \succ \hat{x}_\epsilon$. It follows that the closure, $cl \, S^\lambda(e(p, \cdot)) \subset \{x \in K \mid x \succeq \hat{x}_\epsilon\}$, as this last set is closed. Since $\hat{x}_\epsilon \succ \hat{x}_0 \succeq x_0$, we obtain that $x_0 \notin cl \, S^\lambda(e(p, \cdot))$. This proves that $S^\lambda(e(p, \cdot))$ is closed, i.e., the upper-semicontinuity of $e(p, \cdot)$.

We next show lower-semicontinuity. Consider the set $\hat{S}^\lambda(e(p, \cdot)) = \{x \in K \mid e(p, x) > \lambda\}$. If $\hat{S}^\lambda(e(p, \cdot))$ is not open, then there exist $\epsilon > 0$ and x such that $e(p, x) = \lambda + \epsilon$ and for every open neighborhood of x there is some y such that $e(p, y) \leq \lambda$. Consider now the set $K_\lambda = \{y \in K \mid p \cdot y \leq \lambda\}$. By assumption, K_λ is a compact set and $S^x_\succeq \cap K_\lambda = \phi$. Moreover, by Assumptions (A)-(C) and $int(K) \neq \phi$, we obtain $S^x_\succeq = \bigcap_{x' \prec x} S^{x'}_\succeq$. Hence, by the compactness of K_λ, there is $x' \prec x$ such that $S^{x'}_\succeq \cap K_\lambda = \phi$. Also, by Assumption (C), $x \in int(S^{x'}_\succeq)$. Therefore, there is an open neighborhood of x such that for every z in such neighborhood $p \cdot z > \lambda$. Thus, for every y in the same neighborhood, $e(p, y) \geq e(p, x') = \min \{p \cdot z \mid z \succeq x'\} > \lambda$. This contradiction then shows that the mapping $e(p, \cdot)$ is lower-semicontinuous. The proposition is thus established.

THEOREM 4.6: *Let $\succeq$ satisfy (A)-(B). Assume that $int[graph(\succeq)] = graph(\succ)$ and $graph(\succ) \cap (int(K))^2 \neq \phi$. Then the following statements are equivalent.*

(1) $e(p, \cdot)$ *is convex for all $p \in K^*$.*

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(2) *graph$(\succeq)$ is the intersection of K^2 with a closed half-space.*

(3) *There exists $q_0 \in K^*$ and a non-decreasing, linearly homogeneous, weak* upper-semicontinuous, concave function $f : K^* \to R_+$ such that $e(p,x) = f(p)q_0 \cdot x$ for all (p,x) in $K^* \times K$.*

(4) $\succeq$ *admits a continuous, linear utility representation.*

PROOF: (1) implies (2). By Proposition 3.4, graph$(\succeq) = \{(x',x) \in K^2 \mid x' \in \partial e(\cdot,x)(0)\}$, and by (1) this is a convex set. Also, graph$(\succ) = int[\text{graph}(\succeq)]$ is convex. By symmetry, graph$(\prec)$ is convex as well, and this is the complement of graph$(\succeq)$ in K^2. By a separation theorem, there exist q' and $q \in X^*$ and $k \in R$ such that $q' \cdot x' + q \cdot x \geq k$ for all $(x',x) \in$ graph$(\succeq)$ and $q' \cdot x + q \cdot x' \leq k$ for all $(x,x') \in$ graph$(\prec)$. Let us prove that the second inequality is strict.

Suppose $(x,x') \in K^2$ with $q' \cdot x + q \cdot x' = k$. Choose $(x_0',x_0) \in$ graph$(\succ) \bigcap (int(K))^2$. Clearly, $q' \cdot x_0' + q \cdot x_0 > k$. Then for all $t \in [0,1)$, $q' \cdot [(1-t)x_0' + tx] + q \cdot [(1-t)x_0 + tx'] = (1-t)(q' \cdot x_0' + q \cdot x_0) + t(q' \cdot x + q \cdot x') > (1-t)k + tk = k$. Hence, $(1-t)(x_0',x_0) + t(x,x') = [(1-t)x_0' + tx, (1-t)x_0 + tx'] \notin$ graph$(\prec)$ for all $t \in [0,1)$. Since graph$(\prec)$ is open, $(x,x') \notin$ graph$(\prec)$. This proves the assertion. We then have

$$graph(\succeq) = \{(x',x) \in K^2 \mid q' \cdot x' + q \cdot x \geq k\} \qquad (*)$$

(2) implies (3). Suppose $(*)$ holds. Let $p \in K^*$ and define $e_p : R_+ \to R_+$ by

$$e_p(\lambda) = \inf \{p \cdot x' \mid x' \in K, q' \cdot x' \geq \lambda\}$$

Then

$$
\begin{aligned}
e_p(\lambda) &= \inf\{p \cdot x' \mid x' \in K, q' \cdot \frac{x'}{\lambda} \geq 1\} = \\
&= \lambda \inf \{p \cdot \frac{x'}{\lambda} \mid x' \in K, q' \cdot \frac{x'}{\lambda} \geq 1\} = \\
&= \lambda \inf \{p \cdot x'' \mid x'' \in K, q' \cdot x'' \geq 1\} = \\
&= \lambda e_p(1)
\end{aligned}
$$

On the other hand, $e_p(0) = 0$. Therefore, e_p is linear. Also, from $(*)$, $e(p,x) = e_p(k-q \cdot x) = (k - q \cdot x)e_p(1)$. Hence, we must have $e(p,x) = -e_p(1)q \cdot x$ for all (p,x) in $K^* \times K$. Moreover, $e \neq 0$ (given that graph$(\succ) \neq \phi$) and so $e_p(1) > 0$. We then get that $-q \in K^*$ It suffices now to define $q_0 = -q$ and $f : K^* \to R_+$ by $f(p) = e_p(1)$. Function f has

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the same properties as $e(\cdot, x)$ for x satisfying $q_0 \cdot x = 1$, whose existence is guaranteed as $e \neq 0$.

(3) implies (4). For every x, x' in K we have $x' \succeq x$ if and only if $e(\cdot, x') \geq e(\cdot, x)$. And if and only if $(q_0 \cdot x')f \geq (q_0 \cdot x)f$. As f is non negative and different from zero, this is equivalent to $q_0 \cdot x' \geq q_0 \cdot x$.

(4) implies (2). let q_0 be a utility function for $\succeq$ with $q_0 \in X^*$. Then graph$(\succeq) = \{(x', x) \in K^2 \mid q_0 \cdot (x' - x) \geq 0\}$.

(3) implies (1). This step is obvious.
The theorem is thus established.

4.3. Extension of Some Classical Results

We now show some simple generalizations of certain classical results, concluding with a derivation of Slutsky's equation as in the pioneering work of McKenzie (1957). In the following proposition, $\succeq$ is an arbitrary (non necessarily complete) preorder on K, and we shall always assume that $K \neq \{0\}$.

PROPOSITION 4.7: *For given $p \in K^*$ and $x \in K$, assume that p has a unique minimizer $\bar{x}$ over $S_{\succeq}^x$. Then $\bar{x}$ is maximal in the budget set*

$$\{x' \in K \mid p \cdot x' \leq e_{\succeq}(p, x)\}$$

PROOF: Assume that $x' \in K$ is such that $p \cdot x' \leq e_{\succeq}(p, x)$ and $x' \succeq \bar{x}$. As $\bar{x} \in S_{\succeq}^x$, we also have $x' \in S_{\succeq}^x$. Hence, from the uniqueness of $\bar{x}$ as a minimizer, it follows that $x' = \bar{x}$. This proves the maximality of $\bar{x}$ over the above budget set.

REMARK: The preceding proposition remains valid even if $\bar{x}$ is not a unique minimizer, as long as all other minimizers under p on $S_{\succeq}^x$ are either uncomparable with or indifferent to $\bar{x}$.

PROPOSITION 4.8: *Let $\succeq$ be a complete preorder on K. For given $p \in K^*$ and $x \in K$, assume the existence of some minimizer under p on $S_{\succeq}^x$. If $\bar{x}$ is a maximal element in the budget set $\{x' \in K \mid p \cdot x' \leq e_{\succeq}(p, x)\}$ then $\bar{x}$ is one such minimizer.*

PROOF: Assume that $x \succ \bar{x}$. Then, in view of the maximality of $\bar{x}$, for every $x' \in S_{\succeq}^x$ we must have $p \cdot x' > e_{\succeq}(p, x)$. Since this is a contradiction to the existence of a minimizer under p on $S_{\succeq}^x$, it follows that $\bar{x} \in S_{\succeq}^x$. Hence, the inequality $p \cdot \bar{x} \leq e_{\succeq}(p, x)$ simply states that $\bar{x}$ minimizes p on $S_{\succeq}^x$, and in fact such weak inequality must hold with equality.

PROPOSITION 4.9: *Let $\succeq$ be a complete preorder on K. Let $\bar{x} \in K$ and $p \in K^*$.*

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Then $\bar{x}$ is the unique maximal element in the budget set $\{x \in K \mid p \cdot x \leq p \cdot \bar{x}\}$ if and only if it is the unique minimizer under p on $S_\succeq^{\bar{x}}$.

PROOF: If $\bar{x}$ is the unique maximal element in $\{x \in K \mid p \cdot x \leq p \cdot \bar{x}\}$ and $x \in S_\succeq^{\bar{x}}$ is such that $p \cdot x \leq p \cdot \bar{x}$, then x is also maximal in the budget set; moreover, by the asserted uniqueness, $x = \bar{x}$. This proves that $\bar{x}$ is the unique minimizer under p on $S_\succeq^{\bar{x}}$.

Conversely, suppose that $\bar{x}$ is the unique minimizer under p on $S_\succeq^{\bar{x}}$ and let $x \in K$ be such that $p \cdot x \leq p \cdot \bar{x}$. If $x \succeq \bar{x}$, then by the asserted uniqueness of $\bar{x}$, one again has $x = \bar{x}$. This concludes the proof of the proposition.

REMARK: The first part of the proof of this proposition does not require the completeness of $\succeq$.

In the following result we assume that $X = R^n$, $p \in int(K^*)$ and $\succeq$ is a complete preorder on K, $K \neq \{0\}$. Observe that these assumptions imply that, for each λ in R, sets of the form $\{x \in K \mid p \cdot x \leq \lambda\}$ are compact. Let $\partial e_\succeq(\cdot, x)(p)$ be the superdifferential of the concave mapping $e_\succeq(\cdot, x) : K^* \to R_+$ at p, i.e., $\partial e_\succeq(\cdot, x)(p) = \{\bar{x} \in X \mid e_\succeq(p', x) - e_\succeq(p, x) \leq \bar{x} \cdot (p' - p) \text{ for all } p' \text{ in } K^*\}$.

LEMMA 4.10: *Assume that for a given x in K the upper-contour set $S_\succeq^x$ is closed. Then*

$$\partial e_\succeq(\cdot, x)(p) = \bar{co}\{x' \in K \mid x' \succeq x, p \cdot x' = e_\succeq(p, x)\}$$

PROOF: One can write

$$e_\succeq(p, x) = \inf \{p \cdot x' \mid x' \succeq x, p \cdot x' \leq p \cdot x\}$$

Since $\{x' \in K \mid x' \succeq x, p \cdot x' \leq p \cdot x\}$ is a compact set, the lemma is just a consequence of the corresponding concave version of Theorem 1.9 in Auslender (1976).

COROLLARY 4.11: *Assume that for a given x in K the upper-contour set $S_\succeq^x$ is closed and convex. Then*

$$\partial e_\succeq(\cdot, x)(p) = \{x' \in K \mid x' \succeq x, p \cdot x' = e_\succeq(p, x)\}$$

Let $\nabla e_\succeq(\cdot, x)(p)$ denote the derivative or gradient vector of the mapping $e_\succeq(\cdot, x)$ at p.

COROLLARY 4.12: *Assume that for a given x in K the upper-contour set $S_\succeq^x$ is closed. Then $e_\succeq(\cdot, x)$ is differentiable at p if and only if there is a unique vector $\bar{x} = h(p, x)$ for which p attains a minimum on $S_\succeq^x$. In this case,*

$$\nabla e_\succeq(\cdot, x)(p) = h(p, x)$$

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PROOF: It is well known [e.g., Rockafellar (1970)] that a concave function is differentiable at a given point if and only if the superdifferential at such point consists of a unique element. Hence, from the previous lemma, $e_\succeq(\cdot, x)$ is differentiable at p if and only if the set of minimizers $\{x' \in K \mid x' \succeq x,\ p \cdot x' = e_\succeq(p, x)\}$ reduces to a singleton, $\{\bar{x}\}$, in which case the gradient vector $\nabla e_\succeq(\cdot, x)(p) = \bar{x} = h(p, x)$.

The implicitly defined function, $h(p, x)$, corresponds in the preference context to the so called Hicksian or compensated demand function. As in the utility case, if the mapping $e_\succeq(\cdot, x)$ is C^2 at p it follows that the matrix of partial derivatives or substitution terms, $\left(\frac{\partial h_i}{\partial p_j}(p, x) \right)$ is symmetric and negative semi-definite. Similarly, we have the following version of the Slutsky equation.

THEOREM 4.13: *Let $X = R^n$. Let $\succeq$ be a non-decreasing complete preorder on K, such that all upper-contour sets $S^x_\succeq$ are closed. Assume that, for every $p \in int(K^*)$ and $\lambda > 0$, there exists a unique maximal element $m(p, \lambda)$ in the budget set $\{x \in K \mid p \cdot x \leq \lambda\}$ and that the function $m : int(K^* \times R_+) \to R^n$ is C^1. Then for all $i, j = 1, ..., n$ and all (p, λ) in $int(K^* \times R_+)$ it must hold that,*

$$\frac{\partial m_i}{\partial p_j}(p, \lambda) = \frac{\partial h_i}{\partial p_j}(p, m(p, \lambda)) - m_j(p, \lambda) \frac{\partial m_i}{\partial \lambda}(p, \lambda)$$

PROOF: Let $(p, \lambda) \in int(K^* \times R_+)$, and define $\bar{x} = m(p, \lambda)$. Since $\succeq$ is non-decreasing, $K \neq \{0\}$, and $\bar{x}$ is uniquely detemined by (p, λ), we must have $p \cdot \bar{x} = \lambda$. Hence, by Proposition 4.9, $\bar{x} = h(p, \bar{x})$. Consequently, $p \cdot \bar{x} = e_\succeq(p, \bar{x})$. Thus, we obtain

$$m(p, e_\succeq(p, \bar{x})) = m(p, p \cdot \bar{x}) = m(p, \lambda) = \bar{x} = h(p, \bar{x}) \qquad (*)$$

Moreover, from Corollary 4.12, $e_\succeq(\cdot, \bar{x})$ is differentiable at p with $\nabla e_\succeq(\cdot, \bar{x})(p) = h(p, \bar{x})$. Considering the ith component function in (*) and partially differentiating with respect to p_j, we get

$$\frac{\partial m_i}{\partial p_j}(p, e_\succeq(p, \bar{x})) + \frac{\partial m_i}{\partial \lambda}(p, e_\succeq(p, \bar{x})) h_j(p, \bar{x}) = \frac{\partial h_i}{\partial p_j}(p, \bar{x}) \qquad .$$

The desired result is now obtained after rearranging terms and making use of the equalities $e_\succeq(p, \bar{x}) = p \cdot \bar{x} = \lambda$, $h_j(p, \bar{x}) = m_j(p, e_\succeq(p, \bar{x})) = m_j(p, \lambda)$ and $\bar{x} = m(p, \lambda)$.

5. CONCLUDING REMARKS

In this paper we offered characterizations of the expenditure function for both utility representations and preference structures. Also, we have established under minimal conditions duality theorems between expenditure functions and utility representations, and

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between expenditure functions and preference structures. With respect to previous work, our results extend to infinite-dimensional spaces and non-complete preference orders, and are free of particular continuity and boundary conditions. These latter conditions are foreign to the nature of the analysis.

For the purposes of microeconomic theory, there are two lessons to be learned from these results:

(a) The conditions that characterize an expenditure function are not the same as those that guarantee the duality between expenditure functions and utilities. This is a departure with respect to indirect utilities, where the conditions that characterize an indirect utility are the same as those that guarantee the duality of direct and indirect utilities [cf. Martínez-Legaz and Santos (1993, Th.1)]. For preference structures, the asymmetry has already been observed for indirect preferences [op. cit., Th. 2 and Prop. 3], albeit for different reasons. As illustrated in an example at the end of Section 2, some utility representations may yield expenditure functions that cannot be generated by quasi-concave, non-decreasing and upper-semicontinuous utility representations.

(b) The conditions that guarantee the duality between expenditure functions and utility representations are not the same as those that guarantee the duality between direct and indirect utilities. It follows from Theorem 2.5 that, to recover the original utility function from the expenditure function, such utility must be quasiconcave, non-decreasing, and upper-semicontinuous. However, to recover the original utility function from the indirect utility function such function must be evenly quasiconcave, non-decreasing, and satisfy certain continuity conditions at the boundary [cf., Martínez-Legaz and Santos (1993, Th. 1)]. For instance, the utility function u on R_+,

$$u(x) = \begin{cases} x & \text{if } x \leq 1 \\ x + 1 & \text{otherwise} \end{cases}$$

satisfies the required conditions for the duality of direct and indirect utilities. Such function, however, fails to be upper-semicontinuous, and hence does not satisfy the required conditions for the duality of expenditure and utility functions (Th. 2.5).

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