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WHEN ARE SIGNALS COMPLEMENTS OR SUBSTITUTES?*

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Abstract

The paper introduces a notion of complementarity (substitutability) of two signals which requires that in all decision problems each signal becomes more (less) valuable when the other signal becomes available. We provide a general characterization which relates complementarity and substitutability to a Blackwell-comparison of two auxiliary signals. In a special setting with a binary state space and binary, symmetric signals, we find an explicit characterization that permits an intuitive interpretation of complementarity and substitutability. We demonstrate how these conditions extend to the general case. Finally, we study implications of complementarity and substitutability for information acquisition and in a second price auction.

Keywords: Complementarity, substitutability, value of information, Blackwell- ordering, statistical decision problem, information acquisition, second price auction

JEL Classification: C00, C44, D81, D83

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1 Introduction

Suppose that two signals are available to a decision maker, and that each signal contains some information about an aspect of the world that is relevant to a future decision. In this paper we ask under which conditions these two signals are *substitutes*, and under which conditions they are *complements*. Roughly speaking, we mean by this that the incentive to acquire one signal decreases as the other signal becomes available (in the case of substitutes), or that it increases as the other signal becomes available (in the case of complements).

Now the incentives to acquire signals depend, of course, on the decision for which the information will be used. When we call signals complements or substitutes in this paper, then we mean that the conditions described above are satisfied *in all decision problems*. Hence we say in this paper that signals are substitutes if in all decision problems the value of each signal decreases as the other signal becomes available. The signals are complements if in all decision problems the value of each signal increases as the other signal becomes available.

The conditions that we shall provide will thus not refer to any particular decision problem, but only to the joint distribution of signals, conditional on the various possible states of the world. We thus identify statistical features of signals which imply that these signals are substitutes or complements.

We now give a simple example that indicates how signals can be complements. Suppose that you can observe in a war the enemy's coded communication, and that you have access to the enemy's encryption code. Then observing the enemy's communication is of no use if you do not know how it is coded, and understanding the encryption code is of no use if you don't have any access to communication that uses this code. However, together the two pieces of information are potentially valuable. Your incentive to acquire any one is larger if you already have the other.

One of the main results in this paper shows that, in a particular and special setting, complementary signals are characterized by a property that is very closely related to the main feature of the above example. This property is that the meaning of a realization of one signal depends on the realization of the other signal. The second signal thus provides the key that is needed to unlock the first signal. More technically, the result shows that, in a specific setting, signals are complements if and only if there is a realization of one signal that may increase, but also decrease, the decision maker's subjective probability that some event has occurred, depending on what the other signal is.

The result described in the previous paragraph will be shown for a special setting only. However, we also explore the extent to which it generalizes. We show that in many cases it is necessary for complementarity of signals that the meaning of the realization of one signal can be reverted by a realization of the other signal. We cannot show, however, that this condition is sufficient.

We also explore whether there are cases where pairs of signals are complements without having the property that one signal's meaning can be reverted by the other signal. A case in point is when one signal is completely uninformative about the state of the world and is thus useless by itself. Yet, as we shall demonstrate, such a signal might still enhance the value of the other signal by providing information about the other signal's quality.

A simple example that indicates how signals can be substitutes is also easily constructed. Suppose you have two advisors, and you know that they both work with the same sources, and will tell you exactly the same thing. Then each of them will have positive value, but once you have heard what one of them says, you do not derive any additional benefit from hearing what the other one says.

For the special setting referred to earlier we shall show that signals are substitutes if and only if they share one important feature with the example described in the previous paragraph, namely that the value of a second signal will always be zero. In a more general setting, a related, less stringent condition is a necessary condition for signals to be substitutes. This necessary condition is that the additional signal cannot reinforce the decision maker's most extreme belief that he can have after observing one signal alone.

The results described so far provide interesting, yet partial insights into the nature of the complementarity and substitutability relations among signals. We also offer in this paper completely general characterizations of complements and substitutes. These results show that two signals are complements (resp. substitutes) if and only if, among two other signals that are derived from the two original signals, one dominates the other in the sense of Blackwell (1951), that is, is more valuable in all decision problems. We thus reduce the problem of determining whether two signals are complements (resp. substitutes) to the problem of determining whether among two other signals one Blackwell- dominates the other. This is useful because it allows us to use the well-known characterizations of Blackwell-dominance, including, of course, Blackwell's own characterization, to find out whether two signals are complements (resp. substitutes).

Complementarity and substitutability of signals are also important if different signals are accessible to different decision makers. We will elaborate below the economic relevance of the relations among signals that we investigate in this paper in two examples of environments with decentralized information. In the first example, we consider a strategic information acquisition game in which each decision maker can observe the other player's information before making a decision. In the second example, we study the bidding behavior in a second price, common value action when bidders' private signals are complements or substitutes.

Many pairs of signals are neither complements nor substitutes, if our definitions are used. This is because our definition of these terms requires certain conditions to be true in *all* decision problems. This is in the spirit of Blackwell's (1951) work. It seems plausible that more signals will satisfy the conditions for being substitutes or complements if we restrict attention to smaller classes of decision problems. In the context of Blackwell's original work this line of investigation has been taken up by Lehmann (1988), Persico (2000) and Athey and Levin (2001). The analogous research for our problem is left to a future paper.

Radner and Stiglitz (1984) consider a setting in which a one-dimensional real parameter indexes the quality of a signal. They show that the value of the signal in any decision problems is weakly increasing but not everywhere concave as the quality of information increases. In particular, a non-concavity occurs for any decision problem in the neighborhood of the parameter value for which the signal is entirely uninformative. Non-concavity of the value of a signal as the quality of the signal improves indicates increasing returns to scale in information. It may be possible to interpret an improvement in the quality of a signal as "making a further signal available", and one might be able to interpret a non-concavity of the value of information as a complementarity between an existing signal, and a further signal that might be made available. We have not yet explored whether we can make these analogies precise.

Sobel (2006) considers a fixed group decision problem with common interests but decentralized information, and asks about the relation between individual beliefs, and group beliefs, or the relation between optimal actions based on individual beliefs, and optimal actions based on group beliefs. He adopts the position of an outside observer who doesn't know the agents' information structure. By contrast, we keep the information structure fixed, and vary the decision problem. Moreover, our focus is on expected utility rather than the underlying beliefs and optimal actions.

Complementarity and substitutability of signals has previously been referred to in an auction context by Milgrom and Weber (1982b). Their definition is tailored to the auction setting, whereas our definition is independent of the specific underlying economic decision problem.

The paper is organized as follows: Section 2 provides definitions. Section 3 contains our main completely general result. Section 4 studies in detail a binary, symmetric example. Section 5 generalizes intuitive insights that we obtained for the binary, symmetric example. Section 6 describes two economic applications. Section 7 concludes. Some of the proofs are contained in the appendix.

2 Definitions

The state $\tilde{s}$ of the world is a random variable with realizations in a finite set S. Two signals are available: $\tilde{\sigma}_1$ which takes values in the finite set S_1, and $\tilde{\sigma}_2$ which takes values in the finite set S_2. We assume without loss of generality that $S_1 \cap S_2$ is empty. The joint distribution of signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ conditional on the state being equal to $s \in S$ is denoted by p_s. For $i = 1, 2$ the marginal distribution of signal $\tilde{\sigma}_i$ conditional on the state being equal to $s \in S$ is denoted by $p_{i,s}$.

Our objective in this section is to define when two signals are substitutes and when they are complements. We first need some auxiliary definitions.

Definition 1. *A* decision problem *is a triple* (π, A, u) *where* π *is a probability distribution on* S *(the prior distribution),* A *is some finite set (the set of actions), and* u *is a function of the form:* $u : A \times S \to \mathbb{R}$ *(the utility function).*

Definition 2. *For given decision problem* (π, A, u)*:*

- *The* value of not having any signal *is:*

$$V_\emptyset \equiv \max_{a\in A} \sum_{s\in S} (u(a,s)\pi(s)).$$

- *For* $i \in \{1,2\}$ *the* value of having signal $\tilde{\sigma}_i$ alone *is:*

$$V_i \equiv \sum_{\sigma_i \in S_i} \max_{a\in A} \sum_{s\in S} (u(a,s)p_{i,s}(\sigma_i)\pi(s)).$$

- *The* value of having both signals *is:*

$$V_{1,2} \equiv \sum_{\sigma_1\in S_1} \sum_{\sigma_2 \in S_2} \max_{a\in A} \sum_{s\in S} (u(a,s)p_s(\sigma_1,\sigma_2)\pi(s)).$$

We can now offer the two key definitions of this paper.

Definition 3. *Signals* $\tilde{\sigma}_1$ *and* $\tilde{\sigma}_2$ *are* substitutes *if for all decision problems* (π, A, u) *we have:*[1]

$$V_1 - V_\emptyset \geq V_{1,2} - V_2$$

and

$$V_2 - V_\emptyset \geq V_{1,2} - V_1.$$

Definition 4. *Signals* $\tilde{\sigma}_1$ *and* $\tilde{\sigma}_2$ *are* complements *if for all decision problems* (π, A, u) *we have:*

$$V_{1,2} - V_2 \geq V_1 - V_\emptyset$$

and

$$V_{1,2} - V_1 \geq V_2 - V_\emptyset.$$

The motivation for this definition is best understood if one considers a setting in which an agent has to choose whether to purchase either one, or both, of the signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$, and the agent's utility equals the expected utility from a decision problem of the type described in Definition 1 plus money holdings. Thus, the agent's utility is additively separable in the utility from the decision problem and money. In this case the utility differences $V_i - V_\emptyset$ and $V_{1,2} - V_j$ reflect the agent's willingness to pay for signal i if no

[1]Note that the two inequalities in this definition, and also the two inequalities in Definition 4, are equivalent.

signal is available (resp. if signal $j \neq i$ is available). Substitutability means that the willingness to pay for a signal decreases if the other signal becomes available, whereas complementarity means that the willingness to pay for a signal increases if the other signal becomes available.

The requirement that the inequalities in Definition 3 or 4 have to be true for *all* decision problems is very restrictive, and one may well ask whether any signal structures satisfy these requirements. We therefore give two simple examples.

Example 1. *States are:* $S = \{+1, -1\}$. *Signals are:*[2] $S_1 = S_2 = \{+1, -1\}$. *The signal distributions are given by:* $p_s(\sigma_1, \sigma_2) = 1/2 \Leftrightarrow \sigma_1 \cdot \sigma_2 = s$. *Each individual signal's distribution is independent of the true state, yet together the two signals fully reveal the true state. Therefore, these signals are* complements.

Example 2. *States are:* $S = \{+1, -1\}$. *Signals are:* $S_1 = S_2 = \{+1, -1\}$. *The signal distributions are given by:* $p_s(\sigma_1, \sigma_2) = 1 \Leftrightarrow \sigma_1 = \sigma_2 = s$. *Each individual signal completely reveals the true state. Therefore, these signals are* substitutes.

3 A General Result

To obtain a general characterization of signals that are complements or substitutes, we define two auxiliary signals, $\tilde{\sigma}_L$ and $\tilde{\sigma}_R$. Informally, the first of these signals, $\tilde{\sigma}_L$, can be described as follows. An unbiased coin is tossed. If "head" comes up, the decision maker is informed about the realization of $\tilde{\sigma}_1$. If "tails" comes up, the decision maker is informed about the realization of $\tilde{\sigma}_2$. Formally, the second auxiliary signal $\tilde{\sigma}_L$ has realizations in the set $S_L \equiv S_1 \cup S_2$. We assume that $S_1 \cap S_2$ is empty. For given state $s \in S$, the probability that $\tilde{\sigma}_L$ has realization $\sigma_1 \in S_1$ is $p_{L,s}(\sigma_1) \equiv \frac{1}{2} p_{1,s}(\sigma_1)$, and the probability that $\tilde{\sigma}_L$ has realization $\sigma_2 \in S_2$ is $p_{L,s}(\sigma_2) \equiv \frac{1}{2} p_{2,s}(\sigma_2)$. We write P_L for the matrix with $\#S$ rows and $\#S_1 + \#S_2$ columns which is constructed as follows. Each row corresponds to a state in S. Each column corresponds to a realization of the signal σ_L. Each entry of the matrix indicates the probability with which the column signal is observed in the row state.

[2] Examples 1 and 2 violate our assumption that $S_1 \cap S_2 = \emptyset$, but this is without consequence, and could be repaired by relabeling signals.

The second auxiliary signal, $\tilde{\sigma}_R$, is intuitively constructed as follows. An unbiased coin is tossed. If "head" comes up, the decision maker is informed about the realization of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. If "tails" comes up, the decision maker receives no information. Formally, the first signal $\tilde{\sigma}_R$ has realizations in the set $S_L \equiv (S_1 \times S_2) \cup \{N\}$. Here, the symbol N denotes the case that the decision maker receives no information. For given state $s \in S$, the probability that $\tilde{\sigma}_R$ has realization $(\sigma_1, \sigma_2) \in S_1 \times S_2$ is $p_{R,s}(\sigma_1, \sigma_2) \equiv \frac{1}{2} p_s(\sigma_1, \sigma_2)$, and the probability that $\tilde{\sigma}_R$ has realization N is $p_{R,s}(N) \equiv \frac{1}{2}$. We write P_R for the matrix with $\#S$ rows and $\#S_1 \cdot \#S_2 + 1$ columns which is constructed as follows. Each row corresponds to a state in S. Each column corresponds to a realization of the signal σ_R. Each entry of the matrix indicates the probability with which the column signal is observed in the row state. We assume that the ℓ-th row of matrix P_R corresponds to the same state as the ℓ-th row of matrix P_L.

Definition 5. *For given decision problem (π, A, u), and for $k \in \{L, R\}$, the* value of having signal $\tilde{\sigma}_k$ *is:*

$$V_k \equiv \sum_{\sigma_k \in S_k} \max_{a \in A} \sum_{s \in S} (u(a, s) p_{k,s}(\sigma_k) \pi(s)).$$

Definition 6. *Suppose $k, \ell \in \{L, R\}$ and $k \neq \ell$. Signal $\tilde{\sigma}_k$ is* more valuable *than signal $\tilde{\sigma}_\ell$ if for all decision problems (π, A, u) we have:*

$$V_k - V_\emptyset \geq V_\ell - V_\emptyset.$$

Lemma 1. *(i) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes if and only if signal $\tilde{\sigma}_L$ is more valuable than signal $\tilde{\sigma}_R$.*

(ii) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if and only if signal $\tilde{\sigma}_R$ is more valuable than signal $\tilde{\sigma}_L$.

Proof. For part (i) note that the two inequalities that define substitutes, $V_1 - V_\emptyset \geq V_{1,2} - V_2$ and $V_2 - V_\emptyset \geq V_{1,2} - V_1$ are equivalent to each other, and to: $\frac{1}{2}(V_1 + V_2) \geq \frac{1}{2}(V_{1,2} + V_\emptyset)$. But by definition the expression on the left hand side is the same as V_L, and the expression on the right hand side is the same as V_R. Thus (i) follows. The proof of part (ii) is similar. □

Application of Blackwell's theorem on the comparison of experiments (Blackwell, 1951) now shows immediately that Lemma 1 implies the following Proposition.

Proposition 1. *(i) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes if and only if there is a Markov matrix G with $\#S_1 + \#S_2$ rows and $\#S_1 \cdot \#S_2 + 1$ columns such that:*

$$P_L \cdot G = P_R.$$

(ii) Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if and only if there is a Markov matrix G with $\#S_1 \cdot \#S_2 + 1$ rows and $\#S_1 + \#S_2$ columns such that:

$$P_R \cdot G = P_L.$$

4 A Symmetric Binary Example

We now consider the special case that the state space and both signals are binary: $S = \{a, b\}$, $S_1 = \{\alpha, \beta\}$, and $S_2 = \{\hat{\alpha}, \hat{\beta}\}$. We also assume that the signals are symmetric. By this we mean that they are symmetric if only one signal is received, i.e. the posteriors if α or β are received are equal respectively to the posteriors if $\hat{\alpha}$ or $\hat{\beta}$ are received. Formally, $p_{1,a}(\alpha) \cdot p_{2,b}(\hat{\alpha}) = p_{2,a}(\hat{\alpha}) \cdot p_{1,b}(\alpha)$, and $p_{1,a}(\beta) \cdot p_{2,b}(\hat{\beta}) = p_{2,a}(\hat{\beta}) \cdot p_{1,b}(\beta)$, which imply that $p_{1,s}(\alpha) = p_{2,s}(\hat{\alpha})$, $p_{1,s}(\beta) = p_{2,s}(\hat{\beta})$, and $p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha})$ for all $s \in S$. Note that for our binary example this form of symmetry implies that the signals are symmetric when two signals are received, i.e. the posterior if $(\alpha, \hat{\beta})$ is received is the same as the posterior if $(\beta, \hat{\alpha})$ is received.

Our next two assumptions rule out trivial cases. The first assumption is that for each realization of each signal there is strictly positive probability that it occurs in at least some state, i.e.: for every $i \in \{1, 2\}$ and every $\sigma_i \in S_i$ there is some $s \in S$ such that $p_{i,s}(\sigma_i) > 0$. The second assumption is that there is at least one informative signal realization: $p_a(\sigma_1, \sigma_2) \neq p_b(\sigma_1, \sigma_2)$ for at least one $(\sigma_1, \sigma_2) \in S_1 \times S_2$.

Our final assumption is without loss of generality. We assume that: $p_{1,a}(\alpha) \geq p_{1,b}(\alpha)$. Thus, if the decision maker receives signal $\tilde{\sigma}_1 = \alpha$ and no other signal, then his posterior probability of state a will not be lower than his prior probability. This implies that, if the decision maker receives signal $\tilde{\sigma}_1 = \beta$ and no other signal, then his posterior probability of state a is not higher than his prior probability. By symmetry, of course, the same is true for signals $\tilde{\sigma}_2 = \hat{\alpha}$ and $\tilde{\sigma}_2 = \hat{\beta}$.

We shall refer to the example defined in the above paragraphs as the "symmetric, binary example". We now characterize substitutes and complements in the symmetric, binary example.

Proposition 2. *In the symmetric, binary example: (i) Signals are substitutes if and only if:*

$$\textit{(C1)} \quad p_s(\alpha, \hat{\beta}) = p_s(\beta, \hat{\alpha}) = 0 \textit{ for all } s \in S.$$

(ii) Signals are complements if and only if at least one of the following conditions holds:

$$\textit{(C2)} \quad p_a(\alpha, \hat{\alpha}) \leq p_b(\alpha, \hat{\alpha});$$

$$\textit{(C3)} \quad p_a(\beta, \hat{\beta}) \geq p_b(\beta, \hat{\beta}).$$

The proof of Proposition 2 is in the Appendix. Here, we provide a discussion of this result. Condition (C1) says that the two signals are perfectly correlated, that is, if the decision maker knows the the realization of one signal he can deduce with certainty what the realization of the other signal has been. Thus, in this case, in all decision problems $V_{1,2} - V_i = 0$ for $i \in \{1, 2\}$, while $V_i - V_\emptyset \geq 0$ for $i \in \{1, 2\}$, with strict inequality in most decision problems. It is obvious that signals are then substitutes.

The above paragraph has made clear that the "if-part" of part (i) of Proposition 2 is trivial. Therefore, the proof of part (i) of Proposition 2 that is provided in the Appendix deals with the "only if-part" only.

Condition (C2) says that signal $(\alpha, \hat{\alpha})$ induces the decision maker to raise (or at least not to lower) his probability of state b in comparison to his prior probability for this state. This is despite of the fact that, as we have assumed, individually each of the signals $\tilde{\sigma}_1 = \alpha$ or $\tilde{\sigma}_2 = \hat{\alpha}$, if received alone without the other signal, induces the decision maker to raise (or at least not to lower) his subjective probability of state a. In other words, these signals' meaning to the decision maker depends on whether they are received individually or together: Each signal alone is (weakly) indicative of state a, but if received together, they are (weakly) indicative of state b. Condition (C3) is the analogous statement for the signal realizations $(\beta, \hat{\beta})$.

To understand Proposition 2 more fully it helps to be more precise about the notion of the "meaning" of a signal realization $(\sigma_1, \sigma_2) \in S_1 \times S_2$. Let us say that "the meaning of (σ_1, σ_2) is s" (where $s \in S$) if $p_s(\sigma_2, \sigma_2) > p_{s'}(\sigma_2, \sigma_2)$ (where $s' \neq s$), and that "the meaning of (σ_1, σ_2) is $\emptyset$" if $p_s(\sigma_2, \sigma_2) =$

	$\hat{\alpha}$	$\hat{\beta}$
α	a	a
β	a	b

Case 1

	$\hat{\alpha}$	$\hat{\beta}$
α	a	b
β	b	b

Case 2

	$\hat{\alpha}$	$\hat{\beta}$
α	b	a
β	a	b

Case 3

	$\hat{\alpha}$	$\hat{\beta}$
α	$\emptyset$	a
β	a	b

Case 4

	$\hat{\alpha}$	$\hat{\beta}$
α	a	b
β	b	a

Case 5

	$\hat{\alpha}$	$\hat{\beta}$
α	a	b
β	b	$\emptyset$

Case 6

	$\hat{\alpha}$	$\hat{\beta}$
α	a	$\emptyset$
β	$\emptyset$	b

Case 7

Figure 1: Possible meanings of signal realizations in the symmetric binary example

$p_{s'}(\sigma_2, \sigma_2)$ (where $s' \neq s$). Thus, in the latter case the signal realization is uninformative.

For given distributions p_s we can construct a 2×2 matrix which describes the meaning of each signal distribution. Figure 1 lists the seven different forms which this matrix can take under our assumptions. It is easily verified that each of these matrices can arise, and that no other matrix is compatible with our assumptions.

Proposition 2 now tells us that signals are complements if the signal structure is of the type labeled in Figure 1 as "Cases 3-6". Signals are complements if the signal structure is of the type labeled in Figure 1 as "Case 7", and if, moreover, the probability that uninformative signal realizations are observed is zero. Cases 1 and 2 in Figure 1 are neither complements nor substitutes. Note that Figure 1 makes clear that substitutes are non-generic in our framework, whereas complements are robust.

5 Generalizations

In this section we develop necessary (but not sufficient) conditions for signals to be substitutes or complements. The conditions that we present have a similar flavor as the characterizations that we obtained in Section 4 for the symmetric binary example.

Denote by $q_\pi(s \mid \sigma_i)$ the posterior belief that the true state is s if signal realization σ_i was observed, and denote by $q_\pi(s \mid \sigma_1, \sigma_2)$ the posterior belief that the true state is s if signal realization (σ_1, σ_2) was observed.

Proposition 2 showed for the symmetric binary example that signals are substitutes if and only if each signal is redundant given the other signal. The following general result has a similar flavor.

Proposition 3. *If signals are substitutes, then for every prior π and every state s there is at least one signal i such that:*

$$q_\pi(s \mid \sigma_1, \sigma_2) \leq \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i)$$

for every (σ_1, σ_2) that is observed with positive probability; and there is also some signal i such that:

$$q_\pi(s \mid \sigma_1, \sigma_2) \geq \min_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i).$$

for every (σ_1, σ_2) that is observed with positive probability.

In words, the first inequality says that no joint signal realization (σ_1, σ_2) can provide stronger evidence in favor of state s than the strongest realization of signal σ_i. Intuitively, thus, the signal $j \neq i$ provides no new information whenever that realization of signal i is observed that provides strongest evidence of state s. The second inequality has an analogous interpretation.

Proposition 3, unlike Proposition 2, refers to prior distributions π. It would be interesting to obtain a formulation of Proposition 3 that refers to the conditional distribution of signals only. However, we have not been able to find such a formulation. The same remark applies to Proposition 4 below.

Proof. We only prove the first inequality. The proof of the second inequality is analogous. The proof is indirect. Suppose there are π and s such that for every i there exists a signal realization (σ_1, σ_2) that is observed with positive probability such that

$$q_\pi(s \mid \sigma_1, \sigma_2) > \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i).$$

Then:

$$\max_{(\sigma_1,\sigma_2)} q_\pi(s \mid \sigma_1, \sigma_2) > \max_{\sigma_i \in S_i} q_\pi(s \mid \sigma_i)$$

where the maximum on the left hand side is taken over all signal realizations (σ_1, σ_2) that are observed with positive probability. Let $\bar{q}$ denote a probability that is between the left hand side and the right hand side of the above inequality. Consider the decision problem with the prior π, action set $A = \{T, B\}$, and payoff function given by $u(T, s) = 1 - \bar{q}$, $u(T, s') = 0$ if $s' \neq s$, $u(B, s) = 0$, and $u(B, s') = \bar{q}$ if $s' \neq s$. The decision maker will choose T if and only if his posterior probability of state s is at least $\bar{q}$. The prior probability $\pi(s)$ is not more than $\bar{q}$ because, for any i, it is a convex combination of $q_\pi(s \mid \sigma_i)$ for $\sigma_i \in S_i$. Therefore, without information, the agent chooses B. By the definition of $\bar{q}$, no signal realization $\sigma_i \in S_i$ of any signal i will give the decision maker an incentive to switch to T. Therefore, $V_i = 0$ for $i = 1, 2$. However, if the joint signal realization (σ_1, σ_2) is observed, the decision maker switches to T. Because this signal realization has positive prior probability this implies: $V_{1,2} > 0$. It then follows that the signals are not substitutes. □

Proposition 2 showed that signals are complements in the symmetric, binary example if and only if the meaning of each signal realization can be reversed by the realization of the other signal. Proposition 4 is a general result that has a somewhat similar flavor.

Proposition 4. *Suppose signals are complements, and suppose there is a prior π, a state s, a signal i, and a realization σ_i of signal i that is observed with positive probability, such that*

$$q_\pi(s \mid \sigma_i) \neq \pi(s).$$

Then there is at least one realization σ_j of signal $j \neq i$ that is observed with positive probability, and at least one realization σ_i' of signal i such that (σ_j, σ_i') is observed with positive probability, such that one of the following holds:

- $q_\pi(s \mid \sigma_j) > \pi(s)$ *and* $q_\pi(s \mid \sigma_j, \sigma_i') < \pi(s)$

- $q_\pi(s \mid \sigma_j) < \pi(s)$ *and* $q_\pi(s \mid \sigma_j, \sigma_i') > \pi(s)$
- $q_\pi(s \mid \sigma_j) = \pi(s)$ *and* $q_\pi(s \mid \sigma_j, \sigma_i') \neq \pi(s)$

In words, the condition on which the Proposition is based says that at least one realization of signal i alone changes the decision maker's belief that the true state is s. The result is thus based on a weak condition that ensures that signal i is informative. The first and the second bullet points then say that there are realizations of signal j alone, and of signal i and j together, such that the realization of signal i reverses the meaning of the realization of signal j. The third bullet point says that there is a realization of signal j that leaves the prior unchanged, but if the realization of signal i is observed, the prior does change.

Proof. Let π, s, and i be as described in the Proposition. Consider the decision problem with prior π, action set $A = \{T, B\}$, and payoff function given by $u(T, s) = 1 - \pi(s), u(T, s') = 0$ if $s' \neq s$, $u(B, s) = 0, u(B, s') = \pi(s)$ if $s' \neq s$. The decision maker will choose T if and only if his posterior probability of state s is at least $\pi(s)$. Hence, without information, the agent is willing to choose T. By assumption, since $q_\pi(s \mid \sigma_i) \neq \pi$ for some σ_i, there is some realization σ_i' of signal i which induces the agent to switch to action B. Hence signal i has strictly positive value: $V_i - V_\emptyset > 0$.

Now suppose that, contrary to the assertion, $q_\pi(s \mid \sigma_j) > \pi(s)$ implied $q_\pi(s \mid \sigma_j, \sigma_i') \geq \pi(s)$ for all σ_i' such that (σ_j, σ_i') is observed with positive probability, $q_\pi(s \mid \sigma_j) < \pi(s)$ implied $q_\pi(s \mid \sigma_j, \sigma_i') \leq \pi(s)$ for all σ_i' such that (σ_j, σ_i') is observed with positive probability, and $q_\pi(s \mid \sigma_j) = \pi(s)$ implied $q_\pi(s \mid \sigma_j, \sigma_i') = \pi(s)$ for all σ_i' such that (σ_j, σ_i') is observed with positive probability. Then no realization σ_i that the agent observes in addition to any realization σ_j induces a strict preference for changing actions. Therefore, signal i has no additional value when signal j is available: $V_{1,2} - V_j = 0$. But this, together with $V_i - V_\emptyset > 0$, implies that signals are not complements. $\square$

We now explore examples in which signals are complements, yet there is no reversal of the meaning of the signals, i.e. cases in which the third bullet point in Proposition 4 is satisfied. We distinguish two cases. Firstly, it may be that the third bullet point in Proposition 4 holds for all signal realizations σ_j and hence signal j by itself is uninformative. It turns out that in this case signals are always complements.

	H	L
α	rp	$(1-r)q$
β	$r(1-p)$	$(1-r)(1-q)$

State a

	H	L
α	$r(1-p)$	$(1-r)(1-q)$
β	rp	$(1-r)q$

State b

Figure 2: Conditional Signal Distributions in Example 3

Proposition 5. *If there is a signal j such that the marginal distribution of signal j does not depend on the state, i.e. such that*

$$p_{j,s} = p_{j,s'} \text{ for all } s, s' \in S$$

then signals are complements.

Proof. Because signal j is uninformative we have: $V_j = V_\emptyset$ in all decision problems. Because $V_{1,2} - V_i \geq 0$ in all decision problems, the condition defining complements is satisfied. □

One trivial case in which Proposition 5 applies is, of course, the case in which signal j is of no value when combined with signal i. However, there are other cases, as the following example shows.

Example 3. *There are two states: $S = \{a, b\}$. Signal 1 has two realizations: $\{\alpha, \beta\}$, and signal 2 has two realizations: $\{H, L\}$. Let $1/2 \leq q \leq p \leq 1$ and let $r \in [0, 1]$. The joint distributions of the signals conditional on the state are given in Figure 2.*

In Example 3 signal 2 indicates the precision of signal 1, but does not in itself contain information about the true state. Signal 1's precision is high (H) with probability r, and low with probability $1 - r$. When signal 1's precision is high, it indicates the true state with probability p. If signal 2's precision is low, it indicates the true state with probability q.

Because signal 2 is independent of the true state, its value by itself is zero: $V_2 - V_\emptyset = 0$. Moreover, it will always be true that $V_{1,2} - V_1 \geq 0$. Therefore, signals 1 and 2 will be weak complements. For some decision problems, however, signal 1 will be useful only if it shifts player 1's prior sufficiently strongly, and this will only be possible if the decision maker knows signal 1's precision to be high. In such decision problems: $V_{1,2} - V_1 > 0$. In such problems, signals 1 and 2 will be strict complements.

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	p
β	0	0	0
γ	$1-p$	0	0

State a

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
α	0	0	0
β	0	0	p
γ	0	$1-p$	0

State b

Figure 3: Conditional Signal Distributions in Example 4

The second case of complementary signals with no signal reversion is the case in which the third bullet point in Proposition 4 is satisfied for some, but not all signal realizations σ_j. In this case, signal j by itself does potentially contain useful information. We now give an example of this case.

Example 4. *There are two states: $S = \{a, b\}$. Signal 1 has three realizations: $\{\alpha, \beta, \gamma\}$, and signal 2 has three realizations: $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$. Let $p \in (0, 1)$. The joint distributions of the signals conditional on the state are given in Figure 3.*

Suppose the decision maker in Example 4 observes only the realization of signal σ_1. If the decision maker observes that this realization is α she is certain that the state is a. Similarly, if she observes that the realization of signal σ_1 is β, then she is certain that the state is b. These two signal realizations are completely revealing about the state. However, if the decision maker observes signal realization γ, then her prior about the state is unchanged. Analogous statements hold for $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$. Finally, considering the joint distribution of the signals, note that each signal is informative if and only if the other signal is uninformative. As a consequence, the marginal value of each signal is independent of whether the other signal is acquired, and both signals are complements. The condition in Proposition 4 is satisfied although the meaning of no signal is reversed, because each signal has realizations that leave the prior unchanged yet the realization of the other signal may move the prior, and thus the third bullet point in Proposition 4 holds.

Signals in Example 4 are also substitutes. The conditions of Proposition 3 are satisfied because each signal has *some* realization which indicates the true state with certainty. Thus, no realization of the other signal can raise the decision maker's probability of that state, nor can any realization of the other signal lower the decision maker's probability of the other state. Nevertheless, ex ante, each signal has positive marginal value, even if the other signal has been acquired, because each signal has *some* realization which does not indicate the true state with certainty. As part (i) of Proposition 2 showed, in the symmetric binary example if signals are substitutes each signal has zero marginal value if the other signal has been acquired. Example 4 demonstrates how, in a more general setting, signals can be substitutes without being perfectly correlated.

6 Economic Applications

In this section, we explore implications of substitutability and complementarity in two simple examples. When studying examples, we somewhat change the perspective of the previous sections, as in each example we deal with a specific set of decision problems rather than with all decision problems.

The first example considers the effects of substitutability and complementarity in a game of strategic information acquisition. The second example considers the effects of substitutability and complementarity on bidding behavior in a common value second price auction.

6.1 Strategic information acquisition

Consider two players $i = 1, 2$ each of whom faces the same (non-strategic) decision problem. Before making the decision, the players decide independently and simultaneously whether or not to acquire signal $\tilde{\sigma}_i$ at cost $c_i \geq 0$, and then players meet and share the information acquired. One may think of a decision that involves some expert's advice, such as health treatment or financial investments. One player can acquire the expert's advice itself and the other player can acquire information about the expert's credibility.

Suppose c_i is player i's private information, and let c_1 and c_2 be independently and uniformly distributed on $[\underline{c}, \underline{c} + 1]$. We are interested in comparative statics properties of the equilibrium of the information acquisition game with respect to $\underline{c}$. It turns out that these properties depend crucially on whether signals are complements or substitutes.

For simplicity and to permit graphical illustrations, we focus on the symmetric case in which the value of information in the players' (specific) decision problem depends only on the number but not on the identity of the signals observed. Let u_k be the value of having $k \in \{0, 1, 2\}$ observations available. Signals are then complements (resp. substitutes) if and only if $u_2 - u_1 \geq u_1 - u_0$ (resp. $u_2 - u_1 \leq u_1 - u_0$).

We focus on symmetric equilibria in which each player i acquires information if his cost type c_i is smaller than a threshold $\hat{c} \in [\underline{c}, \underline{c}+1]$. Let $\theta = \hat{c} - \underline{c}$ be the ex ante probability with which a player acquires information under such a threshold strategy. With abuse of notation, we refer to the equilibrium information acquisition probability θ^* itself as an equilibrium.

Given a player acquires information with ex ante probability θ, the other player's expected gain from acquiring information if his cost type is c is given as

$$\Delta(\theta, c) \equiv \theta(u_2 - u_1) + (1 - \theta)(u_1 - u_0) - c.$$

Observe that Δ is increasing in θ if signals are complements and decreasing in θ if signals are substitutes. This means that the information acquisition game displays strategic complements (resp. substitutes) if signals are complements (resp. substitutes). Moreover, observe that Δ is decreasing in c. This implies that an interior equilibrium θ^* is given by the mass of cost types below the cost type who is just indifferent, given θ^*, i.e. θ^* solves $\Delta^*(\theta^*) = 0$ where

$$\Delta^*(\theta) \equiv \Delta(\theta, \underline{c} + \theta) = \theta[(u_2 - u_1) - (u_1 - u_0) - 1] - \underline{c}.$$

Furthermore, if $\Delta^*(1) \geq 0$, then even the highest cost type $\underline{c}+1$ has a positive information acquisition incentive given the other player acquires information with probability 1, hence $\theta^* = 1$ is then an equilibrium. Likewise, if $\Delta^*(0) \leq 0$, then $\theta^* = 0$ is an equilibrium.

Next, we illustrate these observations graphically. Note that the slope of Δ^* is positive if and only if there are "strong" complementarities in the sense that the additional value of the second signal exceeds the value of the first signal by more than 1. Figure 4 displays the functions $\Delta^*, \Delta(\cdot, \underline{c}), \Delta(\cdot, \underline{c}+1)$ if signals are substitutes (left panel) and strong complements (right panel). Equilibria are indicated by circles.

The left panel exhibits a unique equilibrium. The highest cost type $\underline{c}+1$ would not acquire a signal even if there was no possibility to observe the other signal for free. Substitutability implies that the other player's presence only lowers this type's information acquisition incentive. Conversely, the lowest cost type $\underline{c}$ would acquire information even if the other signal were free. Thus there is some but not full information acquisition in equilibrium.

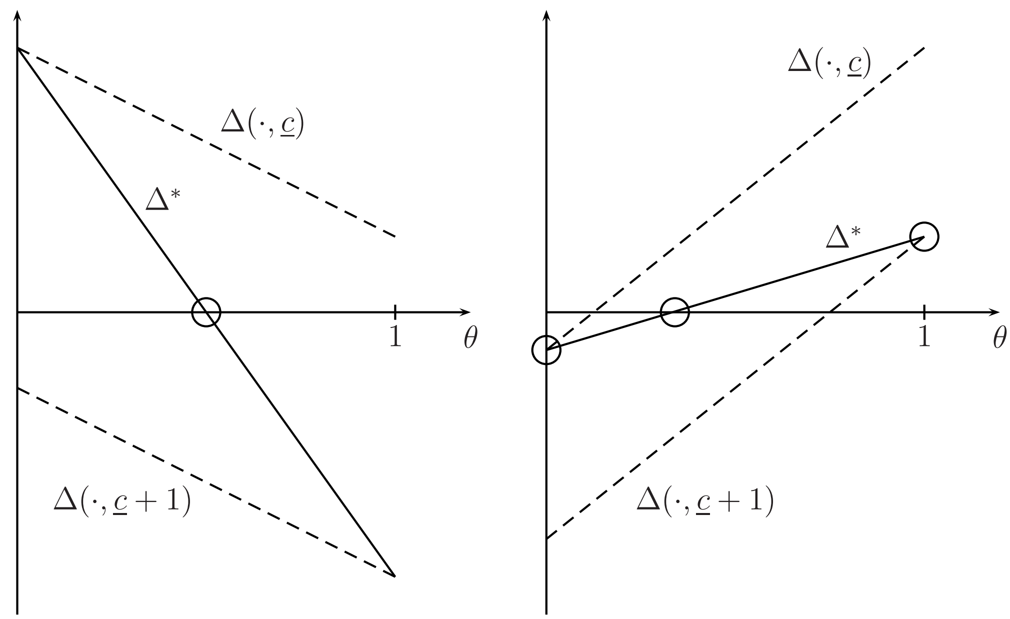

Figure 4: Incentives to acquire information when signals are substitutes (left) and complements (right)

The right panel features multiple equilibria. Here, even the lowest cost type would not acquire information in the absence of the other player. Thus, if each player believes the other player to abstain from information acquisition, there will be no information acquisition in equilibrium. Due to (strong) complementarity however, a player's information acquisition is stimulated if he believes that the other player acquires information. Thus, everyone acquiring information is an equilibrium. Note also that the interior equilibrium is unstable in the sense that a small upward (downward) perturbation of a player's equilibrium information acquisition probability would raise (lower) the other player's information acquisition probability, which, in turn, would raise (lower) the first player's information acquisition probability etc.

Note that the case with "weak" complementarity is a case in–between the depicted cases. Here the functions $\Delta(\cdot, \underline{c}), \Delta(\cdot, \underline{c}+1)$ are increasing in θ, while Δ^* is decreasing in θ. In sum, when signals are substitutes or weak complements, we obtain a unique symmetric equilibrium, while with strong complements, we obtain multiple symmetric equilibria.

In Figure 5, we plot the equilibrium information acquisition probability

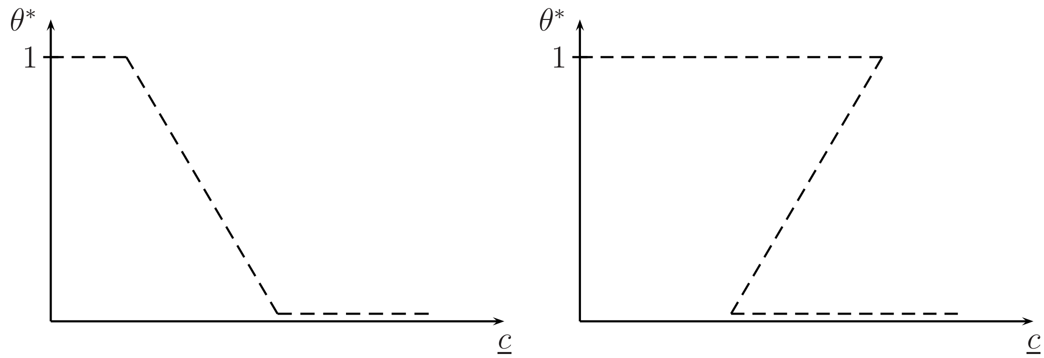

Figure 5: Equilibrium information acquisition probability when signals are substitutes or weak complements (left) and strong complements (right)

as $\underline{c}$ increases. This amounts to shifting up Δ^* in the graphs above.

The left panel displays the case when Δ^* is falling. In this case, we have a smooth downward sloping "demand" for information. As costs $\underline{c}$ go up, the likelihood of acquiring information goes down smoothly. This differs markedly from the right panel where signals are strong complements. Here, small changes in costs might lead to a dramatic drop or rise in information acquisition. Note also that it appears as if along the interior equilibrium, demand for information could go up as costs increase. Recall, however, that this is an unstable equilibrium path so that comparative statics is not necessarily meaningful.

6.2 A second price auction

In this subsection, we study the implications of complementarity and substitutability of signals if these signals are privately observed by bidders in a second price, common value auction. In particular, we show that complementarity of signals may imply that the second price, common value auction has no pure strategy equilibrium, but that it does have a mixed strategy equilibrium, and in this equilibrium the auctioneer's expected revenue equals the expected value of the object. This contrasts sharply with the properties of the second price auction in the standard setting of Milgrom and Weber (1982a).

We assume that a single indivisible auction is sold through a second price auction to two bidders. Bidders submit their bids simultaneously. All non-negative real numbers are allowed as bids. The highest bidder wins the object. She pays the second highest bid. The second highest bidder wins nothing and pays nothing. Ties are resolved by tossing a fair coin.

We consider the binary, symmetric example of Section 4. We assume that the state $s \in \{a, b\}$ represents the true value of the object and set $a = 0$ and $b = 1$. This value is common to both bidders. Bidder i has von Neumann-Morgenstern utility $s - p$ if she wins and pays a price p, and zero, if she does not win. To make the problem interesting, we assume that $\pi(b) \in (0, 1)$. Before submitting a bid, bidder i privately observes signal $\tilde{\sigma}_i$. As in section 5, we denote by $q_\pi(s|\sigma_i)$ and by $q_\pi(s|\sigma_1, \sigma_2)$, for $q_\pi(\sigma_1, \sigma_2) > 0$, the posterior beliefs that the true state is s if realization σ_i or (σ_1, σ_2) were observed, respectively.

We study the symmetric Bayesian Nash equilibrium of the game. We allow players to randomize, and hence admit behavior strategies. Symmetry here means that bidder 1's behavior strategy when she observes α is equal to bidder 2's behavior strategy when she observes $\hat{\alpha}$, and a similar condition for β and $\hat{\beta}$. Thus, a symmetric strategy is characterized by bidder 1's behavior strategies. Whenever the equilibrium is in pure strategies, we denote by $b_i(\sigma_i)$ bidder i's bid when she observes $\sigma_i \in S_i$.

One motivation for restricting attention to symmetric equilibria is that this rules out equilibria in which one bidder bids very high and the other bidder bids very low. Second price auctions always have equilibria of this type. However, they are of less interest than symmetric equilibria.

By Proposition 2, if signals are substitutes, they are perfectly correlated. Thus, bidders' conditional expected values of the good are common knowledge among the bidders, and they are identical. It is immediate that both bidders bidding the correct conditional expected value is then a symmetric equilibrium. We omit the proof.

Proposition 6. *If $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are substitutes, then there exists a symmetric equilibrium in pure strategies characterized by: $b_1(\alpha) = q_\pi(b|\alpha, \hat{\alpha})$, and $b_1(\beta) = q_\pi(b|\beta, \hat{\beta})$.*

Note that the equilibrium described in Proposition 6 implies that the price paid to the auctioneer is identical to the true value of the good, and thus fully reveals the bidders' private information. The auctioneer achieves the maximum revenue that is compatible with individual rationality.

Next, we consider the case that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements. If each individual signal is by itself uninformative, as in Example 1, then it is again easy to find a symmetric equilibrium: each bidder bids the ex ante expected value of the good.

Proposition 7. *If signals are complements and individually uninformative, there is a symmetric equilibrium in pure strategies characterized by:* $b_1(\alpha) = b_1(\beta) = \pi(b)$.

We omit the proof of Proposition 7. It maybe worthwhile to compare the strategies in Proposition 7 to the equilibrium strategies in a two bidder second price, common value auction that have been described in Milgrom and Weber (1982a). According to these strategies, in a symmetric, affiliated signals set-up, each bidder i bids the value that the good would have if the other bidder had the same signal as bidder i. Each bidder's bid thus reveals that bidder's signal. The winning bidder has a positive surplus, and the auctioneer's expected revenue is less than the expected true value of the good. By contrast, in the equilibrium in Proposition 7, the price does not reveal bidders' signals. Yet, the auctioneer's revenue equals the expected value of the good, and is the maximum expected revenue compatible with individual rationality.

It remains to consider the case in which signals are complements and individually informative. We assume that $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ has full support conditional on each state. Moreover, we only consider Cases 3 and 5 of Figure 1. When (C2) of Proposition 2 holds, then generically Case 3 applies, and when (C3) of Proposition 2 holds, then generically Case 5 applies. Finally, we assume that each signal is informative by itself: $p_a(\alpha) \neq p_b(\alpha)$.

Proposition 8. *If* $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ *has full support, both signals are informative by themselves, and Cases 3 or 5 of Figure 1 apply, there is no symmetric equilibrium in pure strategies.*

Note the contrast between Proposition 8 and the case discussed by Milgrom and Weber (1982a). In Milgrom and Weber's (1982a) symmetric model with affiliated signals the second price auction always has a symmetric Bayesian equilibrium in pure strategies. Intuitively, the reason why a similar existence result does not apply in Cases 3 and 5 in our model is that in those cases the signals are not well-ordered. Whether α or β indicates a high or a low value depends on the realization of the signal of the other bidder. By contrast, in Milgrom and Weber (1982a), signals are well-ordered. Their meaning is independent of the other agents' signals.

Proof. Any symmetric equilibrium in pure strategies would have to satisfy the following condition: if bidder 1 bids p when she observes σ_1, the expected value of the good conditional on $\tilde{\sigma_1} = \sigma_1$ and conditional on bidder 2 bidding p when she uses the symmetric bid function must be equal to p. To see why, note that otherwise, bidder 1 would have incentives to deviate. For instance, suppose that the above conditional expected value is less than p. Then, bidder 1 gets negative expected utility when she observes σ_1, bidder 2 bids p, both tie and bidder 1 wins. Thus, bidder 1 can improve by bidding $p-\varepsilon$ when she observes σ_1 for $\varepsilon > 0$ and ε small enough. This deviation only affects bidder 1's payoffs when bidder 1 ties in the above situation. A similar argument applies when bidder 1 obtains positive expected utility conditional on the event that she observes σ_1, bidder 2 bids p, both tie and bidder 1 wins.

Next, note that in a symmetric equilibrium in pure strategies in which $b_1(\alpha) = b_1(\beta)$, the event that bidder 2 submits the same p does not provide information about her private signal. Thus, our necessary condition becomes $b_1(\sigma_1) = q_\pi(b|\sigma_1)$, for $\sigma_1 \in \{\alpha, \beta\}$, which together with $b_1(\alpha) = b_1(\beta)$ gives us a contradiction with our assumption that the signals are individually informative.

Consider next a symmetric equilibrium in pure strategies in which $b_1(\alpha) \neq b_1(\beta)$. Both bidders submit the same bid only if they observe the same signal realization. Thus, our necessary condition becomes $b_1(\alpha) = q_\pi(b|\alpha, \hat{\alpha})$ and $b_1(\beta) = q_\pi(b|\beta, \hat{\beta})$. We show next that if the bid function satisfies this condition, it must also satisfy that:

$$q_\pi(b|\beta, \hat{\alpha}) \in [\min\{b_1(\alpha), b_1(\beta)\}, \max\{b_1(\alpha), b_1(\beta)\}].$$

To see why, suppose, for instance, that $q_\pi(b|\beta, \hat{\alpha}) < \min\{b_1(\alpha), b_1(\beta)\}$. Then bidder 1 obtains zero expected utility if she and bidder 2 submits the higher bid. However, bidder 1 gets negative expected utility if she submits her higher bid and bidder 2 her lower bid. This is because the expected value of the good conditional on this event is equal to $q_\pi(b|\beta, \hat{\alpha})$, which by our starting assumption is less than bidder 2's bid. As a consequence, bidder 1 has an incentive to deviate and submit a bid that loses with probability one. The argument for $q_\pi(b|\beta,\hat{\alpha}) > \max\{b_1(\alpha), b_1(\beta)\}$ is similar. In this case, bidder 1 has an incentive to replace her lower bid by a bid that allows her to win with probability one.

But now observe that the condition that we have derived in the previous paragraph is incompatible with complementary signals. In Case 3 of Figure 1 we have that $q_\pi(b|\beta,\hat{\alpha}) < \max\{b_1(\alpha), b_1(\beta)\}$, and in Case 5 of Figure 1 we have that $q_\pi(b|\beta,\hat{\alpha}) > \max\{b_1(\alpha), b_1(\beta)\}$. Thus, there cannot be a symmetric equilibrium in pure strategies. □

While the second price auction does not have symmetric Bayesian equilibria in pure strategies when signals are complements, it does have symmetric Bayesian equilibria in mixed strategies in this case, as our next result shows. The proof of this result is in the Appendix. While it is not surprising that a symmetric Bayesian equilibrium in mixed strategies exists, it is surprising that in Case 3 the second price auction extracts all the bidders' surplus as revenue for the auctioneer. In Case 5, by contrast, the bidders have positive expected surplus.

Proposition 9. *If $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ has full support, both signals are informative by themselves, and Cases 3 or 5 of Figure 1 apply, there is a symmetric equilibrium in behavior strategies with the following properties:*

(i) When Case 3 applies:

- *Bidder 1 bids p^* with probability one, if she observes α;*
- *Bidder 1 bids p^* with probability k^* and $q_\pi(b|\beta,\hat{\beta})$ with probability $1-k^*$, if she observes β;*

for a $p^* \in (q_\pi(b|\alpha,\hat{\beta}), \min\{q_\pi(b|\alpha,\hat{\alpha}), q_\pi(b|\beta,\hat{\beta})\})$ and $k^* \in (0,1)$. In this equilibrium, each bidder's ex ante expected utility equals zero, and the auctioneer's ex ante expected revenue equals the unconditional expected value of $\tilde{s}$, $E[\tilde{s}]$.

(ii) When Case 5 applies:

- *Bidder 1 bids $q_\pi(b|\alpha,\hat{\alpha})$ with probability $\frac{1}{k^*}$ and p^* with probability $1-\frac{1}{k^*}$, if she observes α;*

– *Bidder 1 bids p^* with probability one, if she observes β;*

for a $p^* \in (\min\{q_\pi(b|\alpha, \hat{\alpha}), q_\pi(b|\beta, \hat{\beta})\}, q_\pi(b|\alpha, \hat{\beta}))$ and $\frac{1}{k^*} \in (0, 1)$. In this equilibrium, each bidder's ex ante expected utility is strictly greater than zero, and the auctioneer's ex ante expected revenue is strictly less than the unconditional expected value of $\tilde{s}$, $E[\tilde{s}]$.

It is easy to see the technical reason why the equilibrium in Proposition 9 must have the property that bidders' expected surplus is zero in Case 3. In Case 3, for low and for high signals, bidders make with positive probability the lowest bid that any bidder makes in the auction. Their expected utility must therefore equal the expected utility of this lowest bid. This bid can only win when when it ties. Its expected utility must be zero because otherwise a bidder would have an incentive to change her bid slightly to break the tie.

In a monotone equilibrium in which high signal bidders submit higher bids than low signal bidders, high signal bidders can obtain positive surplus only when their bid, and therefore their signal, is strictly larger than that of low signal bidders. This means that, ex post, the news is "mixed." In our model, in Case 3, mixed news gives the lowest conditional expected value possible. In contrast to Case 3, in Case 5 mixed news gives the highest conditional expected value possible. This may be the intuitive reason why in Case 3 the high signal bidder does not obtain strictly positive expected utility, whereas in Case 5 she does.

Note that we have not proved that there is a unique symmetric equilibrium. We conjecture that the equilibria are unique.

7 Conclusion

This paper has provided some insights into the nature of complementarity and substitutability relations among signals, but many questions remain open. Firstly, we have not been able to determine to which extent the intuitive insights of Section 4 generalize. Our results in Section 5 are only partial. Secondly, whereas in this paper we have sought characterizations that imply substitutability or complementarity in *all* decision problems, one might restrict attention to smaller classes of decision problems as they typically arise in economics, for example to monotone decision problems as in Athey and Levin (2001).

Complementarity and substitutability relations among signals may matter in economic contexts where agents hold private signals, and each agents' preferences depend on all signals, that is, in contexts with interdependent preferences. Such contexts arise naturally in auctions or in voting games. It seems worthwhile to explore the implications of complementarity and substitutability in those contexts.

Complementarity of signals may also matter when agents acquire signals sequentially. In this case, the second signal may be acquired when the agent already knows the realization of the first signal. By contrast, in our setting, each signal is acquired without knowing the realization of the other signal. Extending our results to a setting where agents evaluate signals knowing the realization of other signals is another project for future work.

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Appendix

Proof of Proposition 2

We begin by introducing additional notation. Throughout we shall assume that the probability distributions p_a and p_b are given and fixed. For given prior distribution π, for $i \in \{1, 2\}$ and $\sigma_i \in S_i$, we denote by $q_\pi(\sigma_i)$ the prior probability of observing σ_i, i.e. $q_\pi(\sigma_i) = \pi(a)p_{i,a}(\sigma_i) + \pi(b)p_{i,b}(\sigma_i)$. For given prior probability distribution π, and for $i \in \{1, 2\}$ and $\sigma_i \in S_i$ we denote by $q_\pi(s \mid \sigma_i)$ the conditional probability of state s if signal realization σ_i is observed. Note that our assumption that for every signal realization there is a state in which this signal realization is observed with positive probability implies that these conditional probabilities are all well-defined. Finally, consider a joint realization of both signals, $(\sigma_1, \sigma_2) \in S_1 \times S_2$. If there is a state in which this joint realization is observed with positive probability, then we write $q_\pi(s \mid \sigma_1, \sigma_2)$ for the conditional probability of state s if the prior is π, and the joint signal realization (σ_1, σ_2) is observed.

Part (i): As mentioned in the main text, the "if-part" of part (i) of Proposition 2 is obvious. Therefore, we focus on the "only if-part". We proceed in two steps.

Step 1: We show that substitutability of the two signals implies that there is a $\lambda \geq 0$ such that:

$$(p_b(\alpha, \hat{\alpha}), p_b(\alpha, \hat{\beta})) \quad = \quad \lambda(p_a(\alpha, \hat{\alpha}), p_a(\alpha, \hat{\beta})). \tag{1}$$

Intuitively, this equality says that when the decision maker receives signal $\sigma_1 = \alpha$, then his or her beliefs will be the same when the second signal is $\sigma_2 = \hat{\alpha}$ and when it is $\sigma_2 = \hat{\beta}$. Before we proceed to the proof, we note that if (1) is true, symmetry implies:

$$(p_b(\alpha, \hat{\alpha}), p_b(\beta, \hat{\alpha})) \quad = \quad \lambda(p_a(\alpha, \hat{\alpha}), p_a(\beta, \hat{\alpha})). \tag{2}$$

Moreover, arguments analogous to those that we use below to show (1) also prove that there is a $\gamma \geq 0$ such that:

$$\begin{aligned} (p_b(\beta, \hat{\alpha}), p_b(\beta, \hat{\beta})) &= \gamma(p_a(\beta, \hat{\alpha}), p_a(\beta, \hat{\beta})); && (3) \\ (p_b(\alpha, \hat{\beta}), p_b(\beta, \hat{\beta})) &= \gamma(p_a(\alpha, \hat{\beta}), p_a(\beta, \hat{\beta})). && (4) \end{aligned}$$

To prove (1) we now distinguish two cases.

Case 1: Suppose that there is only one $\sigma_2 \in S_2$ such that $p_a(\alpha, \sigma_2) > 0$. Our proof of the assertion is indirect. The condition which we seek to prove can only be violated if $p_b(\alpha, \sigma_2') > 0$ for $\sigma_2' \neq \sigma_2$. Consider the prior π given by: $\pi(a) = \pi(b) = 0.5$. The posterior probability of state a after receiving signal α is less than the posterior probability of state a after receiving signal (α, σ_2):

$$q_\pi(a \mid \alpha) < q_\pi(a \mid \alpha, \sigma_2).$$

Pick some number x strictly between these two posterior probabilities, and consider the following decision problem. The set of actions is: $A = \{T, B\}$. The payoff function u is defined in Figure A1.

	a	b
T	$1-x$	0
B	0	x

Figure A1: A payoff function

Observe that the decision maker will choose action T if and only if his posterior probability of state a is at least x. Now suppose $x > 0.5$. Then, without any information, the decision maker will choose B. Now suppose the decision maker had only signal $\tilde{\sigma}_1$ available. No realization of signal 1 can convince the decision maker to switch to action T. Indeed, if $\tilde{\sigma}_1 = \beta$, then the decision maker revises his prior probability of state a (weakly) downwards. Thus, he will choose B. On the other hand, if $\tilde{\sigma}_1 = \alpha$, then by construction of x the posterior probability of a is below x. Again, the decision maker will choose B. Because no realization of $\tilde{\sigma}_1$ can convince the decision maker to change his choice in comparison to the case in which he receives no signal, we have: $V_1 - V_\emptyset = 0$.

On the other hand, if the decision maker receives two signals, then for some realizations of the signal he changes his action. Specifically, if the decision maker observes (α, σ_2), then the probability which he attaches to a rises above x, and he will choose T. Therefore, with positive probability, two signals together will have a realization that induces the decision maker to change his action. This implies: $V_{1,2} - V_\emptyset > 0$. Because: $V_{1,2} - V_1 > V_1 - V_\emptyset$, we have obtained a contradiction to substitutability.

Case 2: We have $p_a(\alpha,\hat{\alpha}) > 0$ and $p_a(\alpha,\hat{\beta}) > 0$. We prove our assertion indirectly. Our assertion is violated if and only if:

$$\frac{p_b(\alpha,\hat{\alpha})}{p_a(\alpha,\hat{\alpha})} \neq \frac{p_b(\alpha,\hat{\beta})}{p_a(\alpha,\hat{\beta})}.$$

Suppose the left hand side were smaller than the right hand side. Then signal $(\alpha,\hat{\alpha})$ raises the posterior probability of state a to a higher level than signal $\sigma_1 = \alpha$ alone. Thus, we can repeat the construction displayed in Case 1 and obtain a contradiction to substitutability. The same argument works if the right hand side is smaller than the left hand side.

Step 2: We now deduce from equations (1)-(4) that Condition (C1) in Proposition 2 must hold. Suppose that $p_a(\alpha,\hat{\beta}) > 0$. Then we would have $\lambda = \gamma$ because by (1):

$$p_b(\alpha,\hat{\beta}) = \lambda p_a(\alpha,\hat{\beta}),$$

and by (4):

$$p_b(\alpha,\hat{\beta}) = \gamma p_a(\alpha,\hat{\beta}).$$

Thus, we would conclude from (1)-(4) that $p_a(\sigma_1,\sigma_2) = p_b(\sigma_1,\sigma_2)$ for all $(\sigma_1,\sigma_2) \in S_1 \times S_2$. But this contradicts our assumption that at least some signal realization is informative. We can thus conclude that $p_a(\alpha,\hat{\beta}) = 0$. By symmetry: $p_a(\beta,\hat{\alpha}) = 0$. By equations (1) and (2) this implies: $p_b(\alpha,\hat{\beta}) = p_b(\beta,\hat{\alpha}) = 0$.

Part (ii): We begin by showing that the assertion is rather trivially true if $p_s(\alpha,\hat{\beta}) = p_s(\beta,\hat{\alpha}) = 0$ for both $s \in \{a,b\}$, i.e. condition (C1) holds. The trivial way in which the assertion is true is that if this condition is true, signals cannot be complements, nor can conditions (C2) or condition (C3) be satisfied.

Lemma 2. *Suppose $p_s(\alpha,\hat{\beta}) = p_s(\beta,\hat{\alpha}) = 0$ for $s \in \{a,b\}$. Then the signals are not complements. Moreover, neither condition (C2) nor condition (C3) holds.*

Proof. Suppose $p_s(\alpha,\hat{\beta}) = p_s(\beta,\hat{\alpha}) = 0$ for $s \in \{a,b\}$. Then signals are perfectly correlated. As explained in the main text, this implies that the value of a second signal, if one signal has already been observed, is zero. Moreover, because by assumption at least some signal realization is informative, the value of one signal is in some decision problems positive. Therefore, the signals cannot be complements.

Now suppose either (C2) or (C3) were true. By assumption $p_{i,a}(\alpha) \geq p_{i,b}(\alpha)$ and $p_{i,a}(\beta) \leq p_{i,b}(\beta)$. Because at least one of (C2) and (C3) holds, one of these inequalities has to hold with equality. Because probabilities add up to one, both inequalities then hold with equality. This implies that all signal realizations occur with the same probability in both states, and hence that all signals are uninformative. We have ruled this out by assumption. Thus we have obtained a contradiction. $\square$

Lemmas 2 implies that it is sufficient to prove part (ii) of Proposition 2 for the case that there is at least one state s such that $p_s(\alpha, \hat{\beta}) > 0$. We shall make this assumption from now on without further mentioning.

In Lemmas 3 and 4 below we show that only a restricted class of decision problems needs to be considered when proving complementarity of signals.

Lemma 3. *In the symmetric, binary example, two signals are complements if and only if they are complements for all decision problems with two actions.*

Proof. The "only if"-part of Lemma 3 is obvious. We only prove the "if-part". Thus, suppose that two signals are complements in all decision problems with two actions only, and consider any arbitrary decision problem (π, A, u). We aim to show that: $V_{1,2} - V_1 \geq V_2 - V_\emptyset$. Note that this is sufficient, because symmetry then implies that $V_{1,2} - V_2 \geq V_1 - V_\emptyset$ is also true.

Starting from the given decision problem we construct a new decision problem $(\bar{\pi}, \bar{A}, \bar{u})$ as follows. Let $\bar{\pi} = \pi$. Define $\bar{A}$ to be a subset of A that consists of two actions, and that includes one action that is optimal in (π, A, u) if signal $\sigma_1 = \alpha$ is received, and one action that is optimal in (π, A, u) if signal $\sigma_1 = \beta$ is received. Finally, define $\bar{u}$ to be the restriction of u to domain $\bar{A} \times S$.

Denote the signal values in the new decision problem by $\overline{V}_\emptyset, \overline{V}_1, \overline{V}_2$, and $\overline{V}_{1,2}$. By assumption the two signals are complements in all two action decision problems. Therefore:

$$\overline{V}_{1,2} - \overline{V}_1 \geq \overline{V}_2 - \overline{V}_\emptyset. \tag{5}$$

Observe that by construction:

$$\overline{V}_\emptyset \leq V_\emptyset,\ \overline{V}_1 = V_1,\ \overline{V}_2 = V_2,\ \text{and}\ \overline{V}_{1,2} \leq V_{1,2}. \tag{6}$$

The two inequalities in (6) are true because the decision problem $(\bar{\pi}, \bar{A}, \bar{u})$ differs from the decision problem (π, A, u) only in that fewer actions are available. Therefore, values can only fall, but not rise. The two equalities in (6) are true because the actions in A that are optimal for the two signal realizations α and β (or, symmetrically, $\hat{\alpha}$ and $\hat{\beta}$) are also contained in $\bar{A}$. Equations (5) and (6) imply:

$$V_{1,2} - V_1 \geq V_2 - V_\emptyset.$$

which is what we sought to prove. $\square$

Lemma 4. *In the symmetric, binary example, two signals are complements if and only if they are complements in decision problems with two actions where $A = \{T, B\}$, and where the utility function u is given by Figure A2 with $x \in (0, 1)$.*

	a	b
T	$1-x$	*0*
B	*0*	x

Figure A2: A payoff function

Proof. The "only if-part" of Lemma 4 is obvious. We prove the "if-part" only. Suppose that the complementarity condition is satisfied in all decision problems of the sort described in Lemma 4. We want to prove that it is then satisfied in all decision problems. By Lemma 3 it is sufficient to consider decision problems where the action set A has only two elements. Without loss of generality we set: $A = \{T, B\}$.

We first note that we need not consider decision problems in which either $u(T, s) \geq u(B, s)$ for both $s \in S$, or the reverse holds for both $s \in S$. In such decision problems differences between the values of different signals are always zero because an optimal action that is independent of the state can be found. Thus, the complementarity conditions are trivially satisfied. In our proof we shall now focus on the case in which $u(T, a) > u(B, a)$ and $u(B, b) > u(T, b)$. The reverse case can be treated similarly.

By Lemma 1 it is sufficient to show that the signal $\tilde{\sigma}_R$ is more valuable than the signal $\tilde{\sigma}_L$. This is the case if and only if:

$$\begin{aligned}
\sum_{\sigma_R \in S_R} \max\{ & u(T,a)p_{R,a}(\sigma_R)\pi(a) + u(T,b)p_{R,b}(\sigma_R)\pi(b), \\
& u(B,a)p_{R,a}(\sigma_R)\pi(a) + u(B,b)p_{R,b}(\sigma_R)\pi(b)\} \quad \geq \\
\sum_{\sigma_L \in S_L} \max\{ & u(T,a)p_{L,a}(\sigma_L)\pi(a) + u(T,b)p_{L,b}(\sigma_L)\pi(b), \\
& u(B,a)p_{L,a}(\sigma_L)\pi(a) + u(B,b)p_{L,b}(\sigma_L)\pi(b)\}.
\end{aligned}$$

Now we subtract $u(B,a)\pi(a) + u(T,b)\pi(b)$ on both sides of this inequality and re-arrange terms to obtain:

$$\begin{aligned}
\sum_{\sigma_R \in S_R} \max\{ & \pi(a)(u(T,a) - u(B,a))p_{R,a}(\sigma_R), \\
& \pi(b)(u(B,b) - u(T,b))p_{R,b}(\sigma_R)\} \quad \geq \\
\sum_{\sigma_L \in S_L} \max\{ & \pi(a)(u(T,a) - u(B,a))p_{L,a}(\sigma_L), \\
& \pi(b)(u(B,b) - u(T,b))p_{L,b}(\sigma_L)\}. \qquad (7)
\end{aligned}$$

Finally, we divide both sides of this equation by $(u(T,a) - u(B,a)) + (u(B,b) - u(T,b))$. By our assumptions, this expression is positive. To simplify notation we define:

$$\hat{x} \equiv \frac{u(T,a) - u(B,a)}{(u(T,a) - u(B,a)) + (u(B,b) - u(T,b))}.$$

Equation (7) is then equivalent to:

$$\begin{aligned}
\sum_{\sigma_R \in S_R} \max\{\pi(a)\hat{x}p_{R,a}(\sigma_R), \pi(b)(1-\hat{x})p_{R,b}(\sigma_R)\} \quad \geq \\
\sum_{\sigma_L \in S_L} \max\{\pi(a)\hat{x}p_{L,a}(\sigma_L), \pi(b)(1-\hat{x})p_{L,b}(\sigma_L)\}. \qquad (8)
\end{aligned}$$

Equation (8) is the condition under which signal $\tilde{\sigma}_R$ is more valuable than signal $\tilde{\sigma}_L$ in a decision problem of the sort described in Lemma 4 if the parameter x equals $\hat{x}$. Because we have assumed that the two signals are complementary in all such decision problems, we can deduce from Lemma 1 that inequality (8) holds. □

Next we provide a formula for value differences in decision problems of the type described in Lemma 4. For this, we introduce some additional notation and terminology. Consider a decision problem of the type described in Lemma 4. Denote by $d_\emptyset \in \{T, B\}$ an action that maximizes expected utility if no signal is available. If both actions are optimal, then we define $d_\emptyset$ to be T. For any $\sigma_1 \in S_1$ we denote by $d_1(\sigma_1) \in \{T, B\}$ an action that maximizes expected utility if only signal $\tilde{\sigma}_1$ is available, and the realization of that signal is σ_1. We apply the same tie breaking rule in favor of T as before. Finally, for any $(\sigma_1, \sigma_2) \in S_1 \times S_2$. we denote by $d_{1,2}(\sigma_1, \sigma_2) \in \{T, B\}$ an action that maximizes expected utility if signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are available, and the realizations of these signals are σ_1 and σ_2. We again use the tie breaking rule in favor of T.

Now consider a $\sigma_1 \in S_1$. We say that σ_1 is *critical* if $q_\pi(\sigma_1) > 0$ and $d_1(\sigma_1) \neq d_\emptyset$. Intuitively, we thus call a realization of $\tilde{\sigma}_1$ critical if it occurs with positive probability, and if it induces the decision maker to take an action that is different from the one chosen without information. Likewise, $(\sigma_1, \sigma_2) \in S_1 \times S_2$ is *critical* if $q_\pi(\sigma_1, \sigma_2) > 0$ and $d_{1,2}(\sigma_1, \sigma_2) \neq d_1(\sigma_1)$. Intuitively, we thus call a realization of $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ critical if it occurs with positive probability, and if it induces the decision maker to take an action that is different from the one chosen if only σ_1 were observed.

The following Lemma shows how value differences can be calculated by focusing on critical signal realizations.

Lemma 5. *Consider the symmetric, binary example, and a decision problem of the type described in Lemma 4. Then:*

$$V_1 - V_\emptyset = \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical}}} \left(q_\pi(\sigma_1) |\ q_\pi(a|\sigma_1) - x| \right),$$

and

$$V_{1,2} - V_1 = \sum_{\substack{(\sigma_1,\sigma_2) \in S_1 \times S_2 \\ (\sigma_1,\sigma_2) \text{ is critical}}} \left(q_\pi(\sigma_1, \sigma_2) |\ q_\pi(a|\sigma_1, \sigma_2) - x| \right).$$

Proof. We only prove the first equation. The second follows by a similar argument. Observe first that by definition:

$$V_1 - V_\emptyset = \sum_{\sigma_1 \in S_1} \sum_{s \in S} \pi(s) p_s(\sigma_1)(u(d_1(\sigma_1), s) - u(d_\emptyset, s)).$$

Clearly, for signal realizations σ_1 that are not critical the term in the sum on the right hand side of this equation is zero. Thus, we can restrict the sum to critical observations:

$$V_1 - V_\emptyset = \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical}}} \sum_{s \in S} \pi(s) p_s(\sigma_1)(u(d_1(\sigma_1), s) - u(d_\emptyset, s)).$$

Taking into account the payoff structure described in Figure 3, we can spell out this sum as follows:

$$\begin{aligned}
V_1 - V_\emptyset = & \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=T}} (\pi(a) p_a(\sigma_1)(1-x) + \pi(b) p_b(\sigma_1)(-x)) \\
& + \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=B}} (\pi(a) p_a(\sigma_1)(-(1-x)) + \pi(b) p_b(\sigma_1) x) \\
= & \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=T}} (\pi(a) p_a(\sigma_1) - q_\pi(\sigma_1) x)) \\
& + \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=B}} (q_\pi(\sigma_1) x - \pi(a) p_a(\sigma_1)) \\
= & \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=T}} q_\pi(\sigma_1) \left(q_\pi(a|\sigma_1) - x\right)) \\
& + \sum_{\substack{\sigma_1 \in S_1 \\ \sigma_1 \text{ is critical} \\ d_1(\sigma_1)=B}} q_\pi(\sigma_1) \left(x - q_\pi(a|\sigma_1)\right).
\end{aligned}$$

It becomes obvious that the last expression can also be written in the form provided in Lemma 5 once one realizes that $d_1(\sigma_1) = T$ implies $q_\pi(a|\sigma_1) \geq x$, and $d_1(\sigma_1) = B$ implies $q_\pi(a|\sigma_1) \leq x$. $\square$

¿From now on we shall focus on the case that $p_a(\alpha, \hat{\beta}) \geq p_b(\alpha, \hat{\beta})$. We shall show that under this assumption, signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements if and only if condition (C2) holds. Identical arguments show that when $p_a(\alpha, \hat{\beta}) \leq p_b(\alpha, \hat{\beta})$, the two signals are complements if and only if condition (C3) holds.

In the argument that follows, it turns out that the crucial case on which we need to focus is the case of decision problems for which the threshold x is between the posterior after receiving signal $\tilde{\sigma}_1 = \beta$, that is $q_\pi(a|\beta)$, and the prior $\pi(a)$. In the following lemma, we calculate value differences for this case.

Lemma 6. *Consider the symmetric, binary example. Consider a decision problem of the type described in Lemma 4, and suppose that $x \in [q_\pi(a|\beta), \pi(a)]$. Then:*

$$V_1 - V_\emptyset = q_\pi(\beta)(x - q_\pi(a|\beta)).$$

Moreover, if the joint signal realization $(\alpha, \hat{\alpha})$ has positive probability in some state, then:

$$V_{1,2} - V_1 = q_\pi(\beta, \hat{\alpha})(q_\pi(a|\beta, \hat{\alpha}) - x) + q_\pi(\alpha, \hat{\alpha})(x - q_\pi(a|\alpha, \hat{\alpha}))^+,$$

where for a real number z, z^+ is defined as z if z is positive and as 0 if z is non-positive. If the joint signal realization $(\alpha, \hat{\alpha})$ has probability zero in both states, then:

$$V_{1,2} - V_1 = q_\pi(\beta, \hat{\alpha})(q_\pi(a|\beta, \hat{\alpha}) - x).$$

Proof. Lemma 6 is a straightforward application of Lemma 5. We therefore provide a proof only for the second of the three equalities in Lemma 6. The other two equalities can be shown using similar arguments. To simplify our arguments, we focus, moreover, on the case that $q_\pi(a|\beta) < \pi(a)$. Thus we rule out the marginal case that $q_\pi(a|\beta) = \pi(a)$. It is easy to extend our argument to that case.

To apply Lemma 5 we need to identify the critical joint signal realizations (σ_1, σ_2). There are four possible joint signal realizations:

(i) Observation $(\alpha, \hat{\alpha})$ is critical if and only if the optimal action conditional on this observation is B, that is, if and only if $q_\pi(a|\alpha, \hat{\alpha}) < x$. This is because observation $(\alpha, \hat{\alpha})$ has in the case that we are considering by assumption positive probability, and because the optimal action is T if only signal α is received. This explains the second term on the right hand side of the equality that we are proving.

(ii) Observation $(\alpha, \hat{\beta})$ is not critical because we assumed earlier that: $p_a(\alpha, \hat{\beta}) \geq p_b(\alpha, \hat{\beta})$. This implies that the decision maker will choose action T if observing $(\alpha, \hat{\beta})$. He would have made the same choice had he had only one observation, namely $\sigma_1 = \alpha$.

(iii) Observation $(\beta, \hat{\alpha})$ is critical. As explained in the previous paragraph, this observation induces the decision maker to choose T, whereas, with observation $\sigma_1 = \beta$ alone, the decision maker would have chosen B. This explains the first term on the right hand side of the equality that we are proving.

(iv) Observation $(\beta, \hat{\beta})$ is not critical, because, if the decision maker makes this observation, he will choose B, the same action that he would have chosen with one observation $\sigma_1 = \beta$ alone. To see that the optimal action following observation $(\beta, \hat{\beta})$ is B, note that our assumption $p_a(\alpha, \hat{\beta}) \geq p_b(\alpha, \hat{\beta})$ implies by symmetry: $p_a(\beta, \hat{\alpha}) \geq p_b(\beta, \hat{\alpha})$. On the other hand, by assumption: $p_{1,a}(\beta) \leq p_{1,b}(\alpha)$. Thus, it follows that: $p_a(\beta, \hat{\beta}) \leq p_b(\beta, \hat{\beta})$, and hence that the optimal choice following observation $(\beta, \hat{\beta})$ is B.

The equality that we have to prove is now an immediate implication of Lemma 5. □

We can now conclude our proof with the following Lemma.

Lemma 7. *Signals $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are complements in all decision problems of the type described in Lemma 4 if and only if condition (C2) holds.*

Proof. Consider first the case in which the joint signal realization $(\alpha, \hat{\alpha})$ has positive probability in at least one state. To prove the "only if"-part of the claim, note that if the signals are complements in all decision problems of the type described in Lemma 4, then, in particular, they are complements in those decision problems in which $x = \pi(a)$. Using the formulae in Lemma 6, we find that $V_{1,2} - V_1 \geq V_1 - V_\emptyset$ is equivalent to:

$$\begin{aligned} & q_\pi(\beta, \hat{\alpha})(q_\pi(a|\beta, \hat{\alpha}) - \pi(a)) + q_\pi(\alpha, \hat{\alpha})(\pi(a) - q_\pi(a|\alpha, \hat{\alpha}))^+ \\ \geq \ & q_\pi(\beta)(\pi(a) - q_\pi(\alpha|\beta)) \end{aligned} \tag{9}$$

Because the expected value of the posterior equals the prior, we have:

$$q_\pi(\alpha)(q_\pi(a|\alpha) - \pi(a)) + q_\pi(\beta)(q_\pi(a|\beta) - \pi(a)) = 0,$$

or, equivalently:

$$q_\pi(\alpha)(q_\pi(a|\alpha) - \pi(a)) = q_\pi(\beta)(\pi(a) - q_\pi(a|\beta))$$

Applying on the left hand side of this equation again the result that the expected value of the posterior equals the prior, we obtain:

$$\begin{aligned}
& q_\pi(\alpha,\hat\alpha)(q_\pi(a|\alpha,\hat\alpha)-\pi(a))+q_\pi(\alpha,\hat\beta)(q_\pi(a|\alpha,\hat\beta)-\pi(a)) \\
= \; & q_\pi(\beta)(\pi(a)-q_\pi(\alpha|\beta))
\end{aligned}$$

Because of symmetry, this is equivalent to:

$$\begin{aligned}
& q_\pi(\alpha,\hat\alpha)(q_\pi(a|\alpha,\hat\alpha)-\pi(a))+q_\pi(\beta,\hat\alpha)(q_\pi(a|\beta,\hat\alpha)-\pi(a)) \\
= \; & q_\pi(\beta)(\pi(a)-q_\pi(\alpha|\beta)) \qquad (10)
\end{aligned}$$

Subtracting (10) from (9), we find:

$$\begin{aligned}
q_\pi(\alpha,\hat\alpha)(\pi(a)-q_\pi(a|\alpha,\hat\alpha))^+ &\geq q_\pi(\alpha,\hat\alpha)(q_\pi(a|\alpha,\hat\alpha)-\pi(a)) \Leftrightarrow \\
(\pi(a)-q_\pi(a|\alpha,\hat\alpha))^+ &\geq q_\pi(a|\alpha,\hat\alpha)-\pi(a) \qquad (11)
\end{aligned}$$

But (11) cannot be true if $q_\pi(a|\alpha,\hat\alpha)>\pi(a)$, because then the left hand side of (11) is zero, and the right hand side is strictly positive. We thus must have $q_\pi(a|\alpha,\hat\alpha)\leq\pi(a)$, which is equivalent to: $p_a(\alpha,\hat\alpha)\leq p_b(\alpha,\hat\alpha)$, i.e. condition (C2). This completes the proof of the "only if"-part of Lemma 7.

To prove the "if" part of Lemma 7 we fix a prior, and we prove that $V_{1,2}-V_1\geq V_1-V_\emptyset$ holds for all $x\in[0,1]$. Observe first that for $x\in[0,q_\pi(a|\beta)]$, and for $x\in[q_\pi(a|\alpha),1]$, signal $\tilde\sigma_1$ has no critical realization. In the first case, no realization of $\tilde\sigma_1$ alone can induce the decision maker to adopt B rather than T, and in the latter case, no realization of $\tilde\sigma_1$ alone can induce the decision maker to adopt T rather than B. Hence it follows from Lemma 5 that $V_1-V_\emptyset$ is zero. Thus the assertion is immediate.

Next consider the case that $x\in(q_\pi(a|\beta),\pi(a)]$. From Lemma 6 we know:

$$(V_{1,2}-V_1)-(V_1-V_\emptyset)\geq q_\pi(\alpha,\hat\beta)(q_\pi(a|\alpha,\hat\beta)-x)+q_\pi(\beta)(q_\pi(a|\beta)-x).$$

Since $x\leq\pi(a)$, the difference on the right hand side is not less than:

$$q_\pi(\alpha,\hat\beta)(q_\pi(a|\alpha,\hat\beta)-\pi(a))+q_\pi(\beta)(q_\pi(a|\beta)-\pi(a)),$$

which by equation (12) is equal to:

$$q_\pi(\alpha,\hat\alpha)\left(\pi(a)-q_\pi(a|\alpha,\hat\alpha)\right),$$

which is non-negative under condition (C2). Thus, the assertion is shown for this case.

Consider finally the case that $x \in [\pi(a), q_\pi(a|\alpha)]$. Arguments of the type shown in the proof of Lemma 7 show that in this case, under condition (C2), the set of critical observations is independent of x. Lemma 5 then implies that the difference $(V_{1,2}-V_1)-(V_1-V_\emptyset)$ is linear in x. $(V_{1,2}-V_1)-(V_1-V_\emptyset) \geq 0$ is then true for all x if and only if it holds for the boundaries: $x = \pi(a)$, and $x = q_\pi(a|\alpha)$. But both boundary cases are covered by arguments provided above. Thus the claim follows.

It remains to consider the case in which the joint signal realization $(\alpha, \hat{\alpha})$ has zero probability in both states. Clearly, this implies (C2), and hence the "only if" part of Lemma 7 is trivially true. For the "if" proof, we can use the same arguments as above, with the only difference that equation (9) has to be modified because the conditional probability $q_\pi(a|\alpha, \hat{\alpha})$ to which the equation refers is not well-defined. Instead, the following version of this equation can be used:

$$q_\pi(\alpha, \hat{\beta})(q_\pi(a|\alpha, \hat{\beta}) - \pi(a)) + q_\pi(\beta)(q_\pi(a|\beta) - \pi(a)) = 0.$$

□

Proof of Proposition 9

We first define p^* and k^* that are referred to in the proposition. This is done in the next lemma:

Lemma 8. *Suppose that the signals are individually informative and $(\tilde{\sigma}_1, \tilde{\sigma}_2)$ has full support. Then, there exists a p^* that solves the following equation in p:*

$$\frac{q_\pi(b|\alpha, \hat{\alpha}) - p}{p - q_\pi(b|\alpha, \hat{\beta})} \cdot \frac{q_\pi(\alpha, \hat{\alpha})}{q_\pi(\alpha, \hat{\beta})} = \frac{p - q_\pi(b|\beta, \hat{\alpha})}{q_\pi(b|\beta, \hat{\beta}) - p} \cdot \frac{q_\pi(\beta, \hat{\alpha})}{q_\pi(\beta, \hat{\beta})},$$

Moreover, for:

$$k^* \equiv \frac{q_\pi(b|\alpha, \hat{\alpha}) - p^*}{p^* - q_\pi(b|\alpha, \hat{\beta})} \cdot \frac{q_\pi(\alpha, \hat{\alpha})}{q_\pi(\alpha, \hat{\beta})},$$

it holds that:

- $p^* \in (q_\pi(b|\alpha, \hat{\beta}), \min\{q_\pi(b|\alpha, \hat{\alpha}), q_\pi(b|\beta, \hat{\beta})\})$ *and* $k^* < 1$ *if Case 3 applies;*

- $p^* \in (\max\{q_\pi(b|\alpha,\hat{\alpha}), q_\pi(b|\beta,\hat{\beta})\}, q_\pi(b|\alpha,\hat{\beta}))$ *and* $k^* > 1$ *if Case 5 applies.*

Proof. We only consider the case in which Case 3 applies. The other case is symmetric. Note first that the equation that defines p^* can be written as:

$$(q_\pi(b|\alpha,\hat{\alpha})-p)q_\pi(\alpha,\hat{\alpha})(q_\pi(b|\beta,\hat{\beta})-p)q_\pi(\beta,\hat{\beta})-(p-q_\pi(b|\alpha,\hat{\beta}))^2 q_\pi(\alpha,\hat{\beta})^2 = 0.$$

By the definition of Case 3, we know that $q_\pi(b|\alpha,\hat{\beta}) < q_\pi(b|\alpha,\hat{\alpha}), q_\pi(b|\beta,\hat{\beta})$. To see why there exists a solution to our equation between $q_\pi(b|\alpha,\hat{\beta})$ and $\min\{q_\pi(b|\alpha,\hat{\alpha}), q_\pi(b|\beta,\hat{\beta})\}$ note the following: the left hand side of our equation is strictly greater than zero when p is equal to the lower bound, it is strictly less than zero when p is equal to the upper bound and it is strictly decreasing when p is in between.

Finally, we show that $k^* < 1$ if individual signals are informative. To see why, note that $k^* \geq 1$ implies by definition of k^* that:

$$p^* \leq \sum_{\sigma_2 \in \{\hat{\alpha},\hat{\beta}\}} q_\pi(b|\alpha,\sigma_2)\frac{q_\pi(\alpha,\sigma_2)}{\sum_{\sigma_2' \in \{\hat{\alpha},\hat{\beta}\}} q_\pi(\alpha,\sigma_2')},$$

and hence that $p^* \leq q_\pi(b|\alpha)$.

Moreover, we can also deduce from the definition of k^* and p^* that $k^* \geq 1$ implies that:

$$\frac{p^* - q_\pi(b|\beta,\hat{\alpha})}{q_\pi(b|\beta,\hat{\beta}) - p^*} \cdot \frac{q_\pi(\beta,\hat{\alpha})}{q_\pi(\beta,\hat{\beta})} \geq 1.$$

which implies by similar arguments as above that $p^* \geq q_\pi(b|\beta)$. This contradicts that when individual signals are informative $q_\pi(b|\alpha) < q_\pi(b|\beta)$. □

Next, we show that our proposed equilibrium is indeed an equilibrium for the values of k^* and p^* given by Lemma 8. Consider first Case 3. We show that bidder 1 does not have incentives to deviate from our proposed strategy when $\tilde{\sigma}_1 = \alpha$. Note that according to our proposed equilibrium, bidder 1 bids p^* whereas bidder 2 may bid either p^* or $q_\pi(\beta,\hat{\beta})$. Since $p^* < q_\pi(\beta,\hat{\beta})$ by Lemma 8, it is sufficient to show: (i) that bidder 1 does not find it profitable to win the auction when bidder 2 bids $q_\pi(\beta,\hat{\beta})$, and (ii) that bidder 1 is indifferent between winning and losing when bidder 2 also bids p^*.

To see why (i) is true, note that bidder 2 bids $q_\pi(\beta, \hat{\beta})$ only if she observes $\hat{\beta}$. Consequently, the conditional expected value of the good in this case is equal to $q_\pi(\alpha, \hat{\beta})$, which is less than the bid of bidder 2.

To prove (ii), we show that bidder 1's expected value of the good conditional on observing α and conditional on bidder 2 bidding p^* is equal to p^*. Note that bidder 2 bids p^* when either she observes $\hat{\alpha}$ or $\hat{\beta}$. By Bayes' rule, the probability that $\tilde{\sigma}_2 = \sigma_2 \in \{\hat{\alpha}, \hat{\beta}\}$ when $\tilde{\sigma}_1 = \alpha$ and bidder 2 bids p^* is equal to $\frac{\mu_2(p^*|\sigma_2)q_\pi(\alpha,\sigma_2)}{\sum_{\sigma_2' \in \{\hat{\alpha},\hat{\beta}\}} \mu_2(p^*|\sigma_2')q_\pi(\alpha,\sigma_2')}$, where $\mu_2(p^*|\sigma_2)$ denotes the probability that bidder 2 bids p^* when she observes $\sigma_2 \in S_2$. Thus, the expected value of the good conditional on $\tilde{\sigma}_1 = \alpha$ and on bidder 2 bidding p^* is equal to:

$$\sum_{\sigma_2 \in \{\hat{\alpha},\hat{\beta}\}} q_\pi(b|\alpha, \sigma_2) \frac{\mu_2(p^*|\sigma_2)q_\pi(\alpha, \sigma_2)}{\sum_{\sigma_2' \in \{\hat{\alpha},\hat{\beta}\}} \mu_2(p^*|\sigma_2')q_\pi(\alpha, \sigma_2')}.$$

We now show that this conditional expected value is equal to p^*. To prove this we replace $\mu_2(p^*|.)$ by its corresponding values and subtract p^*, and obtain the following expression:

$$(q_\pi(b|\alpha, \hat{\alpha}) - p^*) \frac{q_\pi(\alpha, \hat{\alpha})}{q_\pi(\alpha, \hat{\alpha}) + k^* q_\pi(\alpha, \hat{\beta})} + (q_\pi(b|\alpha, \hat{\beta}) - p^*) \frac{k^* q_\pi(\alpha, \hat{\beta})}{q_\pi(\alpha, \hat{\alpha}) + k^* q_\pi(\alpha, \hat{\beta})}$$

It can easily be seen that the definition of k^* implies that this expression equals zero.

Finally, we show that bidder 1 does not have an incentive to deviate when she observes β. A sufficient condition is that the expected value of the good conditional on $\tilde{\sigma}_1 = \beta$ and on bidder 2's bid is equal to bidder 2's bid for any possible bid of bidder 2. In this case, bidder 1 gets zero expected utility if she wins and thus is indifferent between winning and losing. Note that bidder 2 bids either p^* or $q_\pi(b|\beta, \hat{\beta})$. That our condition is satisfied when bidder 2 bids $q_\pi(b|\beta, \hat{\beta})$ is straightforward. We can show that our condition also holds true when bidder 2 bids p^* by a similar argument as the one used in (ii) above. In this case, we can prove that the corresponding conditional expected value is equal to p^* using the following consequence of Lemma 8:

$$k^* = \frac{p^* - q_\pi(b|\beta, \hat{\alpha})}{q_\pi(b|\beta, \hat{\beta}) - p^*} \cdot \frac{q_\pi(\beta, \hat{\alpha})}{q_\pi(\beta, \hat{\beta})}.$$

Case 5 is symmetric. The only remarkable difference is that since $p^* > q_\pi(b|\alpha, \hat{\alpha})$, bidder 1 may win the auction when she observes β bids p^* and bidder 2 bids $q_\pi(b|\alpha, \hat{\alpha})$. Note that in this case bidder 1 gets strictly positive expected utility. The reason is that bidder 2 bids $q_\pi(b|\alpha, \hat{\alpha})$ only if she observes $\hat{\alpha}$ and in this case the value of the good is equal to $q_\pi(b|\beta, \hat{\alpha})$ which, by the definition of Case 5, is greater than bidder 2's bid.