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A DISCRETE APPROACH TO CONTINUUM ECONOMIES

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Abstract

We consider a perfectly competitive economy in which only a finite number of different agents' characteristics can be distinguished. We associate this economy with an n-agents' economy with an ordered Banach commodity space, and we prove that the continuum and the discrete approach to the equilibrium problem can be considered equivalent.

Key words: Continuum economies, core, Walrasian equilibrium.

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1 Introduction

In economies where the mechanism of perfect competition prevails, and in particular in problems related to Walrasian equilibrium, the participating agents should be modelled so that they act as price takers. Aumann (1964) emphasizes this fact to criticize the classical n–agents model of Arrow-Debreu because, mathematically, it is not possible to assume that the influence of each agent is negligible. Alternatively, he proposed the study of economies with a continuum of agents, and the use of the integral instead of the sum to represent the average allocation rather than the aggregate.

Following this approach, for economies whit finitely many commodities, he showed (1964) that the core and the set of equilibrium allocations coincide (core–Walras equivalence), and he established (1966) the existence of a competitive equilibrium in exchange economies where the preferences of the agents need not be convex. Bewley (1970) proved a core–Walras equivalence result for economies in which the commodity space is $\mathcal{L}_\infty(M, \mathcal{M}, \mu)$, the space of essentially bounded, real-valued, measurable functions on the measure space $(M, \mathcal{M}, \mu)$. Ostroy (1984) established the existence of equilibrium in more general economies with a continuum of agents and infinitely many commodities. Kehoe, Levine, Mas-Collel and Zame (1989) analyzed the problem of determinacy of equilibrium in the context of separable Hilbert spaces. More recently, Rustichini and Yannelis (1991) generalized Aumann's results for economies in which the commodity space is an ordered separable Banach space. Moreover, they provided a counterexample which shows that the above result fails if an "extremely desirable commodity" does not exist.

In all these models, the influence of each agent (or of a set of measure zero) is null because the integral does not change if the behavior of such a set of agents is modified. The mathematical elegance of this approach may not be inmune to the criticism that, frecuently enough, economic reality only allows us to distinguish a finite number of participants.

In this paper we will consider a perfectly competitive economy (i.e., an economy with a continuum of agents) with infinitely many commodities, in which only a finite number of different agents' characteristics can be distinguished.

This continuum economy can be interpreted as a discrete economy with n agents in which each agent is repeated infinitely many times. The aim of this paper is to emphasize that the continuum and the discrete treatment can be considered equivalent. To be precise, under the hypothesis of convexity of preferences, we show that an allocation belongs to the core (respectively is an equilibrium allocation) in the continuum economy if and only if it is an Edgeworth equilibrium (respectively is an equilibrium allocation) in the discrete economy. By using the above result, we establish the existence of core allocations and Walras equilibrium in the continuum economy under the standard hypothesis for n–agents' economies. As a consecuence, we obtain a core–Walras equivalence for this particular economy.

The convexity of preferences is indispensable to prove a lemma which sustains the proof of the theorems. This must be so because we know that the core of an n–agents' economy can be empty without this requirement and, on the other hand, as a continuum economy, Aumann's result guarantees the non–emptyness of the core. This means that if we discreticize a continuum economy we ought to suppose that the preferences of the representative agents are convex.

We refer the reader to Aliprantis, Brown and Burkinshaw (1989) for notations and results on n–agents' economies with infinitely many commodities.

2 The model and main results

The commodity space is an ordered Banach space denoted by E. We denote by E_+ and E' the positive cone of E and the space of continuous functionals on E, respectively. We consider an exchange economy in which only a finite number of different agents can be distinguished. To guarantee that each individual agent has no influence, the set of all agents is represented by the continuum [0,1] divided into n pairwise disjoint subintervals each of which represents a type of agent.

We denote this exchange economy by $\mathcal{E}_C$

$$\mathcal{E}_C = (< E, E' >, \preceq_t, \mathrm{w}(t), t \in I = [0,1]),$$

where $< E, E' >$ represents the commodity-price duality. The set of agents is represented by the continuum I = [0,1] $= \bigcup_{i=1}^{n} I_i$, where $I_i = [a_{i-1}, a_i)$ if $i \neq n$ and $I_n = [a_{n-1}, 1]$ denotes the set of agents of type i. Each consumer

$t \in I$ is characterized by her consumption set E_+, her initial endowment $\mathrm{w}(t) = \omega_i \in E_+$ for all $t \in I_i$, and her preference relation $\preceq_t = \preceq_i$ for all $t \in I_i$. The notation $x \preceq_t z$ is read "the bundle z is at least as good as the bundle x for the consumer t" and $x \prec_t z$ is read "z is preferred to x" or "z is better than x for the consumer t".

An allocation is a Bochner integrable function $f : I \longrightarrow E_+$. An allocation is said to be feasible if $\int_I f d\mu = \int_I \mathrm{w} d\mu = \sum_{i=1}^{n} \mu(I_i)\omega_i$, where μ denotes the Lebesgue measure on the Borel subset of [0,1] (see Diestel and Uhl (1977) for the definition of the integral of a Banach-valued function). A coalition of agents is a measurable set $S \subset [0,1]$ such that $\mu(S) > 0$. We say that a coalition S improves upon an allocation f whenever there exists another allocation g such that $\int_S g = \int_S \mathrm{w}$ and $f(t) \prec_t g(t)$ μ–a.e. in S. The core of the economy $\mathcal{E}_C$, denoted by $C(\mathcal{E}_C)$, is the set of all feasible allocations that cannot be improved upon by any coalition of agents. A Walrasian equilibrium for the economy $\mathcal{E}_C$ is a pair (f,p) where f is a feasible allocation and $p \in E'_+$ is a non zero price such that $f(t) \in B_t(p) = \{x \in E_+/p(x) \leq p(\mathrm{w}(t))\}$ and if $f(t) \prec_t x$ then $x \notin B_t(p)$.

This model allows to interpret economies with n agents as continuum economies where the ith agent is the representative of infinitely many identical agents. We can consider this model as a replica of an n–agents' economy in which each agent is repeated infinitely many times. Moreover we can consider this model as representative of an economy with n non–homogeneous agents, where the relative influence of the ith agent is represented by the measure $\mu(I_i)$ of each subinterval I_i.

On the other hand, if we are not able to distinguish the infinitely many different characteristics of the agents in a continuum economy, we will wish to treat such a situation as a discrete economy.

For these reasons, to each economy $\mathcal{E}_C$ we will associate a discrete economy with n agents $\mathcal{E}_n = (< E, E' >, \preceq_i, \omega_i, i \in \{1, \cdots, n\})$. Then, an allocation f in our economy can be interpreted as an allocation $x = (x_1, \cdots, x_n)$, in the n–agents' economy, $\mathcal{E}_n$, where $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f$. Reciprocally, an allocation $x = (x_1, \cdots, x_n)$, in the n–agents' economy can be interpreted as an allocation f in $\mathcal{E}_C$, where f is the step function defined by $f(t) = x_i$ if $t \in I_i$.

Let us consider the discrete economy $\mathcal{E}_n$ associated with our economy

$\mathcal{E}_C$. An n–tuple $x = (x_1, \cdots, x_n)$ is a feasible allocation if and only if the associated step function f is a feasible allocation in $\mathcal{E}_C$, i.e., $(x_1, \cdots, x_n)$ is feasible iff $\sum_{i=1}^{n} \mu(I_i)x_i = \sum_{i=1}^{n} \mu(I_i)\omega_i$.[1]

We will say that a (discrete) coalition $S \subset \{1, \cdots, n\}$ improves upon an allocation $x = (x_1, \cdots, x_n)$ whenever there exists another allocation $z = (z_1, \cdots, z_n)$ such that: $\sum_{i \in S} z_i = \sum_{i \in S} \omega_i$, and $x_i \prec_i z_i$ for all $i \in S$. The core of the economy $\mathcal{E}_n$ is the set of feasible allocations that cannot be improved upon by any coalition of agents.

If r is any positive integer then the r–fold replica economy $r\mathcal{E}_n$ of $\mathcal{E}_n$ is a new exchange economy with $r\,n$ consumers indexed by (i, j), $i = 1, \cdots, n$; $j = 1, \cdots, r$ such that each consumer (i, j) has a preference relation $\preceq_{ij} = \preceq_i$ and an initial endowment $w_{ij} = w_i$. Every allocation $x = (x_1, \cdots, x_n)$ in the economy $\mathcal{E}_n$ leads in a natural way to an allocation $rx = (x_{11}, \cdots, x_{1r}, \cdots, x_{n1}, \cdots, x_{nr})$ for the r–fold replica economy $r\mathcal{E}_n$ by letting $x_{ij} = x_i$ for $j = 1, \cdots, r$ and $i = 1, \cdots, n$. Any such allocation is called an equal treatment allocation for $r\mathcal{E}_n$. In this manner every allocation of $\mathcal{E}_n$ can be considered as an allocation for every r–fold replica economy $r\mathcal{E}_n$.

An Edgeworth equilibrium for the exchange economy $\mathcal{E}_n$ is a feasible allocation $(x_1, \cdots, x_n)$ that belongs to the core of every replica economy. A Walrasian (or a competitive) equilibrium for the economy $\mathcal{E}_n$ is a pair (x, p) consisting of a feasible allocation $x = (x_1, \cdots, x_n)$ and a non-zero price $p \in E'$ such that $x_i \in B_i(p) = \{x \in E_+ / p(x) \leq p(w_i)\}$ and $x_i \prec_i x$ implies that $x \notin B_i(p)$.

The aim of this paper is to emphasize that the continuum and the discrete approach can be considered equivalent. The next theorem makes this equivalence precise:

Theorem 1 *If the preference relation $\preceq_i$ is convex and continuous for each $i = 1, \cdots, n$, then:*

a) If $((x_1, \cdots, x_n), p)$ is a Walrasian equilibrium for $\mathcal{E}_n$, then (f, p) is a Walrasian equilibrium for $\mathcal{E}_C$.

[1] Note that his feasibility condition is standard if $\mu(I_i) = \frac{1}{n}$ for all i. Moreover, if $\mu(I_i) = \frac{a_i}{q}$, we could divide the n subintervals into q subintervals of identical length. If the allocation is an equal treatment allocation, the standard feasibility condition can be formulated as $\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} a_i \omega_i$ which is the same as $\sum_{i=1}^{n} \mu(I_i)x_i = \sum_{i=1}^{n} \mu(I_i)\omega_i$

b) If $(x_1, \cdots, x_n)$ is an Edgeworth equilibrium for $\mathcal{E}_n$, then f is a core allocation for $\mathcal{E}_C$.

c) If (f, p) is an equilibrium for $\mathcal{E}_C$, then $((x_1, \cdots, x_n), p)$ is an equilibrium for $\mathcal{E}_n$.

d) If f is a core allocation for the economy $\mathcal{E}_C$, then $(x_1, \cdots, x_n)$ is an Edgeworth equilibrium for the economy $\mathcal{E}_n$.

To establish this theorem we need the following lemma based in the mean value of the integral:

Lemma *Let $\preceq$ be a convex and continuous preference relation. If $S \subset I$ has positive measure, $g : S \longrightarrow E_+$ is an integrable function and $x \in E_+$ is such that $x \prec g(t)$ for all t, then:*

$$x \prec \frac{1}{\mu(S)} \int_S g$$

Proof.- The measurability of g implies that there exists a compact set $K \subset S$ of positive measure such that the restriction of g on K is continuous and thus there exits $b \in E_+$ such that $x \prec b \preceq g(t)$ for all $t \in K$. We have $y = \frac{1}{\mu(K)} \int_K g \in \overline{Co}(g(K))$ (see Diestel and Uhl (1977)), and $\overline{Co}(g(K)) \subset \{z \in E_+ / b \preceq z\}$ because $\{z \in E_+ / b \preceq z\}$ is closed and convex. Thus $x \prec b \preceq y$. Analogously, if $\mu(S \setminus K) > 0$ then $x \preceq \frac{1}{\mu(S \setminus K)} \int_{S \setminus K} g$, and again by convexity:

$$x \prec \frac{\mu(K)}{\mu(S)} \Big(\frac{1}{\mu(K)} \int_K g \Big) + \frac{\mu(S) - \mu(K)}{\mu(S)} \Big(\frac{1}{\mu(S \setminus K)} \int_{S \setminus K} g \Big) = \frac{1}{\mu(S)} \int_S g$$

Q.E.D.

Note that the lemma is true if in the hypothesis and in the statement we substitute $\prec$ for $\preceq$.

Proof of theorem 1.-

a) Let $((x_1, \cdots, x_n), p)$ be a Walrasian equilibrium for $\mathcal{E}_n$, then:

i) $\int_I f = \sum_{i=1}^{n} \int_{I_i} x_i = \sum_{i=1}^{n} \mu(I_i) x_i = \sum_{i=1}^{n} \mu(I_i) \omega_i = \sum_{i=1}^{n} \int_{I_i} \omega_i = \int_I$ w

ii) $f(t)$ is a maximal element for $B_t(p) = B_i(p)$ for all t$\in I_i$

and thus (f,p) is a Walrasian equilibrium for $\mathcal{E}_C$.

b) Let $(x_1,\cdots,x_n)$ be an Edgeworth equilibrium for $\mathcal{E}_n$. Let us assume that a coalition S and a function g such that $\int_S g = \int_S \mathrm{w}$ and $f(t) \prec_t g(t)$ μ–a.e. in S exist. Let $S_i = S \cap I_i$ and $J = \{i/\beta_i = \mu(S_i) > 0\}$. Let us denote $y_i = \frac{1}{\mu(S_i)}\int_{S_i} g$ and, for each $i \in J$ and $k \in \mathbf{N}$, $a_i^k = E[k\beta_i + 1]$ ($E[z]$ denotes the entire part of the real number z). The sequence $y_i^k = \frac{k\beta_i}{a_i^k}(y_i - \omega_i) + \omega_i \in E_+$ converges to y_i for all $i \in J$, and by the lemma $x_i \prec_i y_i$. Thus there exists an index k_0 such that $x_i \prec_i y_i^k$ for all $i \in J$ and for all $k \geq k_0$. (*)

Now, we consider the coalition with a_i^k agents of type $i \in J$ in the k–fold replica economy $k\mathcal{E}_n$, which improves upon the allocation y_i^k, because we have (*) and

$$\begin{aligned}\sum_{i\in J} a_i^k y_i^k &= \sum_{i\in J} k\beta_i(y_i - \omega_i) + a_i^k\omega_i = k\sum_{i\in J}\beta_i y_i - k\sum_{i\in J}\beta_i\omega_i + \sum_{i\in J} a_i^k\omega_i = \\ &= k\int_S g - k\int_S \mathrm{w} + \sum_{i\in J} a_i^k\omega_i = \sum_{i\in J} a_i^k\omega_i.\end{aligned}$$

c) By construction, $(x_1,\cdots,x_n)$ is an allocation in the economy $\mathcal{E}_n$, and $x_i \in B_i(p)$ because $p(x_i) = p\Big(\frac{1}{\mu(I_i)}\int_{I_i} f\Big) = \frac{1}{\mu(I_i)}\int_{I_i} p(f) \leq \frac{1}{\mu(I_i)}\int_{I_i} p(\omega_i) = p(\omega_i)$. If $z \succ x_i = \frac{1}{\mu(I_i)}\int_{I_i} f$, by the lemma, we have $z \succ f(t)$ for all $t \in S \subset I_i$, $\mu(S) > 0$ and thus $p(z) > p(\mathrm{w}(t)) = p(\omega_i)$.

d) Suppose, on the contrary, that $(x_1,\cdots,x_n)$ can be improved upon in a r–fold replica economy $r\mathcal{E}_n$ of the economy $\mathcal{E}_n$ by a coalition with a_i agents of type i via an allocation y_i, $i \in J$ (by convexity of preferences, without loss of generality we may suppose that the consumption vector y_i is the same for all agents of type i). Thus $x_i \prec_i y_i$ for all $i \in J$ and $\sum_{i\in J} a_i y_i = \sum_{i\in J} a_i\omega_i$.

Let $\alpha = \min_{i\in J}\{\mu(I_i)\}$. Now for each $i \in J$, let $S_i \subset I_i$ be a subinterval with Lebesgue measure $\frac{a_i}{r}\alpha$ and consider the coalition $S = \bigcup_{i\in J} S_i$ and the function g defined by $g(t) = y_i$ if $t \in I_i$, then the coalition S improves upon f via g because:

$$i)\ \int_S g = \sum_{i\in J}\Big(\frac{a_i}{r}\alpha\Big)y_i = \Big(\frac{1}{r}\alpha\Big)\sum_{i\in J} a_i y_i = \Big(\frac{1}{r}\alpha\Big)\sum_{i\in J} a_i\omega_i = \sum_{i\in J}\Big(\frac{a_i}{r}\alpha\Big)\omega_i = \int_S \mathrm{w}$$

ii) $f(t) = x_i \prec_i y_i = g(t)$ Q.E.D.

The rest of the section is dedicated to establishing results concerning the existence of core and equilibrium allocations in the economy $\mathcal{E}_C$ under the standard assumptions for n–agents' economies.

Theorem 2 *If for each $i = 1, \cdots, n$ the preference relation $\preceq_i$ is convex, strictly monotone and continuous on E_+ with respect to some linear topology τ on E less fine than the norm topology, and the order interval $[0, \omega]$ is $\tau-$compact, then there exists an allocation $f : I \longrightarrow E_+$, which belongs to the core of the economy $\mathcal{E}_C$.*

Proof.-

Let us denote $\alpha_i = \mu(I_i) > 0 \quad i = 1, \cdots, n$. We choose n sequences $\{\alpha_i^m\}_{m \in \mathbf{N}}$ of rational numbers converging to α_i for all i and such that there exists a real number $r > 0$ with $r < \alpha_i^m$ for all $m \in N$ and $\alpha_1^m + \cdots + \alpha_n^m = 1$.

Given $m \in \mathbf{N}$, there are integer numbers p, q such that $\alpha_i^m = \dfrac{p_i}{q}$, $i = 1, \cdots, n$. We consider the q agents' economy $\mathcal{E}_q = (< E, E' >, \omega_{ij}, \preceq_{ij}, i \in \{1, 2, \cdots, n\}, j \in \{1, 2, \cdots, p_i\})$, where the preference relation and the initial endowment of the ij consumer are identical to those of the ith consumer in the economy $\mathcal{E}_n$. Under the hypothesis of the theorem, the economy $\mathcal{E}_q$ has an Edgeworth equilibrium (see Aliprantis *et al.* (1989)), which, due to the equal treatment property, can be expressed as $(x_1, ^{p_1}, x_1, \cdots, x_n, ^{p_n}, x_n)$.

In this way we construct n sequences $\{x_i^m\}_{m \in \mathbf{N}}$ contained in the order interval $[0, \frac{1}{r}\omega]$ which is compact. Therefore there exists y_i, which is the limit of a subnet of the sequence $\{x_i^m\}$, for $i = 1, \cdots, n$. The step function $f : I \longrightarrow E_+$ defined by $f(t) = y_i$ for all $t \in I_i$ is a core allocation. In fact:

i) $\displaystyle\int_I f = \sum_{i=1}^{n} \alpha_i y_i = \lim_\beta \sum_{i=1}^{n} \alpha_i^{m(\beta)} x_i^{m(\beta)} = \lim_\beta \sum_{i=1}^{n} \alpha_i^{m(\beta)} \omega_i = \sum_{i=1}^{n} \alpha_i \omega_i = \int_I \mathrm{w}.$

ii) f cannot be improved upon by any coalition S of agents, because otherwise there exist a function g such that $\int_S g = \int_S \mathrm{w}$ and $f(t) \prec_i g(t)$ for almost every $t \in S_i = S \cap I_i$ with $\mu(S_i) > 0$. By the lemma we have $y_i \prec_i \dfrac{1}{\mu(S_i)} \displaystyle\int_{S_i} g$ for all $i \in J = \{1 \leq i \leq n / \mu(S_i) > 0\}$. Denote $z_i = \dfrac{1}{\mu(S_i)} \displaystyle\int_{S_i} (g - \mathrm{w})$ then we have $y_i \prec_i z_i + \omega_i$ for all $i \in J$. (*)

For each index $i \in J$ we consider the sequence of real numbers converging to 1 defined by $\{\frac{k\mu(S_i)}{a_i^k}\}_{k\in \mathrm{N}}$ where $a_i^k = E[k\mu(S_i)+1]$, and we denote $z_i^k = (\frac{k\mu(S_i)}{a_i^k})z_i$. There exists a positive entire k_0 such that if $k \geq k_0$, $y_i \prec_i z_i^k + \omega_i$. Therefore there exists an index β_0 such that if β is greater than β_0, then the continuity of the preference relation implies $x_i^{m(\beta)} \prec_i z_i^k + \omega_i$. We consider the k–fold replica economy of the economy $\mathcal{E}_{m(\beta)}$ with k and β greater than k_0 and β_0 respectively, in which the coalition formed by a_i^k agents of type $i \in J$ improves the core allocation $(x_1^m, \cdots, x_1^m, \cdots, x_n^m, \cdots, x_n^m)$ via the vectors $z_i^k + \omega_i$, because in addition to (*) we also have:

$$\begin{aligned}\sum_{i\in J} a_i^k(z_i^k+\omega_i) &= \sum_{i\in J}[k\mu(S_i)z_i + a_i^k\omega_i] = \sum_{i\in J} k\int_{S_i}(g-\mathrm{w}) + \sum_{i\in J} a_i^k\omega_i = \\ &= k\int_S (g-\mathrm{w}) + \sum_{i\in J} a_i^k\omega_i = \sum_{i\in J} a_i^k\omega_i.\end{aligned}$$

Q.E.D.

Theorem 3 *If, in addition to the assumptions used in theorem 2, the preference relations are uniformly τ–proper and the initial endowment $\omega = \sum_{i=1}^{n}\alpha_i\omega_i$ is strictly positive, then the economy $\mathcal{E}_C$ has a competitive equilibrium.*

Proof.-

We know from theorem 2 that the economy has a core allocation, which can be interpreted as an Edgeworth equilibrium for the n agents' economy $\mathcal{E}_n$. The additional assumptions allow us to follow step by step the construction of the equilibrium price as in Aliprantis *et al.* (1989, pp. 138–139) and to obtain a competitive equilibrium in the economy $\mathcal{E}_n$, which corresponds to a competitive equilibrium in the economy $\mathcal{E}_C$.

Q.E.D.

As consecuence of the above results, under the hypothesis of the theorem 3, the economy $\mathcal{E}_C$ has the core–Walras equivalence property. Note that the economy $\mathcal{E}_C$ is not in general included in the work of Rustichini and Yannelis (1991).

References

[1] ALIPRANTIS,C.D.; BROWN,D.J.; BURKINSHAW,O.*1987. "Edgeworth equilibria"* . Econometrica, 55.

[2] ALIPRANTIS,C.D.; BROWN,D.J.; BURKINSHAW,O. *1989. "Existence and Optimality of Competitive Equilibria".* Springer-Verlag.

[3] AUMANN, R. *1964. "Markets with a Continuum of Traders".* Econometrica, 32.

[4] AUMANN, R. *1966. "Existence of competitive equilibria in Markets with a Continuum of Traders".* Econometrica, 34.

[5] BESADA, M.; ESTEVEZ, M.; HERVES, C. *1988. "Una generalización del teorema de Scarf a economías con infinitas mercancías".* Revista Española de Economía, 5.

[6] BEWLEY, T. *1973. "Equality of the Core and Set of Equilibria in Economies with Infinitely Many Commodities and a Continuum of Agents".* International Economic Review,14.

[7] DEBREU,G.; SCARF,H.E. *1963. "A limit theorem on the Core of an Economy".* International Economic Review, 4.

[8] DIESTEL, S.; UHL, J. *1977. "Vector Measures".* Mathematical Surveys, 15.

[9] KEHOE, T.; LEVINE, D.; MAS-COLELL, A.; ZAME,W. *1989. "Determinacy of equilibrium in large-scale economies".* Journal of Mathematical Economics, 18.

[10] MAS-COLELL, A. *1986. "The price equilibrium existence problem in topological vector lattices".* Econometrica, 54.

[11] OSTROY, J. *1984. "The existence of Walrasian equilibrium in large-square economies".* Journal of Mathematical Economics, 13.

[12] RUSTICHINI, A.; YANNELIS, N.C. *1991. "Edgeworth's conjecture in economies with a continuum of agents and commodities"*. Journal of Mathematical Economics, 20.