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ROUGH ANALYSIS IN LATTICES

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Abstract

An outline of an algebraic generalization of the rough set theory is presented in the paper. It is shown that the majority of the basic concepts of this theory has an immediate algebraic generalization, and that some rough set facts are true in general algebraic structures. The formalism employed is that of lattice theory. New concepts of rough order, approximation space and rough (quantitative) approximation space are introduced and investigated. It is shown that the original Pawlak's theory of rough sets and information systems is a model of this general approach.

Key words: Rough set; information system; rough dependence; rough lattice; approximation space.

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0. Introduction. The aim of this paper is to outline an algebraic generalization of the *rough set* theory that would provide a common framework for various rough set applications and models and, possibly, enhance the scope of such applications. The contents of this paper are a result of some earlier works concerning group preferences, multicriterial decisions and human categorizations and concepts, and it is author's hope that it may be useful in modeling and analysis of preferences and coalition structures.

The theory of rough sets and information systems, presented for the first time in 1981 by Pawlak, see [23], has been initially proposed as a framework for a systematic study of imprecise or incomplete knowledge. Pawlak has introduced new concepts of independence, rough dependence and rough approximation, as well as the notion of reduction of information systems. These concepts play an essential role both in further development of the theory and in its various applications, and proved to be useful tools also outside the initially intended field of application.

The scope of successful applications of rough set methods to empirical problems is constantly increasing, and ranges from industrial control systems [19] and expert systems [1], [17], to analysis of empirical data in psychological problems of decision and cognition [4], [13], [14], or in medicine [27]. Various computer programs of rough analysis are commercialized, or distributed by academic channels. First *rough chips* are being manufactured.

The rough set notions aroused also a more theoretical interest, especially in the computer-oriented areas of mathematics, related to expert systems, decision making, artificial intelligence, etc. Various papers have been published since 1982 concerning relations between rough and fuzzy theories of sets, e.g. [5], rough sets theory and evidence theory, e.g. [31], rough and probabilistic approach to indetermined situations, e.g. [28], etc. Several *rough logics* have been constructed and investigated, as well as knowledge representation and machine learning systems (see, for example, [18] and the bibliography of [5]). A number of rough-set-based models has been proposed for empirical sciences, such as, for example, that of contextual structures of natural concepts, [7], [15], or of natural categorization rules, [13], [16]. The concept of *rough dependency* seems to be of some importance for new models and solutions of decision problems, and much has been done in this direction (see, for example, [3], [14], [24]).

Parallelly, several purely theoretical studies of formal structures arising in rough set theory have been performed. Some results concerning algebraic structure of rough sets were presented, for example, in [8] and [30] (see also the bibliography of [20]), and the limit properties of rough approximations have been studied, [18]. Some new results have been obtained by applying algebraic technics to the families of information systems [21], [22].

At the same time, the development of the rough set theory itself is being strongly influenced by practical needs. One of the effects of this influence is the necessity of some generalizations, since the original Pawlak's approach (and the majority of the works mentioned above is based on it) is in a sense a very restrictive one: on the interpretative level, it enables to analyze only such situations in which objects under analysis are - or are not - equivalent (see Appendix) and does not take into account other possible types of relation between objects, such as, for example, similarity or order.

Recently, several papers have been published proposing some generalizations of the theory. For example, Pawlak, Wong and Ziarko present in [28] a theory of probabilistic information systems, Pomykała [29] and Nieminen [20] introduce two different approaches based on tolerance (= similarity) relations, and in [10] and [11] a framework for a theory based on order relations is proposed. In the paper [12] a generalized concept of rough approximation is introduced which allows to propose a general algebraic scheme called *rough order*, and to apply 'rough methods' in the situation in which the approximating objects are not sets but concepts (in the sense of Wille, cf. [32]).

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Present paper can be regarded as belonging to the latter group. As we have already mentioned at the beginning, its main objective is to propose a general and uniform algebraic framework for various *"rough theories"*, as well as to open new possible fields of application such as, for example, analysis of preferences, decisions and coalition structures.

One of the models of the theory presented here is the original Pawlak's approach, which is partially characterized in various examples throughout the text, and in the Section 10. A concise summary of the basic concepts of rough sets and information systems theory is annexed at the end of the paper. For more information, see the original papers [23] - [27].

In the Section 1 we introduce the algebraic counterparts of the basic concepts of the rough set theory.

Sections 2 to 6 are devoted to the problems of approximation in lattices. We define general rough structures called *rough orders* and general approximation operators called *preclosures*, and investigate their properties (sections 2 and 3). Sections 4 and 5 contain some results concerning interdependence between algebraic properties of rough orders, and the properties of approximation operations defined in them; concepts of *approximation space* and *complete rough lattice* are introduced. Section 6 is devoted to lower approximation operators, and some algebraic properties of families of such operators are analyzed; the concept of *approximation system* is introduced.

In the section 7 we introduce some quantitative elements: descriptions of quality of approximations, and the measures of *roughness* of objects which allow to introduce the concept of *partial dependence*. Some properties of partial dependence are analyzed in the section 8.

Section 9 is devoted to what we call *normal families* of sets, and it is intended to provide a uniform conceptual base for those models, applications and generalizations of original rough set theory, in which approximated objects are sets or families of sets.

In the section 10 we show that the basic facts of the theory of rough sets and rough approximation can be derived from the general results presented here.

Throughout the text we shall use a standard lattice theory notation and terminology, following that of Grätzer [6].

An *order* is a partially ordered set, that is, a system $\mathcal{R} = (R,\leq)$ where R is a nonempty set and $\leq\, \subseteq R^2$ is a partial ordering of R: it is reflexive, transitive and antisymmetric.

$\mathcal{L} = (L,\leq)$ will denote a bounded partially ordered set which is a *lattice*. The symbols $\mathbf{0}$, $\mathbf{1}$ denote the bounds of L; $\wedge$ and $\vee$ are the *meet* and *join* operations in L, respectively: $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. For any subset A of L, $\bigwedge A$ and $\bigvee A$ denote its infimum and its supremum. In a complete lattice all subsets of its universe have their suprema and minima; we recall that $\bigwedge\{\emptyset\} = \mathbf{1}$ and $\bigvee\{\emptyset\} = \mathbf{0}$.

We shall also consider *semilattices*. A *meet-semilattice* $\mathcal{S} = (S,\leq)$ is a partially ordered set such that for any pair a,b of its elements there exists their *infimum* $a \wedge b \in S$; it is *complete* iff for any $A \subseteq S$ there exists its infimum $\bigwedge A$. The definition of *join-semilattice* is the dual one. It is a well known fact that any complete meet-semilattice bounded from above is a lattice, and the dual for join-semilattices holds.

The symbol '$\blacksquare$' denotes end of a proof, of an example, or of a remark.

1. Independence in meet-semilattices. Let $\mathcal{S} = (S,\leq)$ be a complete meet-semilattice, and let A, B be subsets of S.

DEFINITION 1. We shall say that A and B are *equivalent* iff $\bigwedge A = \bigwedge B$.

Obviously, the equivalence of sets is an equivalence relation in the general sense of the term. Observe that A is equivalent to $\{\bigwedge A\}$; the empty set is equivalent to the singleton $\{\mathbb{1}\}$. The following observation will be useful later on:

Let $m(A)$ be the set of all minimal elements of a nonempty set A; it is evident that $\bigwedge A \leq \bigwedge m(A)$ since $m(A)$ is a subset of A. If, in addition, for any $a \in A$ there exists a $m \in m(A)$ such that $m \leq a$ (which is true in the case of finite A, for example), then $\bigwedge m(A) = \bigwedge A$: A and $m(A)$ are equivalent sets.

DEFINITION 2. Let A be a subset of S. We shall say that:

1° an element $b \in S$ is *superfluous in* A iff A is equivalent to $A-\{b\}$, that is, iff $\bigwedge(A-\{b\}) = \bigwedge A$;

2° the set A is *independent* iff there are no superfluous elements in it; if a set is not independent, then it is called *dependent.*

Observe that any element not belonging to A is superfluous in it. If $b \geq a$ for some $a \in A$, then b is superfluous in A; it follows that any independent set is an antichain in $\mathcal{S}$.

The empty set is independent, and it is equivalent to the set $\{\mathbb{1}\}$ which, consequently, is not independent. Therefore, a one-element set $\{a\}$ is independent iff $a \neq \mathbb{1}$.

Any subset of an independent set is also independent: if b is superfluous in B and $B \subseteq A$, then

$$\begin{aligned}\bigwedge A &= \bigwedge(A-B) \wedge \bigwedge B = \bigwedge(A-B) \wedge \bigwedge(B-\{b\}) \\ &= \bigwedge(A-\{b\}).\end{aligned}$$

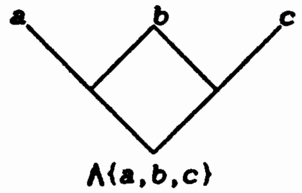
The same argument implies that any superset of a dependent set is dependent as well.

We shall say, by analogy, that a set $B \subseteq S$ is superfluous in A iff A and $A - B$ are equivalent sets.

It follows that if B is superfluous in A, then any element of B is superfluous in A, since $\bigwedge A \leq \bigwedge(A-\{b\}) \leq \bigwedge(A-B) = \bigwedge A$, for any $b \in B$ (the inverse is not true, cf. Example 2).

A set B *depends* on the set A iff $\bigwedge B \geq \bigwedge A$. This fact wil be - when convenient - as $A \longrightarrow B$. Obviously, A and B are equivalent iff $A \longrightarrow B$ and $B \longrightarrow A$, compare Section 8.

EXAMPLE 1. Let $\mathcal{S}$ be a meet-semilattice containing the fragment represented by the following diagram:



The set $A = \{a,b,c\}$ is not independent, since b is superfluous in it: $\bigwedge(A-\{b\}) = \bigwedge A$; any two-element subset of A is independent; $\{a,c\}$ is equivalent to A, the sets $\{a,b\}$ and $\{b,c\}$ depend on A (and on $\{a,c\}$ as well). ■

EXAMPLE 2. Let $\mathcal{L}$ be the Boolean algebra of all subsets of $\mathbf{R}^2$. Let $D=D(0,1)$ be the open unit disc, and let P_c denote the open halfplane containing D and tangent to it at the point c belonging to the unit circle C. The infimum of the family $\mathcal{A} = \{P_c\colon c \in C\}$ is equal to D, and $\mathcal{A}$ is independent, since for any $c \in C$

$$\bigwedge(\mathcal{A} - \{P_c\}) = D \cup \{c\} \supsetneq D.$$

On the other hand, if $\bar{D}$ and $\bar{P}_c$ are the corresponding closed sets, then any element of the family $\mathcal{B}$ of all closed halfplanes $\bar{P}_c$ is superfluous in it:

$$\bigwedge(\mathcal{B} -\{\bar{P}_c\}) = \bar{D} \quad \text{for any } c \in C.$$

(we omit the easy geometric proof). ■

DEFINITION 3. Let $A \subseteq S$ be a set of elements of a meet-semilattice $\mathcal{S}$, and let R be a subset of A. We shall say that R is a *reduct of* A iff R is independent an equivalent to A. The family of all reducts of A will be denoted by $\mathbb{RED}(A)$.

PROPOSITION 1. If R is a reduct of A, then it is a maximal independent subset of A.

Proof. Let R be a reduct of A. Assume that there exists an independent set $M \subseteq A$ such that $R \subsetneq M$. Therefore $\bigwedge A \leqslant \bigwedge M < \bigwedge R$, the latter inequality being implied by independence of M. On the other hand, $\bigwedge A = \bigwedge R$, since R is a reduct of A. It follows that $\bigwedge A \leqslant \bigwedge M < \bigwedge A$, which is a contradiction. ■

Observe that Proposition 1 implies that a reduct is not contained in any other reduct.

REMARK 1. It has been conjectured, [28], that the inverse is also true: a set is a reduct iff it is maximal independent. Example 1 shows that it not so: the set $\{a,b\}$ is a maximal independent subset of $\{a,b,c\}$, but it is not a reduct. (This fact has been observed independently and by other means by Novotny and Pawlak in the paper [22].) ■

We shall say that $a \in A$ is *absolutely superfluous in A* iff it does not belong to any reduct of A. If a subset P of A is maximal independent and it contains an element which is absolutely superfluous in A, then obviously P is not a reduct of A. Maximal independent subsets which are not equivalent to the entire set are called *subreducts*, [22].

An element $a \in A$ is *indispensable in A* iff it belongs to all reducts of A. The set of all indispensable elements is called *the core of A*, and denoted by $core(A)$. Obviously,

$$core(A) = \bigcap \mathbf{RED}(A).$$

PROPOSITION 2. Any finite set has at least one reduct.

Proof. Let A be a finite set of elements of a meet-semilattice $\mathcal{S}$. If A is independent, then it is its own reduct. If not, then there exists an $a_1 \in A$ which is superfluous in A. Let $A_1 = A - \{a_1\}$; the set A_1 is equivalent to A: $\bigwedge A_1 = \bigwedge A$. If A_1 is independent then it is a reduct of A; if not, then there exists an element a_2 superfluous in A_2, and so on. In a finite number of steps we obtain an A_i which is independent and equivalent to A, that is, a reduct of A (we recall that the empty set is independent). ∎

Notice that, in view of the remark following the definition of equivalent sets, any finite set A has a reduct which is a reduct of the subset $m(A)$ of its minimal elements, since a reduct of a subset which is equivalent to A is a reduct of A.

No sufficient conditions are known for an infinite set to have a reduct. The above Example 2 implies that even in complete Boolean algebra there may exist dependent sets with no reducts – the family $\mathcal{B} = \{\bar{P}_C\}$ has this property, since the existence of a reduct of $\mathcal{B}$ would be equivalent to the existence of a subset of the unit circle C that is dense in C and has no condensation points, which is impossible.

2. Rough orders and preclosure maps. Let $\mathcal{R} = (R, \leqslant)$ be an order. It is said that a mapping $c: R \longrightarrow R$ is *closure map* in $\mathcal{R}$ iff it is *idempotent* ($c(c(r)) = c(r)$ for all $r \in R$), is *extensive* ($c(r) \geqslant r$ for all $r \in R$), and is *isotonic* ($c(r) \geqslant c(s)$ for all $r \geqslant s$ in R). If $c(r) = r$ for some $r \in R$, then r is a *closed element* of R (for details, see [2], [6]). A mapping d is a *dual closure* iff it is a closure map in the dual order $\mathcal{R}^d = (R, \geqslant)$.

DEFINITION 4. Let $\mathcal{R} = (R, \leqslant)$ be a bounded order and let p be a mapping of R into itself. We shall say that p is a *preclosure map* in $\mathcal{R}$ iff it is extensive and idempotent. If $p(r) = r$ for some $r \in R$, then r is a *p-exact element* of R.

Observe that $p(\mathbf{1}) = \mathbf{1}$ by the extensivity of p. If, additionally, $p(\mathbf{0}) = \mathbf{0}$, then p will be called *upper approximation map in* $\mathcal{R}$. If p is a dual preclosure in $\mathcal{R}$, that is, a preclosure in $(R, \succ)$, and $p(\mathbf{1}) = \mathbf{1}$, then it will be called *lower approximation map in* $\mathcal{R}$.

Evidently, any $\mathbf{0}$-preserving closure is an upper approximation map, and any $\mathbf{1}$-preserving dual closure is a lower approximation map, since any closure is a preclosure.

Let $\mathcal{R} = (R,\prec)$ be a bounded order, let E be a subset of R such that $\mathbf{0},\mathbf{1} \in E$ and let the system $\mathcal{E} = (E,\prec')$, where $\prec' = \prec|_{\mathcal{R}}$, be the corresponding suborder of $\mathcal{R}$.

DEFINITION 5. If the order $\mathcal{E} = (E, \prec')$ is a complete lattice, then the triple $(R, E, \prec)$ is a *rough order*; the elements of E will be called *exact elements* of R.

If, additionally, $\mathcal{R}$ is a structure: semilattice, lattice, complete lattice, boolean algebra etc., and $\mathcal{E}$ is a corresponding substructure of $\mathcal{R}$, then $(R, E, \prec)$ will be called, respectively, *rough semilattice, rough lattice, rough complete lattice, rough boolean algebra,* etc.

EXAMPLE 3. Let U be a nonempty universe, and let $\mathbf{C} = \{C_1, C_2, \dots , C_n\}$ be a finite partition of U. Let $\mathbf{D}_0(\mathbf{C})$ be the family of all unions of elements of $\mathbf{C}$, and let $\mathbf{D} = \mathbf{D}(\mathbf{C}) = \mathbf{D}_0(\mathbf{C}) \cup \{\emptyset\}$. Obviously, $\mathcal{D} = (\mathbf{D}, \subseteq)$ is a sub-order of the family $\mathbf{P} = \mathbf{P}(U)$ of all subsets of U ordered by inclusion. It is a finite lattice with respect to set union and set intersection, and it contains the bounds of $\mathbf{P}$: $\emptyset \in \mathbf{D}$, and $U = \bigcup \mathbf{C} \in \mathbf{D}$. Therefore the triple $(\mathbf{P}, \mathbf{D}, \subseteq)$ is a rough algebra of sets. Define, now, for any $X \subseteq U$:

$$\overline{\mathbf{D}}(X) = \bigcap\{D\in\mathbf{D}\colon X\subseteq D\} ,$$

$$\underline{\mathbf{D}}(X) = \bigcup\{D\in\mathbf{D}\colon X\supseteq D\} .$$

An elementary verification shows that $\overline{\mathbf{D}}\colon \mathbf{P}\longrightarrow \mathbf{D}$ is an upper approximation map in $\mathbf{P}$, and $\underline{\mathbf{D}}\colon \mathbf{P}\longrightarrow \mathbf{D}$ is a lower approximation; observe that they are a closure and a dual closure, respectively. In both cases the family $\mathbf{D}$ is the set of all exact elements of $\mathbf{P}$. ■

REMARK 2. There is a one-to-one correspondence between partitions of U and equivalence relations in U. If we assume that the elements of the partition $\mathbf{C}$ are equivalence classes of an equivalence relation in U, then the operations $\overline{\mathbf{D}}$ and $\underline{\mathbf{D}}$ are the approximation operators considered in Pawlak's theory of rough sets and approximation spaces, see Appendix or [25]. ■

3. Approximation in rough orders. Let $\mathcal{R} = (R, E, \prec)$ be a rough order. For any $r \in R$ the set $M(r) = \{e \in E: e \succ r\}$ is nonvoid, since $E \ni 1 \succ r$. Let $U(r)$ be the set of all minimal elements of $M(r)$.

We shall say that r is *recognizable from above* (or: *u-recognizable*) in $\mathcal{R}$ iff $M(r)$ has the following property:

(U) *for any $m \in M(r)$ there exists a $u \in U(r)$ such that $u \prec m$.*

Observe that any element of E is recognizable from above, since $r \in E$ implies that $U(r) = \{r\}$: r is the only minimal element of $M(r)$, and, of course, the property (U) holds with $u = r$. Notice, too, that if E is a finite subset of R, then any element of R is recognizable from above.

Let $r \in R$ be u-recognizable in $\mathcal{R}$. Define

$$\overline{E}(r) = \bigvee U(r).$$

The correspondence $r \longrightarrow \overline{E}(r)$ is well defined for all u-recognizable elements, since $\mathcal{E} = (E, \prec)$ is a complete lattice and $\bigvee$ is taken in E.

If R_u is the set of all u-recognizable elements, then $\mathcal{R}_u = (R_u, E, \prec)$ is a rough order (since $E \subseteq R_u$) in which all elements are u-recognizable. A similar construction leads to the concept of l-recognizable elements.

Let, for $r \in R$, $K(r)$ be the set of all lower bounds of r belonging to E: $K(r) = \{e \in E: e \succ r\}$, and let $L(r)$ be the set of all maximal elements of $K(r)$. We shall say that r is *recognizable from below* (or *l-recognizable*) in $\mathcal{R}$ iff the following holds:

(L) *for any $m \in K(r)$ there exists an $l \in L(r)$ such that $l \succ m$.*

It follows that all elements of E are l-recognizable, and that all elements of R are l-recognizable in the case of finite E.

For any l-recognizable $r \in R$ there exists in E the g.l.b of the set $L(r)$. Let

$$\underline{E}(r) = \bigwedge L(r).$$

The correspondence $r \longrightarrow \underline{E}(r)$ is a partial map in R whose domain is the set R_l of all l-recognizable elements. The structure $\mathcal{R}_l = (R_l, E, \prec)$ is a rough order in which all elements are l-recognizable.

Proposition 3. Let $\mathcal{R} = (R, E, \prec)$ be a rough order, and let $\overline{E}$, $\underline{E}$, R_u, and R_l be defined as above. Then

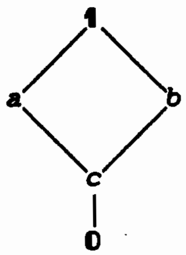
(i) The operation $\overline{E}$ is an upper approximation map in $\mathcal{R}_u$ and E is the set of all $\overline{E}$-exact elements of R_u.

(ii) The operation $\underline{E}$ is a lower approximation map in $\mathcal{R}_u$ and E is the set of all $\underline{E}$-exact elements of R_l.

Proof. (i) If $e \in E$ then, as we have seen above, $\overline{E}(e) = \bigvee\langle e\rangle = e$; therefore $\overline{E}(e) = e$ for any $e \in E$. For any $r \in R_u$ the element $\overline{E}(r)$ belongs to E by definition. Hence, $\overline{E}^2(r) = \overline{E}(\overline{E}(r)) \in E$, and the operation $\overline{E}$ is idempotent in R_u. On the other hand, for any u-recognizable $r \in R_u$ there exists an $u \in U(r)$ such that $r \preceq u \preceq \overline{E}(r)$ (by the condition (U)), which implies that $\overline{E}$ is extensive. Consequently, it is an upper approximation map in $\mathcal{R}_u$. Now, if $\overline{E}(r) = r$ holds for some $r \in R$, then $r \in E$, and it follows that E is the set of all $\overline{E}$-exact elements, which ends the proof of (i). The proof of (ii) is the dual one. ■

Recall that the elements of E are called exact elements (Definition 5). Proposition 3 implies that an $r \in R$ is exact in $(R, E, \preceq)$ iff it is $\overline{E}$-exact and iff it is $\underline{E}$-exact.

Remark 3. In general, the preclosure $\overline{E}$ is not a closure, even if $(R, \preceq)$ is a lattice. For example, if $(R, \preceq)$ is described by the following diagram



and if $E = \{0, a, b, \mathbf{1}\}$, then $(R, E, \preceq)$ is a rough lattice such that $a > c$ and $\overline{E}(a) < \overline{E}(c)$, since

$$\overline{E}(a) = a < \mathbf{1} = a \vee b = \overline{E}(c).$$

The dual example will show that $\underline{E}$ needs not to be a dual closure. Observe that in this example both $(R, \preceq)$ and $(E, \preceq)$ are lattices, but the second is not a sublattice of the first. ■

4. **Approximation spaces.** Any rough order contains a rough sub-order $\mathcal{R}_o$ in which all elements are recognizable, that is, are l-recognizable and u-recognizable: $\mathcal{R}_o = (R_o, E, \preceq')$, where $R_o = R_l \cap R_u$ and $\preceq' = \preceq|_{R_o}$. The structure $\mathcal{R}_o$ is a well-defined rough order, since $R_l \supseteq E$ and $R_u \supseteq E$ by Proposition 3, which implies that $R_o \supseteq E$.

Definition 6. If in a rough order $\mathcal{R} = (R, E, \preceq)$ all elements are recognizable: $R = R_u = R_l$, then R will be called *approximation space*; if $R = R_u$ then it is *upper approximation space*, and it is *lower approximation space* whenever $R = R_l$.

PROPOSITION 4. The following conditions are sufficient for a rough order $\mathcal{R} = (R, E, \prec)$ to be an approximation space:

(i) $(E, \prec)$ is a finite lattice;

(ii) $(R, E, \prec)$ is a complete rough lattice (that is, $(R, \prec)$ is a complete lattice and $(E, \prec)$ is its complete sub-lattice).

Proof. If (i) holds, then the thesis is obvious, since in a finite E both (U) and (L) hold.

Assume now that (ii) holds. Let r be arbitrary element of R, and let, as before, $M(r)$ be the set of all upper bounds of r in E. Let $e_o = \wedge_R M(r)$. The element r is one of the lower bounds of $M(r)$ in R, therefore $r \prec e_o$. The condition (ii) means that $\wedge_R M = \wedge_E M$ for any $M \subseteq E$. In particular, $e_o = \wedge_E M(r) \in E$. It follows that $e_o \in M(r)$, and $e_o \prec m$ for all $m \in M(r)$. Consequently, e_o is the unique minimal element of $M(r)$, and the condition (U) is satisfied with $u = e_o$. This means that r is u-recognizable, and that $\overline{E}(r) = \vee\{e_o\} = e_o$.

A dual reasoning demonstrates that any $r \in R$ is l-recognizable, and that $\underline{E} = \vee_E K(r) = \vee_R K(r)$, where $K(r)$ is the set of all lower bounds of r in E. ∎

COROLLARY. If the condition (ii) of Proposition 4 holds, then upper approximation and lower approximation are closure and dual closure, resp.

Indeed, if $s \prec r$, then $M(s) \supseteq M(r)$. Consequently, $\wedge_R M(s) \prec \wedge_R M(r)$, which means that $\overline{E}(s) \prec \overline{E}(r)$. Therefore the upper approximation operation is a isotonic preclosure, that is, a closure. A dual argument shows that $\underline{E}$ is a dual closure.

For finite approximation spaces the inverse is also true:

PROPOSITION 5. Let $\mathcal{R} = (R, E, \prec)$ be a finite approximation space such that $(R, \prec)$ is a lattice. Then $\mathcal{R}$ is a rough lattice iff the approximation operations $\overline{E}$ and $\underline{E}$ are closure and dual closure, respectively.

Proof. Observe, first, that in the finite case the concepts of sublattice and complete sublattice coincide. The 'only if' part of the thesis is, therefore, a consequence of the previous Corollary.

Let $\wedge'$ be the meet operation in the lattice $(R, \prec)$. Assume that $\overline{E}$ is a closure in R; E is the set of all closed elements of R, by Proposition 3(i): $E = \overline{E}(R)$. It is well known (see, for example, Birkhoff [2], or compare [6], Theorem I.6.4) that for any closure c in R the sub-order $(c(R), \prec)$ of the lattice $(R, \prec)$ is a lattice $(c(R), \wedge, \vee)$ in which $\wedge = \wedge'|_{c(R)}$. It follows that E is closed with respect to $\wedge'$ in $(R, \prec)$, that is, $(E, \prec)$ is a meet-subsemilattice of the lattice $(R, \prec)$. A dual argument shows that if $\underline{E}$ is a dual closure then $(E, \prec)$ is a join-subsemilattice of $(R, \prec)$. Therefore $(E, \prec)$ is a sublattice of $(R, \prec)$. ∎

In an approximation space $(R, E, \leq)$ both $\underline{E}(r)$ and $\overline{E}(r)$ exist for any $r \in R$.

DEFINITION 7. We shall say that the pair $(\underline{E}(r), \overline{E}(r))$ is the *rough approximation* of r, and the function $\overline{\underline{E}}: R \longrightarrow E^2$ defined as follows:

$$\overline{\underline{E}}(r) = (\underline{E}(r), \overline{E}(r))$$

will be called the *rough approximation operator*.

REMARK 4. The image $\overline{\underline{E}}(R)$ of R, which is a subset of E^2, can be 'equipped' with an algebraic structure, since both E and R do posses such structures. The question is, what is the 'natural' way of doing it? This problem has ben investigated only in the case of R being the power set of some universe U. In the paper [8] of Iwiński $\overline{\underline{E}}(R)$ resulted to be a de Morgan algebra, and an alternative approach of Pomykała & Pomykała [30] leads to a Stone algebra. ■

5. **Complete rough lattices.** We recall that $\mathcal{R} = (R, E, \leq)$ is a complete rough lattice iff $(R, \leq)$ is a complete lattice and $(E, \leq)$ is complete sublattice of $\mathcal{R}$. In this case (see Proposition 4 and the corresponding Corollary), $\mathcal{R}$ is an approximation space in which upper and lower approximations $\underline{E}$ and $\overline{E}$ are closure and dual closure, respectively, and they are described by the formulae

$$\underline{E}(r) = \bigvee K(r), \text{ where } K(r) = \{e \in E: e \leq r\},$$

and

$$\overline{E}(r) = \bigwedge M(r), \text{ where } M(r) = \{e \in E: e \geq r\}.$$

PROPOSITION 6. In a complete rough lattice the following conditions are satisfied, for any $r, r' \in R$:

(i) $\underline{E}(r \wedge r') = \underline{E}(r) \wedge \underline{E}(r')$, (i') $\overline{E}(r \vee r') = \overline{E}(r) \vee \overline{E}(r')$,

(ii) $\underline{E}(r \vee r') \geq \underline{E}(r) \vee \underline{E}(r')$, (ii') $\overline{E}(r \wedge r') \leq \overline{E}(r) \wedge \overline{E}(r')$.

Proof. It is sufficient to prove (i) and (ii): (i') and (ii') will hold by duality. Let $r, r' \in R$.

(i) The map $\underline{E}$ is a dual closure, therefore $\underline{E}(r) \leq r$ and $\underline{E}(r') \leq r'$. It follows that $\underline{E}(r) \wedge \underline{E}(r') \leq r \wedge r'$ in R. The element $\underline{E}(r) \wedge \underline{E}(r')$ belongs to$E$, therefore $\underline{E}(\underline{E}(r) \wedge \underline{E}(r')) = \underline{E}(r) \wedge \underline{E}(r')$. Hence, by isotonic property of $\underline{E}$, $\underline{E}(r) \wedge \underline{E}(r') \leq \underline{E}(r \wedge r')$. On the other hand, $\underline{E}(r) \geq \underline{E}(r \wedge r')$ and $\underline{E}(r') \geq$ $\geq \underline{E}(r \wedge r')$ by the same property. Consequently, $\underline{E}(r) \wedge \underline{E}(r') \geq \underline{E}(r \wedge r')$, and it follows that (i) holds.

(ii) $\underline{E}(r) \vee \underline{E}(r') \leq r \vee r'$, because $\underline{E}(r) \leq r$ and $\underline{E}(r') \leq r'$. Both $\underline{E}(r)$ and

$\underline{E}(r')$ belong to E, thus $\underline{E}(r) \vee \underline{E}(r')$ also belongs to E, since E is a sublattice of R. Therefore $\underline{E}(r) \vee \underline{E}(r') = \underline{E}(\underline{E}(r) \vee \underline{E}(r')) \leqslant \underline{E}(r \vee r')$, which ends the proof. ∎

Remark 5. The analogy between $\underline{E}$, $\overline{E}$ and topological operations of interior and closure is obvious (but superficial, see Section 11); the elements of E play the role of clopen (= closed and open) sets here. Notice that in the proof of the properties (i) – (ii') we have used only the fact that $\underline{E}$ is a dual closure, and $\overline{E}$ is a closure in R, and not the specific definitions of approximation operations. The results (ii) and (ii') can not be strengthened, since for any exact r, r' equalities appear instead of inequalities, but in any non-trivial complete rough lattice there exist elements r and r' such that the sharp inequality holds in (ii). ∎

Let $(R, E, \leqslant)$ and $(R, G, \leqslant)$ be two complete rough lattices with $G \subseteq E$. Observe that $(G, \leqslant)$ is a complete sublattice of $(E, \leqslant)$, since $\bigwedge_E M = \bigwedge_R M = \bigwedge_G M$ and $\bigvee_E M = \bigvee_R M = \bigvee_G M$ for any $M \subseteq G$.

If $r \in R$, then $\underline{G}(r) \leqslant \underline{E}(r)$, since $\underline{G}(r)$ is one of the lower bounds of r in E. Observe, too, that $\underline{G}(\underline{E}(r)) \leqslant \underline{G}(r)$ by isotonicity of $\underline{G}$. Moreover,

$$\begin{aligned}(f \in G) \wedge (f \leqslant r) \ &\text{iff} \ ((f \in G) \wedge (f \in E)) \wedge (f \leqslant r)\\ &\text{iff} \ (f \in G) \wedge ((f \in E) \wedge (f \leqslant r))\\ &\text{iff} \ (f \in G) \wedge (f \leqslant \underline{E}(r)),\end{aligned}$$

and it follows that $\underline{G}(r) = \underline{G}(\underline{E}(r))$ for all $r \in R$. Furthermore, it obvious that $\underline{G}(r) = \underline{E}(\underline{G}(r))$, since $\underline{G}(r)$ is E-exact. The dual equalities and inequalities for upper approximations can be obtained in the same way. Thus we have demonstrated the following

Proposition 7. If $(R, E, \leqslant)$ and $(R, G, \leqslant)$ are complete rough lattices and $G \subseteq E$ then, for any $r \in R$,

(i) $\quad \underline{E}(\underline{G}(r)) = \underline{G}(\underline{E}(r)) = \underline{G}(r) \leqslant \underline{E}(r) \leqslant r$, and

(i') $\quad \overline{E}(\overline{G}(r)) = \overline{G}(\overline{E}(r)) = \overline{G}(r) \geqslant \overline{E}(r) \geqslant r$. ∎

In other words, if $G \subseteq E$, then $\underline{E} \circ \underline{G} = \underline{G} \circ \underline{E} = \underline{G}$, and the dual holds.

Remark 6. An example can easily be constructed showing that in general it is not true that $\underline{E} \circ \underline{F} = \underline{F} \circ \underline{E}$. ∎

Let $(R, E, \leqslant)$ and $(R, F, \leqslant)$ be two complete rough lattices. Notice that the complete lattices $(E, \leqslant)$ and $(F, \leqslant)$ are consistent in the sense that $\bigwedge_E M = \bigwedge_F M$ and $\bigvee_E M = \bigvee_F M$ for any $M \subseteq E \cap F$. Let $G = E \cap F$. Then it follows from (i) and (i'), respectively, that for any $r \in R$

$$\underline{G}(\underline{F}(\underline{E}(r))) = \underline{G}(\underline{E}(\underline{F}(r))) = \underline{G}(\underline{F}(r)) = \underline{G}(\underline{E}(r)) = \underline{G}(r)$$

and

$$\overline{G}(\overline{F}(\overline{E}(r))) = \overline{G}(\overline{E}(\overline{F}(r))) = \overline{G}(\overline{F}(r)) = \overline{G}(\overline{E}(r)) = \overline{G}(r)\ .$$

6. Approximation systems. Let $\mathcal{R} = (R, \leq)$ be a complete lattice, an let l: $D_l \longrightarrow \mathcal{P}(R)$ be a function defined on a subset D_l of R.

DEFINITION 8. The pair $(\mathcal{R}, l)$ will be called *approximation system* iff the following conditions are satisfied:

(i) the partial order $(D_l, \leq)$ is a meet-subsemilattice of $(R, \leq)$;

(ii) for any $e \in D_l$ the system $(R, l(e), \leq)$ is a complete rough lattice;

(iii) for any $e \in D_l$, $e \in l(e)$;

(iv) if $e \leq f$, then $l(e) \supseteq l(f)$, for all $e, f \in D_l$.

In other words, the partial function l defines a family of complete sublattices of $(R, \leq)$ in such a way that the 'smaller' is element e, the 'finer' is the corresponding approximation space.

EXAMPLE 4. Let $\bar{\mathbb{R}}^+ = [0, +\infty)\cup\{+\infty\}$ be the 'closed' set of real non-negative numbers ordered by $\leq$, and let $D = \{2^{-n}: n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers. Define $l(n) = \{m2^{-n}: m \in \mathbb{N}\} \cup \{0, +\infty\}$. The pair $((\bar{\mathbb{R}}^+,\leq), l)$ is an approximation system. (More suggestive examples will be presented below, in the section 10).

Let $(\mathcal{R}, l)$ be an approximation system. We shall adopt the following notation: if $e \in D_l$ and $r \in R$, then $e(r)$ will denote the lower approximation of r in the complete rough lattice $(R, l(e), \leq)$:

$$e(r) = \underline{(l(e))}(r).$$

PROPOSITION 8. For arbitrary $e, f \in D_l$ and $r, s \in R$ the following conditions hold:

(1) $e(e) = e$;

(2) $e(e(r)) = e(r)$;

(3) $e(r\wedge s) = e(r)\wedge e(s)$ and $e(r\vee s) \geq e(r)\vee e(s)$;

(4) if $e \leq f$, then $e(r) \geq f(r)$;

(5) if $e \leq f$, then $e(f(r)) = f(e(r)) = f(r)$;

(6) $(e\wedge f)(r) \geq e(r)\vee f(r)$;

(7) $(e\vee f)(r) \leq e(r)\wedge f(r)$ whenever $e \vee_R f \in D_l$.

Proof. The mapping e is a dual closure map in $(R, \prec)$, and $l(e)$ is the set of all e-closed elements of R, by Proposition 5. Therefore (2) holds, and (1) is a consequence of Definition 8(iii). The properties (3) are equivalent to those stated in Proposition 6(ii); (4) is implied by Definition 8(iv), and (5) is equivalent to Proposition 7(i) in view of the same condition. The inequalities (6) and (7) are a simple consequence of (4). ■

7. Quantitative approximation. Let $\mathcal{R} = (R, E, \prec)$ be an approximation space, and let $E^* \subseteq E^2$ be defined as follows:

$$E^* = \{(e, f): e, f \in E \text{ and } e \prec f\} .$$

We shall say that a non-negative real-valued function ν defined on E^*:

$$\nu: E^* \longrightarrow \mathbb{R}^+,$$

is an *estimation function* in $\mathcal{R}$ iff it satisfies the following conditions:

(i) $\nu(e, e) = 0$ for all $e \in E$,

(ii) if $e_1 \prec e_2 \prec e_3 \prec e_4$, then $\nu(e_2, e_3) \leq \nu(e_1, e_4)$;

observe that (ii) and (i) imply that $\nu(e, f) \geq 0$ for all $(e, f) \in E^*$ (it is sufficient to take $e = e_1 = e_2 = e_3$ and $f = e_4$).

Let ν be an estimation function in $(R, E, \prec)$.

DEFINITION 9. The system $Q_{\mathcal{R}} = (R, E, \nu, \prec)$ is a *rough estimation space*, and for any $r \in \mathcal{R}$ the number

$$\rho(r) = \nu(\overline{\underline{E}}) = \nu(\overline{E}, \underline{E})$$

is called the *degree of roughness* of r.

When convenient, we shall say *roughness* instead of *degree of roughness.* Observe that if r is exact, then it follows from the Proposition 3 that its roughness is 0, by the condition (i) of the definition of estimation function. If the estimation function ν is strictly isotonic, that is, if $\nu(e, f) = 0$ implies $e = f$, then inverse is also true: r is exact iff $\rho(r) = 0$.

One can imagine various ways of constructing estimation functions. One of them could be the following one:

If μ is a positive real-valued function defined on E which is isotonic with respect to the ordering relation $\prec$:

$$\text{if } e \prec f \text{ then } \mu(e) \leq \mu(f) \quad \text{for all } e, f \in E,$$

then

$$\nu_\mu: \nu_\mu(e, f) = \mu(f) - \mu(e)$$

is an estimation function. The function μ will be called *approximating measure* in $(R, E, \prec)$. The degree of roughness in this case can be expressed as follows: $\rho(r) = \mu(\overline{E}(r)) - \mu(\underline{E}(r))$.

Observe that the functions μ and $\mu_o = \mu - \mu(0)$ yield the same estimation function; from now on we shall always assume that $\mu(0) = 0$. Notice, too, that μ is a bounded function, since $\mathbb{1} \in e$ and $\mu(\mathbb{1}) \geq \mu(r)$ for all $r \in E$. It follows that $\mu \cdot (\mu(\mathbb{1}))^{-1}$is an approximating measure, too.

DEFINITION 10. If $Q = (R, E, \nu_{\mu}, <)$ is a rough estimation space and ν_{μ} is defined by an approximating measure μ on E, then Q will be called *rough approximation space*. For any $r \in R$ the numbers $\mu(\overline{E})$ and $\mu(\underline{E})$ will be called, respectively, the *upper measure* and the *lower measure* of r.

For example, if $(U, \mathfrak{M}, \mu)$ is a measurable space, R is a family of subsets of U ordered by inclusion, $\mathfrak{M}$ is a σ-algebra of measurable sets, μ is a bounded measure, and all elements of E are measurable sets ($E \subseteq \mathfrak{M}$), then μ is an approximating measure in $(R, E, \subseteq)$. Notice that in this case $\nu_{\mu}(e,f)=\mu(f-e)$, where '$-$' is the set difference symbol. In particular, if μ is a probability measure, then the corresponding function ν_{μ} could be called *probability estimation function*. If U is a finite universe and $\mu(e)$ is the cardinality of e, then the derived estimation function corresponds to the approach adopted in the Pawlak's theory of rough sets.

EXAMPLE 5. Let us return to the situation considered in the Example 3, with the additional assumption that the universe U is a finite set: $(\mathbf{P}, \mathbf{D}, \subseteq)$ is an approximation space in which $\mathbf{P}$ is the power set of U, $\mathbf{D}$ is the lattice of sets generated by the elements of a given partition $\mathbf{C}$ of U, $\mathbf{C}=\{C_1,\dots,C_n\}$, and $\subseteq$ is the set inclusion in $\mathbf{P}$. Let μ be defined as follows:

$$\mu(D) = \frac{card(D)}{card(U)}, \text{ for all } D \in \mathbf{D}.$$

The corresponding estimation function is

$$\nu(D, D') = \frac{card(D-D')}{card(U)}, \text{ for all } D,D' \in \mathbf{D},\ D \subseteq D'.$$

$(\mathbf{P}, \mathbf{D}, ,\nu, \subseteq)$ is a rough approximation space and for any set $X \subseteq U$ (that is, for any $X \in \mathbf{P}$), and the degree of roughness of X is equal to

$$\rho(X) = \frac{card(\overline{\mathbf{D}}(X)-\underline{\mathbf{D}}(X))}{card(U)}.$$

The degree of roughness corresponds in this case to the Pawlak's measure of the 'doubtful region' (or 'boundary') of the set X (see Appendix, A1). ■

The approach presented above can be roughly characterized as 'based on the weights of elements'. An alternative one can be 'based on the weight of relation between elements':

If $\mathcal{P} = (P, <)$ is a partial order then a *chain* in $\mathcal{P}$ is any linear sub-order of $\mathcal{P}$; observe that any single element can be regarded as a chain. A *maximal chain* is a maximal (with respect to the inclusion in P^2) linear sub-order of $\mathcal{P}$. An *edge* in $\mathcal{P}$ is a pair (e, f) such that f *covers* e, that is $e < f$ and there is no g in P such that $e < g < f$. For any $e, f \in P$, if $e \lessdot f$, then there exists at least one *chain* $\mathfrak{C}$ joining e with f, that is, a linear sub-order of E with e being its minimal element and f the maximal one; consequently, there exists at least one maximal chain joining e with f. If $\mathfrak{C}$ is a maximal chain joining e with f, and it is finite:

$$\mathfrak{C}: \quad e = e_0 < e_1 < e_2 < \ldots < e_n = f,$$

then the number n is called the *length* of the chain; the length of the chain $e = f$ is equal to 0. Observe that in this case any pair (e_{i-1}, e_i) is an edge. Therefore the length of $\mathfrak{C}$ is the number of its edges. The length of an infinite chain is $+\infty$. The length of an order is defined as the supremum of the lengths of all its chains; it is equal to the supremum of lengths of all its maximal chains.

Let $(R, E, \lessdot)$ be an approximation space, and let $\nu(e, f)$ be the maximum of the lengths of all chains joining e and f, for any $e, f \in E$ such that $e \lessdot f$. If the length of E is a finite number, then the function ν is a rough mesure; we shall refer to it as the *algebraic* estimation function.

The latter construction can be generalized as follows. Let E be a lattice, and assume that a non negative function π is given which is defined on the set of all edges of E; for any edge (e, f) the value $\pi(e, f)$ is called the *weight* of the edge. For any maximal chain $\mathfrak{C}$ in E we shall define its *weighted length* as 0 if it is trivial, that is, consists of one element; as $+\infty$ if it contains a non-trivial subchain with no edges (any segment of a real line ordered by $\leq$ has this property); and, otherwise, as the (possibly infinite) sum of the weights of all its edges. The *weighted length* of E is the supremum of the weighted lengths of all maximal chains in E.

Now, if $(R, E, \lessdot)$ is an approximation space in which the edges of E are weighted, and if the weighted length of E is finite, then we can define a number $\nu(e, f)$ for any pair $e \lessdot f$ as the supremum of the weighted lengths of all maximal chains joining e with f. The function ν coincides with the algebraic estimation function when $\pi \equiv 1$,

It is easy to see that, in general, ν is an estimation function. Indeed, $\nu(e, e) = 0$, and the condition (i) of the definition of estimation function is satisfied. If $e_1 \lessdot e_2 \lessdot e_3 \lessdot e_4$, and $\mathfrak{C}$ is a maximal chain joining e_2 with e_3 then there exists a maximal chain $\mathfrak{C}_1$ joining e_1 with e_4 and containing $\mathfrak{C}$, which implies that (ii) holds.

8. Partial dependence. If $\mathcal{R} = (R, E, \lessdot)$ is a complete rough lattice and $(R, E, \nu_\mu, \lessdot)$ is a rough approximation space, then, following our terminology, we should say that $(R, E, \nu_\mu, \lessdot)$ is a *complete rough approximation space.* To be short, we shall call it a c.r.a. space, and denote it by $(\mathcal{R}, \mu)$.

Without loss of generality we can assume that the approximating measure μ is a restriction to E of an isotonic (with respect to $\lessdot$) function μ^* defined on the whole R; it is sufficient, for example, to define $\mu^*(r)$ as the mean of $\mu(\overline{E}(r))$ and $\mu(\underline{E}(r))$. The correctness of such an extension is an immediate consequence of the assumption of completeness: both $\overline{E}$ and $\underline{E}$ isotonic.

In this section we shall assume that μ is defined on R.

Let $(\mathcal{R}, \mu)$ be a complete rough approximation space. For any $r \in R$ we have:

$$\mu(\underline{E}(r)) \leq \mu(r) \leq \mu(\overline{E}(r)) ,$$

$$\text{if } r \text{ is exact then } \mu(\underline{E}(r)) = \mu(r) = \mu(\overline{E}(r)) , \quad \text{and}$$

$$\mu(0) = 0.$$

If the approximating measure μ is strictly isotonic in R (that is, $r \lessdot s$ implies $\mu(r) < \mu(s)$), then stronger conditions hold:

$$\mu(\underline{E}(r)) = \mu(r) \text{ iff } \underline{E}(r) = r \text{ iff } \mu(\overline{E}(r)) = \mu(r) ,$$

and

$$\mu(r) = 0 \text{ iff } r = 0.$$

In other words, in this case $\mu(r) = \mu(\underline{E}(r))$ iff r is an exact element.

Let $\gamma_E(r) = \mu(\underline{E}(r))\cdot(\mu(r))^{-1}$ for $r \neq 0$ and $\gamma_E(0) = 1$. The above observations can be resumed as follows.

PROPOSITION 9. If $(\mathcal{R}, \mu)$ is a c.r.a. space and μ is strictly isotonic, then for any $r \in R$

$$\gamma_E(r) = 0 \text{ iff } \underline{E}(r) = 0,$$

and

$$\gamma_E(r) = 1 \text{ iff } r \text{ is an exact element.} \quad \blacksquare$$

Consider, now, an approximation system $\mathcal{A} = (\mathcal{R}, l)$; by Definition 11, for any e in the domain of l, the system $\mathcal{R}_e = (R, l(e), \lessdot)$ is a complete rough lattice. If μ is an isotonic non-negative function on R with $\mu(0) = 0$, then for any $e \in \mathcal{D}_l$ the pair $(\mathcal{R}_e, \mu)$ is a complete rough approximation space.

The triple $(\mathcal{R}, l, \mu)$, which can be identified with the family $\{(\mathcal{R}_e, \mu)$: $e \in D_l\}$ of all c.r.a. spaces generated by l, will be called *rough approximation system.*

Let $(\mathcal{R}, l, \mu)$ be a rough approximation system, let $e \in D_l$, and let $E = l(e)$.

DEFINITION 11. Let $r \in R$. The number $\gamma_E(r)$ will be called *the degree of dependence of r on e* (shortly: *dependence* $e \longrightarrow r$), and it will be denoted by $\gamma(e \longrightarrow r)$.

In other words, $\gamma(e \longrightarrow r)$ is the relative measure of $e(r)$ with respect to r, and it can be interpreted as the measure of the 'lower $l(e)$-exactness' of the rough element r.

In particular, when μ is strictly isotonic, then r fully depends on e iff r is $l(e)$-exact: L

$$\gamma(e \longrightarrow r) = 1 \text{ iff } r \in l(e) .$$

PROPOSITION 10. Let e, f $\in E$ and $r, s \in R$. Then

(i) $$\gamma(e \longrightarrow r \wedge s)\cdot\mu(r \wedge s) \le \min\{\gamma(e \longrightarrow r)\cdot\mu(r), \gamma(e \longrightarrow s)\cdot\mu(s)\} ,$$

and

(ii) $$\gamma(e \wedge f \longrightarrow r) \ge \max\{\gamma(e \longrightarrow r), \gamma(f \longrightarrow r)\} .$$

Proof. (i). $e(r \wedge s) = e(r) \wedge e(s)$ by Proposition 8(3). Therefore $\mu(e(r \wedge s)) \le$ $\le \min\{\mu(e(r)), \mu(e(s))\}$ by isotonicity of μ. Now it is sufficient to observe that $\mu(e(y)) = \gamma(e \longrightarrow y)\cdot\mu(y)$ for any $y \in R$, by definition.

(ii). If $\mu(r) = 0$, then both members of (ii) are equal to 0 by definition of γ. Assume that $\mu(r) \ne 0$. The inequality

$$\mu((e \wedge f)(r)) \ge \mu(e(r) \vee f(r)) \ge \max\{\mu(e(r)), \mu(f(r))\}$$

holds by, consecutively, Proposition 10(6) and isotonicity of μ. Dividing it by the non-negative number $\mu(r)$ we obtain (ii). ∎

REMARK 7. The Proposition 10 is a generalization of results of Novotny and Pawlak, [21], concerning dependence of families of sets. ∎

9. Normal families. Let U be a nonempty universe. We shall say that a family $\mathbf{L} = \{L_t: t \in T\}$ of nonempty subsets of U is a *normal family* iff $L_t \subseteq L_s$ implies that $L_t = L_s$, for all $t, s \in T$. The class of all normal families of subsets of U will be denoted by $\mathbf{L}(U)$, or $\mathbf{L}$. The class of all finite normal families will be denoted by $\mathbf{F}(U)$ (or $\mathbf{F}$), and $\mathbf{C}(U)$ (or $\mathbf{C}$) will stand for the class of all partitions (=classifications) of U. Observe that

$$\mathbf{L}(U) = \mathbf{F}(U) \text{ iff } U \text{ is a finite set.}$$

The symbol $\mathbf{1}$ will stand for the one-element normal family $\{U\}$; $\mathbf{1}$ belongs to $\mathbf{C}(U)$, an therefore to $\mathbf{F}(U)$ and $\mathbf{L}(U)$. The empty family is a normal family; it will be denoted by $\mathbf{0}$: $\mathbf{0} = \emptyset \in \mathbf{F}(U)$. We shall also assume, for the sake of convenience, that $\mathbf{0} \in \mathbf{C}(U)$.

Observe that if $\mathbf{S}$ is arbitrary family of subsets of U, then the family $\mathbf{M}(\mathbf{S})$ of all its maximal elements is a normal one.

Let U be a universe, $\mathbb{L} = \mathbb{L}(U)$, and let $\mathbf{L}, \mathbf{L}' \in \mathbb{L}$.

Definition 12. We shall say that $\mathbf{L}$ is *finer than* $\mathbf{L}'$ (shortly: $\mathbf{L} \prec \mathbf{L}'$) iff for any $L \in \mathbf{L}$ there exists an $L' \in \mathbf{L}'$ such that $L \subseteq L'$.

Proposition 11. The system $\mathcal{L}(U) = (\mathbb{L}(U), \prec)$ is a bounded partial order.

Proof. It is obvious that $\prec$ is reflexive and transitive. To prove anti-symmetry, assume that $\mathbf{L} \prec \mathbf{L}' \prec \mathbf{L}$ for some $\mathbf{L}, \mathbf{L}' \in \mathbb{L}$. It follows that for any set $L \in \mathbf{L}$ there exist $L' \in \mathbf{L}'$ and $L'' \in \mathbf{L}$ such that $L \subseteq L' \subseteq L''$, which implies that, consecutively, $L = L''$ (by normality of $\mathbf{L}$), $L' = L$ and $\mathbf{L} \subseteq \mathbf{L}'$. The same assumption implies that $\mathbf{L}' \subseteq \mathbf{L}$. Therefore $\mathbf{L} = \mathbf{L}'$, what means that $\prec$ is a anti-symmetric relation. To close the proof, it is sufficient to observe that $\mathbf{0} \prec \mathbf{L} \prec \mathbf{1}$ for any $\mathbf{L} \in \mathbb{L}$. ∎

Consequently, also $\mathcal{F} = \mathcal{F}(U) = (\mathbb{F}(U), \prec)$ and $\mathcal{C} = \mathcal{C}(U) = (\mathbb{C}(U), \prec)$ are partial orders.

The following properties are immediate consequence of the definition of the relation *finer*:

if $\mathbf{L} \in \mathbb{L}$ and $L \in \mathbf{L}$ then $\{L\} \prec \mathbf{L}$;

if $\mathbf{L} \in \mathbb{L}$ and $\mathbf{L}' \subseteq \mathbf{L}$, then $\mathbf{L}' \in \mathbb{L}$ and $\mathbf{L}' \prec \mathbf{L}$;

if $L, L' \subseteq U$, then $\{L\}, \{L'\} \in \mathbb{L}$, and: $\{L\} \prec \{L'\}$ iff $L \subseteq L'$.

It is well known, [6], that the set of all equivalence relations on U is a partial order (with respect to inclusion in U^2) which is isomorphic with $\mathcal{C}(U)$ – the empty relation corresponds by definition to the 'empty partition' $\mathbf{0}$. Hence, we can interpret $\mathbb{C}(U)$ as the partially ordered set of all equivalence relations on U, when convenient. It is also known that $\mathcal{C}(U)$ is a complete lattice. The following observations will be useful in the analysis of the algebraic structure of $\mathcal{L}(U)$ and $F(U)$.

Lemma 1. If $\mathbf{L}, \mathbf{L}' \in \mathbb{L}$ and there exist $L \in \mathbf{L}$, $L' \in \mathbf{L}'$ such that $L \supsetneq L'$, then it is not true that $\mathbf{L} \prec \mathbf{L}'$.

Proof. It is a simple consequence of the fact that $\mathbf{L}$ is a normal family. ∎

Lemma 2. If $\mathbf{F}, \mathbf{F}' \in \mathbb{F}$ are finite normal families, then there exist in $\mathbb{L}$ $sup_{\mathbb{L}}\{\mathbf{F}, \mathbf{F}'\} \in \mathbb{F}$ and $inf_{\mathbb{L}}\{\mathbf{F}, \mathbf{F}'\} \in \mathbb{F}$.

Proof. 1). Let $\mathbf{S} = \mathbf{F} \cup \mathbf{F}'$, and let $\mathbf{U} = \mathbf{M}(\mathbf{S})$ be the set of all maximal (with respect to $\subseteq$) elements of $\mathbf{S}$; $\mathbf{U}$ is a finite normal family. The fact that $\mathbf{S}$ is finite implies that for any $S \in \mathbf{S}$ there exists an $M \in \mathbf{U}$ such that $S \subseteq M$

(observe that this may not be true in the case of infinite **S**). Hence, **U** is an upper bound of both **F** and **F'**, and it belongs to **F**.

Let **V** be an upper bound of {**F**, **F'**}. For any $S \in$ **S** there exists an $V \in$ **V** such that $S \subseteq V$. In particular, the same holds for any $S \in$ **U**, since **U** $\subseteq$ **S**. Therefore **U** $\preccurlyeq$ **V**. It follows that **U** is the l.u.b. of {**F**, **F'**}.

2). Let **P** = $\{F \cap F'$: $F \in$ **F**, $F' \in$ **F'**, $F \cap F' \neq \emptyset\}$, and take **I** = **M**(**P**). **I** is a finite normal family, and **I** $\subseteq$ **P**. It follows that for any $I \in$ **I** there exist $F \in$ **F** and $F' \in$ **F'** such that $I \subseteq F$ and $I \subseteq F'$, that is, **I** $\preccurlyeq$ **F** and **I** $\preccurlyeq$ **F'**: **I** is a lower bound of {**F**, **F'**}.

Consider, now, an arbitrary normal family **J** which is a lower bound of {**F**, **F'**}. For any $J \in$ **J** there exist $F \in$ **F** and $F' \in$ **F'** such that $J \subseteq F$ and $J \subseteq F'$. Consequently, $J \subseteq F \cap F'$, and $J \subseteq I$ for some $I \in$ **I**, by the definition of the family **I**. Therefore **I** is the g.l.b. of {**F**, **F'**}. ∎

Observe that if **L**, **L'** $\in$ **C**(U), then also *inf*{**L**, **L'**} is a partition. It is not so in the case of the lowest upper bound: if card(U) ≥ 3 then there exist **L**, **L'** such that *sup*{**L**, **L'**} $\notin$ **C**(U). To see that, it is sufficient to take **L** = $\{\{a, b\}, \{c\}\}$ and **L'** = $\{\{a\}, \{b, c\}\}$, where a, b, c are three different elements of U. In such a case *sup*{**L**, **L'**} = $\{\{a, b\}, \{b, c\}\} \notin$ **C**(U).

LEMMA 3. If U is an infinite universe, then there exist in **L**(U) families **L** and **L'** such that there is no *inf*{**L**, **L'**} in **L**(U).

<u>Proof</u>. Let $A = \{a_n : n \in \mathbb{N}\}$, $B = \{b_m : m \in \mathbb{N}\}$ be two disjoint subsets of U: $a_n \neq b_m$ for all $n, m \in \mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers). Let $A_k = \{a_n : n \leq k\}$ and $B_k = \{b_m : m \geq k\}$ for all $k \in \mathbb{N}$. Consider **L'** = $\{A\}$, and **L** = $\{L_k : k \in \mathbb{N}\}$, where $L_k = A_k \cup B_k$. It is easy to see that **L** is a normal family, that all $\{A_k\}$ are lower bounds of **L**, and that none of them is the greatest lower bound of **L**, since $\{A_k\} < \{A_{k+1}\}$ for all k.

Assume now that **M** is a lower bound of {**L**, **L'**}. It follows that for any $M \in$ **M** there exists a k such that

$$M \subseteq A \cap L_k = \{a_1, a_2, \dots, a_k\} = A_k \subsetneq A_{k+1},$$

which implies, by Lemma 1, that $\{A_{k+1}\}$ is a lower bound of {**L**, **L'**} which is not finer than **M**. Consequently, no lower bound of {**L**, **L'**} is the greatest lower bound of the pair. ∎

LEMMA 4. If U is an infinite universe, then there exists a set of finite normal families with no supremum in **F**(U).

<u>Proof</u>. Let A, B, A_n, B_m and L_k be defined as in the proof of Lemma 3. The set $\mathfrak{L} = \{\{L_k\} : k \in \mathbb{N}\}$ is a set of finite normal families.

Let $\mathbf{L}_k$ be defined as follows:

(*) $\mathbf{L}_k = \{L_1, L_2, \ldots, L_{k-1}, B_k \cup A\}$, $k \in \mathbb{N}$.

Each $\mathbf{L}_k$ is a finite family, and it is normal, since $A_n \subsetneq A$ for all n, and $B_m \supsetneq B_k$ for $m < k$. Moreover, $\mathbf{L}_k > \{L_l\}$ for all $k, l \in \mathbb{N}$, since $L_l \subsetneq B_k \cup A$ for $l \geq k$. Notice that $\mathbf{L}_{k+1} < \mathbf{L}_k$ for all $k \in \mathbb{N}$, since $L_k \subsetneq B_k \cup A$. Thus all $\mathbf{L}_k$ are upper bounds of $\mathfrak{L}$, and none of them is the lowest upper bound.

Suppose, now, that $\mathbf{K} = \{K_1, K_2, \ldots, K_I\}$ is a finite normal family which is an upper bound of $\mathfrak{L}$. For any $k \in \mathbb{N}$ there exists an $i(k) \in \{1, \ldots, I\}$ such that $L_k \subseteq K_{i(k)}$. Consequently, there exists i_o such that $L_k \subseteq K_{i_o}$ for infinitely many k. Therefore

(**) $$K_{i_o} \supseteq \bigcup\{L_k: i(k) = i_o\} = B_{k_o} \cup A \, ,$$

where $k_o = min\{k: L_k \subseteq K_{i_o}\}$, since $\{A_n\}$ is an increasing, and $\{B_n\}$ a decreasing sequence of sets.

Consider $\mathbf{K}' = \mathbf{L}_{k_o}$. The conditions $\mathbf{K} > \{L_k\}$ and (**) imply that $\mathbf{K}' < \mathbf{K}$. It follows that if there exists a supremum of the set $\mathfrak{L}$ in $\mathbb{F}$, then it must be of the form (*), which is a contradiction: $sup_{\mathbb{F}}\mathfrak{L}$ does not exist. ■

REMARK 8. Observe that $sup_{\mathbb{L}}\mathfrak{L} = \{L_m: m \in \mathbb{N}\}$. It is easy to prove that, in general, if $\mathbf{L} = \{L_t: t \in T\}$ is a normal family, then $\mathbf{L} = sup_{\mathbb{L}}\{\{L_t\}: t \in T\}$. ■

Now we are able to formulate the following

PROPOSITION 13. The partial orders $\mathcal{L}(U) = (\mathbb{L}(U), <)$, $\mathcal{F}(U) = (\mathbb{F}(U), <)$ and $\mathcal{C}(U) = (\mathbb{C}(U), <)$ have the following properties:

a). $\mathcal{L}(U)$ is a lattice iff the universe U is a finite set; if it is a lattice, then a complete one.

b). $\mathcal{F}(U)$ is a lattice for any U, and it is a complete lattice iff U is finite.

c). $\mathcal{C}(U)$ is a complete lattice for any U; it is a meet-subsemilattice of $\mathcal{F}(U)$, but not a sublattice if $card(U) \geq 3$.

<u>Proof</u>. We have already observed that $\mathcal{L}(U) = \mathcal{F}(U)$ iff U is a finite universe. It follows that *a)* and the first part of *b)* are implied by Lemmas *2* and *3.*; the second part of *b)* is a consequence of Lemma *4* and of *a)*; the first part of *c)* is a well-known theorem (see, for example, [6]), and the second part of *c)* is a consequence of the observation preceding Lemma *3.* ■

10. Pawlak approximation systems. The aim of this section is to show that the basic facts of the Pawlak's theory of information systems and rough sets are derivable from the general construction presented here, and some new results can be added (cf. Appendix).

Let U be a finite universe.

The set $\mathbf{C} = \mathbf{C}(U)$ of all partitions of U (the empty partition included) is a complete meet-semilattice with respect to the *finer* relation. Therefore all the concepts introduced in the Section 1 (equivalence, independence, superfluousness, reducts, etc) are applicable to the subsets of $\mathbf{C}$ and, in particular, the conclusions of Propositions 1 and 2 are valid. (Compare, for example, the results of [25], [22]; see also sections A2 and A3 of the Appendix).

Let $C \in \mathbf{C}(U)$ be arbitrary partition of U:

$$C \in \{C_t : t \in T\} .$$

Denote by $\mathcal{B}(C)$ the algebra of subsets of U generated by the partition C:

$$X \in \mathcal{B}(C) \quad \text{iff} \quad (\exists_{T' \subseteq T} (X = \bigcup\{C_t : t \in T'\}) \text{ or } X = \emptyset) ;$$

the elements of $\mathcal{B}(C)$ are called *C-definable sets.* $\mathcal{B}(C)$ is a subalgebra of $\mathcal{P}(U)$, and $(\mathcal{P}(U), \mathcal{B}(C), \subseteq)$ is a complete rough algebra of sets, in which all elements of $\mathcal{P}(U)$ are recognizable. Therefore it is an approximation space, and the operations of lower and upper approximation are a closure and dual closure, respectively (see Sections 2, 3 and 4). A subset of U is $\mathcal{B}(C)$-exact iff it is C-definable. The system $(\mathcal{P}(U), \mathcal{B}(C), \subseteq)$ is a complete rough lattice for any C, and it follows that the conclusions of both Proposition 6 and Proposition 7 are valid.[1] (Compare the results of [16], [23], [26]; also A1 and A4).

Furthermore, since U is a finite universe, cardinality is a measure on $\mathcal{P}(U)$ and, consequently, it is an approximating measure in the approximation space $(\mathcal{P}(U), \mathcal{B}(C), \subseteq)$. It follows (compare Example 5 of Section 7) that the degree of roughness of any subset of U can be measured, and, since cardinality is a strictly isotonic on finite sets, a set is C-definable iff its roughness is equal to 0. Observe, too, that the degree of dependence of a set $X \subseteq U$ on a partition $C \in \mathbf{C}(U)$ can be defined (cf. Definition 11).

Let, now, $l(C)$ be the set of all normal families of C-definable sets, and let $l(\emptyset) = \mathbf{L}(U)$.

The mapping $C \longrightarrow l(C)$ is defined on $\mathbf{C}(U) \subseteq \mathbf{L}(U)$, and it is easy to see that the pair

$$(\mathcal{L}(U),\ l)$$

is an approximation system (Definition 8, Section 6), since

- the condition (i) of Definition 8 is implied by Proposition 13,c): $D_l = \mathbf{C}(U)$, and the latter is a meet-subsemilattice of $\mathbf{L}(U)$;
- the pair $(l(C), \lessdot)$ is a complete sublattice of $\mathcal{L}(U)$, which implies that

[1] Notice that in the Example 3 we have not assumed that U is finite.

the condition (ii) is satisfied: $(\mathbb{L}(U), l(\mathbf{C}), \lessdot)$ is a complete rough lattice for any $\mathbf{C}$;

– (iii) is satisfied by definition of $l(\mathbf{C})$, since $\mathbf{C}$ is a normal family of $\mathbf{C}$-definable sets: $\mathbf{C} \in l(\mathbf{C})$;

– (iv) is a simple consequence of the fact that a partition $\mathbf{C}'$ is finer than $\mathbf{C}$ iff it is a sub-partition of $\mathbf{C}$, which implies that $\mathcal{B}(\mathbf{C}') \supseteq \mathcal{B}(\mathbf{C})$ and, consequently, $l(\mathbf{C}') \supseteq l(\mathbf{C})$.

Therefore, the assertions (*1*) – (7) of the Proposition 8 are valid for approximation by partitions; (*1*), (2) and (4) are obvious in this model. As far as the author knows, (3), (*5*), (*6*) and (7) are stated for the first time here (the first part of (3) and (*6*) are implicit in [21])

Cardinality can be extended in a natural way to a function # defined on $\mathbb{L}(U)$: if $\mathbf{L} \in \mathbb{L}(U)$ and $\mathbf{L} = \langle L_t : t \in T \rangle$, then

$$\#(\mathbf{L}) = \Sigma_t card(L_t) \ .$$

Notice that '#' *is not* an approximating measure on $\mathbb{L}(U)$, if only U has more than two elements (if $a, b, c \in U$ then $\mathbf{R} = \langle\langle a, b\rangle, \langle b, c\rangle\rangle$ is finer than $\mathbf{S} = \langle a, b, c\rangle$, but $\#(\mathbf{R}) = 4 > 3 = \#(\mathbf{S})$). The same function is strictly isotonic with respect to the relation *finer* on the set $\mathbb{D} = \mathbb{D}(U)$ of all *disjoint* (and thus normal) families of subsets of U. Observe that $\mathbb{D}(U) \supseteq \mathbb{C}(U)$).

Let $\mathcal{D}(U) = (\mathbb{D}, \lessdot)$, and, for any partition $\mathbf{C}$, let $d(\mathbf{C})$ denote the set of all families of disjoint $\mathbf{C}$-definable subsets of U.

The triple $(\mathcal{D}(U), d, \#)$ will be called *Pawlak approximation system*.

It is easy to see that it is a rough approximation system. Consequently, the concept of partial dependence can be introduced (cf. A5). A disjoint family $\mathbf{L}$ fully depends on a partition $\mathbf{C}$ iff all elements of $\mathbf{L}$ are Cefinable sets, and the inequalities (i) and (ii) of Proposition 10 hold for approximations by partitions (they have been proved for this case in [23] and, in a slightly more general formulation, in [21], Theorems 5.1 and 5.2).

It is worthwhile to observe that it can be demonstrated that the degree of dependence in $(\mathcal{D}(U), d, \#)$ satisfies the triangle inequality:

$$\gamma(\mathbf{L} \longrightarrow \mathbf{M}) + \gamma(\mathbf{M} \longrightarrow \mathbf{N}) \geq \gamma(\mathbf{L} \longrightarrow \mathbf{N})$$

for arbitrary categorizations $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ and, consequently, the function

$$\rho(\mathbf{L}, \mathbf{M}) = \tfrac{1}{2}(\gamma(\mathbf{L} \longrightarrow \mathbf{M}) + \gamma(\mathbf{M} \longrightarrow \mathbf{L}))$$

is a distance function on $\mathbb{C}(U)$ (for details, see [21]).

Remark 9. It is possible to define *relative* superfluousness, independence and reducts in purely algebraic terms in such a way that they coincide in

($\mathcal{D}(U)$, d, $\#$) with the respective Pawlak's concepts (cf.A6). The corresponding results will be presented in the final version of this work. ∎

Concluding remarks. We have left many 'loose ends' in this presentation. This is due to various reasons. One of them was the intention of analyzing the 'most general' situations, which led to constant change of assumptions about structures under consideration. Another one is that there are specific models that have influenced the approach adopted here, but are not explicitly present in the text. For example, the notion of *preclosure* introduced in the section 2 has not been used later on. Nevertheless, it plays an important role in the *rough concept* model, cf. [12] and [15]. Still another is that there are certain ideas that seem to be interesting, but have not been investigated yet: this is the case - and the only reason of existence in this paper - of *recognizable elements* (section 3).

We have devoted much more attention to the lower approximations than to the upper approximations. This is due to the fact that the former seem to play more important role in various known models and applications, and that the concept of dependence is based on them. Nevertheless, some more exhaustive study of upper approximations properties would be interesting, at least from the algebraic point of view.

There is also another aspect of this problem. Pomykała has observed in his paper [29] that the analogy between topological closure-interior operations and lower-upper approximations, respectively, is in a sense characteristic for Pawlak theory. He has demonstrated that in his *general approximation spaces* (based on tolerances instead of equivalences, cf. A1) the upper and lower approximations are not topologically dual: they are dual iff the underlying tolerance relation is an equivalence. It would be interesting to obtain some general characterization of those approximation spaces (in our sense) in which this topological duality holds.

The Section 9 devoted to normal families seems to be excessive - its results practically have not been used in this text. Nevertheless, they constitute a common framework for the ideas present in papers of Novotny & Pawlak [21], Pomykała [29] and Nieminen [20], as well as for the original information system theory.

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APPENDIX:

Rough sets and information systems

(*basic concepts*)

It is assumed that all the sets considered here are finite ones. The terminology we use here has been introduced in the author's paper [9] and differs slightly from the 'orthodox' one, see [24].

A1. Approximation space. Let U be a finite set of *objects* called *universe*. If R is an equivalence relation in the universe U, then the pair (R, U) is called *approximation space*.

Let R^* be the set of all equivalence classes of R. A set $X \subseteq U$ is *R-definable* iff it is empty, or it can be represented as a union of some elements of R^*.

If $X \subseteq U$ is not definable, then it is called *rough set*.

For any rough set X there exists a uniquely determined maximal R-definable subset of X. It is called *the lower approximation of* X with respect to R, and is denoted by $\underline{R}(X)$.

Similarly, the minimal R-definable set containing X is called its *upper approximation* with respect to R, and is denoted by $\overline{R}(X)$.

The set $Bd_R(X) = \overline{R}(X) - \underline{R}(X)$ is called the *boundary region* of X.

Observe that a set X is R-definable iff $X = \underline{R}(X)$ and/or $X = \overline{R}(X)$. It follows that X is R-definable iff

$$\#X = \#R(X),$$

where $\#X$ denotes the cardinality of X. If $\#X < \#\overline{R}(X)$, then X is a rough set. The number

$$\iota_R(X) = \frac{\#\underline{R}(X)}{\#X}$$

can be interpreted as a measure of *internal roughness of* X (the *external roughness* can be defined in a similar way). Notice that $0 \leq \iota_R(X) \leq 1$, and $\iota_R(X) = 1$ iff X is a definable set. *Accuracy measure* of the set X is defined as

$$\mu_R(X) = \frac{\#\underline{R}(X)}{\#\overline{R}(X)};$$

$\mu_R(X) \in [0, 1]$ for any $X \subseteq U$, and $\mu_R(X) = 1$ iff X is exact.

A2. Attributes and information systems. If U is a universe, V_a is a non-void set of *values* and a is a function from U onto V_a, then a is called an *attribute* on U. If $V_a \subseteq Re$ then a is a *numerical attribute.* Numerical attributes are also called *scales* or *variables*, especially in social science.

If A is a nonvoid set of attributes on U, then the pair (U, A) is called *information system.*

Let a be an attribute on U. With each value v of a we can associate the set $a^{-1}(v)$ of all elements x of U such that $a(x) = v$:

$$a^{-1}(v) = \{x \in U\colon a(x) = v\}.$$

This set is called the *indiscernibility class* (or *atom*) corresponding to the value v of the attribute a. Indiscernibility classes corresponding to different values of a are disjoint subsets of U, and each element of U belongs to exactly one indiscernibility class. Hence, the family of all indiscernibility classes is a *partition* (or a categorization) of the universe U. This family will be denoted by a^{*}.

Two attributes a,b are said to be *equivalent* iff $a^{*} = b^{*}$. Any attribute is equivalent to a numerical attribute whose values are positive integers (we recall that by definition an attribute is a finitealued function).

More generally, if A is a family of attributes, then two elements x,y of U are said to be *indiscernible* with respect to A, if for every $a \in A$ holds:

$$a(x) = a(y).$$

Any maximal set of indiscernible elements is called *indiscernibility class,* or *atom,* of A. (On the interpretative level, an indiscernibility class is a set of all elements that have the same description by the scales belonging to A). As above, the set of all indiscernibility classes corresponding to A is a partition of the universe U; it will be denoted by A^{*}.

Let A and B be two families of attributes on U. It is said that A and B are *equivalent* iff the corresponding indiscernibility partitions are are equal: $A \sim B$ *iff* $A^{*} = B^{*}$.

A3. **Independence and reducts.** Let A be a set of attributes on the universe U. An attribute $a \in A$ is said to be *superfluous* in A iff $A \sim (A-\{a\})$. If there are no superfluous attributes in A, then A is *independent.*

A set of attributes B is a *reduct* of A iff $B \subseteq A$, $B \sim A$ and B is independent. It can be demonstrated that any nonempty set of attributes has at least one reduct.

Intersection of all reducts is called the *core* of A, and the elements of the core are *indispensable* attributes.

If (U, A) is an information system, then all the concepts introduced above are referred to the system, rather than to the set A.

A4. Approximation in information systems. Let (U, A) be an information system. The indiscernibility relation $\sim_A$ corresponding to the set A is an equivalence relation and, consequently, the pair $(U, \sim_A)$ is an approximation space..

Therefore $\sim_A$-approximations of any $X \subseteq U$ can be defined. They are referred to as A-approximations and are denoted by $\overline{A}(X)$ and $\underline{A}(X)$. Now all the concepts relative to approximation space can be introduced here and, in particular, A-definability, A-roughness, and A-accuracy are well defined notions.

A5. Dependence of attribute sets. Let A and B be two sets of attributes on the same universe U. If all elements of B^{*} (that is, all indiscernibility classes of B) are A-definable sets, then the set B is said to be *(totally) dependent on* A. This fact is symbolically expressed as $A \longrightarrow B$.

Observe that in this case $\sum_{B \in B^{*}} \#\underline{A}(B) = \#U$, or, equivalently,

$$\Big(\sum_{B \in B^{*}} \#\underline{A}(B) \Big) \cdot (\#U)^{-1} = 1 \, .$$

If A and B are one-element sets $\{a\}$ and $\{b\}$, respectively, and $\{a\} \longrightarrow \{b\}$, then it is said that the attribute b depends on the attribute a, and expressed as $a \longrightarrow b$.

The intuitive justification of this notion of dependence is based on the following fact. If B is dependent on A, and x is arbitrary element of U, then the knowledge of values of attributes belonging to A on x is sufficient for finding this indiscernibility class of B^{*} to which x belongs. In other words, the description of x by attributes belonging to B is uniquely determined by its description by attributes belonging to A.

This notion of dependence can be generalized in the following way.

Let, as before, A and B be two sets of attributes. Let

$$B^{*} = \{X_1, X_2, X_3, \ldots, X_n\}.$$

Obviously, $\bigcup X_i = U$, and $\sum_i \#X_i = \#U$. The number

$$\gamma(A \longrightarrow B) = \frac{\sum_{i=1}^{n} \#\underline{A}(X_i)}{\#U}$$

is a generalization of the measure of internal roughness. It indicates 'how much' of the partition B^{*} can be defined, or explained, in terms of attributes belonging to A.

The index $\gamma(A \longrightarrow B)$ is called *the degree of dependency of B on A*. Observe that $\gamma(A \longrightarrow B) = 1$ iff B is totally dependent on A. If $\gamma(A \longrightarrow B) < 1$, then it is said that B *partially depends* on A.

A6. **Decision systems and relative reduction.** In certain applications the following scheme proved to be useful.

Let (U, A) be an information system and assume that $A = C \cup D$, with $C \cap D = \emptyset$. The triple (U, C, D) is called *decision system*. Elements of C are called *conditions*, and elements of D are *decisions*.

Now, from the applicative point of view, one can be interested in *reduction* of the system (U, C) (elimination of superfluous conditions), and in analysis of *dependence* of decisions on conditions, and that can be done by the means described above in the sections A3, A4 and A5.

Moreover, this scheme leads to the concepts of *relative* equivalence and reduction:

- two sets of conditions $B', B'' \subseteq C$ are said to be *D-equivalent* iff

$$\gamma(B' \longrightarrow D) = \gamma(B'' \longrightarrow D);$$

- a set $B \subseteq C$ is a *D-reduct* of C iff it is a minimal subset of C which is D-equivalent to C.

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