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ON THE EXISTENCE OF BAYESIAN COURNOT EQUILIBRIUM*

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Abstract

We show that even in very simple oligopolies with differential information a (Bayesian) Cournot equilibrium in pure strategies may not exist, or be unique. However, we find sufficient conditions for existence, and for uniqueness, of Cournot equilibrium in a certain class of industries. More general results arise when negative prices are allowed.

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1 Introduction

The Cournot model is widely used in studies of imperfectly competitive industries. Its standard version, which is concerned with the case of firms producing a homogeneous good with *complete* information about demand and production costs, has been extensively studied. However, in the past thirty years a fairly big amount of research has been dedicated to questions that arise when the information is *incomplete*, i.e., when the firms face uncertain market demand and/or cost functions, and possibly have differential information about them. (See, e.g., Gal-Or (1985, 1986), Raith (1996), Sakai (1984, 1985), Shapiro (1986), Vives (1984, 1988, 1999), and Einy et al (2002, 2003).)

In oligopolies with incomplete information, some of the questions that had been addressed concern the *value of information* to a firm (that is, whether and by how much a firm can benefit from receiving additional information), as well as firms' *incentives to share information*. Naturally, treating these questions requires comparisons of the (pure strategy Bayesian) Cournot equilibrium[1] outcomes in industries that differ with respect to the information endowments of the firms. The scope of these exercises is thus limited to classes of industries for which a Cournot equilibrium *exists*. Moreover, sharp and general conclusions are hard to obtain unless Cournot equilibrium is also *unique* under various information endowments of the firms.

For a complete information oligopoly, there is an extensive literature concerned with the existence and uniqueness of Cournot equilibrium under various assumptions on the demand and cost functions. A well known and general condition for existence of equilibrium is found in Novshek (1985), who generalizes earlier results (of, e.g., Szidarovszky and Yakowitz (1977)). More recent developments can be found in Amir (1996), where equilibrium existence and uniqueness results are established by making a connection with the theory of supermodular games (Milgrom and Roberts (1990)).

The issues of existence of Cournot equilibrium in incomplete information oligopolies have so far been largely bypassed in the literature by making strong assumptions. For instance, Gal-Or (1985), Vives (1984, 1988), and Raith (1996) assume that the linear

[1] Henceforth we shall use the expression "Cournot equilibrium" to refer to both the pure strategy Cournot equilibria of an oligopoly with complete information, and to the pure strategy Bayesian Cournot equilibria in oligopolies with incomplete information.

market demand is uncertain, but allow the possibility that *negative* prices may arise for large outputs, in order not to break the linearity of the demand function.[2] While negative prices may make sense in some contexts, a model is of a greater appeal if such a possibility is avoided. As we shall see, even if equilibrium prices are positive, the mere possibility of negative prices in some states of nature has strategic implications that are crucial in sustaining equilibrium behavior.[3]

In other papers (Sakai (1985)), incomplete information is assumed only on firms' linear costs, which again allows to avoid the general problem of equilibrium existence. In a non-linear setting, Einy et al (2003) derive conditions under which the value of public information in an oligopoly is either positive or negative, but assume that the firms are symmetrically informed, which allows to reduce the equilibrium existence question to that in a complete information oligopoly. The assumption of symmetry of information, and a reduction to the complete information case that it allows, also stand behind the existence result of Lagerlöf (2006a). In Einy et al (2002) a categorical approach is used: it is assumed that an equilibrium exists, and then its properties are studied.

In this work we ask whether (and when) a Cournot equilibrium exists, and is unique, in a general oligopoly with differential information. Unfortunately, with regard to existence our findings are disappointing: there are simple examples of duopolies with differential information, including one with *linear* inverse demand and cost functions, that possess no Cournot equilibrium (see Examples 1 and 2).[4] The reason for the non-existence in these examples lies in that, although the inverse demand function is well behaved in all states of nature (it is linear in Example 1, and concave in Example 2, before it reaches zero), the expected payoff functions of firms do not have "nice" properties, such as concavity or submodularity, that would

[2] In these papers, linearity of the demand function is instrumental in the proofs of existence and uniqueness of a Cournot equilibrium.

[3] We are not the first to have noticed the differences brought about by possible negativity of prices. Malueg and Tsutsui (1998) did that in the case of an ex-ante symmetric linear duopoly, showing how the requirement that prices be always non-negative affects the known results on feasibility of information sharing. We will see how the requirement of non-negative prices affects the *existence* of Cournot equilibrium in oligopolies with differential information.

[4] The information structure in these examples is very simple too: one firm is better informed than the other.

guarantee equilibrium existence via known theorems. Despite the good behavior of the demand function when it is positive, once it reaches zero for some level of aggregate output and is forced to stay there for all larger outputs, a discontinuity occurs in its derivative. As a result, the expected payoff functions lose properties conducive to equilibrium existence.

In contrast, when prices are allowed to become negative, the good behavior of the demand and cost functions *does* lead to expected payoff functions that are well behaved, and we obtain existence results (see Theorems 1 and 2) under a variant of Novshek (1985) condition. There is a simple reason why these results cannot be used, in general, to deduce equilibrium existence in oligopolies with non-negative prices: disallowing negative prices changes strategic considerations of the firms, sometimes in a very significant way. Specifically, we may have an equilibrium in an oligopoly with (possibly) negative prices, but the moment only non-negative prices are allowed, it may cease being an equilibrium. One of the firms may find an incentive to deviate from its strategy, because if too large an output in one state of nature previously deterred it from deviating to it[5] due to the possibility of negative prices, now that prices are always non-negative this may be a good move on its part.

Even though the existence of Cournot equilibrium with non-negative prices is a more scarce phenomenon, we characterize some classes of oligopolies with incomplete information in which Cournot equilibrium does exist (see Theorem 4 and Corollary 1). The important feature of these classes is the existence of certain thresholds of output, which no firm will ever desire to exceed, and which guarantee positive prices in every state of nature if firms adhere to them. (Existence of such thresholds can be guaranteed if marginal costs of firms increase sufficiently fast.) This is exactly what excludes from the realm of possibilities the scenario described in the end of the previous paragraph: now every equilibrium of an oligopoly with (possibly) negative prices will translate into an equilibrium of the oligopoly with non-negative prices.

On the front of uniqueness, it turns out that, even in a simple duopoly, Cournot equilibrium may not be unique if none of the firms has better information than

[5] Overproduction in one state of nature may be a potentially good strategy for a firm with a deficient information: it may get negative profits in that state as a result of overproduction, but in exchange gain big times in some other state of nature in the same information set (where other firms are producing very little).

its rival. But if there is at least one firm with a superior information, we establish uniqueness of a Cournot equilibrium in an oligopoly with two types of firms, provided an equilibrium exists (see Theorems 3 and 5).

2 Cournot Oligopoly with Differential Information

Consider an industry where a set of firms, $N = \{1, 2, ..., n\}$, produce a homogeneous good in uncertain conditions. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is a set of *states of nature*, $\mathcal{F}$ is a σ-field of subsets of Ω, and μ is a probability measure on $(\Omega, \mathcal{F})$, which represents the *common prior belief* of the firms about the distribution of the realized state of nature. The information of firms about the state of nature may be asymmetric: the *private information* of firm $i \in N$ is given by a σ-subfield $\mathcal{F}^i$ of $\mathcal{F}$. Denote by $\mathcal{F}^i_+$ the collection of sets in $\mathcal{F}^i$ with positive measure.

Before proceeding further, recall that the set of all integrable and $\mathcal{F}$-measurable functions[6], $L_1(\Omega, \mathcal{F}, \mu)$, is a Banach space with the norm $\|h\|_1 \equiv \int |h(\omega)| \, d\mu(\omega)$. It can be partially ordered by defining $g \geq h$ if and only if $\mu(\{\omega \mid g(\omega) < h(\omega)\}) = 0$. We will also write $g > h$ whenever $\mu(\{\omega \mid g(\omega) \leq h(\omega)\}) = 0$. With the partial order $\geq$, every interval $[g, h]$ in $L_1(\Omega, \mathcal{F}, \mu)$ is a complete[7] lattice (see Theorem 3 in Milgrom and Roberts (1990)). The cone of non-negative functions in $L_1(\Omega, \mathcal{F}, \mu)$ will be denoted by $L_1^+(\Omega, \mathcal{F}, \mu)$. Obviously, all these claims and notations also apply to $L_1(\Omega, \mathcal{F}^i, \mu)$, for every $i \in N$. Finally, denote by $L_\infty(\Omega, \mathcal{F}, \mu)$ the set of all bounded $\mathcal{F}$-measurable functions, and let

$$L_\infty^+(\Omega, \mathcal{F}, \mu) \equiv L_\infty(\Omega, \mathcal{F}, \mu) \cap L_1^+(\Omega, \mathcal{F}, \mu).$$

If $q^i(\omega)$ denotes the quantity of the good produced by firm i at state $\omega \in \Omega$, and $Q(\omega) \equiv \sum_{i=1}^n q^i(\omega)$, then the *profit* of firm i at ω is given by

$$\pi^i\left(\omega, \left(q^j(\omega)\right)_{j=1}^n\right) = q^i(\omega) P(\omega, Q(\omega)) - c_i\left(\omega, q^i(\omega)\right),$$

[6] Or, more, precisely, the space of equivalence classes of these functions, where two functions that are equal μ-almost everywhere are identified.

[7] By this we mean that for any $T \subset S^i$, $\inf T$ and $\sup T$ are well defined.

where $P(\omega,\cdot)$ is the inverse demand function at ω, and $c_i(\omega,\cdot)$ is the cost function of firm i. The following conditions will be assumed on the inverse demand function and the cost functions:

(i) $P(\cdot,0)\in L_\infty(\Omega,\mathcal{F},\mu)$, and for every $\omega\in\Omega$ $P(\omega,\cdot)$ is a non-increasing and continuous function;

(ii) there exists $\overline{Q}\in L_\infty^+(\Omega,\mathcal{F},\mu)$ such that $P(\omega,Q)=0$ for every $\omega\in\Omega$ and every $Q\geq\overline{Q}(\omega)$. For the sake of convenience, we assume that $\overline{Q}(\omega)$ is the *minimal* Q-intercept of P;

(iii) for every $\omega\in\Omega$ and $i\in N$ $c_i(\omega,\cdot)$ is a strictly increasing and continuous function, and $c_i(\cdot,q)\in L_\infty^+(\Omega,\mathcal{F},\mu)$ for every $q\geq 0$.

An output decision of firm i is summarized by its *strategy*, i.e., a function $q^i\in L_1^+(\Omega,\mathcal{F}^i,\mu)$, which determines the output $q^i(\omega)$ of i at every state of nature ω, subject to measurability with respect to i's private information. Given an n-tuple $q=(q^j)_{j=1}^n$ of firms' strategies, the *expected payoff* of firm i,

$$\Pi^i(q)=E\left[\pi^i\left(\cdot,\left(q^j(\cdot)\right)_{j=1}^n\right)\right]=E\left[q^i(\cdot)\,P(\cdot,Q(\cdot))-c_i\left(\cdot,q^i(\cdot)\right)\right],$$

is well defined (with values in $[-\infty,\infty)$).

An n-tuple $q_*=(q_*^j)_{j=1}^n\in\prod_{j=1}^n L_1^+(\Omega,\mathcal{F}^j,\mu)$ constitutes a *Cournot equilibrium* if no firm finds it profitable to unilaterally deviate to another strategy, i.e., for every $(q^j)_{j=1}^n\in\prod_{j=1}^n L_1^+(\Omega,\mathcal{F}^j,\mu)$ and every $i\in N$,

$$\Pi^i(q_*)\geq\Pi^i\left(q_*\mid q^i\right), \tag{1}$$

where $(q_*\mid q^i)$ stands for the n-tuple of strategies which is identical to q_* in all but the ith strategy, which is replaced by q^i. This is obviously equivalent to requiring that

$$E\left(\pi^i(\cdot,q_*(\cdot))\mid A\right)\geq E\left(\pi^i\left(\cdot,\left(q_*\mid q^i\right)(\cdot)\right)\mid A\right) \tag{2}$$

for every $A\in\mathcal{F}_+^i$, or that

$$E\left(\pi^i(\cdot,q_*(\cdot))\mid\mathcal{F}^i\right)\geq E\left(\pi^i\left(\cdot,\left(q_*\mid q^i\right)(\cdot)\right)\mid\mathcal{F}^i\right) \tag{3}$$

when both sides are viewed as functions in $L_1(\Omega,\mathcal{F}^i,\mu)$. (Here $E(h(\cdot)\mid A)$ stands for the expectation of random variable h conditional on event A, and $E(h(\cdot)\mid\mathcal{F}^i)$ stands for the conditional expectation of h given the field $\mathcal{F}^i$.)

3 Examples of Industries without a Cournot Equilibrium

3.1 Example 1: Linear Duopoly

As we shall see in Section 5 (Example 3), if the inverse demand function $P(\omega, \cdot)$ is *linear* on the interval $\left[0, \overline{Q}(\omega)\right]$, the duopoly does possess a Cournot equilibrium, provided the Q-intercept of P is known to both firms at every state of nature, i.e., $\overline{Q}$ is measurable with respect to both $\mathcal{F}^1$ and $\mathcal{F}^2$. In particular, if the Q-intercept is the same in all states of nature, the duopoly has a Cournot equilibrium.

However, if the Q-intercept is sufficiently variable across the states of nature, and firms *differ* in their information about it (including when $\mathcal{F}^2 \subset \mathcal{F}^1$, i.e., firm 1 is generally better informed than firm 2), a linear duopoly may not have a Cournot equilibrium as we show next.

Consider the following duopoly with asymmetric information. The set of states of nature Ω consists of just two states, ω_1 and ω_2, where the probability of ω_1 is $\frac{1}{4}$, and the probability of ω_2 is $\frac{3}{4}$. Firm 1 is *completely informed* about the realized state of nature, while firm 2 has *no information* about it. Thus, $\mathcal{F}^2 = \{\Omega, \varnothing\} \subset \mathcal{F}^1 = \{\{\omega_1\}, \{\omega_2\}, \Omega, \varnothing\}$. The inverse demand function is given by

$$P(\omega_1, Q) = \max\{1 - \frac{Q}{4}, 0\}, \text{ and } P(\omega_2, Q) = \max\{1 - Q, 0\}. \qquad (4)$$

Thus, here both $P(\omega_1, \cdot)$ and $P(\omega_2, \cdot)$ are linear till they reach zero; the Q-intercept of P equals 4 at ω_1, and 1 at ω_2. Only firm 1 knows the Q-intercept of P at every state of nature.

Firm 2 has a constant marginal cost of 0.001 at both states of nature. The marginal cost for firm 1 is 0.001 at ω_2, but it is 2 at ω_1.

We will show that *no Cournot equilibrium exists* in this duopoly. Indeed, suppose to the contrary that an equilibrium does exist, and denote it by $q_* = (q_*^1, q_*^2)$. Since at ω_1 the marginal revenue of firm 1 is always below its marginal cost, at this state it always produces zero in equilibrium. It follows that both q_*^1 and q_*^2 can be regarded as scalars: q_*^1 can be viewed as the quantity produced by firm 1 at state ω_2, and q_*^2 as the quantity produced by firm 2 at both states.

Consider first the possibility that

$$q_*^2 < 0.999. \tag{5}$$

Since q_*^1 maximizes $\Pi^1(\cdot, q_*^2)$ and the output is zero at state ω_1, obviously

$$q_*^1 \in \arg\max_{q^1 \in [0,1-q_*^2]} \pi^1\left(\omega_2, (q^1, q_*^2)\right),$$

where $\pi^1(\omega_2, (q^1, q_*^2)) = q^1(1 - q^1 - q_*^2) - 0.001q^1$. The unique maximizer $q^1 = q_*^1$ solves the first order condition:

$$\begin{aligned} 0 &= \frac{\partial}{\partial q^1}\left[q^1(1 - q^1 - q_*^2) - 0.001q^1\right] \\ &= 0.999 - q_*^2 - 2q^1, \end{aligned}$$

and consequently

$$q_*^1 = \frac{0.999 - q_*^2}{2}. \tag{6}$$

It follows that $q_*^1 + q_*^2 < 1$. Thus, similarly,

$$q_*^2 \in \arg\max_{q^2 \in [0,1-q_*^1)} \Pi^2(q_*^1, q^2),$$

where

$$\begin{aligned} \Pi^2(q_*^1, q^2) &= \frac{1}{4}\pi^2\left(\omega_1, (0, q^2)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q_*^1, q^2)\right) \\ &= \frac{1}{4}q^2(1 - \frac{q^2}{4}) + \frac{3}{4}q^2(1 - q_*^1 - q^2) - 0.001q^2. \end{aligned}$$

The unique maximizer $q^2 = q_*^2$ solves the first order condition:

$$\begin{aligned} 0 &= \frac{\partial}{\partial q^2}\left[\frac{1}{4}q^2(1 - \frac{q^2}{4}) + \frac{3}{4}q^2(1 - q_*^1 - q^2) - 0.001q^2\right] \\ &= 0.999 - \frac{3}{4}q_*^1 - \frac{1}{8}q^2 - \frac{3}{2}q^2 \end{aligned}$$

and it follows that

$$q_*^2 = \frac{0.999 - \frac{3}{4}q_*^1}{\frac{13}{8}}. \tag{7}$$

Solving (6) and (7) yields

$$q_*^1 = 0.249\,75 \tag{8}$$

and

$$q_*^2 = 0.499\,5. \tag{9}$$

Now note that

$$\begin{aligned}\Pi^2(q^1_*, q^2_*) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 0.499\,5)\right) + \frac{3}{4}\pi^2\left(\omega_2, (0.249\,75, 0.499\,5)\right) \\ &= \frac{1}{4}\cdot 0.499\,5 \cdot \left(1 - \frac{0.499\,5}{4}\right) + \frac{3}{4}\cdot 0.499\,5\cdot(1 - 0.499\,5 - 0.249\,75) - 0.001\cdot 0.4995 \\ &= 0.202\,72.\end{aligned}$$

However, since $q^1_* + 2 > 1$, we have

$$\begin{aligned}\Pi^2(q^1_*, 2) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 2)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q^1_*, 2)\right) \\ &= \frac{1}{4}\cdot 2\cdot(1 - \frac{2}{4}) - 2\cdot 0.001 \\ &= 0.248.\end{aligned} \tag{10}$$

Thus

$$\Pi^2(q^1_*, 2) > \Pi^2(q^1_*, q^2_*),$$

which shows that (5) does not hold in equilibrium.[8]

Next assume that

$$0.999 \le q^2_* \le 1. \tag{11}$$

Clearly

$$\begin{aligned}\Pi^2(q^1_*, q^2_*) &= \frac{1}{4}\pi^2\left(\omega_1, (0, q^2_*)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q^1_*, q^2_*)\right) \\ &\le \frac{1}{4}\cdot\left(1 - \frac{0.999}{4}\right) + \frac{3}{4}\cdot(1 - 0.999) \\ &= 0.188\,31.\end{aligned}$$

On the other hand, just as in (10), $\Pi^2(q^1_*, 2) = 0.248$. Consequently,

$$\Pi^2(q^1_*, 2) > \Pi^2(q^1_*, q^2_*)$$

and thus (11) does not hold in equilibrium either.

We are left with the possibility that

$$q^2_* > 1. \tag{12}$$

[8] Had we not disallowed negative prices in (4), the pair (q^1_*, q^2_*) given by (8) and (9) would have in fact been a Cournot equilibrium in the duopoy: the deviation of firm 2 from q^2_* to 2 would not have been profitable. We will return to this point in the beginning of Section 4.

But then

$$q_*^2 \in \arg \max_{q^2 \in (1,4)} \Pi^2(q_*^1, q^2),$$

where

$$\begin{aligned}\Pi^2(q_*^1, q^2) &= \frac{1}{4}\pi^2\left(\omega_1, (0, q^2)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q_*^1, q^2)\right) \\ &= \frac{1}{4}q^2(1 - \frac{q^2}{4}) - 0.001q^2.\end{aligned}$$

The unique maximizer $q^2 = q_*^2$ solves the first order condition:

$$\begin{aligned}0 &= \frac{\partial}{\partial q^2}\left[\frac{1}{4}q^2(1 - \frac{q^2}{4}) - 0.001q^2\right] \\ &= 0.249 - \frac{q^2}{8},\end{aligned}$$

which yields $q_*^2 = 1.992$. It follows that $q_*^1 = 0$ (firm 1 would clearly prefer not to produce for revenue zero, and save its costs instead). Now

$$\begin{aligned}\Pi^2(q_*^1, q_*^2) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 1.992)\right) + \frac{3}{4}\pi^2\left(\omega_2, (0, 1.992)\right) \\ &= \frac{1}{4} \cdot 1.992 \cdot \left(1 - \frac{1.992}{4}\right) - 0.001 \cdot 1.992 \\ &= 0.248004.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Pi^2(q_*^1, 0.5) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 0.5)\right) + \frac{3}{4}\pi^2\left(\omega_2, (0, 0.5)\right) \\ &= \frac{1}{4} \cdot 0.5 \cdot \left(1 - \frac{0.5}{4}\right) + \frac{3}{4} \cdot 0.5 \cdot (1 - 0.5) - 0.001 \cdot 0.5 \\ &= 0.29638.\end{aligned}$$

Thus

$$\Pi^2(q_*^1, 0.5) > \Pi^2(q_*^1, q_*^2)$$

and it follows that (12) does not hold in equilibrium.

Since neither (5), (11), or (12) hold in an equilibrium, we have reached a contradiction. We conclude that there exists no Cournot equilibrium in this duopoly.

Remark 1 (Non-Existence of Cournot Equilibrium with Nearly Complete Information). Cournot equilibrium may fail to exist even in an industry with *nearly complete* information. Indeed, suppose that the scenario described in Example 1 takes place with (small) probability $\varepsilon > 0$, and that, with complementary probability $1-\varepsilon$, firms know the inverse demand function and their costs. More precisely, let us assume that $\omega_1, \omega_2 \in \Omega$, $\mu(\{\omega_1\}) = \frac{1}{4}\varepsilon$ and $\mu(\{\omega_2\}) = \frac{3}{4}\varepsilon$, $\{\omega_1, \omega_2\} \in \mathcal{F}^1 \cap \mathcal{F}^2$, firm 1 can distinguish between ω_1 and ω_2 while firm 2 cannot, and at each $\omega \in \Omega \backslash \{\omega_1, \omega_2\}$ both firms know the inverse demand function and their cost functions. On $\{\omega_1, \omega_2\}$, let the inverse demand function and the cost functions be the same as in the previous example. Then the same arguments as above show that no Cournot equilibrium exists in this duopoly. (One has to replace throughout the expected payoff function Π^i of firm i by its expected profit π^i conditional on $A = \{\omega_1, \omega_2\}$, and all the contradictions would now be obtained from the equivalent definition of a Cournot equilibrium in (2).) ■

Remark 2 (Non-Existence of Cournot Equilibrium With a Constant Slope of the Inverse Demand Function). In Example 1, the vertical (P-)intercept of the inverse demand function is fixed at 1 across all states of nature, while the slope of the demand (and the Q-intercept) are variable. A trivial modification of this duopoly can yield an example of equilibrium non-existence, in which at all states of nature the inverse demand function has a constant slope of -1 when it is positive (but its P-intercept is variable). Simply multiply the function $P(\omega_1, \cdot)$ by 4 at state ω_1, to obtain a new $P(\omega_1, Q) = \max\{4-Q, 0\}$ with slope -1 on $\left[0, \overline{Q}(\omega_1)\right]$. To offset the four-fold increase of $P(\omega_1, \cdot)$ and to keep all previous arguments working, divide the probability of ω_1 by 4 (and then normalize the vector $\left(\frac{1}{4}\mu(\{\omega_1\}), \mu(\{\omega_2\})\right) = \left(\frac{1}{16}, \frac{3}{4}\right)$ to obtain new probabilities $\left(\frac{1}{13}, \frac{12}{13}\right)$ for the two states of nature). ■

3.2 Example 2: Piecewise-Linear Duopoly

In the previous subsection, firms differed in their information about the Q-intercept of P. However, outside the linear case, even the complete knowledge of the Q-intercept by both firms does not guarantee equilibrium existence in a duopoly. In this subsection we adopt *the same duopoly* as before, with *just one difference*: the inverse demand function at state ω_1 is given by

$$P(\omega_1, Q) = \begin{cases} 1, & \text{if } Q \leq 0.99; \\ 100(1-Q) & \text{if } 0.99 < Q \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $P(\omega_1, \cdot)$ is *piecewise-linear* and *concave,* before it reaches zero. Additionally, the function $P(\omega_1, \cdot)$ can be made *smooth* in a small neighborhood of $Q = 0.99$, without affecting any of our arguments in the sequel.[9] Note also that each firm knows the Q-intercept of P, $\overline{Q} \equiv 1$, at any state of nature.

We will show that this duopoly does not have a Cournot equilibrium. Indeed, suppose to the contrary that an equilibrium does exist, and denote it by $q_* = (q_*^1, q_*^2)$. As before, firm 1 produces zero quantity at ω_1, and it follows that both q_*^1 and q_*^2 can be regarded as scalars: q_*^1 can be viewed as the quantity produced by firm 1 at state ω_2, and q_*^2 as the quantity produced by firm 2 at both states.

Consider first the possibility that

$$q_*^1 + q_*^2 < 0.99. \tag{13}$$

Just as in the previous subsection,

$$q_*^1 \in \arg \max_{q^1 \in [0, 1-q_*^2]} \pi^1 \left(\omega_2, (q^1, q_*^2)\right), \tag{14}$$

and therefore

$$q_*^1 = \frac{0.999 - q_*^2}{2}. \tag{15}$$

On the other hand,

$$q_*^2 \in \arg \max_{q^2 \in [0, 0.99 - q_*^1)} \Pi^2(q_*^1, q^2),$$

where

$$\begin{aligned} \Pi^2(q_*^1, q^2) &= \frac{1}{4}\pi^2 \left(\omega_1, (0, q^2)\right) + \frac{3}{4}\pi^2 \left(\omega_2, (q_*^1, q^2)\right) \\ &= \frac{1}{4}q^2 + \frac{3}{4}q^2(1 - q_*^1 - q^2) - 0.001q^2. \end{aligned}$$

[9] We have not done this smoothing of $P(\omega_1, \cdot)$ in order to simplify the presentation. Also note that $P(\omega_1, \cdot)$ can be made *strictly* decreasing on $[0, 1]$ by adding to it a function $\varepsilon(Q) \equiv 0.001(1 - Q)$. Again, doing this would not qualitatively affect any of our arguments, but would make the presentation messier.

The unique maximizer $q^2 = q^2_*$ solves the first order condition:

$$\begin{aligned} 0 &= \frac{\partial}{\partial q^2}\left[\frac{1}{4}q^2 + \frac{3}{4}q^2(1 - q^1_* - q^2) - 0.001q^2\right] \\ &= 0.999 - \frac{3}{4}q^1_* - \frac{3}{2}q^2, \end{aligned}$$

and it follows that

$$q^2_* = \frac{0.999 - \frac{3}{4}q^1_*}{\frac{3}{2}}. \tag{16}$$

Solving (15) and (16) yields

$$q^1_* = 0.222 \tag{17}$$

and

$$q^2_* = 0.555. \tag{18}$$

Now note that

$$\begin{aligned} \Pi^2(q^1_*, q^2_*) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 0.555)\right) + \frac{3}{4}\pi^2\left(\omega_2, (0.222, 0.555)\right) \\ &= \frac{1}{4} \cdot 0.555 + \frac{3}{4} \cdot 0.555 \cdot (1 - 0.777) - 0.001 \cdot 0.555 \\ &= 0.23102. \end{aligned}$$

However,

$$\begin{aligned} \Pi^2(q^1_*, 0.99) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 0.99)\right) + \frac{3}{4}\pi^2\left(\omega_2, (0.222, 0.99)\right) \\ &= \frac{1}{4} \cdot 0.99 - 0.99 \cdot 0.001 \\ &= 0.24651. \end{aligned}$$

Thus

$$\Pi^2(q^1_*, 0.99) > \Pi^2(q^1_*, q^2_*),$$

which shows that (13) does not hold in an equilibrium.

Now assume that

$$0.99 \leq q^1_* + q^2_* < 1. \tag{19}$$

Just as before, (14) and thus (15) still hold. It follows that $q^2_* = 0.999 - 2q^1_*$, and thus

$$0.99 \leq q^1_* + (0.999 - 2q^1_*) = 0.999 - q^1_*.$$

Therefore

$$q_*^1 \leq 0.009,$$

and

$$q_*^2 \geq 0.99 - 0.009 = 0.981.$$

Notice that then

$$\begin{aligned}
\Pi^2(q_*^1, q_*^2) &= \frac{1}{4}\pi^2\left(\omega_1, (0, q_*^2)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q_*^1, q_*^2)\right) \\
&\leq \frac{1}{4} + \frac{3}{4} \cdot (1 - 0.981) \\
&= 0.264\,25.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Pi^2(q_*^1, 0.5) &= \frac{1}{4}\pi^2\left(\omega_1, (0, 0.5)\right) + \frac{3}{4}\pi^2\left(\omega_2, (q_*^1, 0.5)\right) \\
&\geq \frac{1}{4} \cdot 0.5 + \frac{3}{4} \cdot 0.5 \cdot (1 - 0.5 - 0.009) - 0.001 \cdot 0.5 \\
&= 0.308\,63
\end{aligned}$$

It follows that

$$\Pi^2(q_*^1, 0.5) > \Pi^2(q_*^1, q_*^2),$$

and thus (19) does not hold in equilibrium either.

We are left with the possibility that

$$q_*^1 + q_*^2 \geq 1. \tag{20}$$

However, this is also impossible. Since the revenue of each firm is zero at this level of aggregate output, one of the firms will be better off by switching its output level to zero, thereby saving its costs.

Since neither (13), (19), or (20) hold in an equilibrium, we have reached a contradiction. We conclude that there exists no Cournot equilibrium in this duopoly.

Remark 3. In both previous examples it was assumed that the marginal cost of production of firm 1 at state ω_1 is 2, which forces firm 1 to produce zero at this state. This assumption is made only for simplicity and clarity of the proof, since it allows to identify the equilibrium strategy of firm 1 with a *scalar*, representing its output at state ω_2. However, what is needed for our arguments to work is that the quantity produced by 1 at ω_1 be sufficiently small. For instance, a Cournot equilibrium cannot exist either when the marginal costs of firm 1 at ω_1 are very low for $q \in [0, 0.001]$, but exceed 2 for $q \in [0.002, \infty)$. ■

4 Existence and Uniqueness of Cournot Equilibrium in Oligopolies with (Possibly) Negative Prices

According to condition (ii), the inverse demand function is *non-negative*: once the price reaches zero for some level $\overline{Q}(\omega)$ of aggregate output at some state of nature ω, it remains zero for all levels of output higher than $\overline{Q}(\omega)$. In this section we do not insist on prices being non-negative, replacing condition (ii) by (ii)' below:

(ii)' there exists a positive real number Z such that $P(\omega, Z) \leq 0$ for every $\omega \in \Omega$ and every $Q \geq Z$; moreover, for every $\omega \in \Omega$ the function $P(\omega, \cdot)$ is twice continuously differentiable and

$$QP''(\omega, Q) + P'(\omega, Q) \leq 0 \tag{21}$$

for every $Q \in \mathbb{R}_+$ (at $Q = 0$ we have in mind the right-side derivatives of P and P').

Inequality (21) is borrowed from Novshek (1985)[10]; it is satisfied, e.g., by all inverse demand functions which are concave on $\mathbb{R}_+$. Typically, condition (ii)' implies that the price P is *negative* when the aggregate output is sufficiently high. This assumption is reasonable in certain contexts, and, as we shall see below, it is important in guaranteeing existence of Cournot equilibrium. Results that we present in this section will be instrumental in proving equilibrium existence under certain conditions in oligopolies with non-negative prices, that will be the topic of the next section, where condition (ii) will be reinstated.

[10] We could not have assumed instead the log-concavity of each $P(\omega, \cdot)$, thereby adopting the condition of Amir (1996) from complete information duopolies. As we shall see in the proofs of Theorems 1 and 2, (21) implies certain properties of the state-dependent revenue function (decreasing differences, concavity), that fully translate into the same properties of the *expected* revenue function. This would not have been the case for log-concave $P(\omega, \cdot)$.

Allowing prices to be negative by means of condition (ii)' has far reaching implications for the existence of Cournot equilibrium. Take, for instance, the two examples of duopolies without an equilibrium in Section 3. The "candidates" $q_* = (q_*^1, q_*^2)$ for an equilibrium (given by (8) and (9) for the duopoly in Example 1; and by (17) and (18) for the duopoly in Example 2), that were found by solving equations obtained from the first-order conditions, were later disqualified from being equilibria. This was because a profitable deviation existed for the least informed firm 2, in which it chose a large output, for which the price became zero at one of the states of nature. But, had we not restricted the prices not to fall below zero in our definition of the inverse demand function P, $q_* = (q_*^1, q_*^2)$ would have been an equilibrium in each duopoly! Firm 2 would have been *deterred* from deviations to very large outputs, because negative prices in some states of nature would ensue. This deterrence from choosing large outputs, achieved by means of possibly negative prices, provides some intuition as for why quite general positive results for equilibrium existence can be obtained for oligopolies satisfying (ii)'. More formally, it is the "nice" behavior of the expected payoff functions (submodularity, or concavity) that stands behind these equilibrium existence results for oligopolies with possibly negative prices. When prices are constrained to be non-negative, the differentiability conditions embodied in (ii)' break up very abruptly at the point where the price becomes zero, and the expected payoff functions lose some of these "nice" properties as a result.

The following condition will be used in the sequel:

(iv) P is bounded on the interval $[0, nZ]$ uniformly in ω, i.e., $\sup_{\omega \in \Omega, 0 \leq Q \leq nZ} |P(\omega, Q)| < \infty$.

Theorem 1. In a *duopoly* satisfying (i), (ii)', (iii), and (iv), a Cournot equilibrium *exists.*

Proof. We will show first that for each $\omega \in \Omega$ the profit function $\pi_\omega^1(\cdot) = \pi^1(\omega, \cdot)$ of firm 1 has *decreasing differences* in the first coordinate, that is, if $x_1 \geq x_2 \geq 0$ and $y_1 \geq y_2 \geq 0$, then

$$\left[\pi_\omega^1(x_1, y_2) - \pi_\omega^1(x_2, y_2)\right] - \left[\pi_\omega^1(x_1, y_1) - \pi_\omega^1(x_2, y_1)\right] \geq 0, \tag{22}$$

i.e.,

$$[x_1 P(\omega, x_1 + y_2) - x_2 P(\omega, x_2 + y_2)] - [x_1 P(\omega, x_1 + y_1) - x_2 P(\omega, x_2 + y_1)] \geq 0.$$

Since $P(\omega, \cdot)$ is continuously differentiable, this condition is equivalent to

$$\frac{\partial}{\partial y_2}[x_1 P(\omega, x_1 + y_2) - x_2 P(\omega, x_2 + y_2)] \leq 0,$$

or

$$x_1 P'(\omega, x_1 + y_2) - x_2 P'(\omega, x_2 + y_2) \leq 0,$$

for every $x_1 \geq x_2 \geq 0$ and $y_2 \geq 0$. This condition, in turn, is equivalent (since $P'(\omega, \cdot)$ is also continuously differentiable) to

$$\frac{\partial}{\partial x_2}[x_2 P'(\omega, x_2 + y_2)] \leq 0,$$

or

$$x_2 P''(\omega, x_2 + y_2) + P'(\omega, x_2 + y_2) \leq 0, \tag{23}$$

for every $x_2 \geq 0$ and $y_2 \geq 0$. However, (23) is implied by conditions (i) and (ii)' on P.

From (22) it follows that the expected profit function Π^1 of firm 1 also has decreasing differences in the first coordinate: for every $(q^1, q^2), (\widetilde{q}^1, \widetilde{q}^2) \in L^+_\infty(\Omega, \mathcal{F}^1, \mu) \times L^+_\infty(\Omega, \mathcal{F}^2, \mu)$ such that $q^1 \geq \widetilde{q}^1$, $q^2 \geq \widetilde{q}^2$,

$$\left[\Pi^1\left(q^1, \widetilde{q}^2\right) - \Pi^1\left(\widetilde{q}^1, \widetilde{q}^2\right)\right] - \left[\Pi^1\left(q^1, q^2\right) - \Pi^1\left(\widetilde{q}^1, q^2\right)\right] \geq 0.$$

Similarly, the expected payoff function Π^2 of firm 2 has decreasing differences in the second coordinate.

Now denote the constant function on Ω which is fixed at the level Z by the same symbol, Z. Consider intervals $S^1 = [0, Z] \subset L^+_1(\Omega, \mathcal{F}^1, \mu)$, and $S^2 = [0, Z] \subset L^+_1(\Omega, \mathcal{F}^2, \mu)$. Note that each function Π^i, restricted to $S^1 \times S^2$, is $\|\cdot\|_1$-continuous in both coordinates. Indeed, if a sequence $\{(q^1_n, q^2_n)\}^\infty_{n=1}$ in $S^1 \times S^2$ converges to (q^1, q^2) with this norm on the coordinates, it has a subsequence $\left\{\left(q^1_{n_k}, q^2_{n_k}\right)\right\}^\infty_{k=1}$ that converges (μ-)almost everywhere to (q^1, q^2). But then $\lim_{k\to\infty} \Pi^i\left(q^1_{n_k}, q^2_{n_k}\right) = \Pi^i(q^1, q^2)$ by the bounded convergence theorem, since the state-dependent revenue and cost functions are continuous and uniformly bounded (by conditions (iii) and (iv)). This obviously implies continuity of Π^i. According to Theorem 3 of Milgrom and Roberts (1990), this in turn implies *order continuity* of Π^i, i.e., Π^i converges along all convergent nets of functions in $S^1 \times S^2$.

Now reverse the order in S^2, i.e., replace the order "$\geq$" with "$\geq'$" according to which $g \geq' h$ if and only if $h \geq g$. Then both Π^1 and Π^2 exhibit *increasing*, rather than decreasing, differences. The reversal of order has no effect on order continuity of Π^1 and Π^2. Since both S^1 and S^2 are complete lattices[11], Theorem 5 of Milgrom and Roberts (1990) applies[12]: there exists a Cournot equilibrium when strategy profiles of the firms are restricted to be in $S^1 \times S^2$. Denote one such equilibrium by $\left(q_*^1, q_*^2\right)$. If $\left(q^1, q^2\right) \in L_1^+\left(\Omega, \mathcal{F}^1, \mu\right) \times L_1^+\left(\Omega, \mathcal{F}^2, \mu\right)$, notice that $\Pi^1\left(q^1, q_*^2\right) \leq \Pi^1\left(\min(q^1, Z), q_*^2\right)$ and $\Pi^2\left(q_*^1, q^2\right) \leq \Pi^2\left(q_*^1, \min(q^2, Z)\right)$ as follows from the definition of Z and condition (iii). Therefore

$$\Pi^1\left(q_*^1, q_*^2\right) \geq \Pi^1\left(\min(q^1, Z), q_*^2\right) \geq \Pi^1\left(q^1, q_*^2\right)$$

and

$$\Pi^2\left(q_*^1, q_*^2\right) \geq \Pi^2\left(q_*^1, \min(q^2, Z)\right) \geq \Pi^2\left(q_*^1, q^2\right),$$

since $\left(q_*^1, q_*^2\right)$ is a Cournot equilibrium when the strategy profiles of the firms are restricted to $S^1 \times S^2$. But these inequalities show that $\left(q_*^1, q_*^2\right)$ is actually a Cournot equilibrium without any restrictions on strategies. ■

When the set of states Ω is finite and all cost functions are convex, the existence result of Theorem 1 holds for any number of firms.

Theorem 2. In an *oligopoly* satisfying (i), (ii)', and (iii), a Cournot equilibrium *exists*, provided Ω is a finite set and $c_i(\omega, \cdot)$ is convex for every $i \in N$ and $\omega \in \Omega$.

Proof. This is a direct consequence of the Nash existence theorem. First, for each $\omega \in \Omega$, $\pi_\omega^i(\cdot) = \pi^i(\omega, \cdot)$ is concave in strategies of firm i. Indeed, the second

[11] As was mentioned already, this is due to Theorem 3 of Milgrom and Roberts (1990).

[12] One more thing needs to be verified before applying this theorem, namely that Π^1 is supermodular in q^1 for fixed q^2, i.e., for every $q^1, \widetilde{q}^1 \in S^1$ and $q^2 \in S^2$, $\Pi^1\left(q^1, q^2\right) + \Pi^1\left(\widetilde{q}^1, q^2\right) \leq \Pi^1\left(\max(q^1, \widetilde{q}^1), q^2\right) + \Pi^1\left(\min(q^1, \widetilde{q}^1), q^2\right)$, and similarly for Π^2. However, it can be easily checked that this inequality actually holds as equality.

derivative of $q^i(\omega) P(\omega, Q(\omega))$ with respect to $q^i(\omega)$ is equal to $q^i(\omega) P''(\omega, Q(\omega)) + 2P'(\omega, Q(\omega))$, which is non-positive as follows from (i) and (ii)'. Thus, $q^i(\omega) P(\omega, Q(\omega))$ is concave in $q^i(\omega)$, and from convexity of $c_i(\omega, \cdot)$ it follows that $\pi^i_\omega(q(\omega)) = q^i(\omega) P(\omega, Q(\omega)) - c_i(\omega, q^i(\omega))$ is also concave in $q^i(\omega)$. The expected payoff function Π^i clearly inherits concavity in q^i from π^i_ω.

Second, following notations of the proof of Theorem 1, restrict the strategy set of each firm i to $S^i = [0, Z] \subset L^+_1(\Omega, \mathcal{F}^i, \mu)$, which is compact due to the finiteness of Ω. As in the proof of Theorem 1, Π^i is $\|\cdot\|_1$-continuous[13] in all coordinates simultaneously on the compact cube $[0, Z]^N$. Thus, all ingredients for the existence of Nash equilibrium are in place, with the above restriction of strategies. However, the restricted equilibrium is an equilibrium in the unrestricted oligopoly as well, which can be shown again exactly as in the proof of Theorem 1. ■

The following theorem establishes uniqueness of Cournot equilibrium in an oligopoly in which there are *two types* of firms, one of which possesses *superior information.* Before stating the theorem, we introduce the following strengthened versions of conditions (i) (combined with (iv)), and of (iii).

(i)' For every $\omega \in \Omega$ $P(\omega, \cdot)$ is strictly decreasing; moreover $\sup_{\omega \in \Omega, 0 \le Q \le nZ+1} |P(\omega, Q)| < \infty$ and[14] $\sup_{\omega \in \Omega, 0 \le Q \le nZ+1} |P'(\omega, Q)| < \infty$.

(iii)' For every $\omega \in \Omega$ and $i \in N$, $c_i(\omega, \cdot)$ is a strictly increasing, twice continuously differentiable and convex function, and $c_i(\cdot, q) \in L^+_\infty(\Omega, \mathcal{F}, \mu)$ for every $q \ge 0$; moreover $\sup_{i \in N, \omega \in \Omega, 0 \le q \le Z+1} c'_i(\omega, q) < \infty$.

Theorem 3. Consider an oligopoly satisfying (i)', (ii)', and (iii)'. Suppose further that the set N of firms can be partitioned into two disjoint sets, K and M, such that $1 \in K$, $2 \in M$, and such that $\mathcal{F}^i = \mathcal{F}^1$, $c_i = c_1$ for every $i \in K$, $\mathcal{F}^j = \mathcal{F}^2$, $c_j = c_2$ for

[13] Now condition (iv) is not needed, since due to the finiteness of Ω $P(\omega, \cdot)$ is clearly bounded uniformly in ω on any finite interval.

[14] Condition (i)' will always be assumed in conjunction with (ii)' or (ii)" (to be introduced in the next section), according to which $P(\omega, \cdot)$ is differentiable.

every $j \in M$, and $\mathcal{F}^2 \subset \mathcal{F}^1$. Then, *if* a Cournot equilibrium exists (e.g., when either $n = 2$ or the set Ω is finite)[15], it is *unique*.

Proof. Let $q_* = (q_*^i)_{i=1}^n$ be a Cournot equilibrium. The strategies in q_* are clearly bounded by Z (beyond which all prices are non-positive by (ii)'). Now pick a firm i, and a set $A \in \mathcal{F}_+^i$. Denote by $A_+ \in \mathcal{F}^i$ the subset of A on which $q_*^i > 0$, and let $A_0 = A \backslash A_+$. Obviously,

$$E\left(q^i(\cdot)\, P\left(\cdot, \sum_{j \neq i} q_*^j(\cdot) + q^i(\cdot)\right) - c_i\left(\cdot, q^i(\cdot)\right) \mid A_+\right) \tag{24}$$

and

$$E\left(q^i(\cdot)\, P\left(\cdot, \sum_{j \neq i} q_*^j(\cdot) + q^i(\cdot)\right) - c_i\left(\cdot, q^i(\cdot)\right) \mid A_0\right) \tag{25}$$

are maximized (and in particular locally maximized) at $q^i = q_*^i$, provided $A_+, A_0 \in \mathcal{F}_+^i$. Consequently, the boundedness of strategies in q_*, and our assumptions on the uniform boundedness and continuity of P and the derivatives of P and c_i (conditions (i)' and (iii)'), imply by the bounded convergence theorem that Kuhn-Tucker conditions are satisfied:

$$E\left(q_*^i(\cdot)\, P'\left(\cdot, Q_*(\cdot)\right) + P\left(\cdot, Q_*(\cdot)\right) - c_i'\left(\cdot, q_*^i(\cdot)\right) \mid A_+\right) = 0$$

if $A_+ \in \mathcal{F}_+^i$, and

$$E\left(q_*^i(\cdot)\, P'\left(\cdot, Q_*(\cdot)\right) + P\left(\cdot, Q_*(\cdot)\right) - c_i'\left(\cdot, q_*^i(\cdot)\right) \mid A_0\right) \leq 0$$

if $A_0 \in \mathcal{F}_+^i$. Since this holds for *every* $A \in \mathcal{F}_+^i$, it follows that in fact

$$E\left(q_*^i(\cdot)\, P'\left(\cdot, Q_*(\cdot)\right) + P\left(\cdot, Q_*(\cdot)\right) - c_i'\left(\cdot, q_*^i(\cdot)\right) \mid \mathcal{F}^i\right)(\omega) = 0 \tag{26}$$

for almost every ω at which $q_*^i > 0$, and

$$E\left(q_*^i(\cdot)\, P'\left(\cdot, Q_*(\cdot)\right) + P\left(\cdot, Q_*(\cdot)\right) - c_i'\left(\cdot, q_*^i(\cdot)\right) \mid \mathcal{F}^i\right)(\omega) \leq 0 \tag{27}$$

for almost every ω at which $q_*^i = 0$.

[15] These assumptions, together with (i)', (ii)', and (iii)', imply conditions of either Theorem 1 or Theorem 2.

Note that for each $\omega \in \Omega$ the function

$$F(q, Q) = qP'(\omega, Q) + P(\omega, Q) - c_i'(\omega, q)$$

is decreasing in q and non-increasing in Q when $q \leq Q$. Indeed, $\frac{\partial F}{\partial q} = P'(\omega, Q) - c_i''(\omega, q) < 0$ since P is decreasing and c is convex by (i)' and (iii)', and $\frac{\partial F}{\partial Q} = qP''(\omega, Q) + P'(\omega, Q) \leq 0$ as follows from (i)' and (ii)'. Now suppose that q_* and q_{**} are two Cournot equilibria. That F is decreasing in q and non-increasing in Q implies that one cannot have

$$\left(q_*^i, Q_*\right) < \left(q_{**}^i, Q_{**}\right) \text{ or } \left(q_*^i, Q_*\right) > \left(q_{**}^i, Q_{**}\right)$$

(inequality in both coordinates and *strict* inequality[16] in the first coordinate) on a set $A \in \mathcal{F}_+^i$. This is because otherwise conditions (26) and (27) would not hold simultaneously for $\max\left(\left(q_*^i, Q_*\right), \left(q_{**}^i, Q_{**}\right)\right)$. To summarize, any firm's equilibrium strategy and the aggregate output in equilibrium cannot move in the same direction:

$$\left(q_*^i, Q_*\right) \not< \left(q_{**}^i, Q_{**}\right) \text{ and } \left(q_*^i, Q_*\right) \not> \left(q_{**}^i, Q_{**}\right) \tag{28}$$

on *any* set $A \in \mathcal{F}_+^i$.

We will next show that every Cournot equilibrium satisfies the equal treatment property, i.e., that strategies of firms of the same type are equal. Indeed, if q_* is a Cournot equilibrium, and $q_*^i \neq q_*^j$ where i and j are firms of the same type (say, 1) then consider an n-tuple q_{**} obtained from q_* by interchanging i and j. Clearly, q_{**} is also a Cournot equilibrium. However, if $A \in \mathcal{F}_+^1$ is a set on which w.l.o.g. $q_*^i > q_*^j = q_{**}^i$, then the obvious fact that $Q_* = Q_{**}$ leads to contradiction with (28). Thus, the equal treatment property holds in any Cournot equilibrium.

Now suppose that q_* and q_{**} are Cournot equilibria in the oligopoly. We will show that they coincide. Due to the equal treatment property, $Q_*(\omega) = |K|\, q_*^1(\omega) + |M|\, q_*^2(\omega)$, and it will suffice to establish that $q_*^i = q_{**}^i$ for $i = 1, 2$. If q_*^2 and q_{**}^2 are *not* equal almost everywhere, then w.l.o.g.

$$q_*^2 > q_{**}^2 \text{ on some } A \in \mathcal{F}_+^2. \tag{29}$$

Consequently,

$$q_*^1 \leq q_{**}^1 \text{ on } A. \tag{30}$$

[16] Recall that a strict inequality $g > h$ (on A) means that $\mu\left(\{\omega \in A \mid g(\omega) \leq h(\omega)\}\right) = 0$.

Indeed, if (30) does not hold, there is $B \subset A$, $B \in \mathcal{F}^1_+$, with[17] $q^1_* > q^1_{**}$ on B. From (29) also $Q_* > Q_{**}$ on B, contradicting (28)).

Consider now the set $D = \{\omega \in A \mid q^1_*(\omega) < q^1_{**}(\omega)\} \in \mathcal{F}^1$. If $D \in \mathcal{F}^1_+$, $Q_* \geq Q_{**}$ on D (otherwise $Q_* < Q_{**}$ on some $E \subset D$, $E \in \mathcal{F}^1_+$, contradicting (28)). And on $A \backslash D$, $q^1_* = q^1_{**}$, and so $Q_* \geq Q_{**}$ because of (29). We conclude that

$$Q_* \geq Q_{**} \text{ on the entire } A. \tag{31}$$

But then (29) and (31) contradict (28). Thus, strategies q^2_* and q^2_{**} must coincide almost everywhere. However, if q^1_* differs from q^1_{**} on some $A \in \mathcal{F}^1_+$, and w.l.o.g. $q^1_* > q^1_{**}$ on A, then $Q_* > Q_{**}$ on A since $q^2_* = q^2_{**}$, contradicting (28) again. We conclude that $q^1_* = q^1_{**}$ as well. ■

5 Existence and Uniqueness of Cournot Equilibrium in Oligopolies with Non-negative Prices

In this section we bring back the condition of non-negativity of prices in the following form, which combines (ii) with the Novshek (1985) condition:

(ii)" there exists $\overline{Q} \in L^+_\infty(\Omega, \mathcal{F}, \mu)$ such that $P(\omega, Q) = 0$ for every $\omega \in \Omega$ and every $Q \geq \overline{Q}(\omega)$; moreover, for every $\omega \in \Omega$ the function $P(\omega, \cdot)$ is twice continuously differentiable[18] on $[0, \overline{Q}(\omega)]$ and $QP''(\omega, Q) + P'(\omega, Q) \leq 0$ on this interval.

According to the next Theorem 4, if for each firm i there exists a certain state-dependent threshold of output, $\overline{q}^i \in L^+_\infty(\Omega, \mathcal{F}^i, \mu)$, which the firm will never desire to exceed (it can never hurt firm i to switch to a strategy that does not exceed the level $\overline{q}^i$), and if adhering to the thresholds by all firms guarantees positive prices at every state of nature, then a Cournot equilibrium exists in an oligopoly under conditions of Theorems 1 and 2 in the previous section. (Existence of such thresholds can be guaranteed if marginal costs of firms increase sufficiently fast.)

[17] Here we use the fact that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$.

[18] At the endpoints of the interval this refers to the continuity of one-side derivatives.

Theorem 4. Consider an oligopoly satisfying (i), (ii)", and (iii). Suppose further that there exists $\left(\overline{q}^i\right)_{i=1}^n \in \prod_{i=1}^n L_\infty^+ \left(\Omega, \mathcal{F}^i, \mu\right)$ such that

$$\sum_{i=1}^n \overline{q}^i \leq \overline{Q}, \tag{32}$$

and such that

$$E\left(\pi^i\left(\cdot, q\left(\cdot\right)\right) \mid \mathcal{F}^i\right) \leq E\left(\pi^i\left(\cdot, q\left(\cdot\right) \mid \min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right)\right) \mid \mathcal{F}^i\right) \tag{33}$$

for every n-tuple of strategies $q = \left(q^i\right)_{i=1}^n$ and every $i \in N$. Assume also that either (1) $n = 2$, or (2) Ω is a finite set and $c_i\left(\omega, \cdot\right)$ is convex for every $i \in N$ and $\omega \in \Omega$. Then a Cournot equilibrium *exists.*

Proof. First, restrict strategy sets of firms to be $S^i = \left[0, \overline{q}^i\right]$. Note that for every n-tuple of strategies $q = \left(q^i\right)_{i=1}^n \in S^1 \times ... \times S^n$, $Q \leq \sum_{i=1}^n \overline{q}^i \leq \overline{Q}$. Hence, strategy profiles in $S^1 \times ... \times S^n$ have exactly the same properties as if the differentiability condition in (ii)" held for all $Q \geq 0$ (i.e., as if (ii)" had the original form (ii)'). Thus, as in the proofs[19] of Theorems 1 and 2 (where strategy sets of each firm were restricted to have the form $[0, Z]$), there is a Cournot equilibrium $q_* = \left(q_*^i\right)_{i=1}^n \in S^1 \times ... \times S^n$ in the oligopoly, provided all unilateral deviations of i considered in (1) are in S^i.

To show that q_* is a Cournot equilibrium in the unrestricted oligopoly as well, we now prove that unilateral deviations of i to strategies outside S^i are not profitable. Indeed, if q^i is ith strategy which is not in S^i, then

$$E\left(\pi^i\left(\cdot, q_*\left(\cdot\right) \mid q^i\left(\cdot\right)\right) \mid \mathcal{F}^i\right) \leq E\left(\pi^i\left(\cdot, q_*\left(\cdot\right) \mid \min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right)\right) \mid \mathcal{F}^i\right)$$

by (33), and

$$E\left(\pi^i\left(\cdot, q_*\left(\cdot\right) \mid \min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right)\right) \mid \mathcal{F}^i\right) \leq E\left(\pi^i \cdot, q_*\left(\cdot\right) \mid \mathcal{F}^i\right)$$

by (3), since $\min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right) \in S^i$. This proves via (3) that q_* is indeed a Cournot equilibrium of the oligopoly without restriction on strategies. ■

[19] We do not need to assume condition (iv) here, although it was needed in the proof of Theorem 1. The reason is that it is implied by the combination of (i) and (ii)".

Remark 4. A closer look into the proof of Theorem 4 reveals that, in fact, a weaker condition would suffice for the existence of a Cournot equilibrium in an oligopoly. Condition (33) could be replaced by the following:

Given any n-tuple of strategies $q = (q^i)_{i=1}^n$, for every firm i there exists a strategy $r^i \leq \overline{q}^i$ such that

$$E\left(\pi^i\left(\cdot, q\left(\cdot\right)\right) \mid \mathcal{F}^i\right) \leq E\left(\pi^i\left(\cdot, q\left(\cdot\right) \mid r^i\left(\cdot\right)\right) \mid \mathcal{F}^i\right). \tag{34}$$

Thus, condition (34) is different from (33) in that $\min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right)$ is replaced by a general $r^i \leq \overline{q}^i$: firm i would prefer *some* strategy r^i over $q^i \notin [0, \overline{q}^i]$, and not necessarily the strategy $\min\left(q^i\left(\cdot\right), \overline{q}^i\left(\cdot\right)\right)$ that simply reduces output to the level $\overline{q}^i$ whenever it exceeds $\overline{q}^i$. ■

Remark 4 leads to the following corollary.

Corollary 1. Consider an oligopoly satisfying (i), (ii)", and (iii). Suppose further that there exists $\left(\overline{q}^i\right)_{i=1}^n \in \prod_{i=1}^n L_\infty^+\left(\Omega, \mathcal{F}^i, \mu\right)$ such that (32) holds and such that

$$E\left(\pi^i\left(\cdot, \left(\overline{q}^i, 0^{-i}\right)\right) \mid \mathcal{F}^i\right) \leq 0 \tag{35}$$

for every $i \in N$. (Here 0^{-i} stands for the zero-output strategies of all firms but i, and accordingly (35) is saying that the conditional expectation of the *monopoly profit* of firm i under strategy $\overline{q}^i$, given its information, is non-positive in all states of nature.) Assume also that either (1) $n = 2$, or (2) Ω is a finite set and $c_i\left(\omega, \cdot\right)$ is convex for every $i \in N$ and $\omega \in \Omega$. Then a Cournot equilibrium *exists*.

Proof. Note that (35) implies condition (34) for the strategy r^i which is equal to 0 on $A = \left\{\omega \mid q^i\left(\omega\right) > \overline{q}^i\left(\omega\right)\right\} \in \mathcal{F}^i$, and to q^i on A^c. Existence of a Cournot equilibrium then follows by Remark 4. ■

The following theorem gives a condition for equilibrium uniqueness, and is a counterpart of Theorem 3 for oligopolies with possibly negative prices. Let us first restate (i)' and (iii)' in the form appropriate in the current setting of non-negative prices:

(i)" For every $\omega \in \Omega$ $P(\omega, \cdot)$ is strictly decreasing on $[0, \overline{Q}(\omega)]$; moreover $\sup_{\omega\in\Omega, 0\leq Q\leq \overline{Q}} |P(\omega, Q)| < \infty$ and $\sup_{\omega\in\Omega, 0\leq Q\leq \overline{Q}} |P'(\omega, Q)| < \infty$.

(iii)" For every $\omega \in \Omega$ and $i \in N$, $c_i(\omega, \cdot)$ is a strictly increasing, twice continuously differentiable and convex function, and $c_i(\cdot, q) \in L^+_\infty(\Omega, \mathcal{F}, \mu)$ for every $q \geq 0$; moreover $\sup_{i\in N, \omega\in\Omega, 0\leq q\leq \sup \overline{Q}+1} c_i'(\omega, q) < \infty$.

Theorem 5. Consider an oligopoly satisfying (i)", (ii)", and (iii)", and assume also that $\overline{Q}$ is strictly positive and measurable with respect to $\mathcal{F}^2$. [20] Suppose further that the set N of firms can be partitioned into two disjoint sets, K and M, such that $1 \in K$, $2 \in M$, and such that $\mathcal{F}^i = \mathcal{F}^1$, $c_i = c_1$ for every $i \in K$, $\mathcal{F}^j = \mathcal{F}^2$, $c_j = c_2$ for every $j \in M$, and $\mathcal{F}^2 \subset \mathcal{F}^1$. Then, *if* a Cournot equilibrium exists (e.g., under conditions of either Theorem 4 or Remark 4) it is *unique*.

Proof. Note that if $q_* = (q^i_*)_{i=1}^n$ is a Cournot equilibrium, then

$$Q_* < \overline{Q}. \tag{36}$$

Indeed, if not, consider the set $A \in \mathcal{F}^1_+$ on which $Q_* \geq \overline{Q}$. If there exists a firm $i \in K$ with $q^i_* > 0$ on some $B \in \mathcal{F}^1_+$, $B \subset A$, then i would benefit by switching its output to zero on B and saving its costs, contradicting (2). And if for all $i \in K$ $q^i_* = 0$ on A, then $\sum_{j\in M} q^j_* = Q_* \geq \overline{Q}$ on A. But since both $\sum_{j\in M} q^j_*$ and $\overline{Q}$ are $\mathcal{F}^2$-measurable, there exists $C \in \mathcal{F}^2_+$ on which $\sum_{j\in M} q^j_* \geq \overline{Q}$ (> 0). Accordingly, there exists a firm $i \in M$ with $q^i_* > 0$ on some $D \in \mathcal{F}^2_+$, $D \subset C$, and just as before this means that i has a profitable deviation from q_* on D, contradicting (2). We conclude that (36) holds.

But if q_* and q_{**} are two Cournot equilibria, it follows from (36) that q_*, q_{**}, and all strategy profiles close to them[21] have exactly the same properties as if the

[20] The last condition is new, and it did not appear in the statement of Theorem 3. It is needed only when prices are not allowed to be negative. Indeed, without $\overline{Q}$'s measurability with respect to both fields, there are counterexamples to uniqueness even if all firms have the same information, see Lagerlöf (2006b).

[21] What we have in mind are strategy profiles that constitute, at each state of nature, small unilateral deviations from q_* or q_{**}.

differentiability condition in (ii)" held for all $Q \geq 0$ (i.e., as if (ii)" had the original form (ii)'). We can therefore show that q_* and q_{**} coincide, just as in the proof of Theorem 3, using the first-order conditions derived from maximization of (24) and (25). ■

Example 3 (Duopoly with Linear Demand and Complete Information on the Q-intercept). Let α, $\beta \in L_\infty^+ (\Omega, \mathcal{F}, \mu)$ be strictly positive functions, and assume moreover that $\beta \in L_\infty^+ (\Omega, \mathcal{F}^1, \mu) \cap L_\infty^+ (\Omega, \mathcal{F}^2, \mu)$. Suppose that we have a duopoly in which for any $\omega \in \Omega$,

$$P(\omega, Q) = \max \{\alpha (\omega) (\beta (\omega) - Q), 0\}.$$

Here $\overline{Q} = \beta$. Since β is both $\mathcal{F}^1$- and $\mathcal{F}^2$-measurable, *both firms know the Q-intercept of P* at every state of nature. This is different from the scenario in Example 1, where a linear duopoly without a Cournot equilibrium was described. In that example $\overline{Q}$ was not measurable with respect to the information field of firm 2. As we shall see, here the assumption of $\overline{Q} = \beta$'s measurability with respect to both fields leads to a different conclusion.

We claim that this duopoly possesses a Cournot equilibrium. Let $\overline{q}^1 = \overline{q}^2 \equiv \frac{1}{2}\beta \in L_\infty^+ (\Omega, \mathcal{F}^1, \mu) \cap L_\infty^+ (\Omega, \mathcal{F}^2, \mu)$. Clearly $(\overline{q}^1, \overline{q}^2)$ satisfies (32) in Theorem 4. We show next that condition (33) of that theorem also holds.

Let $q = (q^1, q^2) \in L_1^+ (\Omega, \mathcal{F}^1, \mu) \times L_1^+ (\Omega, \mathcal{F}^2, \mu)$. If $q^i \in [0, \overline{q}^i]$ for every i then (33) is trivial, and so suppose that $q^i \notin [0, \overline{q}^i]$ for some firm i. We claim that then

$$E \left(\pi^i (\cdot, q (\cdot)) \mid \mathcal{F}^i\right) (\omega) \leq E \left(\pi^i(\cdot, q (\cdot) \mid \overline{q}^i (\cdot)) \mid \mathcal{F}^i\right) (\omega) \tag{37}$$

for almost every $\omega \in A = \left\{\omega \mid q^i (\omega) > \overline{q}^i (\omega)\right\} \in \mathcal{F}^i$. Note that for every $\omega \in \Omega$ and every $x \geq \overline{q}^i (\omega) = \frac{1}{2}\beta (\omega)$, the following holds: either

$$\frac{\partial}{\partial x} [xP(\omega, x + y) - c(\omega, x)] \leq \alpha (\omega) (\beta (\omega) - 2x) - \frac{\partial}{\partial x} c(\omega, x) < 0,$$

if $x + y < \beta (\omega)$, or

$$\frac{\partial}{\partial x} [xP(\omega, x + y) - c(\omega, x)] = -\frac{\partial}{\partial x} c(\omega, x) < 0,$$

if $x + y \geq \beta (\omega)$. Accordingly,

$$\pi^i (\omega, q (\omega)) < \pi^i \left(\omega, q (\omega) \mid \overline{q}^i (\omega)\right)$$

on A, and so (37) holds for almost every $\omega \in A$. Note that this establishes (33) on A. On A^c, $\min\left(q^i(\cdot), \overline{q}^i(\cdot)\right) = q^i(\cdot)$ and thus (33) is trivial. We conclude that in a duopoly with linear prices (33) holds for every $q = (q^1, q^2)$ and every $i = 1, 2$.

But (32) and (33) imply, by Theorem 4, that *the duopoly has a Cournot equilibrium.*

Now assume in addition that $\mathcal{F}^2 \subset \mathcal{F}^1$, and that the cost function of every firm i is convex and twice continuously differentiable. Then, by Theorem 5, this duopoly's *Cournot equilibrium is unique.* ■

Example 4 (Non-Uniqueness of Cournot Equilibrium when no Firm Holds Superior Information). Consider the duopoly in which Ω consists of three states, ω_1, ω_2, and ω_3; each one is chosen by nature with equal probability. In terms of information endowments, firm 1 cannot distinguish only between ω_1 and ω_2, and firm 2 cannot distinguish only between ω_1 and ω_3. Thus $\mathcal{F}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \varnothing, \Omega\}$, and $\mathcal{F}^2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}, \varnothing, \Omega\}$.

In both states of nature the firms face the same quadratic inverse demand function

$$P(Q) = \max\{1 - Q^2, 0\}.$$

Thus, *firms know the inverse demand* at every state of nature; they are, however, asymmetrically informed about the costs. Lagerlöf (2006b) already showed an example of equilibrium non-uniqueness with symmetrically informed firms but with incomplete information on the inverse demand; our example will show that knowing the inverse demand cannot guarantee uniqueness.

Assume further that firm 1 has a constant marginal cost of 0.01 at states ω_1 and ω_2, while its marginal cost is 2 at ω_3. Firm 2 has a constant marginal cost of 0.01 at states ω_1 and ω_3, while its marginal cost is 2 at ω_2.

Since at ω_3 the marginal revenue of firm 1 is always below its marginal cost, it produces zero at this state in any equilibrium. Similarly, firm 2 produces zero at ω_2 in any equilibrium. It follows that in an equilibrium $q_* = (q_*^1, q_*^2)$ both q_*^1 and q_*^2 can be regarded as scalars: q_*^1 can be viewed as the quantity produced by firm 1 at state ω_1 (and thus also at ω_2), and q_*^2 as the quantity produced by firm 2 at state ω_1 (and thus also at ω_3).

We claim that both

$$q_* = (q_*^1, q_*^2) = \left(\frac{3}{10}\sqrt{2}, \frac{3}{10}\sqrt{2}\right) \approx (0.424\,26, 0.424\,26)$$

and

$$q_{**} = (q_{**}^1, q_{**}^2) = \left(\frac{7}{30}\sqrt{6}, \frac{7}{30}\sqrt{6}\right) \approx (0.571\,55, 0.571\,55)$$

are Cournot equilibria in this duopoly.

First let us show that q_* is a Cournot equilibrium. For $q \in [0, 1-q_*^1]$ the expected payoff function of firm 2 when it uses strategy q is

$$\begin{aligned}\Pi^2(q_*^1, q) &= \frac{1}{3}\pi^2\left(\omega_1, (q_*^1, q)\right) + \frac{1}{3}\pi^2\left(\omega_2, (q_*^1, 0)\right) + \frac{1}{3}\pi^2\left(\omega_3, (0, q)\right) \\ &= \frac{1}{3}q\left(1 - \left(\frac{3}{10}\sqrt{2} + q\right)^2\right) + \frac{1}{3}q\left(1-q^2\right) - \frac{2}{3}0.01q,\end{aligned}$$

and its unique maximum on $[0, 1-q_*^1]$ is indeed attained at $q = q_*^2 = \frac{3}{10}\sqrt{2}$. Thus firm 2 has no incentive to deviate from q_*^2 to another strategy in $[0, 1-q_*^1]$. Now, for $q \in [1-q_*^1, 1]$,

$$\Pi^2(q_*^1, q) = \frac{1}{3}q\left(1-q^2\right) - \frac{2}{3}0.01q.$$

The maximum of $\frac{1}{3}q\left(1-q^2\right) - \frac{2}{3}0.01q$ on $[1-q_*^1, 1]$ is attained at $q = \frac{7}{30}\sqrt{6} \approx 0.571\,55$. This maximum is equal to $\frac{343}{6750}\sqrt{6} \approx 0.124\,47$, and therefore firm 2 has no incentive to deviate from q_*^2 (that gives it a payoff $\Pi^2(q_*^1, q_*^2) \approx 0.152\,74$) to a strategy in $[1-q_*^1, 1]$. Since strategies higher than 1 yield negative expected payoff, we have shown that firm 2 will not deviate unilaterally from q_*. By symmetry, the same holds for firm 1, and thus q_* is indeed a Cournot equilibrium.

We show next that q_{**} is a Cournot equilibrium. For $q \in [1-q_{**}^1, 1]$ the expected payoff function of firm 2 is

$$\Pi^2(q_{**}^1, q) = \frac{1}{3}q\left(1-q^2\right) - \frac{2}{3}0.01q,$$

and as was said it is maximized at $q = \frac{7}{30}\sqrt{6} = q_{**}^2$ (and $\Pi^2(q_{**}^1, q_{**}^2) = \frac{343}{6750}\sqrt{6} \approx 0.124\,47$). Thus firm 2 has no incentive to deviate from q_{**}^2 to another strategy in $[1-q_{**}^1, 1]$. Additionally, for $q \in [0, 1-q_{**}^1]$,

$$\Pi^2(q_{**}^1, q) = \frac{1}{3}q\left(1 - \left(\frac{7}{30}\sqrt{6} + q\right)^2\right) + \frac{1}{3}q\left(1-q^2\right) - \frac{2}{3}0.01q$$

has a maximum at $q \approx 0.367\,92$ and the maximum is equal to $\approx 0.117\,98$. Again, 2 has no incentive to deviate from q^2_* to a strategy in $[0, 1 - q^1_{**}]$. Since strategies higher than 1 yield expected negative payoff, this shows that firm 2 will not deviate unilaterally from q_{**}. By symmetry, the same holds for firm 1, and thus q_{**} is another Cournot equilibrium of the duopoly. ■

Remark 5 (Cournot Equilibrium Existence in Mixed Strategies). If we were to allow *mixed strategies* of firms, existence of Cournot equilibrium would follow from the Nash existence theorem under conditions (i)-(iii), at least when the number of states of nature is finite. Indeed, no firm would ever choose with positive probability, as its best response, quantities exceeding $\max \overline{Q}$ at any state of nature, in order to avoid zero revenue (due to assumption (ii)) and to save its costs (due to (iii)). Thus, w.l.o.g. we can assume that the set of (behavioral) mixed strategies of each firm i is $S^i = \left[M(\left[0, \max \overline{Q}\right])\right]^{f_i}$, where $M(\left[0, \max \overline{Q}\right])$ is the set of all probability distributions on the interval $\left[0, \max \overline{Q}\right]$, and f_i is the maximal set of disjoint subsets in the finite field $\mathcal{F}^i$. The set $M(\left[0, \max \overline{Q}\right])$ is compact in the weak topology on measures (see Billingsley (1995)), and hence so is the product set S^i. Since the inverse demand function and the cost functions are continuous by (i) and (iii), the expected payoff function of each firm is continuous in the product (weak) topology on $\prod_{i=1}^{n} S^i$. The expected payoff function Π^i of firm i is also (clearly) concave in its own mixed strategies, $s^i \in S^i$. Consequently, a Cournot equilibrium exists in an oligopoly due to the Nash existence theorem. ■

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