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IDEAL POINTS IN MULTIOBJECTIVE PROGRAMMING

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Abstract

The main object of this paper is to give conditions under which a minimal solution to a problem of mathematical programming can be transformed into a minimum solution in the usual sense of the order relations, or in every case, conditions under which that solution is adherent to the set of the points wich verify this last property. The interest of this problem is clear, since many of the usual properties in optimization (like, for instance, the analysis of the sensitivity of the solutions) are studied more easily for minimum solutions than for minimal solutions.

Key Words

Ordered Banach Space; Multiobjective Programming; Ideal Point; Strongly Proper Optimum; Maximal Optimization.

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INTRODUCTION.

Along this work we will consider a multiobjective program of the type

$$\left.\begin{array}{c}\text{Min } f(x)\\ x\in D\end{array}\right\},$$

where D will be a set and the function f is valued ia an ordered Hausdorff locally convex space Y. One of the differences between the multi objective and the scalar programming is that in the first one the searched solutions are minimal and not minimums in the usual sense of the order relations. Nevertheless, if it is achieved to transform minimal solutions into minimums, then interesting properties appear about important questions such as quality and sensitivity (see for instance [4],[5] and [6]).

One open problem is to determine under what conditions this transformation can be attained, question which is treated in the first section of the present paper where it is stated, between other results, the theorem 6 which assures under very general conditions that for a given point $y_o\in Y$ there exists a Hausdorff locally convex space W, a pointed closed cone W_+ on W and $T:Y\rightarrow W$ such that $T(y_o)\leq Tf(x)$ for every $x\in D$ and (W,W_+,T) is maximal in the sense of the definition 1. Moreover, in the theorem 9 it is proved that in the case of being Y a Banach space and under very natural conditions (which are very easily verified in practice) the last operator T is a topological isomorphism, which motivates in a natural way the concept of ideal point. The proof of the theorem 9, above mentionned, has moreover the advantage of being a constructive one being constructed W, W_+ and T.

The theorem 9 is used in the second section to prove the theorems 11 and 13 which stablish conditions to guarantee that the family of the ideal points is dense in the efficient line of the problem. This result allows that the techniques of the sensitivity analysis introduced in [5] can be applied to all the points of the efficient line.

PRELIMINARIES AND NOTATIONS.

Let D be a set and Y a Hausdorff locally convex (real) space ordered by a pointed closed convex cone Y_+ and consider the following problem of multiobjective programming:

$$\left.\begin{array}{c}\mathrm{Min}\ f(x)\\ x\in D\end{array}\right\}\quad (1),$$

where $f:D\longrightarrow Y$ is an arbitrary function.

A usual Y' will denote the dual space of Y and Y'_+ will be the dual cone of Y_+ (i.e. $Y'_+=\{y'\in Y' : y'(y)\geq 0$ for every $y'\in Y_+\}$). Let also consider

$$\mathcal{Y}_+'=\{\ y'\in Y'_+ : y'(y)>0 \text{ for every } y\in Y_+-\{0\}\ \}.$$

1. MAXIMAL OPTIMIZATION AND IDEAL POINTS.

From now on in this section, we will consider two fixed points $y_0\in Y$ and $y'_0\in\mathcal{Y}_+'$ such that $y'_0(y_0)\leq y'_0f(x)$ for every $x\in D$. Let $\mathcal{S}$ be the set of all trios (W,W_+,T) where W is a Hausdorff locally convex space ordered by the pointed closed convex cone W_+ and $T:Y\longrightarrow W$ is a surjective linear and continuous positive (i.e. $T(Y_+-\{0\})\subset W_+-\{0\}$) function such that $T(y_0)\leq Tf(x)$ for every $x\in D$. It is obvious that $\mathcal{S}\neq\emptyset$ since $(\mathbb{R},\mathbb{R}_+,y'_0)\in\mathcal{S}$.

Definition 1. If $(W,W_+,T),(U,U_+,S)\in\mathcal{S}$, we say that (W,W_+,T) precedes to

(U,U_+,S) and we write $(W,W_+,T) \gg (U,U_+,S)$, if there exists a surjective linear and continuous mapping $\pi: W \longrightarrow U$ such that it is non negative (i.e. $\pi(W_+) \subset U_+$) and $\pi T = S$.

Proposition 2. If $(W,W_+,T),(U,U_+,S) \in \mathcal{S}$ then the following assertions are equivalent:

2.1. $(W,W_+,T) \gg (U,U_+,S)$ and $(U,U_+,S) \gg (W,W_+,T)$.

2.2. There exists a topological isomorphism $\pi: W \longrightarrow U$ such that $\pi(W_+) = U_+$ and $\pi T = S$.

Proof. Clearly it will be enough to prove that 2.1 implies 2.2, so let us suppose that 2.1 holds, then there exists two surjective linear and continuous mappings $\pi_1: W \rightarrow U$ and $\pi_2: U \rightarrow W$ such that $\pi_1(W_+) \subset U_+$, $\pi_2(U_+) \subset W_+$, $\pi_1 T = S$ and $\pi_2 S = T$. Then for every $y \in Y$ we have that $\pi_2 \pi_1 T(y) = \pi_2 S(y) = T(y)$ and therefore, since T is surjective, it follows that $\pi_2 \pi_1$ is the identity on W. In a similar way it is proved that $\pi_1 \pi_2$ is the identity on U, and 2.2 follows immediately taking $\pi = \pi_1$.

Proposition 3. Let us consider on $\mathcal{S}$ the following relation: $(W,W_+,T) \simeq (U,U_+,S)$ if and only if $(W,W_+,T) \gg (U,U_+,S)$ and $(U,U_+,S) \gg (W,W_+,T)$. Then $\simeq$ is an equivalence relation on $\mathcal{S}$. Let us denote as usual by $\mathcal{S}/\simeq$ the quotient space and consider on $\mathcal{S}/\simeq$ the following relation $[(W,W_+,T)] \gg [(U,U_+,S)]$ (where $[(W, W_+,T)]$ and $[(U,U_+,S)]$ denote respectively the equivalence classes of (W,W_+,T) and (U,U_+,S)) if and only if $(W,W_+,T) \gg (U,U_+,S)$, then this relation $\gg$ is an order on $\mathcal{S}/\simeq$.

Proof. It is an immediate consequence of the proposition 2 and the definition 1.

Proposition 4. $(\mathcal{S}/\simeq, \gg)$ is inductive.

Proof. If $\{[W_i,W_{i+},T_i] : i\in I\}$ is a chain in $\mathcal{S}/\simeq$ then let us consider $W_o= \prod_{i\in I} W_i$ endowed with the product topology, $T:Y\to W_o$ the mapping defined by $T(y)=(T_i(y))_{i\in I}$ for every $y\in Y$ and $W=T(Y)$ with the relative topology. Clearly, T is surjective, linear and continuous, and $W_{o+}=\prod_{i\in I} W_{i+}$ is a pointed closed convex cone of W_o and therefore, $W_+=W\cap W_o$ is also a pointed closed convex cone of W and T is positive. Moreover, if $x\in D$ then $T_if(x)\geq T_if(y_o)$ for every $i\in I$ and so we have that $Tf(x)\geq T(y_o)$ and $(W,W_+,T)\in\mathcal{S}$. Let us prove now that $[(W,W_+,T)] \gg[(W_i,W_{i+},T_i)]$ for every $i\in I$. In fact, let be a fixed $i\in I$ and the natural projection $\pi_i:W\longrightarrow W_i$, since π_i is linear, continuous and non negative, and $\pi_iT=T_i$, the proof will be finished proving that it is also surjective. Let be $w_i\in W_i$, then since T_i is surjective there exists $y\in Y$ such that $T_i(y)=w_i$ and therefore, $T(y)\in W$ and $\pi_iT(y) = T_i(y) = w_i$.

Remark 5. Let us remark that in the last proof we have not used the fact of being $\{[(W_i,W_{i+},T_i)] : i\in I\}$ a chain and thus it has been proved in fact that given a family of $\mathcal{S}/\simeq$ there exists an element of $\mathcal{S}/\simeq$ which precedes to all the elements of the family.

Theorem 6. There exists a maximum element of $(\mathcal{S}/\simeq, \gg)$.

Proof. It follows from the proposition 4 and the Zorn's lemma the existence of a maximal element $[(W,W_+,T)]\in\mathcal{S}/\simeq$, whcih is clearly a maximum as the remark 5 states.

Lemma 7. If there exist $(U,U_+,S)\in\mathcal{S}$ such that $S:Y\longrightarrow U$ is a topological isomorphism and $[(W,W_+,T)]$ is the maximum class of $\mathcal{S}/\!\propto$, then $T:Y\rightarrow W$ is also a topological isomorphism.

Proof. Since $[(W,W_+,T)]\gg[(U,U_+,S)]$ we have that there exists a surjective, linear, continuous and non negative mapping $\pi:W\longrightarrow U$ such that $\pi T=S$. Then for every $y\in Y$ we have that $S^{-1}\pi T(y) = S^{-1}S(y) = y$ and for every $w\in W$ there exists (since T is surjective) $y\in Y$ such that $T(y)=w$ and therefore, $TS^{-1}\pi(w) = TS^{-1}\pi T(y) = TS^{-1}S(y) = T(y) = w$ and T is a topological isomorphism.

Definition 8. We say that y_o is an ideal point of the program (1) if the maximal element $[(W,W_+,T)]$ of $(\mathcal{S}/\!\propto,\gg)$ verifies that T is a topological isomorphism. Let us remark that the lemma 7 assures that the fact of being T a topological isomorphism does not depend of the choice of the representative element of the equivalence class.

If y_o is an ideal point of the program (1) and there exists $x_o\in D$ such that $y_o=f(x_o)$ then we say that x_o is an strongly proper optimum of the program (1).

We present now a sufficient condition to be y_o an ideal point of the program (1). This condition will be very useful in the next section and it presents a big capacity to be verified in concret examples of practic type.

Theorem 9. Suppose that the space Y is a Banach space. If the set

$$A' = \{y' \in Y'_+ : \|y'\| \leq 1 \text{ and } y'f(x) \geq y'(y_o) \text{ for every } x \in D\}$$

distinguishes points of Y, then y_o is an ideal point of the program (1), and if $[(W,W_+,T)]$ is the maximum of $(\mathscr{S}/\simeq,\gg)$ then W is a Banach space.

Proof. Suppose that it is already proved the existence of $(U,U_+,S) \in \mathscr{S}$ such that S is a topological isomorphism, then it follows from the lemma 7 that T is also a topological isomorphism and therefore, y_o is an ideal point of the program (1) and W is a Banach space.

Let us construct now the element $(U,U_+,S) \in \mathscr{S}$. Let B' be the closed unit ball of Y' endowed with the weak* topology, then it follows from the Alaoglu-Bourbaki theorem that B' is compact and so we can consider the space $\mathscr{C}(B')$ of the weak* continuous real functions defined on B' with the supremum norm. Consider now the function $S:Y \longrightarrow \mathscr{C}(B')$ such that $S(y)(y') = y'(y)$ for every $y \in Y$ and every $y' \in B'$, then clearly $S(y) \in \mathscr{C}(B')$ for every $y \in Y$ and S is linear. As it is well known, it follows from the Hahn-Banach theorem that

$$\begin{aligned} \|y\| &= \sup \{|y'(y)| : y' \in B'\} \\ &= \sup \{|S(y)(y')| : y' \in B'\} \\ &= \|S(y)\|_\infty \end{aligned}$$

for every $y \in Y$, and therefore, $S:Y \to U$ is a linear and bijective isometry (and in particular continuous) and $U=S(Y)$ is complete (since Y is a Banach space).

The set $\mathscr{C}(B')_+ = \{ h \in \mathscr{C}(B') : h(a') \geq 0 \text{ for every } a' \in A'\}$ is obviously a closed convex cone of $\mathscr{C}(B')$ and so $U_+ = U \cap \mathscr{C}(B')_+$ is also a closed convex cone, let us see that it is also pointed. In fact, if there exists $u \in U$ such that $u,-u \in U_+$ then there exists $y \in Y$ such that $S(y)=u$ and therefore,

$a'(y) = S(y)(a') \geq 0$ for every $a' \in A'$. Moreover, $T(-y)=-u \in U_+$ and then $a'(y) \leq 0$ for every $a' \in A'$. Therefore, since A' distinguishes points of Y, it follows that $y=0$ and so $u=0$.

Let us prove now that S is positive. In fact, if $y \in Y_+ - \{0\}$ then $S(y) \neq 0$ since S is bijective, and $S(y)(a') = a'(y) \geq 0$ for every $a' \in A'$ (because $A' \subset Y'_+$).

Moreover, it follows from the definition of A' that the inequality $a'(y_0) \leq a'(f(x))$ holds for every $a' \in A'$ and every $x \in D$, and therefore, $S(y_0) \leq Sf(x)$ for every $x \in D$ and $(U, U_+, S) \in \mathcal{S}$.

Remark 10. First remark that A' distinguishes points of Y if and only if the set $\{y' \in Y'_+ : y'(y_0) \leq y'f(x)$ for every $x \in D\}$ distinguishes points of Y. Also, proceeding like in the proof of the lemma 7 it can be proved that if $[(W, W_+, T)]$ is the maximum of $(\mathcal{S} /\simeq, \gg)$ and π is the projection of W into the space U constructed in the last proof, then π is a topological ismorphism, and so it can be taken $W=U$ and the last proof is a constructive one with respect W. Also taking $W=U$, it can be proved without any difficulty that $W_+ = \bigcap_{C \in \mathcal{C}} C$ being $\mathcal{C}$ the family of the all pointed closed convex cones C of W such that for the order induced by C on W the inequality $S(y_0) \leq Sf(x)$ holds for every $x \in D$, and $S(Y_+) \subset C$.

2. DENSITY OF THE IDEAL POINTS IN THE EFFICIENT LINE.

Suppose that $y_0 \in Y$ is an ideal point of the program (1) and $x_0 \in D$ verifies that $f(x_0) = y_0$. If $[(W, W_+, T)]$ is the maximum element of $(\mathcal{S} /\simeq, \gg)$ (which has been constructed in the proof of the theorem 9), then from the condition of being T a topological isomorphism which verifies $Tf(x)\geq Tf(x_o)$ for every $x\in D$, it is not difficult to endow the program (1) with a dual program which measures the sensitivity of $f(x_o)$ with respect the changes in D (see [5]). Nevertheless, there are "few" optimums of (1) $x\in D$ verifying that f(x) is an ideal point, so we dedie this section to prove that in any case, if $x\in D$ is an optimum of the program (1) then, under certain conditions, there exists an ideal point of (1) which is "so near to f(x)" as we want, and obviously, for this ideal point it is possible a satisfactory duality theory (see [5]).

Along this section let us assume that the space Y is a Banach space.

Theorem 11. Suppose that $\mathcal{Y}_+'$ is non void and that Y_+' distinguishes points of Y. If $x_o\in D$ is such that the set $\{y'\in Y_+'-\{0\} : y'f(x_o)\leq y'f(x)$ for every $x\in D\}$ is not void and f(D) is a bounded subset (in norm) of Y, then $f(x_o)$ is adherent to the set of the ideal points of the program (1).

Proof. Let be $y_o'\in Y_+'-\{0\}$ such that $y_o'f(x_o)\leq y_o'f(x)$ for every $x\in D$, $y_o\in Y$ such that $\|y_o\|=1$ and $y_o'(y_o)=-\alpha<0$, and finally $\varepsilon>0$, and consider $y_1=f(x_o)+\varepsilon y_o$, then $\|f(x_o)-y_1\|=\varepsilon$ and so if we prove that y_1 is an ideal point of the program (1) then the proof will be finish. First let us remark that

$$y_o'f(x) \geq y_o'(y_1) + \alpha\varepsilon \tag{11.1}$$

for every $x\in D$.

Since f(D) is bounded (in norm) there exists $k>0$ such that $\|f(x)\|\leq k$ for every $x\in D$. Let be

$$r = \begin{cases} \mathrm{Min}\left\{\frac{\alpha\varepsilon}{2k}, \frac{\alpha\varepsilon}{2\|y_1\|}\right\} & \text{if } \|y_1\| > 0 \\ \frac{\alpha\varepsilon}{2k} & \text{if } \|y_1\| = 0 . \end{cases}$$

Let be now $y' \in Y'$ such that $\|y'-y_o\| \leq r$, then

$$|y'(y_1)-y_o'(y_1)| \leq \|y'- y_o'\| \; \|y_1\| \leq \frac{\alpha\varepsilon}{2}$$

and therefore,

$$y'(y_1) \in (-\infty, y_o'(y_1)+ \frac{\alpha\varepsilon}{2}]. \tag{11.2}$$

Moreover, for every $x \in D$ we have that

$$|y'f(x)-y_o'f(x)| \leq \|y'-y_o'\| \; k \leq \frac{\alpha\varepsilon}{2}$$

and

$$y'f(x) \in [y_o'f(x)- \frac{\alpha\varepsilon}{2}, +\infty). \tag{11.3}$$

From (11.1), (11.2) and (11.3) it is deduced that

$$y'f(x) \geq y_o'f(x) - \frac{\alpha\varepsilon}{2} \geq y_o' (y_1) + \frac{\alpha\varepsilon}{2} \geq y'(y_1) \tag{11.4}$$

is verified for every $x \in D$ and every $y' \in Y'$ such that $\|y'-y_o'\| \leq r$.

Since $\mathcal{Y}_+'$ is non void, adding to y_o' an element of $\mathcal{Y}_+'$ with sufficiently little norm, it is obtained a new element of $\mathcal{Y}_+'$ which verifies the conditions of (11.4), and therefore, the set $\mathcal{S}$ associated to y_1 is non void and the theorems 6 and 9 can be applied.

To prove that y_1 is an ideal point of the program (1), it is enough to see that the set $M = \{y' \in Y_+' : y'(y_1) \leq y'f(x) \text{ for every } x \in D\}$ distnguishes points of Y (as it follows from the theorem 9 and the remark 10). Let be y_2, y_3

$\in Y$ with $y_2 \neq y_3$, since it follows immediately from (11.1) that $y'_o \in M$, then if we have that $y'_o(y_2) \neq y'_o(y_3)$ the proof is finished. Suppose that $y'_o(y_2)=y'_o(y_3)$, then there exists $y'_1 \in Y'_+$ such that $\|y'_1\|=1$ and $y'_1(y_2) \neq y'_1(y_3)$. Let be $y'_2=y'_o+ry'_1$ then $y'_2 \in Y'_+$ and $\|y'_2-y'_o\|=r$, and it follows from (11.4) that $y'_2 \in M$. Moreover, from $y'_o(y_2)=y'_o(y_3)$, $y'_1(y_2) \neq y'_1(y_3)$ and $r>0$, it follows that $y'_2(y_2) \neq y'_2(y_3)$.

Remark 12. The (non very hard) hypothesis, assumed in the last theorem, of being $\mathcal{Y}'_+ \neq \emptyset$ can be removed, since it follows from the proof of the theorem that this hypothesis is only used to assure that the class $\mathcal{S}$ associated to the element y_1 is non void, and following the proof of the theorem 9, it is noticed that the construction of the element (U,U_+,S) proves that this class is non void.

If in addition of the hypothesis of the theorem 11, D is a convex subset of some vector space X, the function $f:D \longrightarrow Y$ is convex and the cone Y_+ has non void interior (i.e. $\mathring{Y}_+ \neq \emptyset$), then it is known that from the separation theorems it is deduced that every optimum x_o of the program (1) verifies that $\{y' \in Y'_+-\{0\} \ : \ y'f(x_o) \leq y'f(x) \text{ for every } x \in D\}$ is non void, and therefore, for every optimum x_o of (1), $f(x_o)$ is adherent to the set of the ideal points of the program (1).

When the set f(D) is non bounded and in the context of the convex programming, we have the following result:

Theorem 13. Let us assume the following hypothesis:

13.1. The space Y is a reflexive Banach space.

13.2. D is a convex subset of a vector space X and the function $f:D \longrightarrow Y$ is convex.

13.3. Y'_+ distinguishes points of Y.

13.4. There exits $x_o \in D$ and $y'_o \in \mathcal{Y}'_+$ such that $y'_o f(x_o) \leq y'_o f(x)$ for every $x \in D$.

13.5. $f(x_o)$ is a denting point of the set $f(D)+Y_+$.

13.6. $y'_o f(x_o) < y'_o(y)$ holds for every $y \in \overline{f(D)+Y_+} - \{f(x_o)\}$.

Then, $f(x_o)$ is adherent to the set of the ideal points of the program (1).

Proof. Like in the proof of the theorem 11, let be $\varepsilon>0$, $y_o \in Y$ verifying that $\|y_o\|=1$ and $y'_o(y_o)=-\alpha<0$, and $y_1=f(x_o)+\varepsilon y_o$.It is clear that $\|y_1-f(x_o)\|=\varepsilon$ and the theorem is proved showing that y_1 is an ideal point of the program (1), and for this it is enough, following the theorem 9, to verify that the set M = $\{y' \in Y'_+ : y'(y_1) \leq y'f(x)$ for every $x \in D\}$ distinguishes points of Y. Let be y_2 and y_3 two different points of Y. If $y'_o(y_2) \neq y'_o(y_3)$ we have finished since $y'_o \in M$. Suppose that $y'_o(y_2)=y'_o(y_3)$, then since Y'_+ distinguishes points of Y, there exists $u' \in Y'_+$ such that $u'(y_2) \neq u'(y_3)$. Evidently, for every $t \in (0,1)$ we have that the element $u'_t = tu'+(1-t)y'_o$ belongs to Y'_+ and it distinguishes y_2 and y_3. So let us prove that $u'_t \in M$ for some $t \in (0,1)$. Consider now $D_n=f^{-1}(B_n)$ where B_n denotes the closed ball of Y with center $f(x_o)$ and radius n ($\in \mathbb{N}$). Proceeding like in the proof of thetheorem 11 it is proved the existence of $r_n>0$ ($n \in \mathbb{N}$) such that if $\|u'_t - y'_o\| \leq r_n$ then

(13.1) $$u'_t f(x) \geq u'_t(y_1)$$

for every $x \in D_n$. In particular let us take a decreasing sequence (s_n) which converges to 0 in $\mathbb{R}$ and such that $\|u'_n - y'_o\| \leq r_n$ with $u'_n = u'_{s_n}$ ($n \in \mathbb{N}$). If $u'_n \notin M$ for every $n \in \mathbb{N}$ then it follows from (13.1) the existence of $(x_n) \subset D$ such that

(13.2) $$u'_n f(x_n) < u'_n(y_1) \quad \text{and} \quad \|f(x_n)-f(x_o)\|>n$$

for every $n\in\mathbb{N}$. Since

$$\lim_n \|f(x_n)-f(x_o)\| = +\infty ,$$

we can find $(t_n)\subset(0,1)$ such that

(13.3) $$2 \geq \|t_n f(x_n)+(1-t_n)f(x_o)-f(x_o)\| > 1$$

for every $n\in\mathbb{N}$. Since the set D and the function f are convex we have that $\hat{x}_n = t_n x_n+(1-t_n)x_o \in D$ and $f(\hat{x}_n)\leq t_n f(x_n)+(1-t_n)f(x_o)$ $(n\in\mathbb{N})$. If

(13.4) $$\hat{y}_n = t_n f(x_n)+(1-t_n)f(x_o)-f(\hat{x}_n)$$

then $\hat{y}_n\in Y_+$ and it follows from (13.2) that

(13.5) $$\begin{cases} u'_n[f(\hat{x}_n)+\hat{y}_n] = t_n u'_n f(x_n) + (1-t_n)\, u'_n f(x_o) \\ \qquad\qquad \leq t_n u'_n(y_1) + (1-t_n)\, u'_n f(x_o) \end{cases} \quad (n\in\mathbb{N}).$$

Moreover, since $u'_n = s_n u'+(1-s_n)y'_o$ and $s_n \to 0$, we have that

$$\lim_n u'_n(y_1) = y'_o(y_1) < y'_o f(x_o) = \lim_n u'_n f(x_o)$$

and therefore, there exists $n_o\in\mathbb{N}$ such that $u'_n(y_1) \leq u'_n f(x_o)$ for every $n\geq n_o$. Now, it follows from (13.5) that

(13.6) $$u'_n(f(\hat{x}_n)+\hat{y}_n) \leq t_n u'_n f(x_o) + (1-t_n)\, u'_n f(x_o) = u'_n f(x_o)$$

for every $n\geq n_o$. From (13.3) and (13.4), it is deduced that $f(\hat{x}_n)+\hat{y}_n \in B_2$ for every $n\in\mathbb{N}$ and since the space Y is reflexive, B_2 is $\sigma(Y,Y')$-compact and there exists an accumulation point (in $\sigma(Y,Y')$) $z\in Y$ of the sequence $\{f(\hat{x}_n)+\hat{y}_n\}_{n\in\mathbb{N}}$. Therefore, it is easily proved (recall that $u'_n \longrightarrow y'_0$ in the strong topology of Y') that $y'_0(z)$ is an accumulation point of the sequence $\{u'_n(f(\hat{x}_n)+\hat{y}_n)\}_{n\in\mathbb{N}}$ and it follows from (13.6) that

$$y'_0(z) \le \lim_n u'_n f(x_0) = y'_0 f(x_0). \tag{13.7}$$

From (13.3) and (13.4) it follows that $f(\hat{x}_n)+\hat{y}_n \notin B_1$ and then $z\in \overline{[f(D)+Y_+]-B_1}^{\sigma}$, where the adherence is taken in the topology $\sigma(Y,Y')$, and therefore $z \in \overline{co}\,[(f(D)+Y_+)-B_1]$ where the adherence is taken now in the strong topology, since the Hahn-Banach theorem and the convexity of the set $co[(f(D)+Y_+)-B_1]$ assures that the weak and the strong adherence of this set coincide. Since $f(x_0)$ is a denting point of the set $f(D)+Y_+$ it is clear that $z\neq f(x_0)$ and then (13.7) is a contradiction wiyh the hypothesis of the theorem, contradiction which appears on supposing that $u'_n\notin M$ for every $n\in\mathbb{N}$.

Remark 14. If in the last proof we change in (13.3) the numbers 1 and 2 by suitable scalars, then the hypothesis of being $f(x_0)$ a denting point of $f(D)+Y_+$ can be changed by the following much more weak condition: There exists $\lambda>0$ such that $f(x_0) \notin \overline{co}\,\{[f(D)+Y_+]-B(f(x_0),\lambda)\}$.

In the particular case of being $Y=\mathbb{R}^n$, in the last proof it can be proved that $z\neq f(x_0)$ using the (strong) compactness of the closed balls, whithout assuming the condition about the denting point. Also, changing in (13.3) the numbers 1 and 2 by suitable scalars, the hypothesis 13.6 can be replaced by the condition of being $\{y\in \overline{f(D)+Y_+} : y'_0(y)=y'_0 f(x_0)\}$ a bounded (in norm) subset of Y.

Also, if $Y=\mathbb{R}^n$ and Y_+ is the usual (positive) cone of $\mathbb{R}^n$, then reasonning like in the preceding paragraph, the hypothesis 13.5 and 13.6 can be substituted by the condition of being $\{f(x)\in Y : x\in D$ and $y_0'f(x)=y_0'f(x_0)\}$ a bounded (in norm) subset of Y.

Finally, the condition of being the Banach space Y reflexive can be changed by being the Banach space Y a dual space (Y=Z'). Then the proof holds the same changing in 13.6 the adherence of $Y_+ + f(D)$ by $\overline{Y_+ + f(D)}^{\sigma(Y,Z)}$.

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