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Games of Capacities: A (Close) Look to Nash Equilibria*

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Abstract

The paper studies two games of capacity manipulation in hospital-intern markets. The focus is on the stability of Nash equilibrium outcomes. We provide minimal necessary and sufficient conditions guaranteeing the existence of pure strategy Nash Equilibria and the stability of outcomes.

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1 Introduction

This paper considers capacity reporting games in many-to-one matching markets. A matching market consists of two finite and disjoint sets of agents, say medical interns and hospitals. Each hospital has a capacity that limits the maximum number of interns it can employ. Each agent has a preference relation over the other side of the market. A matching assigns interns to hospitals. The stability criterion is central in studies concerning two-sided matching problems. The ability of a mechanism to achieve stable allocations is decisive in its success and its endurance (see Roth and Sotomayor 1990 and Roth 2002). A matching is stable if it is individually rational for hospitals and for interns and there is no unmatched hospital-intern pair such that the intern prefers the hospital to her assignment and the hospital prefers the intern to one of its interns or keeping a vacant position.

Unfortunately stable matchings are prone to different kinds of manipulation. Dubins and Freedman (1981) show that hospital-optimal stable matching is manipulable via preferences. Roth (1982) shows that no stable matching rule is immune to preference manipulation.[1] Gale and Sotomayor (1985 a and b) study preference manipulation under the hospital-optimal stable rule. Sönmez (1997a) shows that there is no stable matching rule immune to capacity manipulation. Finally Sönmez (1999) show that no stable rule is immune to manipulation by early contracting.

The intern-optimal and the hospital-optimal stable matching rules are of particular interest. They are used in the United States and in the United Kingdom to match medical interns and hospitals (see Roth 1984, 1991, Roth and Peranson 1999, Niederle and Roth 2003). The hospital-intern model has also been used to model school admissions. Balinski and Sönmez 1999 study the admissions to Turkish universities. The preferences of the colleges are derived by the results of a public examination according to a publicly known formula. Abdulkadiroğlu and Sönmez (2003) consider primary and secondary school choice in the United Stated. In many districts students are allocated to schools on the basis of priorities, that are determined by the school district. Schools have not control over priorities, but they can manipulate their capacities. Abdulkadiroğlu, Pathak, and Roth (2005) argue that manipulation via capacities is not only a theoretical possibility: under-reporting of capacities was a source of major concern in the school choice program in NYC before it was redesigned.

In the capacity reporting games studied here hospitals report their capacities and the outcome is determined according either to the hospital-optimal rule or to the intern-optimal rule. Information is complete. Konishi and Ünver (2006) observe that these games might fail to have a pure strategy Nash equilibrium. They show that under strong-monotonicity in population, a pure strategy Nash equilibrium exists and under the intern-optimal matching rule truthful capacity reporting is a dominant strategy.[2] If the interests of one side of the market are aligned truthful capacity reporting is a dominant strategy, too.

Our analysis complements and extends the work of Konishi and Ünver (2006).[3] We focus on the stability of pure strategy Nash equilibrium outcomes. We connect equilibrium stability and incentives in truthful capacity reporting. If an hospital is part of a pair blocking some equilibrium outcome, then it has incentive for under-reporting its true capacity. We provide conditions sufficient to guarantee the stability (and the existence) of pure strategy Nash equilibrium outcomes. The first one is acyclicity. Under acyclicity truthful capacity reporting is a dominant strategy and every Nash equilibrium outcome is stable. Furthermore, under acyclicity the stable set is a singleton (an extension of Eeckhout 2000 result). It is the minimal condition able to guarantee the stability of Nash equilibrium outcome when the hospital-optimal stable rule is used. There is an important difference between the two games. Under the hospital-optimal stable rule, capacity

[1] Alcalde and Barbera (1994) extend the result to individually rational and Pareto optimal rules.

[2] Strong-monotonicity in population is satisfied if hospitals prefer larger set of interns to smaller ones.

[3] See also Kojima (2006, 2007).

under-reporting prevents the creation of harmful cycles of rejection. Under the intern-optimal stable rule, capacity under-reporting engenders new cycles of rejection. Therefore, it is harder to manipulate. We prove that manipulability of this rule needs that at least one hospital has non-monotonic preferences and it is involved in multiple cycles. If none of such cycles exists the game yields the intern-optimal stable matching at equilibrium and true capacity reporting is a dominant strategy. Furthermore, no weaker conditions can be found.

The structure of the article is the following: in Section 2, we present the model, in Section 3 we present the results and Section 4 concludes. The proofs are in the Appendix.

2 The Model

An hospital-intern market is a quadruple (H, I, q, P). The set$H = \{h_1, ..., h_m\}$ is the set of hospitals, $I = \{i_1, ..., i_n\}$ is the set of interns, $q = (q_1, ..., q_m)$ is the vector of hospitals capacities where q_h is the capacity of hospital hospital h. Finally$P = (P_{h_1}, ..., P_{h_m}, P_{i_1}, ..., P_{i_n})$ is the list of agents preferences, where P_h is the preference of hospital h and P_i is the preferences of intern i.

For any $h \in H$, P_h is a linear order on 2^I.[4] Let $I' \subset I$ be a set of interns. If $\varnothing P_h I'$ I' is *unacceptable to i*. Otherwise I' is *acceptable to h*. $A(h)$ denotes the set of interns who are individually acceptable to h. All along the paper we assume that every hospital has responsive preferences. An hospital has responsive preferences if, for any two assignments that differ in one intern only, it prefers the assignment containing the most preferred intern. FormallyP_h are *responsive* if for all $J \subseteq I$ we have: (i) for all $i, j \in I \setminus J$, $J \cup \{i\} P_h J \cup \{j\} \Leftrightarrow i P_h j$ and (ii) for all $i \in I \setminus J$, $J \cup \{i\} P_h J \Leftrightarrow i \in A(h)$. An hospital h has *strong monotonic preferences* if it prefers group of acceptable interns of larger cardinality to sets of acceptable interns of smaller cardinality: if,f or all $J, K \subset A(h)$, $\sharp J > \sharp K \Rightarrow J P_h K$.

For any $i \in I$, P_i is a linear order on $H \cup \{i\}$. Any hospital i such that $i P_i h$ is *unacceptable to i*. Otherwise i is *acceptable to i*. For every $i \in I$, $A(i)$ denotes the set of hospitals that are acceptable to i. For every agent $i \in H \cup I$ let R_x be x's weak preference relation.

A matching assigns each hospital h to a set of at most q_h interns and assigns each intern to at most one hospital. Formally, a *matching* is a function $\mu : H \cup I \to 2^I \cup I$, such that, for every $(h, i) \in H \times I$ (i) $\mu(h) \in 2^I$, (ii) $\sharp\mu(h) \leq q_h$, (iii) $\mu(i) \in F \cup \{i\}$, (iv) i$\mu(i) = f \Leftrightarrow i \in \mu(h)$.
We introduce two binary relations on the set of matchings, P_H and P_I. Let μ, ν be matchings. Let $\mu P_H \nu$ if and only if and $\mu(h) R_h (h)$ for all $h \in H$ and $\mu(h) P_h (h)$ for at least one h. Let $\mu P_I \nu$ if and only if $\mu(i) R_i \nu(i)$ for all $i \in I$ and $\mu(i) P_i \nu(i)$ for at least one $i \in I$. A matching μ is *individually rational* if (i) $\mu(h) P_h \emptyset$, $\forall h \in H$, (ii) $\mu(w) R_i i$ $\forall i \in I$. μ *is blocked by the pair* $(h, i) \in H \times I$ if (i) $i P_i \mu(i)$, (i) $\exists J \subseteq \mu(h)$ such that $J \cup \{i\} P_h \mu(h)$. Finally, μ is *stable in* (H, I, q, P) if it is individually rational and if no pair blocks it. Otherwise μ is *unstable*. $\Gamma(I, H, q, P)$ denotes the *stable set*, the set of matchings that are stable in market (I, H, q, P). If the hospitals have responsive preferences the stable set is not empty.
There is a stable matching, the *hospital-optimal* stable matching that is (weakly) preferred to any other stable matching by every hospital. Another stable matching the *intern-optimal* stable matching is (weakly) preferred to any other stable matching by every intern. We denote by $\varphi^H(H, I, q, P)$ and $\varphi^I(H, I, q, P)$ the hospital-optimal and the intern-optimal stable matching of (H, I, q, P), respectively. When there is on ambiguity we will use $\varphi^H(q)$ and $\varphi^I(q)$instead than $\varphi^H(H, I, q, P)$ and $\varphi^I(H, I, q, P)$, respectively.
The *hospital-proposing deferred acceptance algorithm* (Gale and Shapley 1962) generates the hospital-optimal stable matching of (H, I, q, P) and *intern-proposing deferred acceptance algorithm* generates the the intern-optimal stable matching of (H, I, q, P).

[4]As usual, for all h,i,i' , $i P_h i'$, $i P_h \varnothing$ and $\varnothing P_h i$ denote $\{i\} P_h \{i'\}$, $\{i\} P_h \varnothing$ and $\varnothing P_h \{i\}$, respectively.

2.1 Capacity-reporting games

In a capacity reporting game, each hospital h simultaneously reports a capacity q_h and the outcome is determined according to φ^Hor φ^I. Interns are passive players and information is complete. Let $V \in \{H, I\}$. The *capacity reporting game* induced by φ^V is a normal form game of complete information. The set of players is H and the strategy space of hospital h is $\{1, ..., q_h\}$ (see also Hurwicz et al 1995). The outcome functions is φ^V. The preferences over outcomes are naturally induced by their preferences over subsets of interns.

2.2 Cycles

We finally introduce the notion of cycle in agents preferences that generalizes the notion of common preferences presented in Konishi and Ünver.

Definition 1 *A cycle (of length T+1) is given by* $h_0, ..., h_T$*with* $h_l \neq h_{l+1}$[5] *for* $i = 0, ...T$ *and distinct* $i_0, i_1, ..., i_T$*such that*

1. $i_0 P_{h_0} i_T P_{h_T} i_{T-1} .... i_1 P_{h_1} i_0$,
2. *for every* l, $i_{l+1} \in A(h_l) \cap A(h_{l+1})$.

Hospitals preferences are acyclical if they have no cycles of length 2.

Assume that a cycle exists. If every i_l is initially assigned to h_{l+1} every hospital is willing willing to exchange the its assigned interns with its successor.

Similarly.

Definition 2 *A cycle (of length* $T+1$*) in interns preferences is given by* $h_0, ..., h_T$ *and* $i_0, i_1, ..., i_T$ *and*

1. $h_0 P_{i_T} h_T P_{i_{T-1}} h_{T-1} .... h_1 P_{i_0} h_0$.
2. *for every* l, $h_l \in A(i_{l-1}) \cap A(i_l)$.

Interns preferences are acyclical if they have no cycles of length 2.

Observe that under acyclicity the interests of one side of the market are almost aligned: if interns (resp. hospitals) preferences are acyclical any two interns (resp. hospitals) have the same preferences over every two hospitals (resp. interns) that are acceptable to both.

A *simultaneous cycle* is given by $h = h_0, ..., h_T$ and $i_0, i_1, ..., i_T$ that constitute both a cycle in hospitals and interns preferences.

We next define a generalized cycle.

Definition 3 *A generalized cycle (of length* $T+1$*) at* h *is given by a cycle in interns preferencesh* $= h_0, ..., h_T$, $i_0, i_1, ..., i_T$ *and by* i_{-1} *such that:* $i_0 P_{h_0} i_{-1} P_{h_0} i_T$. *Hospitals preferences are weakly acyclical if, there is no generalized cycle at any* h.

[5] From now on indexes are considered modulo $T + 1$.

I can be shown (see Ergin (2002)) that any generalized cycle can be reduced to a generalized cycle of length 2. Assume that a generalized cycle of length 2 at h exists. Let h_0 be assigned with two interns i_{-1} *and* i_1. and let $h_1$be assigned with i_0. Assume also that $i_0 P_h \{i_{-1}, i_1\}$. Hospital h_0 would be willing to exchange its two interns for i_0 only and h_1 would accept the proposal (maybe hiring i_1 only). [6]

Definition 4 *A non-monotonic cycle at* h *is given by* $M, M' \subseteq I$, *with* $\sharp M < \sharp M'$ *such that:*

1. $M' P_h M$
2. *Let* $M' \setminus M = \left\{i^1, ..., i^s\right\}$. *For* $k = 1, ..., s$ *there is a generalized cycle at* h, $h_0^k, ..., h_{T^k}^k, i_{-1}^k, i_0^k, i_1^k, ..., i_{T^k}^k$, $T^k \geq 1$ *such that* $i^k = i_0^k$ *and* $i_{-1}^k, i_{T^s}^k \in M \setminus M'$.
3. *For* $k \neq k'$, $i_l^k \neq i_{l'}^{k'}$ *for all* $l = 0, ...T^k$, $l' = 0, ..., T^{k'}$.

3 A look to Nash Equilibrium

The literature on capacity manipulation games has devoted a lot of attention to strong-monotonicity in population.[7]

While intuitively linked to capacity manipulation, the notion of strong monotonicity does not guarantees the stability of pure strategy Nash Equilibrium outcomes in the under the hospital-optimal rule.

Example 1 *There are two hospitals* h_1, h_2 *and two interns* i_1, i_2. *Let*$P_{h_1} : i_1, i_2$ *and*$P_{h_2} : i_2, i_1$. *Let*P_{i_1} : h_2, h_1 *and let*$P_{i_2} = h_1, h_2$. *Then there are cycles in interns and in hospitals preferences. When quotas are* $(2,2), (1,2)\,(2,1)$, *the unique stable matching is*

	h_1	h_2
μ_1	$\{i_2\}$	$\{i_1\}$

When quotas are $(1,1)$ *the hospital-optimal stable matchings is*

	h_1	h_2
μ_2	$\{i_1\}$	$\{i_2\}$

When quotas is $(2,2)$ *the capacities revelation game induced by* φ^H *has two Nash Equilibrium,* $(1,1)$ *and* $(2,2)$. *The former yields* μ_2 *as outcome which is blocked by* (h_1, i_1). *The latter yields the hospital-optimal stable matching.*

When hospitals state their true capacities, each intern will receive the offers of both hospitals, along the deferred acceptance algorithm. She can choose her favorite hospital and every hospital hires its worst intern. When both hospitals understate their capacities each one makes an offer to its favorite intern, only. Each intern accepts it so each hospital can hire its favorite intern. Observe that the equilibrium outcome of the game induced by the intern-optimal stable matching is stable.

3.1 General Results

The presence of cycles in agents preferences makes capacity manipulation profitable. Under acyclicity stating the true capacities is dominant strategy for hospitals. All Nash equilibria yields stable matchings. Furthermore, the set of stable matchings is a singleton, a result that extends Eeckhout 1999. The result holds under both rules.

[6]An alternative interpretation is provided in Ergin (2002), who prove that if a generalized cycle exists then the intern-optimal stable rule is not Pareto optimal for interns.

[7]Every counterexample in Konishi and Ünver (2006) and in Sönmez (1997b) use non-strong monotonic preferences and involves at least three interns.

The first preliminary result links instability of Nash equilibrium outcomes and incentives to capacity manipulation.

Lemma 1 *Let q be a Nash Equilibrium of the capacity revelation game induced by φ^V at (H, I, q^*, P). If h belongs to a pair blocking $\varphi^V(q)$, then $\varphi_h^V(q)\, P_h \varphi^V(q_h^*, q_{-h})$.*

It turns out that if some hospital has incentive to understate its capacity a simultaneous cycle exists.

Lemma 2 *If $\varphi_h^V(q)\, P_h \varphi^V(q_h^*, q_{-h})$ for some h and some $q_h > q_h^*$ there exists a simultaneous cycle.*

.

Theorem 1 *Assume that no simultaneous cycle exist. Then:*

1. *Stating the true capacities is a dominant strategy under φ^V.*
2. *The stable set of (H, I, q, P) is a singleton for every q.*
3. *The capacity revelation games induced by φ^H and by φ^I have the same pure strategy Nash Equilibrium outcome for every q: the unique stable matching of (H, I, q, P).*

Checking the existence of cycles of any possible length is very complex. But any cycle can be reduced to a cycle of length 2.

Lemma 3 *Let $h = h_0, ..., h_T$ and $i_0, i_1, ..., i_T$, $T \geq 1$ be such that $h_k \neq h_{k+1}$ and $i_k \neq i_{k+1}$.*

1. *If $i_0 P_{h_0} i_{-1} P_{h_0} i_T P_{h_T} i_{T-1} .... i_1 P_{h_1} i_0$ hospitals preferences have a cycle of length 2.*
2. *If $h_0 P_{i_T} h_T P_{i_{T-1}} h_{T-1} .... h_1 P_{i_0} h_0$, interns preferences have a cycle of length 2.*

Unfortunately, it is not true that any simultaneous cycle can be reduced to a simultaneous cycle of length 2 as the reader can easily check. From Theorem 1 and Lemma 3 follows a more operative result.

Corollary 1 *Assume that either the preferences of the hospitals or the preference of the interns are acyclical. Then*

1. *Stating the true capacities is a dominant strategy under φ^V.*
2. *The stable set of (H, I, q, P) is a singleton for every q.*
3. *The capacity revelation games induced by φ^H and by φ^I have the same Nash Equilibrium outcome for every q: the unique stable matching of (H, I, q, P).*

In particular, the result holds when the interests of one side of the market are aligned (Theorem 6 and 7 in Konishi and Ünver 2006).

If we restrict our attention to the hospital-optimal mechanism, acyclicity is the weakest condition guaranteeing that stating the true capacities is a dominant strategy and that every Nash Equilibrium yields a (unique) stable matching. It can also be stated as a maximal domain result.

Proposition 1 *The domain of acyclical preferences is the maximal one that guarantees 1, 2, and 3.*

1. *Assume that hospitals preferences have a cycle. There exists a vector of interns preferences such that the capacity revelation game induced by φ^H has an unstable Nash Equilibrium outcome.*
2. *Assume that interns preferences have a cycle. There exists a vector of hospitals preferences such that the capacity revelation game induced by φ^H has an unstable Nash Equilibrium outcome.*

3.2 The intern-optimal stable matching

The game induced by the intern-optimal matching is more resistant to manipulation via capacities (see also Example 1) First of all, in order to manipulate the intern optimal stable matching there is at least three interns are needed. The result is straightforward. Consider the case where at least one hospital has quota two. If there the two interns are assigned to only one hospital this hospital cannot benefit from rejecting one of them because preferences are responsive. If the interns are assigned to two different hospitals, reducing capacities does not affect the outcome of the game.

Result 1 *If there are only two interns the capacity revelation game induced by φ^I yields the intern-optimal stable matching as Nash equilibrium outcome.*

The next example provides the basic intuition that explains how the intern-optimal stable rule can result in unstable matchings.

Example 2 *Let* $I = \{i_1, i_2, i_3, i_4\}, H = \{h_1, h_2\}$.
Let P_{h_1} *be such that* $\{i_1, i_2, i_3\} P_{h_1} \{i_1, i_2\} P_{h_1} \{i_1, i_3\} P_{h_1} i_1 P_{h_1} \{i_2, i_3\} P_{h_1} i_2 P_{h_1} i_3 P_{h_1} i_4$, *let* P_{h_2} *be strongly monotonic in population and such that* $i_4 P_{h_2} i_3 P_{h_2} i_2 P_{h_2} i_1$. *Let* $P_{i_1} = h_2 h_1$, $P_{i_2} = h_1 h_2$, $P_{i_3} = h_1 h_2$ *and* $P_{i_4} = h_2 h_1$. *There is a non-monotonic cycle at* h_1: $i_1 P_{h_1} \{i_2, i_3\}$, $i_1 P_{h_1} i_3 P_{h_1} i_2 P_{h_2} i_1$. *When the capacity is* $(2, 2)$ *the intern-optimal stable matching is*

$$\mu_1 \begin{array}{cc} h_1 & h_2 \\ \{i_2, i_3\} & \{i_1, i_4\} \end{array}$$

When the capacity is $(1, 2)$ *the intern-optimal stable matching is*

$$\mu_2 \begin{array}{cc} h_1 & h_2 \\ \{i_1\} & \{i_3, i_4\} \end{array}.$$

When the capacity is $(2, 2)$ *the unique Nash Equilibrium under the intern-optimal rule matching is* $(1, 2)$, *which yields an unstable matching,* μ_2.

If h_1 states her true capacity it only receives the offer by i_2 and i_3 and it never receives an offer by i_1 during the the intern-proposing deferred acceptance algorithm. If it understates its capacity, at the first stage of the deferred acceptance algorithm it rejects the offer by i_3 . At the second stage of the deferred acceptance algorithm i_3 applies to h_2 and induces the rejection of i_1 by h_2. Finally h_1 receives the offer by i_1 and rejects i_2. Non-monotonicity of h_1's preferences is necessary to generate the instability. The generalized cycle at h_1 makes it possible the chain of rejections.

There is an important difference between the manipulation of the hospital-optimal stable matching and the manipulation of the interns optimal stable matching. In the former case by understating capacities hospitals restrain from applying to some interns (in the deferred acceptance algorithm) and prevents potential cycles of rejections of hospitals by interns. In the latter by understating capacities they generate a chain of rejections of interns by hospitals. Some hospitals will receive more offers from interns, but they will be able to cover less positions, which makes non-monotonic preferences necessary for manipulation.

Proposition 2 *Assume that no non-monotonic cycle exists. Then:*

1. *Stating the true capacities is a dominant strategy under* φ^I.
2. *The capacity revelation game induced by* φ^I *yields the intern-optimal stable matching at equilibrium.*

In particular if preferences are strongly monotonic in population no non-monotonic cycle exists and the result implies Theorem 5 in Konishi and Ünver (2006).

Corollary 2 *Under the following conditions stating the true capacity is a dominant strategy in capacity revelation game induced by φ^I. The game yields the intern optimal stable matching at every Nash equilibrium.*

1. *The preferences of the hospitals are strongly monotonic in population.*

2. *The maximum length of every preference cycle is two.*

The absence of non-monotonic cycles is the minimal condition able to prevent capacity manipulation and equilibrium instabilities under φ^I. The same applies to the profile of interns preferences having only cycle of length less than 3.

Proposition 3 1. *If a non-monotonic cycle exists there is a profile of interns preferences and a vector of capacities q such that the capacity reporting game yields an unstable matching at equilibrium.*

2. *If interns preferences have a cycle of length at least 3, there exists a profile of hospitals preferences and and a vector of capacities q such that the capacity reporting game yields an unstable matching at equilibrium.*

4 Conclusions

In the paper we have considered two capacity manipulation games using the intern-optimal and the hospital optimal-stable rules. Generalizing Konishi and Ünver (2006) we have provided minimal conditions that guarantee existence and stability of pure strategy Nash equilibrium outcomes and the strategy-proofness of truthful capacity revelation. We have also underlined the difference between the two mechanisms.

Appendix

Proof of Lemma 1: Let q be a Nash Equilibrium when the quota vector is q^* and let $\mu = \varphi^V(q)$ be the outcome matching. Assume μ is unstable. Let (h,j) blocking μ and set $\mu^* = \varphi^V(q^*_h, q_{-h})$. It must be the case that $q_h < q^*_h$ otherwise (h,j) would block μ in (H,I,P,q). For the same reason $\sharp\mu(h) = q_h$. Consider the related one-to-one matching market. From Proposition 2 in Gale and Sotomayor (1985b) it follows that $\mu^* R_I \mu$ and $\mu(h'_c) R_{h_c} \mu^*(h'_c)$ for every $h' \neq h$ and $\mu(h_c) R \mu^*(h_c)$ for every $h_c$such that $\mu(h_c) \neq h_c$. Furthermore, $\mu(h) R_h \mu^*(h)$ because q is a Nash Equilibrium and$\mu \neq \mu^*$because μ is unstable $(H,I,P,(q^*_h, q_{-h}))$. So $\mu P_H \mu^*$ and$\mu^* P_I \mu$. Finally $\mu(h) P_h \mu^*(h)$, otherwise (h,j) would block μ in (H,I,q,P).

Proof of Lemma 3: We prove only 1, the proof of 2 being identical. Let $\overline{T}$ be the largest T satisfying $i_0 P_{h_0} i_{-1} P_{h_0} i_T P_{h_T} i_{T-1} .... i_1 P_{h_1} i_0$. By contradiction assume $\overline{T} > 1$. We prove that the cycle can be reduced to one of length at most $\overline{T} - 1$. If $i_0 P_{h_2} i_1$, then $i_0 P_{h_2} i_1 P_{h_1} j_0$ a contradiction. If $i_1 P_{h_2} i_0$, then $i_0 P_{h_0} j_0 P_{h_{\overline{T}-1}} i_{\overline{T}-1} ...... i_2 P_{h_2} i_0$ and we have a cycle of length $\overline{T} - 1$, a contradiction.

Proof of Theorem 1: 1. Let $h \in H$. Let $q_h < q^*_h$ and let q_{-h} be a vector of capacities for the other hospitals. Set $\mu = \varphi^V(q)$ and set $\mu^* = \varphi^V(q^*_h, q_{-h})$. Assume that $\mu(h) P_h \mu^*(h)$. Proposition 2 in Gale and Sotomayor (1985b) (applied to the related one-to-one matching market) implies that $\mu^* P_I \mu$ and $\mu P_H \mu^*$ and that for all h' such that $\mu(h') \neq \mu^*(h')$, for all $i \in \mu(h'') \setminus \mu^*(h')$ and for all $j \in \mu^*(h') \setminus \mu(h')$, $i P_{h'} j$. Set $I' = \{i : \mu^*(i) P_i \mu(i)\} \neq \emptyset$. Let $h_0 \in \mu(I')$, then $\mu(h_0) P \mu^*(h_0)$ and let $i_0 \in \mu(h_0) \setminus \mu^*(h_0)$, $i_0 \in I'$. For all $n \geq 1$ set $h_{n+1} = \mu^*(i_{n+1})$ if $h_n \neq h_t$ for every $t < n$ and set $h_{n+1} = h_n$ otherwise. Observe that

$h_0 \neq h_1$. Let $i_n = max_{P_{n-1}}\mu(h_{n-1}) \setminus (\mu^*(h_{n-1}) \cup \{i_1,...,i_{n-1}\})$ if $\mu^*(h_{n-1}) \cup \{i_1,...,i_{n-1}\} \nsupseteq \mu(h_{n-1})$ and set $i_{n+1} = i_n$ otherwise . The sequence is stationary because I' is finite and it stops with at some $\bar{n} > 1$ such that $h_{\bar{n}} = h_{\bar{n}+1}$. Let k be the such that $h_k = h_{\bar{n}}$. Set $j_n = j_{n+k}$ and $r_n = h_{n+k}$for every $n \leq \bar{n} - k$. The sequence is made of different interns and two consecutive hospitals are distinct. It satisfies $\mu(j_n) = r_n = \mu^*(j_{n+1})$ for $n \leq \bar{n} - k$, and $\mu^*(j_k) = r_0$. It follows that (i) $j_0 P_{r_0} j_k P_{r_{k-1}} j_{k-1} ......j_2 P_{r_2} j_1 P_{r_1} j_0$ and (ii) $r_0 P_{j_k} r_k P_{j_k} r_{k-1} ... P_{j_0} r_0$. So $h_0, ...h_k, r_0, ...r_k$ are a simultaneous cycle.
2. Let μ, μ^* be different stable matchings of (H, I, q, P) for some q and assume that $\mu P_H \mu^*$. Set $I' = \{i : \mu^*(i) P_i \mu(i)\} \neq \emptyset$. The proof of the existence of simultaneous cycles is exactly like in 1.
3. From 1 a Nash equilibrium yielding a stable matching exists. From Lemma 1 and 1 the game does not yields unstable matchings at equilibrium. From 2 such outcome is unique.

Proof of Proposition 1: 1. Let h_1, h_2, i_1, i_2 be like in Definition 1 . Define a profile of preferences for interns preferences as follows. Let $h_2 P_{i_1} h_1 P_{i_2} h_2$ and let $A(i_1) = A(i_2) = \{h_1, h_2\}$. Let $P'_{I' \setminus \{i_1, i_2\}}$ be any vector of preferences. Consider the market $\left(H \setminus \{h_1, h_2\}, I \setminus \{i_1, i_2\}, q_{-\{h_1,h_2\}}, P_{H \setminus \{h_1,h_2\}}, P'_{I' \setminus \{i_1,i_2\}}\right)$ and let μ' be the hospital optimal stable matching. Let $P_{I \setminus \{i_1,i_2\}}$ such that $A(i) = \mu(i)$for every $i \in I$. When $q_{h_1} = q_{h_2} = 2$ the market (H, I, q, P)has a unique stable matching: $\mu(i) = \mu'(i)$ for every $i \neq i_1, i_2$, $\mu(i_1) = h_2, \mu(i_2) = h_1$. It is easily seen that when $q = (2, 2, q_{-\{h_1,h_2\}})$, the message $(1, 1, q_{-\{h_1,h_2\}})$ is a Nash Equilibrium. The outcome matching is μ^*, where $\mu^*(i) = \mu'(i)$ for every $i \neq i_1, i_2$, $\mu^*(i_1) = h_1, \mu^*(i_2) = h_2$, and is blocked by (h_1, i_2) and (h_2, i_1) . The proof of 2 is similar.

Proof of Proposition 2: 1. Let $h \in H$. Let $q_h < q_h^*$ and let q_{-h} be a vector of capacities for the other hospitals. Set $\mu = \varphi^I(q)$ and set $\mu^* = \varphi^I(q_h^*, q_{-h})$. Assume that $\mu(h) P_h \mu^*(h)$. Proposition 2 in Gale and Sotomayor (1985) implies that $\mu^* P_I \mu$ and, in the associated one-to one matching market $\mu(h'_c) R_{h_c} \mu^*(h'_c)$ for every $h' \neq h$ that and $\mu(h_c) R \mu^*_{h_c}(h_c)$ for every $h_c$such that $\mu(h_c) \neq h_c$. It follows that for every h such that $\mu(h) \neq \mu^*(h)$, for all $i \in \mu(h) \setminus \mu^*(h)$ and for all $j \in \mu^*(h) \setminus \mu(h)$, $i P_h j$ otherwise (h, j)would block μ^* in market $(H, I, (q_h^*, q_{-h},) P)$. There is no loss of generality in assuming that $\mu^*(i)$ is i's favorite firm, for very $i \in I$, because $\mu P_H \mu^*$ and $\mu P_I^* \mu$. Consider the deferred acceptance algorithm where interns apply with quotas q. Let i the first intern to be rejected by $\mu^*(i) = h'$. It must be the case that i is rejected in favor of some intern in $\mu^*(h')$. It follows that $\sharp\mu(h') < \sharp\mu^*(h')$ and that $h = h'$ so h' preferences are not monotonic. Also, $\mu(h) P_H \mu^*(h)$.
Set $M' = \mu(h)$ and set $M' = \mu^*(h)$. Let $M' \setminus M = \{i^1, ..., i^s\}$ and let $M \setminus M' = \{j^1, ..., j^q\}$. Set $r = \sharp M - \sharp M' = \sharp M - q_h$. We have assumed that $\mu^*(i)$ is i's favorite hospital. Then $\mu^*(i)$ is the first firm i proposes to in the deferred acceptance algorithm. So it must be the case that exactly r interns are rejected by h at the first stage of the deferred acceptance algorithm. Furthermore $r < q$, otherwise during the deferred acceptance algorithm no other intern in M' would be rejected by h and h would end up with $M' = M \setminus \{j^1, ..., j^r\}$, a contradiction. Assume that such intern are $j^{q-r+1}, ..., j^q$.
Consider $i_1^1 = i^1$. Let $t_0 > 1$ the step at which i_1^1 has been accepted by h and let i_1 be an intern that has been rejected in favor of i^t. If $i_1 \in M$ stop and set $i^t_{-1} = i^1_1$, otherwise at stept_1, $1 < t_1 < t_0$, i_1 has been accepted and some i_2 has been rejected in favor of i_1^1. For all $k \geq 2$, If $i_k \in M$ stop, at step t_k, $1 < t_k < t_0$, $i_k$has been accepted by $h$and some i_{k+1} has been rejected by h in favor of i_{k+1}. The sequence eventually stop at some $i_{K^1} \in M$ who has been rejected in a step $t_{K^1} > 1$ of the deferred acceptance algorithm.[8] There is no loss of generality in assuming that $i_{K^1} = j^1$. We have $i^1 P_h j^1$. For$t \geq 2$, define i_{K^2}as before, but choose any $i_k^t \neq i_s^l$ for every $l < t$. It can be done because if at the same stage (after the first one) of the deferred acceptance algorithm p interns are accepted by h then p who were engaged to h are rejected. So $i_{K^t} \neq i_{K^l}$if

[8]Every intern in the sequence is rejected because of the arrival of a proposal from another intern.

$t \neq l$. There is no loss of generality in assuming that $i_{-1}^t = j^t$ for $t = 1, ..., s$, because every i_{-1}^t has been rejected at a step of the deferred acceptance algorithm successive to the first one.
For $t = 1, ..., s$ set $i_0^t = i^t$. Let i_1^t be the intern in favor of which i_0^t has been rejected by $\mu^*(i_0^t) = h_1^t$. For every $p \geq 1$ at step t_p of the deferred acceptance algorithm j_p^t has been rejected by $h_{p+1}^t = \mu^*\left(j_p^t\right) \neq h_p^t$ in favor of some $j_{p+1}^t \notin \mu^*\left(h_p^t\right)$ i_1. If $h_p^s = h_l^{s'}$ and $t_p^t = t_l^{t'}$ for some $t' < t$ then h_p^t has received at least $\sharp\left\{t' < t : h_p^t = h_l^{t'} for\ some\ l\ and\ t_p^t = t_l^{t'}\right\}+1$ better proposal than j_p^t. So we can choose j_p^t different from every other $j_l^{t'}$, $0 \leq t' < t$. We have $h_{p+1}^t = \mu^*\left(i_p^t\right)$ for all t, $h_{p+1}^t P_{i_t} h_p^t$[9] and $i_p^t P_{h_{p+1}^t} i_p^t$. The sequence stops at some K^t where $h_{K^t}^t = h$ rejects some some intern at the first stage of the algorithm. Then, there $h \in H, M, M'$, satisfy $\sharp M < \sharp M'$ $M' P_h M$ and $h_0^k, ..., h_{T^k}^k, i_{-10}^k, i_0^k, i_1^k, ..., i_{K^t}^k$ with $K^t \geq 1$ such that $h = h_0^k, i^k = i_0^k$, $i_{-1}^k, i_{T^s}^k \in M \setminus M'$ and $i_{-1}^k \neq i_{-1}^{k'}$, for $k \neq k'$. Then there is a non-monotonic cycle at h.
2. By 1 a Nash equilibrium yielding a stable matching exists. By Lemma 1 there are not unstable equilibria, so every equilibrium outcome is stable. By contradiction assume that the outcome is not the intern optimal stable matching. It must be the case that some hospital has misrepresented capacity. Let q be the Nash equilibrium of the game and let $q^* \geq q$ be the true capacity vector. Set $\mu = \varphi^I(q)$ and set $\mu^* = \varphi^I(q^*)$. From 1 μ is stable in (H, I, q^*, P) so $\mu P_H \mu^*$ and $\mu^* P_I \mu$. There is no loss of generality in assuming that $\mu^*(i)$ is intern i's favorite hospital. The matching μ is obtained through the intern-proposing deferred acceptance algorithm. It must be the case that at least one i is rejected by $\mu^*(i) = h$ at the first stage of the deferred acceptance algorithm. Because every intern applies to her hospital under μ^* at this stage it is because h has manipulated its true capacity. Then h has less interns under μ than under μ^*. This yields a contradiction because both matching are stable in (H, I, q^*, P).

Proof of Proposition 3: 1. Assume that there is a non-monotonic cycle at h. Using the notation of definition 4 let $I' = \left\{i_{T^k}^1 : h_l^k = h\right\} \cap M' \cup \left\{i_{-1}^1, ..., i_{-1}^s\right\}$. Set $M^* = M' \cap M \cup I'$. Note that $\sharp M^* > \sharp M$ and $M P_h M^*$. Set the preferences of interns as follows. Let $A\left(i_l^k\right) = \left\{h_l^k, h_{l+1}^k\right\}$ and $h_{l+1}^k P_{h_k} h_l^k$ for all k and for all l. Let $A(i) = \{h\}$ if $i \in M' \cap M$. For all other interns let $A(i) = \{h(i)\}$ for some $h(i) \notin \left\{h_l^k : k = 1, ...s, l = 1, ...T^k\right\}$. Let $q_{h_0} = q_h = \sharp M^*$ and let $q_{h_l^k} = 1$ for all k, l such that $h_l^k \neq h$. Set all other capacities arbitrarily. We have $\varphi_h^I(q) = M^*$. From the property of the non-monotonic cycle at h, $\varphi_h^I(q_h', q_{-h}) R_h M P_h M^*$. Let q_h' be h's best response to q_{-h}. We have $q_h' < q_h$. It is easily seen that (q_h', q_{-h}) is a Nash equilibrium at (H, I, q, P). It yields a matching that is unstable because in any stable matching of (H, I, q, P) h is matched to $\sharp M^* > q_h'$ interns.

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[9]Because i_t first proposes to h_{t+1}in the deferred acceptance algorithm.

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