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DECENTRALIZED TRADE, RANDOM UTILITY AND THE EVOLUTION OF SOCIAL WELFARE *

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Abstract

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Decentralized Trade, Random Utility and the Evolution of Social Welfare*

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Abstract

We study decentralized trade processes in general exchange economies and house allocation problems with and without money. The processes are subject to persistent random shocks stemming from agents' maximization of random utility. By imposing structure on the utility noise term –logit distribution–, one is able to calculate exactly the stationary distribution of the perturbed Markov process for any level of noise. We show that the stationary distribution places the largest probability on the maximizers of weighted sums of the agents' (intrinsic) utilities, and this probability tends to 1 as noise vanishes.

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1 Introduction

This paper considers the allocation of indivisible durable goods through decentralized trading processes. A simple example is the allocation of N offices among N students. Even if the size of the problem, N, is relatively small, the number of possible allocations can be quite large. With ten students and offices, the number of allocations is about 3.6 million. When there are other goods to be allocated besides offices (such as parking permits), the problem of finding efficient allocations becomes even more complex. We examine how successful decentralized trading processes are in solving those complex combinatorial problems.

We consider a situation where agents randomly meet over time. When a group of agents meet, they exchange their goods in the following simple way. First, a new allocation for them is randomly proposed, and it is accepted if it provides a higher utility for all of them. Otherwise, the agents continue to hold their endowments. When they assess the proposed allocation, we assume that their utility is subject to random shocks. The shocks can be interpreted as mistakes, or temporal changes in tastes. Alternatively, the shocks may represent speculation based on "animal spirits," that is, an agent may accept a bad bundle of goods for him, betting that it will be exchanged for a better bundle in the future.[1]

Incorporating random terms in utility functions has been found to be quite useful in econometric studies of discrete choice problems (such as the choice of occupation or means of transportation), and we employ one of the leading specifications in econometrics, the logit model, for the distribution of the noise term. Thanks to the special structure of the model, we obtain the closed form solution of the stationary distribution, for any level of noise. This is in contrast to the traditional stochastic stability methodology, first introduced to economics and game theory in Kandori, Mailath and Rob (1993) and Young (1993). The method identifies those states –allocations– in which the economy spends most of its time in the long run, when the noise in the system is made negligible. Negligible noise implies a fairly long waiting time to see the long run effects, and this begs the question about the relevance of the model. The present paper, in contrast, allows us to analyze the case where the noise level is reasonably large, so that the stationary distribution

[1] Our "animal spirits" interpretation is that agents do not always hold rational expectations about the future course of exchange.

provides useful predictions over an economically relevant time horizon. It turns out that in our model, the selected states under vanishing noise remain to be the most likely states in the stationary distribution, for any level of noise. Specifically, we show that, for any level of noise, the states that maximize a weighted sum of the agents' intrinsic utilities receive the largest probability in the stationary distribution and, as the randomness vanishes, the limiting stationary distribution assigns probability one to such states.

Our result sheds light on the previous contribution by Ben-Shoham, Serrano and Volij (2004). They considered house allocation problems and found that, with vanishing noise, the minimum envy allocation is selected when serious mistakes are less likely. An agent's envy level is the number of other agents who have better houses, and the minimum envy allocation is the one that minimizes the aggregate envy level. We show that this somewhat mysterious result can be derived from a more general principle, namely, that evolutionary dynamics with logit noise maximize the aggregate utility level (see Section 4 for the details).

Note that our results imply, in particular, that the most likely state is efficient. One may think that the fact that the trading process reaches an efficient state is not surprising because agents agree to trade only if their payoffs increase. The important point to note, however, is that with no noise the process may be stuck on an inefficient state. For example, when only bilateral trades are possible, this will happen once the economy reaches an inefficient state where there is no double coincidence of wants.[2] In this respect, decentralized trading processes for indivisible goods resemble the algorithms that are used to solve combinatorial optimization problems with multiple local maxima, where the process may get stuck at one of them. For this, it has been found that random search algorithms, notably the ones based on simulated annealing methods (see Aarts and Korst (1989)) are quite effective. Since stochastic evolutionary game theory relies on the same basic idea as simulated annealing, its application to the allocation of indivisible goods should be particularly fruitful. Just like randomness in simulated annealing helps to escape from a local maximum, so does randomness in utility ensure that our trading process is not stuck at an inefficient state.

[2] Ben-Shoham, Serrano and Volij (2004) showed that an inefficient state can be stochastically stable, when all mistakes are equally likely. Hence adding noise does not always help escape from an inefficient state. Our model provides a set of sufficient conditions for the noise term to knock out inefficient states.

Our result is obtained for barter economies, but we also study exchange of goods with monetary transfers. It turns out that our assumption on the noise term and the quasi-linearity of utility in money allows us to extend the same techniques to this case, thereby yielding a similar result. This may be of independent interest: despite money being a continuous variable in the model, we are able to use the methods and framework developed mostly for discrete variables.

The paper is organized as follows. Section 2 analyzes the dynamic model for discrete barter economies, and Section 3 introduces money. The final section discusses related literature.

2 Decentralized Barter: Exchange Economies

There are K durable and indivisible commodities in the economy. The set of agents is $N = \{1, \ldots, I\}$. Agent i's consumption set is $X_i \subset \{0, 1, 2, ...\}^K$. This allows for the possibility that an agent consumes an arbitrary number of units of each good, as in general exchange economies, or only one unit of one of the goods and zero of the others, as in house allocation problems. At time $t \in \{1, 2, \ldots, \}$ agent i holds a bundle of commodities denoted by $z_i(t)$. Although the individuals' holdings may change over time, the aggregate endowment of goods remains fixed, i.e. $\sum_{i \in N} z_i(t) = \overline{z}$. A coalition is a non-empty subset of agents. For any coalition $S \subset N$, a feasible allocation for S at time t is a distribution of their endowments at t. Thus, the set of feasible allocations for S at t is

$$A_S(z_S(t)) = \{z'_S \in \times_{i \in S} X_i \mid \sum_{i \in S} z'_i = \sum_{i \in S} z_i(t)\}$$

and in particular, the set of feasible allocations in the economy is given by

$$Z = A_N(\overline{z}) = \{z'_N \in \times_{i \in N} X_i \mid \sum_{i \in N} z'_i = \overline{z}\}.$$

There is an exogenously given set of allowable coalitions, denoted $\mathsf{S} \subset 2^N$ that may meet and trade in each period. For example, when only pairwise meetings are possible (a particular case of our model), we have $\mathsf{S} = \{S \subset N \mid |S| = 2\}$. At period $t = 1, 2, \ldots$ a coalition $S \in \mathsf{S}$ is selected with probability $q(S) > 0$ (independent of time), and has the opportunity to reallocate their holdings of commodities. We assume that from any initial feasible allocation z, any feasible allocation z' can be reached through a series of feasible proposals by a finite sequence of allowable coalitions $S^1, \dots, S^T \in \mathsf{S}$.

Suppose that, in the current period, a coalition $S \in \mathsf{S}$ is selected, and let $z_S \equiv z_S(t)$ be the allocation of goods for this coalition at the beginning of the current period. A new allocation for this coalition is chosen according to a probability distribution, which may depend on the current allocation, over the set of feasible allocations $A_S(z_S)$. We assume that there is certain symmetry in the proposal distribution.

Assumption 1 For any $z_S, z'_S \in A_S(\cdot)$, the probability that allocation z'_S is chosen when the current allocation is z_S is the same as the probability that allocation z_S is chosen when the current allocation is z'_S.

There are some instances where this requirement is naturally satisfied. For example, this assumption holds when proposals are completely random (a new allocation is drawn from the uniform distribution over the set of feasible allocations for the coalition). Another example is a house allocation problem with pairwise trade: Assumption 1 is satisfied if a pair of players, whenever they meet, always propose to exchange their houses.

We assume that agents' utilities are subject to random shocks, so that agent i's utility is given by

$$v_i(z_i) = u_i(z_i) + \eta_i(z_i), \tag{1}$$

where $u_i(z_i)$ and $\eta_i(z_i)$ stand for the intrinsic utility derived from the bundle z_i and noise, respectively. We assume that, when coalition S is formed, they adopt a (myopic) unanimity rule: when allocation z'_S is proposed instead of z_S, it is adopted if and only if $\forall i \in S$, $u_i(z'_i) + \eta_i(z'_i) \geq u_i(z_i) + \eta_i(z_i)$, with a strict inequality for at least one agent. We assume that the noise term has the following distribution.

Assumption 2 The noise term is independently distributed over time and across agents according to the type I extreme value distribution (or Gumbel distribution) with precision parameter $\beta_i > 0$, whose cumulative distribution function F_i is given by

$$F_i(x) = \exp(-\exp(-\beta_i x - \gamma_i)), \tag{2}$$

where γ_i is a constant so that the resulting mean equals zero.

Note that agent i's preferences over z_i' and z_i depend on random variable $\eta_i(z_i') - \eta_i(z_i)$, and the above assumption basically implies that it has a bell-shaped distribution which is quite similar to normal distribution. When the noise term $\eta_i(z_i)$ is distributed according to (2), it is known that the probability that agent i agrees to receive z_i' in exchange for z_i is given by

$$\Pr(v_i(z_i') > v_i(z_i)) = \frac{\exp[\beta_i u_i(z_i')]}{\exp[\beta_i u_i(z_i')] + \exp[\beta_i u_i(z_i)]}. \tag{3}$$

From this formula it can be seen that, as $\beta_i \to \infty$, noise vanishes and the agent maximizes u_i without any error. That is,

$$\lim_{\beta_i \to \infty} \Pr(v_i(z_i') > v_i(z_i)) = \begin{cases} 1 & \text{if } u_i(z_i') > u_i(z_i) \\ 1/2 & \text{if } u_i(z_i') = u_i(z_i) \\ 0 & \text{if } u_i(z_i') < u_i(z_i) \end{cases}$$

This distributional assumption is what is behind the **logit model** in econometrics in econometrics.

The above description defines a Markov process on the set of feasible allocations of the economy. At every period, the economy can transit from one allocation to another and, since we assumed that it is always possible to go from any allocation to any other through a finite sequence of feasible reallocations, the resulting Markov process is irreducible. Moreover, there is a chance that the state does not change, which makes the process aperiodic. For such a process, there is a unique stationary distribution with the following two properties. Firstly, starting from any initial allocation, the probability distribution on period t allocations is known to approach that stationary distribution as $t \to \infty$. Secondly, the stationary distribution also represents the proportion of time spent on each state over an infinite time horizon. Our first result characterizes this stationary distribution.

Proposition 1 In the barter model with random utility, the stationary distribution over the set of allocations is given by

$$\mu(z) = \frac{\exp \sum_{i \in N} \beta_i u_i(z_i)}{\sum_{z' \in Z} \exp \sum_{i \in N} \beta_i u_i(z_i')}.$$

Before we present the proof, a few remarks are in order. First, the denominator is a normalizing constant, common to all z, to ensure that $\sum_{z \in Z} \mu(z) = 1$, so that only the numerator contains relevant information. The formula tells us that the stationary distribution is "exponentially proportional" to the social welfare function $\sum_{i \in N} \beta_i u_i(z_i)$. In particular, the most likely states (for any level of noise) are the ones that maximize that social welfare. Second, recall that β_i is the precision parameter of agent i's noise term, meaning that a larger β_i implies a smaller level of noise. The formula is easiest to understand when we regard the noise term as the representation of mistakes; an agent who makes fewer mistakes (i.e., who has a higher β_i) has a higher weight in the long run distribution, all other things equal. That is, changes in his utility level have a bigger effect on the long run prediction of the model. Third, the stationary distribution is independent of the matching probabilities, represented by $q(s)$. Suppose that we have two players with identical utility functions and precision parameters, and assume that one has more opportunities to trade than the other. Although one might expect that the one with more opportunities to trade does better than the other, in the long run they receive the same payoff distribution.

Proof. Let $\mathrm{Pr}(z, z')$ be the transition probability from z to z'. It is enough to show that

$$\mu(z) \,\mathrm{Pr}(z, z') = \mu(z') \,\mathrm{Pr}(z', z) \qquad \forall z, z' \in Z. \tag{4}$$

To see that this is sufficient, note that by summing both sides over all $z' \in Z$ we get

$$\mu(z) = \sum_{z' \in Z} \mu(z') \,\mathrm{Pr}(z', z) \qquad \forall z \in Z,$$

which means that μ is a stationary distribution. Equation (4) is what is known as the **detailed balance condition**, and it says that the probability inflows and outflows are balanced for any pair of states. Our symmetric proposal assumption 1 implies $\mathrm{Pr}(z, z') = 0 \Leftrightarrow \mathrm{Pr}(z', z) = 0$, so that (4) is satisfied in such a case. In the remaining case, the closed form formula of $\mu(z)$ implies that the detailed balance condition is satisfied if

$$\frac{\exp \sum_{i \in S'} \beta_i u_i(z_i)}{\exp \sum_{i \in S'} \beta_i u_i(z'_i)} = \frac{\mathrm{Pr}(z', z)}{\mathrm{Pr}(z, z')}, \tag{5}$$

where $S' \equiv \{i \in N \,|\, z'_i \neq z_i\}$ is the set of agents who have different bundles at z and z'. Now let us calculate the transition probabilities $\mathrm{Pr}(z, z')$ and $\mathrm{Pr}(z', z)$. Let $\mathsf{S}' \equiv \{S \in \mathsf{S} \,|\, S' \subset S\}$ be the set of feasible coalitions containing S'. Starting with z, the new allocation z' is obtained if and only if a coalition $S \in \mathsf{S}'$ is selected, proposal z'_S is made, and all members of S' prefer z'_i to z_i.[3] Recalling that $q(S)$ is the probability that coalition S is selected to make a proposal, and denoting by $r_{z_\mathsf{S}}(z'_S)$ the probability that S proposes z'_S, we have, using (3),

$$\begin{aligned} \Pr(z, z') &= \sum_{S \in \mathsf{S}'} q(S) r_{z_\mathsf{S}}(z'_S) \prod_{i \in S} \Pr(v_i(z'_i) > v_i(z_i)) \\ &= \sum_{S \in \mathsf{S}'} q(S) r_{z_\mathsf{S}}(z'_S) \frac{\exp[\sum_{i \in S'} \beta_i u_i(z'_i)]}{H}, \end{aligned}$$

where $H = \prod_{i \in S'} \{\exp[\beta_i u_i(z'_i)] + \exp[\beta_i u_i(z_i)]\}$. Similarly, we have

$$\Pr(z', z) = \sum_{S \in \mathsf{S}'} q(S) r_{z'_\mathsf{S}}(z_S) \frac{\exp[\sum_{i \in S'} \beta_i u_i(z_i)]}{H}.$$

By our symmetric proposal assumption 1, we have $r_{z_\mathsf{S}}(z'_S) = r_{z'_\mathsf{S}}(z_S)$, and the condition (5) is satisfied. ■

Note that the detailed balance equation (4) fails when the proposal distribution does not satisfy assumption 1, as the proof shows: without this assumption, the clean closed form solution cannot be obtained.

Thus, Proposition 1 allows one to obtain the exact proportion of time that the system would spend at each feasible allocation in the long run. Let us now examine how the stationary distribution changes with the level of noise. For simplicity, consider the symmetric case with $\beta_1 = \cdots = \beta_I = \beta$. When the precision parameter of the noise terms β is close to 0, the system is subject to large random shocks, and the expression in Proposition 1 shows that the stationary distribution is close to the uniform distribution. As the level of noise decreases (i.e., as β increases), states with higher social welfare $\sum_{i \in N} u_i(z_i)$ receive higher probabilities. When noise is vanishing ($\beta \to \infty$), each term $\exp \beta \sum_{i \in N} u_i(z_i)$, $z \in Z$ diverges to infinity, but the one that corresponds to the maximizer of the social welfare $\sum_{i \in N} u_i(z_i)$ does so with the highest speed. Hence we have the following characterization.

[3] Agents in $\mathsf{S} \setminus \mathsf{S}'$ are proposed the same bundles as before, so they are indifferent between z'_S and z_S.

Corollary 1 In the barter model with random utility, if the noise is symmetric $\beta_1 = \cdots = \beta_I = \beta$, then as $\beta \to \infty$, the limiting stationary distribution places probability 1 on the set of allocations that maximize the sum of the agents' intrinsic utility functions.

One can generalize the above corollary as follows: if for all $i \in N$ the noise parameter is $\beta_i = \lambda_i \beta$ for some $\lambda_i > 0$, then as $\beta \to \infty$, the limiting stationary distribution places probability 1 on the set of allocations that maximize the weighted utilitarian social welfare function $\sum_{i \in N} \lambda_i u_i(z_i)$.

Remark 1 Note that a monotone increasing transformation of the intrinsic utility functions u_i affects the stationary distribution of the dynamic process. This is so because a transformation of the intrinsic utility function does affect the random preferences. That is, the stationary distribution is invariant only to transformations that preserve the random preferences. For example, the random utility functions $v_i(z_i) = u_i(z_i) + \eta_i(z_i)$ and $w_i(z_i) = au_i(z_i) + a\eta_i(z_i)$ represent the same random preferences for all $a > 0$. In fact, letting $\xi_i = a\eta_i$ we can write $w_i(z_i) = au_i(z_i) + \xi_i(z_i)$ and check that if $\eta_i(z_i)$ is distributed according to the type I extreme value distribution with precision parameter $\beta_i > 0$, then $\xi_i(z_i)$ is distributed according to the type I extreme value distribution with precision parameter $\beta_i/a > 0$.

Remark 2 Wondering about robustness of our results, we have conducted some numerical simulations for our assumed logit noise and for the normal noise case. In the case of the logit model, the simulations reveal the dynamic paths in greater detail. The normal case, which does not admit an analytical solution, yields similar results, although convergence to the efficient allocation appears to be slower. This may come from the fact that the logit distribution has fatter tails, so that large shocks are more likely. The interested reader can find more details in Kandori, Serrano and Volij (2004).

Remark 3 Given a representation $(u_i)_{i \in N}$ of the agents' intrinsic preferences, the way the noise term is introduced affects the evolution of the allocations, and hence also its long run distribution. For instance, assume the noise term enters the utility function in the following multiplicative form:

$$v_i(z_i) = u_i(z_i)\xi_i(z_i), \tag{6}$$

where $\eta_i(z_i) \equiv \ln \xi_i(z_i)$ has type I extreme value distribution with parameter β_i. Then we can repeat the analysis, and by replacing in it u_i with $\ln u_i$, obtain that the stationary distribution $\mu(z)$ is proportional to the weighted Nash social welfare function $\prod_{i \in N} u_i(z_i)^{\beta_i}$. The two models are different if there is a cardinal meaning attached to the intrinsic utility (for example, when $u_i(z_i)$ is interpreted as a von Neumann-Morgenstern (or Bernoulli) utility function, or the monetary value of z_i in the case where agent i has a quasi-linear utility function).

3 Trade with Money: House Allocation Problems with Side Payments

We now consider the case where indivisible goods are traded with (divisible) money. While the barter model of the previous section may be a good approximation of the office allocation in a department, where no monetary transfers are associated with the office assignment, in order to describe a housing market it would be more realistic to introduce monetary transfers.

Specifically, we consider an economy with a set H of houses, and a set N of agents. The number of houses is the same as the number of agents: $|H| = |N|$. An agent's consumption bundle consists of only one house and money. Therefore, a house allocation is an assignment $(z_i)_{i \in N}$ of the houses in H to the agents in N. A typical allocation is an object of the form $((z_i, m_i))_{i \in N}$ such that $(z_i)_{i \in N}$ is a house allocation, and for each $i \in N$, m_i is agent i's money holdings, which for simplicity are allowed to be negative.

Each agent $i \in N$ is assumed to have quasi-linear utility:

$$\pi_i(z_i, m_i) \equiv v_i(z_i) + m_i = u_i(z_i) + \eta_i(z_i) + m_i. \tag{7}$$

As before, $\eta_i(z_i)$ is the random component of utility and it is distributed according to the type I extreme value distribution with precision parameter β_i. Here we assume that the agents have the same parameter: $\beta_i = \beta$ for all $i \in N$. This turns out to be essential for the analysis in this section.

For now we only consider bilateral meetings: in each period a pair of agents (i, j) is selected with probability $q(i, j) > 0$. At the end of the section we shall discuss the extension of our analysis to general exchange economies with money and to a trading process involving coalitions other than pairs of agents. We consider the following bargaining procedure. Suppose agents i and j meet, with the current endowments z_i and z_j. Let $p \in \Re$ be a monetary transfer from i to j, where $i < j$. In other words, we follow a convention that p denotes the payment made by the agent with a lower index, and note that this is without loss of generality, as p can be negative. We suppose that the matched pair first come up with p randomly, and then choose to trade at that price if this is mutually beneficial (according to their utilities with realized noise term). More specifically, let $f_{ij}(p)$ be the density of p for pair (i, j). Its support may be a finite interval, which may vary across different pairs. We assume that this distribution is symmetric: $f_{ij}(p) = f_{ij}(-p)$. When i and j meet, first p is realized according to $f_{ij}(p)$, and then exchange their current holdings at price p if and only if

$$u_i(z_j) + \eta_i(z_j) - p > u_i(z_i) + \eta_i(z_i), \text{ and} \tag{8}$$

$$u_j(z_i) + \eta_j(z_i) + p > u_j(z_j) + \eta_j(z_j). \tag{9}$$

Then, as the random utility shocks $\eta_i(z_j)$ and $\eta_i(z_i)$ have extreme value distribution, condition (8) is satisfied with probability

$$\frac{\exp(\beta(u_i(z_j) - p))}{\exp(\beta(u_i(z_j) - p)) + \exp(\beta u_i(z_i))}. \tag{10}$$

Similarly, given the distributional assumption on $\eta_j(z_i)$ and $\eta_j(z_j)$, condition (9) is satisfied with probability

$$\frac{\exp(\beta(u_j(z_i) + p))}{\exp(\beta(u_j(z_i) + p)) + \exp(\beta u_j(z_j))}. \tag{11}$$

Hence, given p, trade occurs with the product of the above probabilities, which is equal to

$$\frac{\exp[\beta(u_i(z_j) + u_j(z_i))]}{H(p)}. \tag{12}$$

Here, $H(p)$ is the product of the denominators of (10) and (11), and it is equal to

$$\begin{aligned} H(p) \quad = \quad & \exp(\beta(u_i(z_j) + u_j(z_i))) + \exp(\beta(u_i(z_j) + u_j(z_j) - p) \\ & + \exp(\beta(u_i(z_i) + u_j(z_i) + p) + \exp(\beta(u_i(z_i) + u_j(z_j))). \end{aligned} \tag{13}$$

Note that the equality of noise parameters $\beta_i = \beta$ for all $i \in N$ is essential to eliminate p from the numerator of (12). After trade takes place, i possesses z_j and j possesses z_i and the monetary transfer p takes place from i to j. When these agents meet again, the probability of trade (to restore the original endowments) given p is obtained by exchanging z_i and z_j in the above expressions when the transfer is $-p$. Namely, the probability of trade is

$$\frac{\exp[\beta(u_i(z_i) + u_j(z_j))]}{H(-p)}. \tag{14}$$

The above description defines a Markov chain over the set of house allocations. This means that, despite the presence of the divisible commodity "money," we can restrict our attention to the allocation of houses, whose evolution can be described as a Markov chain on a finite state space. Intuitively, this is due in part to the absence of income effects of the quasi-linear utility: the preferences over goods, and therefore the law of motion, are not affected by how much income each agent possesses. In addition, the symmetry of precision parameters and of the price distribution, along with the trading procedure (in which the determination of the price and the swapping of houses are not simultaneous) are the other factors that make this possible.

At this juncture, let us make a couple of remarks about our formulation of money. A possible alternative formulation is to treat money as a medium of exchange to obtain a desirable bundle z_i (a search model a la Kiyotaki and Wright (1989)). There are several reasons why we do not take such a formulation. Firstly, the search model is most fruitfully analyzed when we assume forward-looking, rational players, while our focus here is on myopic, boundedly rational agents. Secondly, it is essential that goods are consumed (or "eaten") and produced over time in the search models. This makes sure that at each moment in time, players have potential demand for fiat money in order to obtain consumption goods in the future. In contrast, our focus is on the allocation of durable goods (whose service flow, not the good itself, is consumed) with fixed supply. In this setting, once a Pareto efficient allocation is reached, there is no intrinsic need for further exchange, and the demand for the medium of exchange disappears. This circumstance would make it hard to derive a positive value for fiat money.[4]

Another possibility is to treat money as one of the durable goods which are traded among the agents (a component of bundle z_i, which enters the utility function u_i). This formulation presupposes that an agent enjoys flow utility from monetary balance even though she does not use it to purchase goods and services, and we find this rather unrealistic. Instead, we adopt a "partial

[4] At least in the benchmark case where agents' utility is subject to no noise.

equilibrium" formulation, where the monetary term m_i represents the flow utility of monetary exchanges that lie outside our model of durable goods allocations.

Let us now denote the Markov chain's stationary distribution by μ^z. Just like in the barter model with random utility, we have the following result.

Proposition 2 In the house allocation problem with money, the stationary distribution for the allocation of houses is given by

$$\mu^z(z) = \frac{\exp\left[\beta \sum_{i\in N} u_i(z_i)\right]}{\sum_{z'\in Z} \exp\left[\beta \sum_{i\in N} u_i(z'_i)\right]}.$$

Proof. Let $\Pr(z, z')$ be the transition probability from state z to z'. Again we will show the detailed balance condition:

$$\mu^z(z)\Pr(z, z') = \mu^z(z')\Pr(z', z). \tag{15}$$

(Recall that summing both sides over z' shows that μ^z is the stationary distribution). To show (15), it is sufficient to prove that

$$\exp[\beta \sum_{k\in N} u_k(z_k)]\Pr(z, z') = \exp[\beta \sum_{k\in N} u_k(z'_k)]\Pr(z', z). \tag{16}$$

If z' cannot be obtained by a pairwise trade from z, then (16) is satisfied because $\Pr(z, z') = \Pr(z', z) = 0$. Otherwise, z' is obtained from z when a pair of agents trade, and let us denote the pair by (i, j), where $i < j$. Hence, we have

$$\begin{aligned} z'_k &= z_k \text{ for } k \neq i, j \text{ and} \\ z'_i &= z_j \text{ and } z'_j = z_i. \end{aligned} \tag{17}$$

Then, we have

$$\Pr(z, z') = q(i, j)\int_{-\infty}^{\infty} \frac{\exp[\beta(u_i(z_j) + u_j(z_i))]}{H(p)} f_{ij}(p)dp,$$

where $H(p)$ is given by (13). Recall that $q(i, j)$ is the probability that the pair (i, j) meets, that given p the exchange occurs with probability (12), and that p is proposed according to density f_{ij}.

Similarly, using (14),

$$\Pr(z', z) = q(i, j)\int_{-\infty}^{\infty} \frac{\exp[\beta(u_i(z_i) + u_j(z_j))]}{H(-p)} f_{ij}(p)dp.$$

By the symmetry assumption on $f_{ij}(\cdot)$, i.e., $f_{ij}(p) = f_{ij}(-p)$, we have

$$T \equiv \int_{-\infty}^{\infty} \frac{f_{ij}(p)}{H(p)} dp = \int_{-\infty}^{\infty} \frac{f_{ij}(-p)}{H(p)} dp = \int_{-\infty}^{\infty} \frac{f_{ij}(p)}{H(-p)} dp,$$

so that

$$\Pr(z, z') = q(i,j) T \exp[\beta(u_i(z_j) + u_j(z_i))],$$

and

$$\Pr(z', z) = q(i,j) T \exp[\beta(u_i(z_i) + u_j(z_j))].$$

Hence, the desired condition (16) holds since, using (17), either side of this condition is equal to

$$q(i,j) T \exp\big[\beta\big(\sum_{k \in N} u_k(z_k) + u_i(z_j) + u_j(z_i)\big)\big].$$

■

Therefore, one obtains the same limiting results as in Corollary 1 as $\beta \to \infty$:

Corollary 2 In the house allocation model with money where the noise is symmetric, $\beta_1 = \cdots = \beta_I = \beta$, the limit stationary distribution, as $\beta \to \infty$, places probability 1 on the goods allocation(s) that maximizes $\sum_{i \in N} u_i(z_i)$.

The following remarks are in order:

Remark 4 The results can be extended to an exchange economy in which there are K indivisible goods (apart from money) and where an agent can hold any subset of the indivisible goods. To do this, as in Section 2, one needs to assume that the proposal distribution in each meeting is "symmetric."

Remark 5 The results can also be extended to a process in which coalitions, not only pairs, trade. To do this, the bargaining procedure played by coalition S begins with the draw of transfers $p = (p_i)_{i \in S}$ that is balanced, i.e., $\sum_{i \in S} p_i = 0$. One should continue to assume that the transfer density is symmetric around 0: $f_S(p) = f_S(-p)$. Then, one can replicate the same steps in the above proof to reach identical conclusions.

4 Related Work

Shapley and Scarf (1974) present basic properties of a special case of discrete allocation problems we considered, known as the house allocation problem, where each agent is assigned exactly one object. Uzawa (1962) studies a deterministic barter process for divisible goods, in a setting where the trading process never gets stuck on inefficient states.

Our work generalizes a result due to Ben-Shoham, Serrano and Volij (2004). In that paper, only pairwise trade in the house allocation problem without money is considered, and the persistent shocks are "mistakes" in decision-making.[5] In particular, they assume that, when an agent has his kth best house, the probability of accepting her mth best house ($m > k$) has the order of ε^{m-k}, where $\varepsilon \in (0, 1)$ is a small number. This is a particular formulation of mistake probabilities, where more serious mistakes are less likely. They showed that, when the randomness is vanishingly small (as $\varepsilon \to 0$), the allocation that minimizes envy is selected in the long run. Agent i's envy level is the number of people who have better houses than agent i (according to i's preferences). The envy in the society is the sum of individual agents' envy levels. The current paper shows that there is a more general mechanism at work operating behind the Ben-Shoham et al result. First, we note that their specification of noise can be related to the logit model. Let N be the number of houses/agents and let us assume that agent i's utility for her kth best house x_i is $u_i(x_i) = N - k + 1$ (so that the utilities of the N houses are $1, 2, \ldots, N$, where N is the utility of the best house). A straightforward calculation shows that we obtain their specification of mistake probabilities, when we add the logit noise term to this utility function. Second, one can see that the envy is equal to $\sum_i (N - u(x_i))$, and minimizing this expression is equivalent to maximizing the utilitarian social welfare $\sum_i u(x_i)$. We have found that the driving force of their result is that the logit noise model maximizes the utilitarian social welfare (and this is true for any specifications of utility functions). Furthermore, we are able to derive the stationary distribution not only when the noise is negligible but also when the randomness is large. This addresses the concern that it takes a very long time to see the predictions of stochastic evolutionary models.

[5] In another paper, Serrano and Volij (2003) explore coalitional trade; their exchange process is quite different, though, because it gives a special role to agents' initial endowments. This leads to connections with Walrasian and core allocations.

Several papers have used logit noise in dynamic adjustment processes (see Durlauf (1997) and the references therein). The most closely related work to ours is Blume (1997), who obtained a closed form expression of the stationary distribution for any level of noise when the following two conditions are satisfied: (i) players play a potential game (i.e., each player's best reply function is the same as in a game in which players have an identical payoff ("potential")) and (ii) at each moment of time, only one player can adjust. Blume (1997) shows that the stationary distribution under the logit noise is given by

$$\frac{\beta P(a)}{\sum_{a' \in A} \beta P(a')},$$

where P is the potential, A is the set of strategy profiles, and β is the common parameter measuring the level of noise (in contrast to our model, a common β is necessary to derive the closed form). Young and Burke (2001) and Sandholm (2005) present applications of Blume's result to the geographical distribution of agricultural contracts in Illinois and Pigouvian pricing under externalities. Our models are different from Blume's in that they do not satisfy the above two conditions. The technical contribution of our paper is to show that a similar closed form expression can be obtained for a wider class of situations, where those conditions are not satisfied.

References

Aarts, E. and J. Korst (1989), *Simulated Annealing and Boltzmann Machines*, Chichester: John Wiley & Sons.

Ben-Shoham, A., R. Serrano and O. Volij (2004), "The Evolution of Exchange," *Journal of Economic Theory* **114**, 310-328.

Blume, L. (1997), "Population Games," in B. Arthur, S. Durlauf, and D. Lane (eds.), *The Economic System as an Evolving Complex System II*, Reading, MA:Addison-Wesley.

Durlauf, S. N. (1997), "Statistical Mechanics Approaches to Socioeconomic Behavior", in B. Arthur, S. Durlauf, and D. Lane (eds.), *The Economic System as an Evolving Complex System II*, Reading, MA:Addison-Wesley.

Kandori, M., G. Mailath, and R. Rob (1993), "Learning, Mutations and Long Run Equilibria in Games," *Econometrica* **61**, 29-56.

Kandori, M., R. Serrano and O. Volij (2004), "Decentralized Trade, Random Utility and the Evolution of Social Welfare," Working Paper 2004-06, Department of Economics, Brown University.

Kiyotaki, N., R. Wright (1989) "On Money as a Medium of Exchange", Journal of Political Economy **97**, 927-954.

Sandholm, W. H. (2005), "Pigouvian Pricing and Stochastic Evolutionary Implementation", forthcoming in Journal of Economic Theory.

Serrano, R. and O. Volij (2003), "Mistakes in Cooperation: the Stochastic Stability of Edgeworth's Recontracting," Working Paper 2003-23, Department of Economics, Brown University.

Shapley, L. and H. Scarf (1974), "On Cores and Indivisibility," Journal of Mathematical Economics **1**, 23-28.

Uzawa, H. (1962), "On the Stability of Edgeworth's Barter Process", International Economic Review, **3**, 218-232.

Young, H. P. (1993), "The Evolution of Conventions," Econometrica **61**, 57-84.

Young, H. P. and M. Burke (2001), "Competition and Custom in Economic Contracts: A Case Study of Illinois Agriculture", American Economic Review **91**, 559-573.