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AUCTIONS WITH HETEROGENEOUS ENTRY COSTS

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Abstract

We study the impact of public and secret reserve prices in auctions where buyers have independent private values and heterogeneous entry costs. We find that in a standard auction the optimal (i.e., revenue maximizing) public reserve price is typically above the seller's value. Moreover, an appropriate entry fee together with a public reserve price equal to the seller's value generates greater revenue. Secret reserve prices, however, differ across auction formats. In a second-price sealed-bid auction the secret reserve price is above the optimal public reserve price; hence there is less entry, a smaller probability of sale, and both the seller revenue and the bidders' utility are less than with an optimal public reserve price. In contrast, in a first-price sealed-bid auction the secret reserve is equal to the seller's value, and the bidders' expected utility (seller revenue) is greater (less) than with an optimal public reserve price.

Keywords: Standard Auctions, Endogenous Entry, Heterogeneous Entry Costs, Public Reserve, Secret Reserve.

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1 Introduction

When the number of bidders is exogenously fixed and the bidders have independent private values, then the optimal (i.e., revenue maximizing) reserve price is above the seller's value, is the same in every standard auction, and is independent of the number of bidders. (See Riley and Samuelson (1981) and Myerson (1981) for this classic result.) However, in many instances the number of bidders in an auction is determined endogenously, and depends on both the auction format and the reserve price. Indeed, McAfee and McMillan (1987) and Levin and Smith (1994) – henceforth MM and LS, respectively – show that the endogenous entry of bidders has important implications for the seller's choice of a reserve price. Specifically, they show that when all potential bidders have the same cost for entering the auction, then the optimal reserve price is equal to the seller's value, so long as the number of potential bidders is sufficiently large.

A maintained assumption in MM and LS is that all potential bidders have the same entry cost. In the present paper we study optimal reserve prices in a setting with endogenous entry but where bidders have heterogenous entry costs. Our model is identical to that of MM and LS, except that prior to deciding whether to enter the auction the bidders privately observe their entry costs, which are drawn independently from a common distribution. As in MM and LS, bidders then simultaneously chose whether to enter the auction. Each bidder who enters the auction observes his value for the item and then bids. In this setting we characterize optimal reserve prices both when the reserve price is *public* (as in MM and LS), and when it is *secret*. Our results show that heterogeneity in entry costs alters many of the conclusions obtained for the homogenous entry cost case.

In order to understand the intuition for our results, it is useful to review the intuition for homogeneous entry costs. When entry costs are homogeneous and the public reserve price is the seller's value, then the contribution to social surplus of the entry of an additional bidder is exactly equal to the bidder's expected utility of entering.[1] Thus, the interests of bidders and society are aligned, and the number of entering bidders maximizes social surplus. Moreover, since bidders enter until all bidder surplus is competed away, the seller captures the entire social surplus. Therefore a reserve price equal to the seller's value maximizes both seller revenue and

[1] A version of this result is established in Engelbrech-Wiggans (1993)'s Proposition 1, and is also observed in both MM and LS.

social surplus, independently of the distribution of values and the number of potential bidders (so long as is sufficiently large).

We show that when entry costs are heterogeneous and the public reserve price is the seller's value the marginal (expected) utility to a bidder of a change of her entry threshold equals the marginal (expected) social surplus; i.e., that the interests of bidders and society are aligned. Therefore a standard auction with a public reserve price equal to the seller's value maximizes social surplus whether entry costs are homogeneous or heterogeneous. Heterogeneity in entry costs, however, implies that not all bidder surplus is competed away by entry. Hence, even though raising the reserve price above the seller's value reduces social surplus, it increases the seller's share of the surplus and may increase revenue.

Consequently, when entry costs are heterogeneous the optimal public reserve price may be above the seller's value. Indeed, this is the case when the bidders' values are uniformly distributed, regardless of the distribution of entry costs (so long as its *c.d.f.* is differentiable). In addition, when the optimal reserve price is above the seller's value, then an even greater revenue can be obtained by employing an appropriately chosen entry fee and setting a reserve price equal the seller's value. (In contrast, it is well-known that an entry fee is equivalent to a reserve price when the number of bidders is fixed, and that the optimal entry fee is zero when bidders have homogenous entry costs.) Further, simple examples show that the optimal reserve price (and the optimal entry fee) does depend on the distribution of values and entry costs as well as on the number of potential bidders. Nevertheless, the optimal reserve price is always below the reserve price that is optimal when the number of bidders is exogenously fixed. These results for public reserve prices hold for any standard auction.

When the reserve price is secret and entry is endogenous, then the equilibrium reserve price depends on the auction format. We show that in a second-price sealed-bid auction the equilibrium secret reserve price is the same as when the number of bidders is exogenously fixed. Thus, in a second-price sealed-bid auction there is more entry and a higher probability of sale when the reserve price is public (and set optimally by the seller) than when it is secret. Interestingly, not only does an optimal public reserve generate more revenue for the seller, but it is also preferred by bidders. In a first-price sealed-bid auction the equilibrium secret reserve price is equal to the seller's value. Hence this auction maximizes social surplus, and is more (less) favorable to bidders (seller) than a first-price sealed-bid auction with an optimal public reserve price. Elyakime *et al.* (1994) also establish that the equilibrium secret reserve price is the seller's value in a model with an exogenously fixed number of bidders.

Our result for public and secret reserve prices are consistent with the empirical results reported in Katkar and Lucking-Reiley (2000) – K&L-R henceforth. K&L-R conducted a field experiment comparing secret and public reserves, in which they auctioned pairs of identical Pokemon cards on eBay. For each pair of cards, one card was auctioned using a secret reserve price, while the second card was auctioned with a public reserve price; in each case the reserve price was equal to 30% of the card's book value. It seems unlikely that the 30% reserve price used by K&L-R was an equilibrium reserve price. Nonetheless, our theoretical results are consistent with their empirical findings, provided that the reserve price they used was lower than the equilibrium secret reserve price. Our model predicts less entry when the reserve price is secret than when it is public.[2] Since the reserve price is the same in both cases, less entry implies fewer serious bids (i.e., fewer bids above the reserve), a smaller probability of sale, and less revenue to the seller, which is what K&L-R found. When the public and secret reserve prices are the same but above the equilibrium secret reserve price, our model predicts a revenue reversal – the public reserve price should yield less revenue than the secret reserve price. This prediction of our model is presently untested.

Several papers have established that a secret reserve is advantageous to the seller in some settings. Li and Tan (2000) show, in Elyakime *et al.* (1994)'s setting, that if bidders are sufficiently risk-averse, then a secret reserve price may raise more revenue than a public reserve price. Vincent (1995) has established that seller revenue may also be greater with a secret reserve price than with a public reserve price in second-price auctions with common values.

Other models of auctions with endogenous entry have been studied in the literature. Samuelson (1985) and Menezes and Monteiro (2001), for example, study optimal public reserve prices in a model where all buyers have the same entry cost and make their entry decisions knowing their values. We are no aware of other models of auctions with endogenous entry and heterogenous entry costs.

The paper is organized as follows. In Section 2 we layout the basic setting. Section 3 considers optimal public reserves, first reviewing the results for homogenous entry costs and then providing our results for heterogenous entry costs. In Section

[2] When the reserve is secret entry decisions depend on the bidders' expectation of the reserve rather than the reserve actually chosen. If the equilibrium secret reserve was more than the 30% public reserve, then there is less entry with the secret reserve.

4 we define the appropriate notion of equilibrium when the reserve price is secret, and then characterize the equilibrium secret reserve for first and second-price sealed-bid auctions. Section 4 provides a numerical example. Proofs are relegated to an Appendix.

2 Preliminaries

Consider a market for a single item for which there are $N > 1$ risk-neutral buyers and a risk-neutral seller. In this market the object is allocated using an unspecified *standard* auction with a reserve price. The seller's value for the object is zero. Each buyer must decide whether to enter the auction, and upon entering the auction how much to bid. We refer to buyers who have entered the auction as bidders. Each buyer i has a privately known entry cost Z_i. Buyers' entry costs $Z_1, \ldots, Z_N$ are independently and identically distributed on $[0, \bar{c}]$, where $0 < \bar{c} \leq \infty$, according to a *c.d.f.* H with $H(0) = 0$. If buyer i enters the auction, then she learns her value X_i. Values $X_1, \ldots, X_N$ are independently and identically distributed on $[0, \omega]$ according to an increasing and differentiable *c.d.f.* F with a decreasing hazard rate. We assume throughout that the number of bidders N is sufficiently large so that $\bar{c} > E(Y_1^{(N)}) - E(Y_1^{(N-1)})$, where $Y_1^{(n)} = \max\{X_1, \ldots, X_n\}$ is the highest of n values.[3]

We study equilibrium in two alternative settings that differ only in whether the seller's reserve price is made "public" or kept "secret." When the reserve price is public, it is announced prior to buyers deciding whether to enter the auction. When the reserve price is secret it is unknown to the buyers prior to entering the auction. We focus on the case where the reserve price is observed when the auction is resolved, which is the most common case in practice. Hence when the reserve price is public both entry decisions and bids can be conditioned on the reserve price, whereas when it is secret neither can be conditioned on the reserve price.

3 Public Reserve Prices

Assume that the seller publicly announces the reserve price $r \in [0, \omega]$ prior to the buyers deciding whether to enter the auction. Thus, buyers make entry decisions

[3] This assumption rules out the uninteresting case where every bidder enters the auction regardless of his cost (see Lemma 1). It says that if all $N - 1$ other bidders were to enter, then the expected social contribution of the Nth bidder is less than $\bar{c}$.

knowing their entry cost $z \in [0, \bar{c}]$ and the reserve price. Upon entering the auction, buyers observe their value, and perhaps the number of bidders, and then bid.

In order to focus on the analysis of the "entry game", throughout this section we assume that buyers' bidding strategies conform to the assumptions required to apply the Revenue Equivalence Principle; that is, we assume that following entry decisions, *for each reserve price the buyers' bidding strategies form an increasing symmetric equilibrium of the auction such that the expected payment of a buyer with value zero is zero* – see Myerson (1981), Riley and Samuelson (1981). It is well-known that the Revenue Equivalence Principle applies even when there is uncertainty about the number of bidders in the auction, provided that bidders have symmetric expectations. (See Krishna (2002), Section 3.2.2, whose notation we follow closely.) Hence, if buyers with identical entry costs make identical entry decisions, then expectations about the number of bidder in the auction are symmetric and therefore the REP applies whether or not buyers observe how many bidders are present in the auction.

If the auction used to allocate the object is a standard auction, then seller revenue and the buyer's expected utility following entry decisions can be calculated as if the auction was a second-price sealed-bid auction. Thus, if the reserve price is $r \in [0, \omega]$ and exactly $n \in \{1, \ldots, N\}$ buyers enter the auction, the seller's expected revenue is

$$\pi(r, n) = n \left[r(1 - F(r))F^{n-1}(r) + (n-1) \int_r^\omega y(1 - F(y))F^{n-2}(y)f(y)dy \right],$$

and the expected utility of a buyer (prior to entering the auction and learning her value) is

$$u(r, n) = \int_r^\omega \left(\int_r^y F(x)^{n-1}dx \right) f(y)dy.$$

Also, the gross (expected) social surplus, i.e., the social surplus ignoring entry costs, can be calculated as

$$s(r, n) = \int_r^\omega y dF^n(y).$$

Note that

$$s(0, n) = E(Y_1^{(n)}).$$

It is easy to see that $\pi(r, n)$ is increasing in n, $u(r, n)$ is decreasing in both r and n, and $s(r, n)$ is decreasing in r and increasing in n. The convention $s(0, 0) = 0$ will be useful in what follows.

In a standard auction with a zero reserve price the expected utility of each buyer and the social surplus are related according to a useful formula stated in the following lemma.

Lemma 1. *For $n \in \{1, \ldots, N\}$: $u(0,n) = s(0,n) - s(0,n-1)$.*

In words: in a standard auction with a zero reserve price and n bidders the expected utility of each bidder is equal to the gross social contribution of the n-th bidder. We provide a simple proof in the Appendix. As will be seen later, this fact is key to understanding the intuition for our results. A version of this formula is established in Proposition 1 of Engelbrecht-Wiggans (1993).

In our setting it will be useful to calculate the expected revenue of the seller and the expected utility of a buyer when the number of bidders in the auction follows a binomial distribution $B(N,p)$, where p is the probability that a single buyer enters, and $p_n^N(p)$ is the probability that exactly $n \in \{0, 1, \ldots, N\}$ buyers enter. The expected revenue of the seller is

$$\Pi(r,p) = \sum_{n=1}^{N} p_n^N(p)\pi(r,n),$$

and the (gross) expected utility of an entering buyer (i.e., the expected utility of a buyer before incurring her entry cost) is

$$U(r,p) = \sum_{n=0}^{N-1} p_n^{N-1}(p)u(r,n+1).$$

It is easy to see that $U(r,p)$ is decreasing in p. If $p'' > p'$, then $B(N,p'')$ first order stochastically dominates $B(N,p')$, and therefore since $u(r,n)$ is decreasing with respect to n, we have $U(r,p'') < U(r,p')$.

3.1 Homogenous entry costs

We begin by discussing and deriving existing results and simple extensions for the case of homogenous entry costs; that is, for the case where all buyers have the same fixed entry cost $\bar{c}$ (i.e., $H(z) = 0$ for $z < \bar{c}$ and $H(z) = 1$ for $z \geq \bar{c}$), where $\bar{c} < u(0,1)$, so that there is entry. For homogenous entry costs McAfee and McMillan (1987) establish that in a pure-strategy equilibrium of a first-price sealed-bid auction with a public reserve price (i) buyers capture none of the surplus, (ii) the optimal reserve price (i.e., the reserve price that maximizes seller revenue) is zero, and (iii) when the reserve price is zero a pure-strategy equilibrium maximizes social surplus (i.e., in equilibrium the optimal number of bidders enters the auction). Levin and Smith (1994) show that results analogous to (i)-(iii) hold for symmetric mixed-strategy equilibria of any standard auction. These results are easily derived in our setting, and extended to any standard auction in the case of McAfee and McMillan (1987)'s results. This exercise will help provide intuition for our results for the more realistic case where entry costs are heterogenous (i.e., H is not degenerate).

The maximum social surplus that can be achieved by *any* mechanism with a fixed number n of buyers is

$$w(n) = E(Y_1^{(n)}) - n\bar{c} = s(0,n) - n\bar{c}.$$

A standard auction with a zero reserve price attains this maximum. Since $u(0,n) = s(0,n) - s(0,n-1)$ by Lemma 1, then the social contribution of the n-th buyer is

$$\begin{aligned} w(n) - w(n-1) &= s(0,n) - s(0,n-1) - \bar{c} \\ &= u(0,n) - \bar{c}. \end{aligned}$$

Since $u(0,n)$ is decreasing in n this contribution is decreasing in n.

Consider the incentives of buyers when they sequentially decide whether to enter a standard auction with a zero reserve price. The n-th buyer enters if her payoff to entering is at least her cost, i.e., if

$$u(0,n) - \bar{c} \geq 0. \tag{1}$$

As shown above, the left hand side of this expression is just the social contribution of the n-th buyer. Hence, when the reserve price is zero a buyer enters if and only if her entry raises social surplus. Therefore in a pure-strategy entry equilibrium the number of entering buyers n^* maximizes the social surplus; i.e., $w(n^*) = \max_{n\in\{0,1,\ldots,N\}} w(n)$. Since $u(0,N) < \bar{c}$ by assumption, then $n^* < N$. If we ignore that n^* must be an integer, then n^* satisfies (1) with equality, and buyers capture none of the surplus.

This argument establishes that a standard auction with a zero reserve price maximizes social surplus, and moreover that the seller captures the entire social surplus. A positive reserve price reduces social surplus, and because the seller revenue is at most the social surplus, also reduces the seller revenue. Hence the optimal public reserve price is zero.

The key insight above was that the private and social benefit of the entry of a buyer coincide in a standard auction with a zero reserve price. The same logic applies to symmetric entry equilibrium in mixed-strategies. If all buyers enter with probability p, then the number of bidders follows the binomial distribution $B(N,p)$ and the maximum social surplus that can be achieved by any mechanism is

$$W(p) = \sum_{n=1}^{N} p_n^N(p)s(0,n) - Np\bar{c}.$$

A standard auction with a zero reserve price attains this maximum. Since $u(0,n) = s(0,n) - s(0,n-1)$, then we have

$$\begin{aligned} W'(p) &= N\left(\sum_{n=1}^{N} p_{n-1}^{N-1}(p)s(0,n) - \sum_{n=1}^{N-1} p_n^{N-1}(p)s(0,n) - \bar{c}\right) \\ &= N\left(\sum_{n=0}^{N-1} p_n^{N-1}(p)u(0,n+1) - \bar{c}\right) \\ &= N(U(0,p) - \bar{c}), \end{aligned}$$

i.e., the marginal social contribution of an increase in the probability of entry is proportional to the payoff of an entering bidder. Since U is decreasing in p, then $W''(p) < 0$, i.e., W is a concave function. In an (symmetric mixed-strategy) entry equilibrium buyers are indifferent between entering and not;[4] i.e., buyers enter with a probability p^* satisfying

$$U(0,p^*) - \bar{c} = 0.$$

Hence $W'(p^*) = 0$, and therefore $W(p^*) = \max_{p\in[0,1]} W(p)$; i.e., a symmetric mixed-strategy entry equilibrium maximizes the social surplus. Since the seller captures all the social surplus, the optimal reserve price is zero.

Note that $W(p^*)$ is a "constrained" maximum surplus; i.e., it is the maximum surplus when all buyers enter with the same probability. A greater surplus can be realized if one can choose different probabilities of entry for different buyers. Indeed, $w(n^*)$ is the "unconstrained" maximum surplus, as shown above. It is easy to see that $w(n^*) > W(p^*)$.

Proposition 0 summarizes these results.

Proposition 0. *Assume that entry costs are homogenous. In a standard auction with a public reserve price equal to zero if buyers follow a (symmetric mixed-strategy) pure-strategy entry equilibrium, the (constrained) maximum social surplus is realized and is captured by the seller. Hence the optimal (i.e., revenue maximizing) reserve price is zero (the seller's value).*

Since the seller captures the entire social surplus with a zero reserve price, there would be no advantage to the seller to setting an entry fee if it were feasible.

[4] It is easy to see that a symmetric mixed-strategy equilibrium p^* exists, is unique, and satisfies $p^* > 0$.

3.2 Heterogenous entry costs

In this section we study the case where buyers have heterogenous entry costs (i.e., H is not a degenerate probability distribution), and for simplicity we assume henceforth that H is absolutely continuous and increasing. Under this assumption the entry strategy of a buyer can be described by a number $t \in [0, \bar{c}]$, indicating the threshold (the maximum entry cost) of a buyer entering the auction; that is, the buyer enters if her entry cost is less than t, and does not enter if it is greater than t – whether the buyer enters when her entry cost is exactly t is inconsequential.[5] If all buyers employ the same threshold t, then the number of bidders in the auction is distributed according to a binomial distribution $B(N, p)$ where $p = H(t)$.

Consider any standard auction with a public reserve price $r \in [0, \omega]$. A *symmetric (Bayes perfect) entry equilibrium* is a threshold $t \in [0, \bar{c}]$ such that for all $z \in [0, \bar{c}]$: $U(r, H(t)) > z$ implies $t > z$, and $U(r, H(t)) < z$ implies $t < z$; i.e., in a symmetric entry equilibrium t a buyer enters if her expected utility to entering is above her entry cost, and does not enter otherwise.

For $r \in [0, \omega]$, let $t^*(r) = t$ where $t \in [0, \bar{c}]$ is the unique solution to the equation

$$t = U(r, H(t)). \tag{2}$$

We show in the Appendix (Lemma 3) that the mapping $t^*(\cdot)$ is a continuous and decreasing function on $[0, \bar{c}]$.

Proposition 1 establishes that a standard auction with a public reserve price has a unique symmetric entry equilibrium, and that in this equilibrium buyers capture a positive share of the surplus.

Proposition 1. *Assume that entry costs are heterogeneous. Then any standard auction with a public reserve price $r \in [0, \omega)$ has a unique symmetric entry equilibrium, $t = t^*(r) \in (0, \bar{c})$. In this equilibrium, seller revenue is less than the social surplus (that is, buyers capture a positive share of the social surplus).*

It is easy to see that buyers capture part of the social surplus. Simply note that the equilibrium expected utility of an entering buyer is $U(r, H(t^*(r))) = t^*(r) > 0$,

[5] In general entry decisions are described by a mapping $e : [0, \bar{c}] \to [0, 1]$ which, for each entry cost $z \in [0, \bar{c}]$, indicates the probability $e(z)$ with which the buyer enters the auction. When H is continuous and increasing, however, it is without loss of generality to restrict attention to entry strategies described by some threshold $t \in [0, \bar{c}]$.

whereas her expected cost is less than $t^*(r)$. Hence buyer surplus is

$$N \int_0^{t^*(r)} (t^*(r) - z) dH(z) > 0.$$

The proof of the remainder of Proposition 1 is given in the Appendix.

If each buyer enters when her entry cost is less than $t \in [0, \bar{c}]$, then the maximum social surplus that can be achieved by any mechanism is

$$W(t) = \sum_{n=1}^{N} p_n^N (H(t)) s(0, n) - N \int_0^t z dH(z).$$

The following lemma characterizes the (unique) threshold that maximizes social surplus when H is differentiable.

Lemma 2. *If H is differentiable then $W(t^*(0)) = \max_{t \in [0,\bar{c}]} W(t)$, i.e., a standard auction with a public reserve price equal to zero maximizes social surplus.*

Since Lemma 2 restricts attention to symmetric entry decisions, $W(t^*(0))$ is a "constrained" maximum surplus.

It is well known that when the number of bidders is exogenously given, then the optimal public reserve price r^* is positive, and is the solution to the equation

$$r = \frac{1 - F(r)}{f(r)}, \tag{3}$$

independently of the number of bidders present in the auction – see Riley and Samuelson (1981) and Myerson (1981).

The (equilibrium) seller revenue in a standard auction with a public reserve price $r \in [0, \omega]$ is $\Pi(r, H(t^*(r)))$. Hence an *optimal public reserve price* r_p satisfies $r_p \in \arg\max_r \Pi(r, H(t^*(r)))$. Proposition 2 establishes that the optimal reserve price is below r^* (strictly below r^* if H is differentiable) since the seller has an incentive to induce additional entry through a lower reserve price. Unlike in the homogeneous entry costs case, where the optimal reserve price is zero, when entry costs are heterogeneous, the optimal public reserve price may be strictly positive since increasing the reserve price may generate a distribution of the social surplus more favorable to the seller. This is the case if, e.g., H is differentiable and bidders' values are uniformly distributed.

Proposition 2. *Assume that entry costs are heterogeneous. Then in a standard auction the optimal public reserve price r_p satisfies $0 \leq r_p \leq r^*$. Further, if H is* *differentiable, then $r_p < r^*$, and if in addition values follow a uniform distribution, then $0 < r_p$.*

Hence the conclusions obtained when entry costs are homogenous, namely that (i) the seller captures the entire social surplus, (ii) the optimal public reserve price is zero, and (iii) the social surplus is maximized, are not robust to the introduction of heterogeneity in entry costs. With heterogenous entry costs, (I) buyers capture a positive share of the social surplus (hence seller revenue is strictly less than the social surplus), (II) the optimal reserve price may be positive (e.g., if H is differentiable and values follow a uniform distribution), and (III) social surplus may not be maximized (both because there may be less than the socially optimal amount of entry – see Lemma 2 – and because the auction outcome may be ex-post inefficient).

Entry fees

Assume that the seller can set both an entry fee (or a subsidy) as well as a public reserve price.[6] It is easy to see that an analog of Proposition 1 holds for a standard auction with an entry fee and a public reserve price (see the proof of Proposition 3 in the Appendix). In particular, a unique symmetric equilibrium exists, and the seller revenue is less than the social surplus.

Proposition 3 below establishes that an entry fee enables the seller to obtain more revenue than he can obtain with a reserve price alone. In fact, when the seller can set both an entry fee and a public reserve price, then the optimal reserve price is zero (the seller's value). Thus, when bidders have heterogenous entry costs, an entry fee is a more effective instrument to increase seller revenue than a public reserve price. In contrast, it is well-known that when the number of bidders is exogenous, reserve prices are equivalent to entry fees. And, as established earlier, when the number of bidders is endogenous but entry costs are homogeneous, the optimal entry fee and public reserve prices are both zero.

Proposition 3. *Assume entry costs are heterogeneous. The seller revenue in a standard auction with a positive public reserve price is less than in a standard auction with an appropriate entry fee and a public reserve price equal zero. Hence if the optimal public reserve price is positive when no entry fee is feasible, then the seller revenue is greater with an optimal entry fee than with an optimal public reserve price and no entry fee.*

[6] Of course, often it is not possible for the seller to charge an entry fee. For example, none of the Internet auction websites allow the seller to charge an entry fee.

The intuition for this result is simple: if the reserve price is positive then the seller can reduce the reserve price to zero and at the same time raise the entry fee so that the expected utility to a buyer to entering the auction is unchanged. This entry fee (and no reserve price) induces the same entry by bidders without incurring the ex-post inefficiencies of a positive reserve price. Seller revenue rises since social surplus rises, while buyer surplus is unchanged.

As established in Proposition 3, the optimal entry fee will be positive if the seller would choose a positive public reserve price when no entry fee is possible; this will be the case if, for example, H is differentiable and values are uniformly distributed. It is easy to see that a standard auction with a positive optimal entry fee induces less entry than would socially optimal, although the outcome is ex-post efficient (because the reserve price is zero). And that a result analogous to Lemma 2 holds for a standard auction with an entry fee and a public reserve price; namely, that the social surplus is maximized when both the entry fee and the reserve price are set up equal to zero.

4 Secret Reserve Prices

Assume now that the reserve price is not observed prior to entry decisions. Various assumptions can be made about the information revealed to players in the course of the game. For example, if the seller chooses the reserve price following the buyers' entry decisions, then the seller might observe the number of bidders prior to choosing the reserve price. One could then further distinguish cases depending on whether or not the reserve price, and/or the number of bidders, is observed by buyers prior to bidding. We focus on what appears to be the most common situation in practice in auctions with a secret reserve price: we assume that the seller chooses the reserve price without knowing the number of bidders, and buyers observe neither the reserve price nor the number of bidders when either entering or bidding. Thus, in the current setting the seller and buyers interact simultaneously, whereas when the reserve price is public the seller has a first-mover advantage.

Formally, a strategy for the seller is a reserve price $r \in [0, \omega]$. A strategy for a buyer is pair (t, β) where $t \in [0, \bar{c}]$ is a threshold specifying, as in section 3, the maximum entry cost for which a buyer enters the auction, and β is a bidding strategy mapping values into bids. Denote by $\Pi^A(r, H(t), \beta)$ and $U^A(r, H(t), \beta)$, where $A = I$ for a first-price auction and $A = II$ for a second-price auction, the seller's revenue and the expected utility of an entering buyer, respectively when the secret reserve price is r, the buyers employ the threshold t, and the bidders bid according to β.

A *symmetric (Bayes Nash) equilibrium of auction A with a secret reserve price* is a triple (r_s, t, β) such that:

$(S1)$ $r_s \in \arg\max_r \Pi^A(r, H(t), \beta)$;

$(S2)$ for $z \in [0, c]$: $U^A(r_s, H(t), \beta) > z$ implies $t > z$ and $U^A(r_s, H(t), \beta) < z$ implies $t < z$;

$(S3)$ β is a symmetric equilibrium of auction A with reserve price r_s.

According to $(S1)$, the reserve price maximizes the seller revenue given the buyers' entry threshold and bidding strategy. Conditions $(S2)$ and $(S3)$ require that the entry decision and bidding strategy of each buyer be optimal given the secret reserve price and entry decisions and bidding strategies of the other buyers.

Second-Price Auctions

Proposition 4 establishes the basic properties of the symmetric equilibria of second-price sealed-bid auctions with a secret reserve price. Denote by β^* the function given by $\beta^*(x) = x$ for all $x \in [0, \omega]$. We refer to β^* as value bidding. In a second-price auction with a secret reserve price, value bidding is a weakly dominant strategy for buyers.[7] Recall that r^*, the solution to equation (3), is the optimal reserve price when the number of bidders is exogenously given, and that t^* is the function defined by equation (2).

Proposition 4. *The unique symmetric equilibrium in undominated strategies of a second-price sealed-bid auction with a secret reserve price is* $(r^*, t^*(r^*), \beta^*)$. *In this equilibrium both seller revenue and buyer surplus are less (strictly less if* H *is differentiable) than in a standard auction with an optimal public reserve price.*

According to Proposition 4, in a second-price sealed-bid auction the equilibrium secret reserve price when there is endogenous entry is the same as the seller's optimal reserve price when the number of bidders is exogenously fixed. This result is a consequence of the fact the reserve price influences neither entry decisions nor bidding strategies. Hence in setting the reserve price the seller regards entry as exogenous, and therefore the revenue maximizing reserve price is r^* independently of the probability distribution over the number of entrants. Propositions 2 and 4 imply that

[7] It's well known that value bidding is a weakly dominant strategy in an auction with a public reserve price. It's easy to show that value bidding continues to be weakly dominant in an auction with a secret reserve price, i.e., every strategy (t, β) is weakly dominated by (t, β^*).

the equilibrium secret reserve price is greater than the optimal public reserve price. Interestingly, seller revenue and buyer surplus are less when the reserve price is secret than when it is public.

There are equilibria in which buyers follow dominated strategies and in which the seller sets a reserve price above r^*. In particular, for any $r > r^*$, there is an equilibrium in which the seller sets a reserve price of r and bidders bid their value if their value is at least r and bid zero otherwise. Given this bidding strategy, lowering the reserve price below r reduces seller revenue.[8]

First-Price Auctions

In a first-price auction a secret reserve price of zero is weakly dominant for the seller.[9] In a first-price auction with a zero reserve price, if bidders expect the number of bidders to follow a binomial distribution $B(N, p)$, then it is a symmetric equilibrium to bid according to

$$\beta_p^I(x) = \sum_{n=0}^{N-1} p_n^{N-1}(p) \frac{F(x)^n}{\sum_{k=0}^{N-1} p_k^{N-1}(p) F(x)^k} E\left(Y^{(n)} \mid Y^{(n)} < x\right),$$

(See Krishna (2002), Section 3.2.2.)

Proposition 5 establishes the basic properties of the symmetric equilibria in weakly undominated strategies of a first-price sealed-bid auction with a secret reserve price.

Proposition 5: *Every symmetric equilibrium in undominated strategies of a first-price sealed-bid auction with a secret reserve price, $(r_s^I, t_s^I, \beta_s^I)$, satisfies $r_s^I = 0$ and $t_s^I = t^*(0)$. Specifically, $(0, t^*(0), \beta_{t^*(0)}^I)$ is an equilibrium of this kind. These equilibria maximize constrained social surplus, whereas the seller revenue (buyer surplus) is less (greater) than or equal to that at the symmetric equilibrium of a standard auction with an optimal public reserve price. Further, if H is differentiable and values follow a uniform distribution, then these inequalities are strict.*

It is easy to see that there are symmetric equilibria in which the reserve price is positive, as noted in the following remark. In these equilibria all bidders bid zero if their value is below the reserve price. Hence there is no gains to the seller to reducing the reserve price.

[8] We conjecture that reserves below r^* cannot be sustained as equilibria of an auction with a secret reserve.

[9] Reducing the reserve to zero from a positive amount has the effect of increasing seller revenue when all the bids are below the reserve and one bid is positive, and has no effect otherwise.

Discussion

In studying auctions with secret reserve prices we could not apply the Revenue Equivalence Principle, as we usefully did when we studied auctions with public reserve prices. To illustrate why, consider two auctions, a first price auction and a second-price auction, in which the reserve price is public. By the REP the two auctions yield the same revenue to the seller for every reserve price: When the reserve price is varied publicly, entry and bidding strategies adjust in a way that equalizes revenue across standard auctions. Suppose now that the seller *secretly* increases the reserve price above the optimal public reserve, r_p. Since the change is secret, buyer's entry and bidding decisions are unchanged. Seller revenue falls in the first-price auction since revenue becomes zero when the highest bid is between the new reserve price and r_p, and remains the same otherwise. In contrast, seller revenue rises in the second-price auction since revenue is increasing in the reserve price until the reserve price equals r^*. (Thus, when the reserve price is varied secretly, since entry and bidding strategies do not depend on the reserve price the revenue effects of varying the reserve price depend on the auction format. Consequently, the equilibrium secret reserve price is different in first-price and second-price auction.)

However, as noted earlier, various assumptions can be made about the information revealed to bidders when the reserve price is secret. Suppose the reserve price is secret when bidders make their entry decisions, but it is observed prior to bidding, and therefore that bidders can condition their bidding strategy on the reserve price. We can once again apply the REP to conclude that given entry decisions every standard auction generates the same revenue to the seller. And since the number of bidders is exogenous from the seller's perspective, the seller's optimal reserve price is r^*, regardless of the auction format. Thus, whether the secret reserve price is observed prior to bidding is irrelevant for a second-price sealed-bid auctions – in either case the optimal reserve price is r^*. On the other hand, in a first-price sealed-bid auction the equilibrium reserve price is r^* if the reserve price is observed following entry but prior to bidding, while by Proposition 5 it is zero if it is not observed.

5 An Illustrative Example

Assume that $N = 2$, and that values and entry costs are uniformly distributed on $[0, 1]$. We calculate first the symmetric entry equilibrium of a standard auction with a public reserve price $r \in [0, \omega]$. For $r \in [0, \omega]$, simple computations yield $u(r, 1) =$

$\frac{1}{2}(1-r)^2$ and $u(r,2) = \frac{1}{6}(2r+1)(1-r)^2$. In a symmetric equilibrium a buyer makes her entry decisions according to $t^*(\cdot)$ defined for $r \in [0,\omega]$ by the solution to Equation (2), given by

$$z = (1-z)\frac{1}{2}(1-r)^2 + z\frac{1}{6}(2r+1)(1-r)^2.$$

Solving this equation we get

$$t^*(r) = \frac{3}{2}\frac{(1-r)^2}{3+(1-r)^3},$$

for $r \in [0,\omega]$.

If buyers use the threshold $t^*(r)$ to make entry decisions, then an optimal public reserve price maximizes the seller's revenue,

$$\Pi(r, H(t^*(r))) = 2t^*(r)(1-t^*(r))\pi(r,1) + t^*(r)^2\pi(r,2),$$

where $\pi(r,1) = r(1-r)$, and $\pi(r,2) = \frac{1}{3}(1-r)(4r^2+r+1)$. Direct calculation yields that the seller's optimal public reserve price is $r_p = \frac{1}{4}$. Note that the optimal public reserve price when the number of bidders is exogenously fixed is the solution to Equation (3),

$$r = 1-r,$$

i.e., $r^* = \frac{1}{2}$.

Now if entry fees are feasible, then the optimal public reserve price is zero, and the buyers make entry decisions using $\tilde{t}^*(\cdot,0)$ given for $\phi \in \mathbb{R}_+$ by the solution to the equation

$$z + \phi = (1-z)\frac{1}{2} + z\frac{1}{6};$$

i.e., $\tilde{t}^*(\phi,0) = \frac{3}{8} - \frac{3}{4}\phi$. Hence seller revenue is

$$\hat{\Pi}(\phi, 0, \tilde{t}^*(\phi,0)) = 2\tilde{t}^*(\phi.0)(1-\tilde{t}^*(\phi,0))\phi + \tilde{t}^*(\phi,0)^2\left(2\phi + \frac{1}{3}\right).$$

The optimal entry fee is then $\phi^* = \frac{3}{14}$.

For the different auctions formats we have considered, Table 1 describes the optimal (or equilibrium) reserve price (r), the equilibrium threshold (t), the seller revenue (Π) and the surplus captured by buyers (BS). (The calculations for first-price and second-price sealed-bid auctions with a secret reserve price correspond to the equilibria in undominated strategies.) Among these alternative auction formats, the first-price sealed-bid auction with a secret reserve price is the most favorable to buyers (which also maximizes constrained social surplus), whereas the standard auction with a optimal entry fee and a zero public reserve price generates the greatest revenue to the seller. Note that, consistent with our findings, in a standard auction with a public reserve price both the seller revenue and a buyers' surplus are greater than in a second-price sealed-bid auction with a secret reserve price.

Auction Format	**Reserve Price**	**Entry Fee**	r	t	Π	BS
Standard	Public	No	.25	.2465	.0924	.0608
Standard	Public	Yes	0	.2142	.1071	.0458
Second-Price SB	Secret	No	.50	.1200	.0588	.0144
First-Price SB	Secret	No	0	.3750	.0468	.1406

Table I: Equilibrium Outcomes with Uniform Values and Entry Costs

In this example, the optimal public reserve price seems to be independent of the number of potential bidders, whereas the optimal entry fee decreases with the number of potential bidders. These features are peculiar to the example since, by modifying the distribution of entry costs, it is easy to generate examples where the optimal public reserve (and optimal entry fee) decreases, or increases, with the number of potential bidders.

6 Appendix

Proof of Lemma 1: For $n > 1$, by interchanging the order of integration we obtain

$$\begin{aligned} u(0,n) &= \int_0^\omega \left(\int_0^y F(x)^{n-1} dx \right) f(y) dy \\ &= \int_0^\omega \left(\int_x^\omega f(y) dy \right) F(x)^{n-1} dx \\ &= \int_0^\omega (1 - F(x)) F(x)^{n-1} dx. \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} \int_0^\omega F(x)^n dx &= xF^n(x)|_0^\omega - \int_0^\omega nxF(x)^{n-1} f(x) dx \\ &= \omega - E\left(Y_1^{(n)}\right). \end{aligned}$$

Hence

$$\begin{aligned} u(0,n) &= \int_0^\omega F(x)^{n-1}dx - \int_0^\omega F(x)^n dx \\ &= \left(\omega - E\left(Y_1^{(n-1)}\right)\right) - \left(\omega - E\left(Y_1^{(n)}\right)\right) \\ &= s(0,n) - s(0,n-1). \end{aligned}$$

For $n = 1$ we have

$$u(0,1) = \int_0^\omega yf(y)dy = E(Y^{(1)}) = s(0,1) = s(0,1) - s(0,0). \ \square$$

In order to prove Proposition 1 we begin by establishing some properties of the mapping t^*.

Lemma 3: *The mapping t^* is a continuous and decreasing function on $[0,\omega]$, and satisfies $t^*(r) \in (0,\bar{c})$ for $r \in [0,\omega)$.*

Proof: Let $r \in [0,\omega)$; we have

$$U(r, H(0)) = \sum_{n=0}^{N-1} p_n^{N-1}(0)u(r,n+1) = u(r,1) > 0.$$

(Note that for $p(0) = H(0) = 0$, and therefore $p_0^{N-1}(0) = 1$, and $p_n^{N-1}(0) = 0$ for $n \in \{1, ..., N-1\}$.) And

$$U(r, H(\bar{c})) = \sum_{n=0}^{N-1} p_n^{N-1}(1)u(r,n+1) = u(r,N) \leq u(0,N).$$

(Note that for $p(\bar{c}) = H(\bar{c}) = 1$, and therefore $p_n^{N-1}(1) = 0$ for $n \in \{0, 1, ..., N-2\}$ and $p_{N-1}^{N-1}(1) = 1$.) Since

$$u(0,N) = E(Y^{(N)}) - E(Y^{(N-1)})$$

by Lemma 1, and by assumption

$$E(Y^{(N)}) - E(Y^{(N-1)}) < \bar{c}$$

we have

$$U(r, H(\bar{c})) < \bar{c}.$$

Hence, since $U(r, H(\cdot))$ is continuous (because H is absolutely continuous) the equation

$$t = U(r, H(t))$$

has a solution on $[0, \bar{c}]$; and since $U(r, H(\cdot))$ decreasing (because $U(r, p)$ is decreasing in p and H is increasing), there is a unique solution. Therefore the function $t^*(\cdot)$ is well defined, and since $U(r, t)$ is continuous (because each $u(\cdot, n)$ for $n \in \{1, ..., n\}$ is continuous), then $t^*(\cdot)$ is also continuous. We show that $t^*(\cdot)$ is decreasing. Let $r', r'' \in [0, \omega]$ be such that $r'' > r'$. Write $t^*(r') = t'$, and $t^*(r'') = t''$. Suppose by way of contradiction that $t'' \geq t'$. Then since $U(r, H(\cdot))$ is decreasing in both r and t we have

$$t' = U(r', H(t')) > U(r'', H(t'')) = t'',$$

which is a contradiction.

Let $r \in [0, \omega)$. We show that $t^*(r) > 0$. Suppose that $t^*(r) = 0$. Then

$$U(r, H(t^*(r))) = U(r, H(0)) = u(r, 1) > 0 = t^*(r) = U(r, H(t^*(r))),$$

which is a contradiction. We show that $t^*(r) < \bar{c}$. Suppose that $t^*(r) = \bar{c}$. Then

$$U(r, H(t^*(r))) = U(r, 1) = u(r, N) \leq u(0, N) < \bar{c} = t^*(r) = U(r, H(t^*(r))),$$

which is a contradiction. □

Proof of Proposition 1: Consider a standard auction with a public reserve price $r \in [0, \omega]$. We show that $t^*(r)$ is the unique symmetric equilibrium. Clearly $t^*(r)$ is a symmetric equilibrium. We show that no other symmetric equilibrium exists. Suppose not; let $\bar{t} \neq t^*(r)$ be a symmetric equilibrium. Assume that $\bar{t} < t^*(r)$. Then since $U(r, H(\cdot))$ is decreasing we have

$$U(r, H(\bar{t})) > U(r, H(t^*(r))) = t^*(r) > \bar{t}.$$

Then there is $z \in (U(r, \bar{t}), \bar{t})$; i.e., there is $z \in [0, \bar{c}]$ such that $z > U(r, H(\bar{t}))$ and $z < \bar{t}$, contradicting that $\bar{t}$ is an equilibrium. Analogously $\bar{t} > t^*(r)$ also leads to a contradiction. □

Proof of Lemma 2: Differentiating $W(t)$ yields

$$W'(t) = \sum_{n=1}^{N} \frac{dp_n^N(H(t))}{dt} s(0, n) - Nth(t).$$

For $n \leq N-1$ we have

$$\frac{dp_n^N(H(t))}{dt} = N(p_{n-1}^{N-1} - p_n^{N-1})h(t),$$

and

$$\frac{dp_N^N(H(t))}{dt} = Np_{N-1}^{N-1}h(t).$$

(All binomial probabilities are calculated for $p = H(t)$.) Substituting these expressions and using Lemma 1, we have

$$\begin{aligned} W'(t) &= Nh(t)\left(p_{N-1}^{N-1}s(0,N) + \sum_{n=1}^{N-1}(p_{n-1}^{N-1} - p_n^{N-1})s(0,n) - t\right) \\ &= Nh(t)\left(\sum_{n=0}^{N-1}p_n^N u(0,n+1) - t\right) \\ &= Nh(t)\left(U(0,H(t)) - t\right). \end{aligned}$$

Hence

$$W'(t^*(0)) = 0.$$

Moreover, since $h(t) > 0$ and $U(0, H(\cdot))$ is decreasing on $[0, \bar{c}]$, then $W'(t) > 0$ for $t < t^*(0)$, and $W'(t) < 0$ for $t > t^*(0)$. Hence $t = t^*(0)$ uniquely maximizes $W(t)$ on $[0, \bar{c}]$. That a public reserve price equal to zero generates a constrained socially optimal follows directly from Proposition 1. □

Proof of Proposition 2: Follows immediately from lemmas 2-4 below.

Lemma 4. *For all $r \in (r^*, \omega]$: $\Pi(r^*, H(t^*(r^*))) > \Pi(r, H(t^*(r)))$.*

Proof: For $r \in (r^*, \omega]$ we have

$$\begin{aligned} \Pi(r^*, H(t^*(r^*))) &= \sum_{n=1}^{N} p_n^N(H(t^*(r^*)))\pi(r^*, n) \\ &> \sum_{n=1}^{N} p_n^N(H(t^*(r)))\pi(r^*, n) \\ &> \sum_{n=1}^{N} p_n^N(H(t^*(r)))\pi(r, n) \\ &= \Pi(r, H(t^*(r))). \end{aligned}$$

The first inequality follows from the fact that $t^*(\cdot)$ is strictly decreasing by Lemma 1 and therefore the *c.d.f.* of the binomial $B(N, p(H(t^*(r^*))))$ first order stochastically dominates the *c.d.f.* of the binomial $B(N, p(H(t^*(r))))$, and because π is strictly increasing with respect to n. The second inequality follows from the fact that r^* uniquely maximizes $\pi(\cdot, n)$ on $[0, \omega]$ for all $n \in \{1, ..., N\}$ – see Riley and Samuelson (1981) and Myerson (1981). □

Lemma 5. *If H is differentiable then* $\left.\frac{d\Pi(r,t^*(r))}{dr}\right|_{r=r^*} < 0$.

Proof: Clearly if H is differentiable, then both $t^*(\cdot)$ and $\Pi(r, H(t^*(r)))$ are differentiable. We have

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=r^*} = \sum_{n=1}^{N} \left(\left.\frac{dp_n^N(H(t^*(r)))}{dr}\right|_{r=r^*} \pi(r^*, n) + p_n^N(H(t^*(r^*))) \left.\frac{d\pi(r, n)}{dr}\right|_{r=r^*} \right).$$

Since r^* maximizes $u(\cdot, n) \in [0, \omega]$ for all $n \in \{1, ...N\}$ – see Riley and Samuelson (1981) and Myerson (1981) – we have

$$\left.\frac{d\pi(r, n)}{dr}\right|_{r=r^*} = 0$$

for all $n \in \{1, ...N\}$. Denote by $p^* = p(H(t^*(r^*))) = H(t^*(r^*))$ the binomial probability at $t^*(r^*)$. Hence

$$\begin{aligned}
\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=r^*} &= \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t^*(r)))}{dr}\right|_{r=r^*} \pi(r^*, n) \\
&= \sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \left.\frac{dp(H(t))}{dt}\right|_{t=t^*(r^*)} \left.\frac{dt^*(r)}{dr}\right|_{r=r^*} \pi(r^*, n) \\
&= h(t^*(r^*)) \frac{dt^*(r^*)}{dr} \left(\sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \pi(r^*, n) \right).
\end{aligned}$$

In this expression, $h(t^*(r^*)) > 0$, and $\frac{dt^*(r^*)}{dr} < 0$ by Lemma 1. The last term,

$$\sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=p^*} \pi(r^*, n),$$

measures the effect of a marginal variation of the binomial probability around p^* on the seller revenue. This term positive: an increase in the binomial probability induces a new binomial distribution whose *c.d.f.* first order stochastically dominates the *c.d.f.* of $B(N, p^*)$ which, because π is increasing with respect to n, increases the seller revenue. Therefore

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=r^*} < 0. \ \Box$$

Lemma 6. *If H is differentiable and values are distributed uniformly on $[0, \omega]$, then*

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} > 0.$$

Proof: Normalize $\omega = 1$. We have

$$\begin{aligned}
\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} &= \left.\frac{\partial\Pi(r, H(t^*(r)))}{\partial r}\right|_{r=0} + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \left.\frac{\partial\Pi(r, H(t^*(r)))}{\partial t}\right|_{r=0} \\
&= \sum_{n=1}^{N} p_n^N(H(t^*(0))) \left.\frac{\partial\pi(0, n)}{\partial r}\right|_{r=0} \\
&\quad + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t))}{dt}\right|_{t=t^*(0)} \pi(0, n).
\end{aligned}$$

Since buyers values are distributed uniformly on $[0, 1]$, direct calculation yields

$$\pi(0, n) = \frac{n-1}{n+1}$$

for $n \in \{1, ..., N\}$, and

$$\left.\frac{\partial\pi(r, n)}{\partial r}\right|_{r=0} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Hence

$$\left.\frac{d\Pi(r, H(t^*(r)))}{dr}\right|_{r=0} = p_1^N(H(t^*(0))) + \left.\frac{dt^*(r)}{dr}\right|_{r=0} \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t))}{dt}\right|_{t=t^*(0)} \pi(0, n).$$

Now

$$\frac{dt^*(r)}{dr} = \frac{\frac{\partial U(r, H(t))}{\partial r}}{1 - \frac{\partial U(r, H(t))}{\partial t}},$$

where

$$\frac{\partial U(r, H(t))}{\partial r} = \sum_{n=0}^{N-1} p_n^{N-1}(H(t)) \frac{\partial u(r, n+1)}{\partial r},$$

and

$$\frac{\partial U(r,H(t))}{\partial t}=\sum_{n=0}^{N-1}\frac{dp_n^{N-1}(H(t))}{dt}u(r,n+1).$$

Since values are uniformly distributed on $[0,1]$, direct calculation yields

$$u(0,n)=\frac{1}{n\,(n+1)}$$

for $n\in\{1,...,N\}$, and

$$\left.\frac{\partial u(r,n)}{\partial r}\right|_{r=0}=\begin{cases}-1 & \text{if } n=1\\ 0 & \text{if } n>1.\end{cases}$$

Thus

$$\left.\frac{\partial U(r,H(t))}{\partial r}\right|_{r=0}=p_0^{N-1}(H(t))\frac{\partial u(r,1)}{\partial r}=-\left(1-H(t)\right)^{N-1}.$$

Substituting and simplifying notation by writing $p=H(t^*(0))$ and $\frac{dp_n^{N-1}}{dt}=\frac{p_n^{N-1}(H(t))}{dt}$, we get

$$\begin{aligned}\left.\frac{\Pi(r,H(t^*(r)))}{dr}\right|_{r=0} &= Np\,(1-p)^{N-1}\\ &\quad-(1-p)^{N-1}\left(1-\sum_{n=0}^{N-1}\frac{dp_n^{N-1}}{dt}u(0,n+1)\right)^{-1}\sum_{n=1}^{N}\frac{dp_n^N}{dt}\pi(0,n)\\ &= (1-p)^{N-1}\left(1-\sum_{n=0}^{N-1}\frac{dp_n^{N-1}}{dt}u(0,n+1)\right)^{-1}\Delta_N,\end{aligned}$$

where

$$\Delta_N=Np-Np\sum_{n=0}^{N-1}\frac{dp_n^{N-1}}{dt}u(0,n+1)-\sum_{n=1}^{N}\frac{dp_n^N}{dt}\pi(0,n)$$

Note that $\frac{dt^*(r)}{dr}<0$ and $\frac{\partial U(r,H(t))}{\partial r}<0$ imply $1-\frac{\partial U(r,H(t))}{\partial t}>0$. Hence

$$1-\left.\frac{\partial U(r,t)}{\partial t}\right|_{r=0}=1-\sum_{n=0}^{N-1}\frac{dp_n^{N-1}(t)}{dt}u(0,n+1)>0.$$

Since $t^*(0)\in(0,\bar{c})$ by Lemma 3, and since H is increasing, we have $p=H(t^*(0))\in(0,1)$. We prove that

$$\left.\frac{\Pi(r,H(t^*(r)))}{dr}\right|_{r=0}>0$$

by showing that

$$\Delta_N = Np > 0.$$

We have

$$\sum_{n=0}^{N-1} \frac{dp_n^{N-1}}{dt} u(0, n+1) = \sum_{n=1}^{N} \frac{dp_{n-1}^{N-1}}{dt} u(0, n),$$

and therefore

$$\Delta_N = Np - Np \sum_{n=1}^{N} \frac{dp_{n-1}^{N-1}}{dt} u(0, n) - \sum_{n=1}^{N} \frac{dp_n^N}{dt} \pi(0, n).$$

Since

$$\frac{dp_n^N}{dt} = h(t) \frac{dp_n^N}{dp},$$

and $u(0,n) = \frac{1}{n(n+1)}$ and $\pi(0,n) = \frac{n-1}{n+1}$, we have

$$\Delta_N = Np - h(t^*(0)) \left(Q_N + R_N\right),$$

where

$$Q_N = Np \sum_{n=1}^{N} \frac{dp_{n-1}^{N-1}}{dp} \frac{1}{n\,(n+1)},$$

and

$$R_N = \sum_{n=1}^{N} \frac{dp_n^N}{dp} \frac{n-1}{n+1}.$$

Now

$$\begin{aligned} Q_N &= Np \sum_{n=1}^{N} \frac{1}{(n+1)n} \frac{(N-1)!}{(n-1)!(N-n)!} \\ &\quad \times [(n-1)p^{n-2}(1-p)^{N-n} - (N-n)p^{n-1}(1-p)^{N-n-1}] \\ &= N!p \sum_{n=1}^{N} \frac{(n-1)p^{n-2}(1-p)^{N-n} - (N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!}. \end{aligned}$$

Similarly,

$$\begin{aligned} R_N &= \sum_{n=1}^{N} \frac{n-1}{(n+1)} \frac{N!}{n!(N-n)!} [np^{n-1}(1-p)^{N-n} - (N-n)p^n(1-p)^{N-n-1}] \\ &= N!p \sum_{n=1}^{N} \frac{(n-1)\left[np^{n-2}(1-p)^{N-n} - (N-n)p^{n-1}(1-p)^{N-n-1}\right]}{(n+1)!(N-n)!}. \end{aligned}$$

Hence

$$Q_N + R_N = N!p \sum_{n=1}^{N} \frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n} - n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!}.$$

We have

$$\begin{aligned}
\sum_{n=1}^{N} \frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n}}{(n+1)!(N-n)!} &= \sum_{n=2}^{N} \frac{(n+1)(n-1)p^{n-2}(1-p)^{N-n}}{(n+1)!(N-n)!} \\
&= \sum_{n=1}^{N-1} \frac{np^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n-1)!},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{N} \frac{n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!} &= \sum_{n=1}^{N-1} \frac{n(N-n)p^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n)!} \\
&= \sum_{n=1}^{N-1} \frac{np^{n-1}(1-p)^{N-n-1}}{(n+1)!(N-n-1)!}.
\end{aligned}$$

Hence

$$Q_N + R_N = 0,$$

and therefore

$$\Delta_N = Np. \ \square$$

Proof of Proposition 3: Consider a standard auction with an entry fee $\phi \in \mathbb{R}$ and a public reserve price $r \in [0, \omega]$. An entry strategy for a buyer is also described by a threshold $t \in [0, \bar{c}]$. Given (ϕ, r) if all bidders follow the same entry strategy $t \in [0, \bar{c}]$, then the gross expected utility of an entering buyer is

$$\tilde{U}(\phi, r, H(t)) = U(r, H(t)) - \phi,$$

and the revenue of the seller is

$$\tilde{\Pi}(\phi, r, H(t)) = \Pi(r, H(t)) + NH(t)\phi.$$

Note that since the gross surplus is distributed between the seller and buyers, we have

$$\tilde{\Pi}(\phi, r, H(t)) + NH(t)\tilde{U}(\phi, r, H(t)) \equiv \sum_{n=1}^{N} p_n^N(H(t))s(r, n).$$

A *symmetric (Bayes perfect) entry equilibrium* is a threshold $t \in [0, \bar{c}]$ such that for all $z \in [0, \bar{c}]$: $\tilde{U}(\phi, r, H(t)) > z$ implies $t > z$ and $\tilde{U}(\phi, r, H(t)) < z$ implies $t < z$. For $\phi \in \mathbb{R}$ and $r \in [0, \omega]$ write $\tilde{t}^*(\phi, r) = t$, where $t \in [0, \bar{c}]$ is the solution to the equation

$$t = \tilde{U}(\phi, r, H(t)). \tag{4}$$

The mapping $\tilde{t}^*$ can be shown to be a continuous function on $\mathbb{R} \times [0, \omega]$ using arguments analogous to those used in the proof of Lemma 3 above. It is easy to see that an analog of Proposition 1 holds; i.e., *any standard auction with an entry fee ϕ and a public reserve price r has a unique symmetric entry equilibrium $t = \tilde{t}^*(\phi, r) \in [0, \bar{c}]$.*

Let $(\phi, r) \in \mathbb{R} \times [0, \omega]$ with $r > 0$. We establish Proposition 3 by showing that there is $\bar{\phi}$ such that

$$\tilde{\Pi}(\bar{\phi}, 0, H(\tilde{t}(\bar{\phi}, 0))) > \tilde{\Pi}(\phi, r, H(\tilde{t}(\phi, r))).$$

Define $\bar{\phi}$ by the equation

$$\tilde{t}(\phi, r) = \tilde{U}(\bar{\phi}, 0, H(\tilde{t}(\bar{\phi}, 0))).$$

Thus

$$\tilde{U}(\phi, r, H(\tilde{t}(\phi, r))) = \tilde{t}(\phi, r) = \tilde{U}(\bar{\phi}, 0, H(\tilde{t}(\bar{\phi}, 0))) = \tilde{t}(\bar{\phi}, 0).$$

Write $H(\tilde{t}\,(\phi, r)) = H(\tilde{t}(\bar{\phi}, 0)) = p$. Then, since $r > 0$ implies $s(0, n) > s(r, n)$ for each n we have

$$\begin{aligned}
\tilde{\Pi}(\bar{\phi}, 0, H(\tilde{t}(\bar{\phi}, 0))) &= \left(\sum_{n=1}^{N} p_n^N(p) s(0, n)\right) - Np\tilde{U}(\phi, 0, H(\tilde{t}(\bar{\phi}, 0))) \\
&= \left(\sum_{n=1}^{N} p_n^N\,(s(r, n) - s(r, n) + s(0, n))\right) - Np\tilde{U}(\phi, r, H(\tilde{t}(\phi, r))) \\
&= \tilde{\Pi}(\phi, r, H(\tilde{t}(\phi, r))) + \sum_{n=1}^{N} p_n^N\,(s(0, n) - s(r, n)) \\
&> \tilde{\Pi}(\phi, r, H(\tilde{t}(\phi, r))).\square
\end{aligned}$$

Proof of Proposition 4: Let $(r^{II}, t^{II}, \beta^{II})$ be a symmetric equilibrium in undominated strategies. Then $\beta^{II} = \beta^*$. The expected utility of bidder that expects the reserve price to be r^{II} and the other buyers to enter when their entry cost are less than or equal to t^{II}, and upon entry bid their value, is

$$U^{II}(r^{II}, t^{II}, \beta^*) = \sum_{n=1}^{N} p_n^N(t^{II}) u(r^{II}, n) = U(r^{II}, t^{II}).$$

Hence equilibrium condition $S2$ implies

$$U(r^{II}, t^{II}) = t^{II};$$

i.e., $t^{II} = t^*(r^{II})$. Hence for $r \in [0, \omega]$ seller revenue is

$$\begin{aligned} \Pi^{II}(r, H(t^{II}), \beta^*) &= \sum_{n=1}^{N} p_n^N(H(t^{II}))\pi(r, n) \\ &= \Pi(r, H(t^{II})). \end{aligned}$$

Clearly $r = \omega$ is a dominated strategy for the seller. Hence $r^{II} < \omega$, and therefore $t^{II} = t^*(r^{II}) > 0$ by Lemma 3. And since r^* uniquely maximizes $\pi(r, n)$ for every $n \in \{1, ..., N\}$ – see Riley and Samuelson (1981) and Myerson (1981) – $t^{II} > 0$ implies that r^* uniquely maximizes $\Pi^{II}(r, H(t^{II}), \beta^*)$. Therefore equilibrium condition $S1$ implies $r^{II} = r^*$.

Now let r_p be an optimal public reserve price. Then

$$\Pi^{II}(r^*, H(t^*(r^*), \beta^*) = \Pi(r^*, H(t^*(r^*)) \leq \Pi(r_p, H(t^*(r_p));$$

i.e., seller revenue in a symmetric equilibrium in undominated strategies of a second-price sealed-bid auction is less than or equal to his revenue in any standard auction with an optimal public reserve price. Since $r_p \leq r^*$ by Proposition 2, buyer surplus is

$$\begin{aligned} N \int_0^{t^*(r^*)} \left(U^{II}(r^*, t^*(r^*), \beta^*) - z\right) dH(z) &= N \int_0^{t^*(r^*)} (t^*(r^*) - z)\, dH(z) \\ &\leq N \int_0^{t^*(r_p)} (t^*(r_p) - z)\, dH(z); \end{aligned}$$

i.e., buyer surplus in a symmetric equilibrium in undominated strategies of a second-price sealed-bid auction is less than or equal to that in any standard auction with an optimal public reserve price. Moreover, by Proposition 2 when H is differentiable, then $r_p > r^*$, and therefore both inequalities above are strict. $\square$

Proof of Proposition 5: Clearly $r = 0$ is the unique (weakly) dominant strategy for the seller. Hence if (r^I, t^I, β^I) is a symmetric equilibrium in undominated strategies, we have $r^I = 0$. Now given $r^I = 0$, then β^I is a symmetric increasing equilibrium of the auction with a reserve price $r = 0$ when the number of bidders follows a binomial distribution $B(N, p(t^I))$. Thus, we can appeal to the R.E.P. to calculate the bidders expected utility as

$$U^I(r^I, H(t^I), \beta^I) = \sum_{n=1}^{N} p_n^N(H(t^I))u(r^I, n) = U(r^I, t^I).$$

Condition $S2$ implies $t^I = t^*(r^I) = t^*(0)$. Hence buyer surplus is

$$\begin{aligned} N\int_0^{t^*(0)} \left(U^I(r^I, H(t^I), \beta^I) - z\right) dH(z) &= N\int_0^{t^*(0)} (U(0, t^*(0)) - z)\, dH(z) \\ &= N\int_0^{t^*(0)} (t^*(0) - z)\, dH(z) \\ &\geq N\int_0^{t^*(r_p)} (t^*(r_p)) - z)\, dH(z) \\ &= N\int_0^{t^*(r_p)} (U(r_p, t^*(r_p)) - z)\, dH(z). \end{aligned}$$

The last term on the right hand side of the inequality is buyer surplus is a standard auction with an optimal public reserve r_p. Recall that $r_p \geq 0$ by Proposition 2. Seller revenue is

$$\Pi^I(r^I, H(t^I), \beta^I) = \Pi(0, t^*(0)) \leq \Pi(r_p, t^*(r_p)).$$

Further, if H is differentiable and values follow a uniform distribution then $r_p > 0$, and therefore both the above inequalities are strict. The fact that the equilibrium surplus is maximum follows from Lemma 2. Finally, it is clearly $(0, t^*(0), \beta^I_{t^*(0)})$ is a symmetric equilibrium with entry. □

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