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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624 98 75

STRATEGY-PROOF COALITION FORMATION *

Carmelo Rodríguez-Álvarez [1]

Abstract:

We analyze coalition formation problems in which a group of agents is partitioned into coalitions and agents' preferences only depend on the coalition they belong to. We study rules that associate to each profile of agents' preferences a partition of the society. We focus on strategy-proof rules on restricted domains of preferences, as the domains of additively representable or separable preferences. In such domains, only single-lapping rules satisfy strategy-proofness, individual rationality, non-bossiness, and flexibility. Single-lapping rules are characterized by severe restrictions on the set of feasible coalitions. These restrictions are consistent with hierarchical organizations and imply that single-lapping rules always select core-stable partitions. Thus, our results highlight the relation between the non-cooperative concept of strategy-proofness and the cooperative concept of core-stability. We analyze the implications of our results for matching problems

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[1] Universidad Carlos III de Madrid and Universidad de Málaga. Current address: Departamento de Teoría e Historia Económica, Universidad de Málaga. Plaza El Ejido s/n, Ap. Of. Suc. 4. 29071 Málaga SPAIN.
Phone: (+34) 952 13 12 50, Fax: (+34) 952 13 12 99. E-mail: carmelo@uma.es.
URL: http://www.eco.uc3m.es/cralvare

1 Introduction

We study simple coalition formation problems in which a group of agents is partitioned into coalitions and agents have preferences over the coalitions they are members of. Following the terminology proposed by Drèze and Greensberg [10], we focus on problems characterized by the "hedonic" aspect of coalition formation. Agents' preferences only depend on the identity of the members of the coalition they belong to. Hence, we exclude the existence of externalities among different coalitions. Relevant examples of such problems are matching problems such as marriage and roommate problems, or the formation of social clubs, teams, and societies.

The formation of coalitions is a relevant phenomenon in a wide variety of social and economic environments. The rationale behind the formation of coalitions is that agents form groups in order to exploit the joint benefits of cooperation. The literature on Coalitional Game Theory has extensively analyzed the existence of stable partitions in hedonic coalition formation problems.[1] Instead, we propose a social choice and implementation approach. We study coalition formation rules that associate to each profile of agents' preferences a partition of the group of agents. A coalition formation rule can be interpreted as an optimal recommendation for the society that represents an optimal compromise between the conflicting preferences of the agents. However, since preferences are not observable, they must be elicited from the agents. Thus, given a coalition formation rule, a fundamental concern is whether or not agents have the incentive to reveal their *true* preferences. In this paper, we analyze the possibility of devising coalition formation rules that always give agents such an incentive. Hence, we are interested in rules that satisfy *strategy-proofness*. Strategy-proofness is the strongest decentrability property. It implies that it is a dominant strategy for the agents to straight-forwardly reveal their preferences. Moreover, each agent needs to know only her own preferences to compute her best choice.

It is well known that the requirements of strategy-proofness are hard to meet. In the abstract model of social choice, Gibbard [12] and Satterthwaite [17] show that –provided there are more than two alternatives at stake– every strategy-proof social choice rule is dictatorial. However, reasonable strategy-proof rules exist if appropriate restrictions are imposed on agents' preferences. In coalition formation problems, such domain restrictions

[1]For further references, see the recent works by Banerjee, Konishi, and Sönmez [3], Barberà and Gerber [4], Bogomolnaia and Jackson [6], and Pápai [15].

arise naturally. On the one hand, while coalition formation rules select a partition for each preference profile, each agent only cares about the coalition she is a member of. On the other hand, additional restrictions on how an agent may compare different coalitions can be easily justified. For instance, an interesting class of problems consists of situations in which there are no complementarities among the members of a coalition. That is, the preferences of an agent i regarding the convenience of an agent j joining the coalition i belongs to, do not depend on the coalition to which i is assigned.[2] Then, agents' preferences are additively representable or separable. These domains of preferences have been studied in the general context of abstract social choice by Barberà, Sonnenschein, and Zhou [5] and Le Breton and Sen [13], among others, and positive results have been obtained. Yet, the possibility of constructing strategy-proof coalition formation rules when agents' preferences are additively representable or separable has not been addressed in the literature.

Besides strategy-proofness, we would like our rules to satisfy three additional properties. Our rules should be *individually rational, non-bossy*, and *flexible*. Individual rationality is a participation constraint. It means that no agent should ever be worse-off than she would be if staying alone. Non-bossiness is a collusion-proof requirement. It says that if a change in an agent's preferences does not affect the coalition to which this agent is assigned, then the remaining agents are also unaffected by this change of preferences. Flexibility is a minimal efficiency requirement. It says that every partition formed by a collection of feasible coalitions belongs to the range of the rule. Hence, flexibility implies that feasible disjoint coalitions are mutually compatible.

We characterize a family of rules, the family of single-lapping rules, that fulfill the previous axioms in minimally rich domains of preferences (as the domain of additively representable preferences). Single-lapping rules are characterized by strong restrictions over the set of feasible coalitions –the *single-lapping property*–. These restrictions can be justified by the initial existence of a hierarchical structure of the society that implies that some coalitions can never be formed. These restrictions imply a trade-off between strategy-proofness and efficiency. However, single-lapping rules always select core-stable

[2] Think, for example, in the preferences of a senior member of an Economics Department about the job–candidates for two tenure–track positions that are available (but that need not to be filled). Suppose that there are two candidates, a macroeconomist and an econometrician. If the senior economist prefers hiring the macroeconomist rather than not hiring anybody, then the senior economist should also prefer hiring the macroeconomist and the econometrician rather than hiring the econometrician alone.

partitions of the society, in the sense that no feasible coalition of agents unanimously prefer joining each other rather than staying at the coalition they are assigned to. Hence, our main result provides further evidence on the relation between the non-cooperative game theory concept of strategy-proofness and the cooperative game theory concept of the core-stability.

Before proceeding with the formal analysis, we review the most related literature. Pápai [15] introduces the single-lapping property, and shows that it is a necessary and sufficient condition for the existence of a unique core-stable partition of the society. This author also shows that, given an initial set of coalitions that satisfy the single-lapping property, its associated single-lapping rule is the unique rule that satisfies strategy-proofness, individual rationality, and a weak version of efficiency when agents' preferences over coalitions are restricted to prefer any coalition in the initial set to any other coalition. Our analysis complements Pápai's results. We show that the single-lapping structure of the set of feasible coalition is implied directly by strategy-proofness and the remaining axioms. Moreover, the results also hold in restricted domains of preferences over coalitions.

The manipulability of coalition formation rules has also been studied by Alcalde and Revilla [2], Cechlárová and Romero-Medina [7], Sönmez [19], and Takamiya [20]. However, these works focus on different domains of preferences that are not consistent with additively representable or separable preferences. More specifically, Alcalde and Revilla [2], Cechlárová and Romero-Medina [7] assume that agents' preferences over coalitions are based on the best or the worst group of agents in each coalition. In these environments, they prove the existence of strategy-proof rules that always select core-stable partitions. Finally, Sönmez [19] proposes a general model of allocation of indivisible goods which includes our coalition formation model as a special case. This author focuses on problems for which there always exist core-stable partitions. Under some assumptions on agents' preferences, Sönmez [19] shows that there exist strategy-proof, individually rational, and Pareto efficient rules only if the set of core-stable partitions is always essentially single-valued. Takamiya [20] proves that the converse result also holds under additional assumptions on preferences –such as strict preferences and no consumption externalities– that are fulfilled in coalition formation problems.

The remainder of the paper is organized as follows. In Section 2, we present the model and basic notation. In Section 3, we present different domains of preferences over coalitions and the notion of minimally rich domain. In Section 4, we introduce the main axioms while in Section 5 we present single-lapping rules and provide the characterization results. In Section 6, we prove Theorem 2. In Section 7, we conclude by analyzing the implications of our results for matching problems. We include the proofs of some intermediate results and supplemental material in the Appendices.

2 Basic Notation

Let $N \equiv \{1, \ldots, n\}$ be a society consisting of a finite set of at least 3 agents, $(n \geq 3)$. We call a non-empty subset $C \subseteq N$ a ***coalition***. Let $\mathcal{N}$ denote the set of all non-empty subsets of N. For each $C \in \mathcal{N}$, let $[C] \equiv \{\{i\} : \ i \in C\}$. A ***collection of coalitions*** is a set of coalitions $\Pi \subseteq \mathcal{N}$ that contains all singleton sets, $[N] \subseteq \Pi$. Let σ be a partition of N and let Σ denote the set of all partitions of N. For each $i \in N$ and each $\sigma \in \Sigma$, we denote by $\sigma_i \in \sigma$ the coalition in σ to which i belongs.

For each $i \in N$, let $\mathcal{C}_i \equiv \{C \subseteq N, \ i \in C\}$. That is, $\mathcal{C}_i$ is the set of all coalitions to which i belongs. A ***preference*** for i, $\succsim_i$, is a complete order on $\mathcal{C}_i$.[3] For each $i \in N$, we denote by $\mathcal{D}_i$ the set of all preferences for i. Note that preferences are strict. Hence, for each $i \in N$, each $\succsim_i \in \mathcal{D}_i$, and each $C, C' \in \mathcal{C}_i$, we write $C \succ_i C'$ to indicate that i strictly prefers C to C', and $C \succsim_i C'$ to indicate that either $C \succ_i C'$ or $C = C'$. We assume that agents only care about the coalition they belong to, then agents' preferences over partitions are completely defined by their preferences over coalitions. Thus, abusing notation, we say that for each $i \in N$, each $\succsim \in \mathcal{D}_i$, and each σ, $\sigma' \in \Sigma$, σ is at least as good as σ', $\sigma \succsim_i \sigma'$, if and only if $\sigma_i \succsim_i \sigma'_i$.

For each $i \in N$, each set of coalitions $\mathcal{X} \subseteq \mathcal{N}$ with $\mathcal{X} \cap \mathcal{C}_i \neq \varnothing$, and each $\succsim_i \in \mathcal{D}_i$, let $\text{top}(\mathcal{X}, \succsim_i)$ be the coalition in $\mathcal{X} \cap \mathcal{C}_i$ that is ranked first according to $\succsim_i$.

Let $\mathcal{D} \equiv \times_{i \in N} \mathcal{D}_i$. We call $\succsim \in \mathcal{D}$ a preference profile. For each $C \subset N$, $\mathcal{D}_C = \times_{i \in C} \mathcal{D}_i$, while for each $\succsim \in \mathcal{D}$, $\succsim_C \in \mathcal{D}_C$ denotes the restriction of profile $\succsim$ to the preferences of the agents in C.

Let $\tilde{\mathcal{D}} \subseteq \mathcal{D}$, we say that $\tilde{\mathcal{D}}$ is a ***cartesian domain*** if for each $i \in N$ there is $\tilde{\mathcal{D}}_i \subseteq \mathcal{D}_i$ such that $\tilde{\mathcal{D}} = \times_{i \in N} \tilde{\mathcal{D}}_i$.

We are interested in rules that associate a partition of the society to each profile of agents' preferences.

[3] An order is a reflexive, transitive, and antisymmetric binary relation.

Let $\tilde{\mathcal{D}} \subset \mathcal{D}$ be a cartesian domain. A ***(coalition formation) rule*** defined on the domain $\tilde{\mathcal{D}}$ is a mapping $\varphi : \tilde{\mathcal{D}} \to \Sigma$.

Naturally, for each $i \in N$ and each $\succsim \in \tilde{\mathcal{D}}$, $\varphi_i(\succsim)$ denotes the coalition in $\varphi(\succsim)$ to which i belongs.

Finally, R^φ denotes the range of φ, that is, the set of feasible partitions,

$$R^\varphi \equiv \{\sigma \in \Sigma, \text{ such that there is } \succsim \in \tilde{\mathcal{D}}, \ \varphi(\succsim) = \sigma\},$$

while, F^φ denotes the set of feasible coalitions,

$$F^\varphi \equiv \{C \in \mathcal{N}, \text{ such that for some } \sigma \in R^\varphi, \ C \in \sigma\}.$$

3 Restricted Domains of Preferences over Coalitions

We start by presenting two classes of preferences over coalitions – *top* and *bottom preferences*– that play a crucial role in our analysis. Both domains are contained in other domains of preferences that have been extensively analyzed in the social choice literature, namely, the domains of additively representable and separable preferences. Top and bottom preferences are obtained by extending orders over single agents to orders over coalitions. The basic idea behind top and bottom preferences is that each agent i divides the society into two groups according to some order over the set of agents: the agents that she likes and the agents she dislikes. An agent equipped with top preferences prioritizes (lexicographically) joining the agents she likes the most with respect to avoiding the agents she dislikes. On the other hand, an agent equipped with bottom preferences prioritizes (lexicographically) avoiding the agents she dislikes the most with respect to joining the agents she likes.

Let $\mathcal{P}$ be the set of all complete orders over N. For each $P \in \mathcal{P}$, R denotes the weak order associated to P and it is defined in the usual way. For each $C \subseteq N$ and each $P \in \mathcal{P}$, $\max(C, P)$ and $\min(C, P)$ denote, respectively, the first-ranked and the last-ranked agent of C according to P. Next, for each $i \in N$, each $P \in \mathcal{P}$, and each $C \in \mathcal{C}_i$, let $C_i^+(P) \equiv \{j \in C, \text{ s.t. } j \; R \; i\}$, and $C_i^-(P) \equiv \{j \in C \text{ s.t. } i \; R \; j\}$. Now, define $C_i^+(1, P) \equiv \max(C_i^+(P), P)$ and $C_i^-(1, P) \equiv \min(C_i^-(P), P)$. Once $C_i^+(t, P)$ and

$C_i^-(t, P)$ are defined for some $t \geq 1$, iteratively, let

$$C_i^+(t+1, P) \equiv \max\left(\left[C_i^+(P) \setminus \cup_{k=1}^{t} C_i^+(k, P)\right], P\right), \text{ and}$$
$$C_i^-(t+1, P) \equiv \min\left(\left[C_i^-(P) \setminus \cup_{k=1}^{t} C_i^-(k, P)\right], P\right).$$

For each $i \in N$ and each $P \in \mathcal{P}$, the preference $\succsim_i \in \mathcal{D}_i$ is the ***top preference associated to** P **by** i*, $\succsim_i = \succsim_i^+ (P)$ if for each two distinct coalitions $C, C' \in \mathcal{C}_i$, $C \succ_i C'$ if and only if

- $C_i^+(P) \neq C_i'^+(P)$ and $C_i^+(t, P)\ P\ C_i'^+(t, P)$, where t is the first integer such that $C_i^+(t, P) \neq C_i'^+(t, P)$.
- $C_i^+(P) = C_i'^+(P)$ and $C_i^-(t', P)\ P\ C_i'^-(t', P)$, where t' is the first integer such that $C_i^-(t', P) \neq C_i'^-(t', P)$.

Let $i \in N$, $P \in \mathcal{P}$, and let $C, C' \in \mathcal{C}_i$ be such that $C \neq C'$. When comparing the coalitions C and C', if agent i is equipped with preference $\succsim_i^+ (P)$, then she focuses on the sets of agents who are ranked above i according to P, $C_i^+(P)$ and $C_i'^+(P)$. First, i compares C and C' on the basis of the agents who are first-ranked according to P in $C_i^+(P)$ and $C_i'^+(P)$. If these agents are the same, then i compares the second-ranked agents and so on. If $C_i^+(P) = C_i'^+(P)$, then i turns her attention to the agents who are ranked below i according to P, $C_i^-(P)$ and $C_i'^-(P)$, and applies the same lexicographic logic, but starting from the bottom. She compares first the last ranked agents in $C_i^-(P)$ and $C_i'^-(P)$, and she proceeds iteratively in the case that they are the same agent.

The logic behind bottom preferences mimics top preferences.

For each $i \in N$ and each $P \in \mathcal{P}$,, the preference $\succsim_i \in \mathcal{D}_i$ is the ***bottom preference associated to** P **by** i*, $\succsim_i = \succsim_i^- (P)$ if for each two distinct coalitions $C, C' \in \mathcal{C}_i$, $C \succ_i C'$ if and only if

- $C_i^-(P) \neq C_i'^-(P)$, and $C_i^-(t, P)\ P\ C_i'^-(t, P)$, where t is the first integer such that $C_i^-(t, P) \neq C_i'^-(t, P)$.
- $C_i^-(P) = C_i'^-(P)$ and $C_i^+(t', P)\ P\ C_i'^+(t', P)$, where t' is the first integer such that $C_i^+(t', P) \neq C_i'^+(t', P)$.

Let $i \in N$, $P \in \mathcal{P}$, and let $C, C' \in \mathcal{C}_i$ be such that $C \neq C'$. When comparing the coalitions C and C', if agent i is equipped with the preference $\succsim_i^- (P)$, then i focuses on the sets of agents who are ranked below i according to P, $C_i^-(P)$ and $C_i'^-(P)$. First, i compares C and C' on the basis of the agents who are last-ranked according to P in $C_i^-(P)$ and $C_i'^-(P)$. If these agents are the same, then i compares the next-to-the-last ranked agents and so on. If $C_i^-(P) = C_i'^-(P)$, then i turns her attention to the agents who are ranked above i according to P, $C_i^+(P)$ and $C_i'^+(P)$, and applies the same lexicographic logic, but starting from the top. First, she compares the first-ranked agents in $C_i^+(P)$ and $C_i'^+(P)$, and she proceeds iteratively in the case that they are the same agent.

For each $i \in N$, let

$$\begin{gathered}
\mathcal{D}_i^+ \equiv \{\succsim_i \in \mathcal{D}_i \text{ such that for some } P \in \mathcal{P},\ \ \succsim_i = \succsim_i^+ (P)\}\ , \\
\mathcal{D}_i^- \equiv \{\succsim_i \in \mathcal{D}_i \text{ such that for some } P \in \mathcal{P},\ \ \succsim_i = \succsim_i^- (P)\}\ , \\
\mathcal{D}_i^* \equiv \mathcal{D}_i^+ \cup \mathcal{D}_i^- \ \text{ and,} \\
\mathcal{D}^* \equiv \times_{i \in N} \mathcal{D}_i^*.
\end{gathered}$$

Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$. We say that $\bar{\mathcal{D}}$ is ***minimally rich*** if $\bar{\mathcal{D}}$ is cartesian and $\mathcal{D}^* \subseteq \bar{\mathcal{D}}$.

We consider that a domain of preferences over coalitions is minimally rich if it contains top and bottom preferences. Minimal richness also requires that the domain is cartesian. That is, an agent's set of admissible preferences does not depend on the preferences of the remaining agents.

The following remark shows that in minimally rich domains, the preferences of an agent regarding the way in which she may compare the coalition in which she stays on her own and any two other different coalitions she may belong to are not restricted.

Remark 1. *For each $i \in N$ and each two distinct C, $C' \in \mathcal{C}_i \setminus \{i\}$, there exist $\succsim, \succsim', \succsim'' \in \mathcal{D}_i^*$ such that:*

$$\begin{gathered}
\{i\} \succ C \succ C', \\
C \succ' \{i\} \succ' C', \ \textit{and} \\
C \succ'' C' \succ'' \{i\}.
\end{gathered}$$

Proof. See Appendix A. □

It can be argued that top and bottom preferences reflect rather extreme preferences over coalitions. However, the domains of additively representable and separable preferences are minimally rich. These domains exclude the possibility of (negative or positive) complementarities among the members of a coalition.

Let $i \in N$. A ***utility function*** for agent i is a mapping $u_i : N \to \mathbb{R}$ such that $u_i(i) = 0$. A preference for agent i, $\succsim_i \in \mathcal{D}_i$ is ***additively representable*** if there is a utility function u_i such that for each $C, C' \in \mathcal{C}_i$, $C \succsim_i C'$ if and only if $\sum_{c \in C} u_i(c) \geq \sum_{c' \in C'} u_i(c')$. For each $i \in$N, $\mathcal{A}_i$ denotes the set of all i's additively representable preferences for agent i and let $\mathcal{A} \equiv \times_{i \in N} \mathcal{A}_i$.

A preference for i, $\succsim_i \in \mathcal{D}_i$, is ***separable*** if for each $j \in N$ and each $C \in \mathcal{C}_i$ such that $j \notin C$, $\{i, j\} \succ_i \{i\}$ if and only if $(C \cup \{j\}) \succ_i C$. Let $\mathcal{S}_i$ be the set of all agent i's separable preferences and let $\mathcal{S} \equiv \times_{i \in N} \mathcal{S}_i$.

The following remark shows that the domain of additively representable preferences and the domain of separable preferences are indeed minimally rich domains. Moreover, for small societies both domains coincide with the smallest minimally rich domain.

Remark 2. *Let $i \in N$.*

(a) *If $n \geq 4$, then $\mathcal{D}_i^* \subset \mathcal{A}_i \subset \mathcal{S}_i$.*

(b) *If $n = 3$, then $\mathcal{D}_i^* = \mathcal{A}_i = \mathcal{S}_i$.*

Proof. See Appendix A. □

4 Axioms

This section introduces four properties that rules may satisfy. Let $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ be a cartesian domain and let $\varphi : \tilde{\mathcal{D}} \to \Sigma$ be a rule defined on $\tilde{\mathcal{D}}$.

Our main axiom is an incentive constraint. A rule should never provide an incentive for an agent to misreport her preferences. Only if a rule elicits the true preferences from the agents the social choice will be based upon the correct information. Of course, this property refers to the specific domain in which the rule is defined.

Strategy-Proofness. For each $i \in N$, each $\succsim \in \tilde{\mathcal{D}}$, and each $\succsim_i' \in \tilde{\mathcal{D}}_i$, $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim_{N \setminus \{i\}}, \succsim_i')$. Conversely, we say that $i \in N$ ***manipulates*** φ if there exist $\succsim \in \tilde{\mathcal{D}}$ and $\succsim_i' \in \tilde{\mathcal{D}}_i$ such that $\varphi_i(\succsim_{N \setminus \{i\}}, \succsim_i') \succ_i \varphi_i(\succsim)$.

The Gibbard-Satterthwaite Theorem states that every *strategy-proof* rule on an unrestricted domain either is dictatorial or its range contains only two elements.[4] As we assume that agents' preferences over social outcomes are restricted to depend only on the coalitions they are members of and we focus on minimally rich domains, the negative consequences of the Gibbard-Satterthwaite Theorem do not apply to our framework.

We also consider a minimal participation constraint. Agents should not prefer to stay on their own rather than to belong to the coalition that the rule assigns them.

Individual Rationality. For each $i \in N$ and each $\succsim \in \tilde{\mathcal{D}}$, $\varphi_i(\succsim) \succsim_i \{i\}$.

Note that, for every *individually rational* rule, its set of feasible allocations is a collection of coalitions.

We consider rules such that whenever a change in an agent's preference does not change the coalition she is assigned to, then the assignment for the remaining agents does not change.

Non-Bossiness. For each $i \in N$, each $\succsim \in \tilde{\mathcal{D}}$, and each $\succsim_i' \in \tilde{\mathcal{D}}_i$, $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim_i')$ implies $\varphi(\succsim) = \varphi(\succsim_{N\setminus\{i\}}, \succsim_i')$.

We can interpret *non-bossiness* as a collusion-proof or bribe-proof condition. Imagine that there exists a transferable private good and that agents preferences over coalitions and private good allocations are lexicographic. Agents focus first on the coalition they are assigned, and then in the private good allocation. A violation of *non-bossiness* implies a possibility of collusion because an agent might have incentives to misrepresent her preferences in exchange for a positive transfer of the private good from those who benefit from the change in her preference report.

We also introduce a minimal flexibility condition on the range of the rule. We assume that the range of a rule is determined by the set of feasible coalitions.

Flexibility. For each $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$, $C_t \in F^{\varphi}$ for each $t = 1, \ldots, m$, implies $\sigma \in R^{\varphi}$.

[4]A rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ is ***dictatorial*** if there is $i \in N$ (a dictator) such that for each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i(\succsim) = \text{top}(F^{\varphi}, \succsim_i)$.

Flexibility can be interpreted also as a minimal requirement of efficiency. *Flexibility* means that any two disjoint feasible coalitions are mutually compatible. Hence, it implies that the range of the rule is completely determined by the set of feasible coalitions. However, *flexibility* does not implies *onto-ness*.[5] That is, it may be the case that a *flexible* rule does not admit every conceivable coalition as feasible. On the other hand, with *flexibility* we rule out some coalition formation problems. For instance, any rule defined in a four-agent society, in which every couple of agents is a feasible coalitions but partitions containing two couples are not admissible would violate *flexibility*.

Finally, we introduce a weak version of efficiency. This notion of efficiency for coalition formation problems is introduced in Pápai [15].

Pareto efficiency. For each $\succsim \in \tilde{\mathcal{D}}$, there is no $\sigma \in \Sigma$ such that for each $C \in \sigma$, $C \in F^{\varphi}$, and for every $i \in N$, $\sigma_i \succsim_i \varphi_i(\succsim)$, and for some $j \in N$, $\sigma_j \succ_j \varphi_j(\succsim)$.

Note that *Pareto efficiency* is a version of efficiency restricted to the range of the rule. It is clear that *Pareto efficiency* implies *flexibility*, but it does not implies that every coalition is feasible.

5 Characterization Results

In this section we analyze the implications of the axioms listed above over rules defined on rich domains. First, we introduce additional notation due to Pápai [15]. This author proposes a property over sets of coalitions – the single-lapping property– that ensures the existence of a unique core-stable partition for every preference profile.[6] We make use of this property to define a class of rules.

A collection of coalitions Π satisfies the ***single-lapping property*** if

Condition (a)**:** For each $C, C' \in \Pi$, $C \neq C'$ implies $\#(C \cap C') \leq 1$.

Condition (b)**:** For each $\{C_1, \ldots, C_m\} \subseteq \Pi$ with $m \geq 3$ and for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ (where $m + 1 = 1$), there is $i \in N$ such that for each $t = 1, \ldots, m$, $C_t \cap C_{t+1} = \{i\}$.

[5] A rule $\varphi : \tilde{\mathcal{D}} \to \Sigma$ satisfies ***onto-ness*** if $F^{\varphi} = \mathcal{N}$.

[6] Given a preference profile $\succsim \in \mathcal{D}$ and a collection of coalitions $\Pi \subseteq \mathcal{N}$, the partition $\sigma \in \Sigma$ is ***core-stable*** if there is no $C \in \Pi$ such that for each $j \in C$, $C \succ_j \sigma_j$.

Condition (a) states that if there is an overlap between any two coalitions in the collection, there cannot be more than one agent who is member of these two coalitions. *Condition (b)* is a non-cycle condition. It requires that if a set of coalitions in the collection form a cycle in which every two neighbor coalitions have a common member, then all these coalitions have the same common member.

The following remark presents a prominent property of single-lapping collections of coalitions. For every single-lapping collection of coalitions and for every preference profile, there is a coalition in the collection such that all its members think that this coalition is the best coalition in the collection.

Remark 3 (Theorem 1, Pápai [15].)**.** *Let* Π *be a single-lapping collection of coalitions. For each* $\succsim \in \mathcal{D}$ *there is* $C \in \Pi$ *such that for each* $i \in C$*,* $C = \text{top}(\Pi, \succsim_i)$*.*

Remark 3 implies that for every single-lapping collection of coalitions and every preference profile there is a unique core-stable partition of the society. Pápai [15] presents the following algorithm to find such partition.

For each $\succsim \in \mathcal{D}$ and each single-lapping collection of coalitions $\Pi \subset \mathcal{N}$, the ***core-stable partition associated to*** Π ***at profile*** $\succsim$, $\bar{\sigma}^{\Pi}(\succsim)$, can be identified by the following algorithm:

Algorithm: **Pápai [15].** Let $\succsim \in \mathcal{D}$ and let Π be a single-lapping collection of coalitions. Find $C \in \Pi$ such that for each $i \in C$, $\text{top}(\Pi, \succsim_i) = C$. As Π is single-lapping, such coalition exists. Note that there may be several such coalitions, and all these coalitions are disjoint. Let

$$\Pi(1, \succsim) \equiv \Pi,$$
$$M^{\Pi}(1, \succsim) \equiv \{C \in \Pi \text{ such that for each } i \in C, \text{top}(\Pi, \succsim_i) = C\}$$
$$T^{\Pi}(1, \succsim) \equiv \cup_{C \in M^{\Pi}(1,\succsim)} C$$

Hence, $M^{\Pi}(1, \succsim)$ denotes the set of all the coalitions that are formed in this first stage and $T^{\Pi}(1, \succsim)$ denotes the set of agents that are matched in the first stage. Once $\Pi(t, \succsim)$, $M^{\Pi}(t, \succsim)$, and $T^{\Pi}(t, \succsim)$ are defined for some $t \geq 1$, let,

$$\Pi(t+1, \succsim) \equiv \{C \in \Pi \text{ such that } C \cap T^{\Pi}(t, \succsim) = \{\varnothing\}\},$$
$$M^{\Pi}(t+1, \succsim) \equiv \{C \in \Pi(t+1, \succsim) \text{ such that for each } i \in C, \text{top}(\Pi(t+1, \succsim), \succsim_i) = C\} \text{ and,}$$
$$T^{\Pi}(t+1, \succsim) \equiv \cup_{C \in M^{\Pi}(1,\succsim) \cup \ldots \cup M^{\Pi}(t+1,\succsim)} C.$$

Note that, for each $t = 1, \ldots, m$, $\Pi(t, \succsim) \subset \Pi$, $\Pi(t, \succsim)$ is a collection of coalitions for the reduced society $N \setminus T^{\Pi}(t, \succsim)$. Moreover, $\Pi(t, \succsim)$ satisfies the single-lapping property. Let $m \leq n$ be the smallest integer such that $T^{\Pi}(m, \succsim) = N$. Then, the algorithm identifies a unique partition,

$$\bar{\sigma}^{\Pi}(\succsim) \equiv \{C \in \Pi \text{ such that for some } t \leq m,\ C \in M^{\Pi}(t, \succsim)\}.$$

For each single-lapping collection of coalitions and each preference profile there is a unique core-stable partition. Thus, each single-lapping collection of coalitions defines a unique rule.

Let $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ be a cartesian domain of preferences and let φ be a rule defined on $\tilde{\mathcal{D}}$. The rule φ is a ***single-lapping rule*** if there is a single-lapping collection of coalitions Π such that for each $\succsim \in \tilde{\mathcal{D}}$, $\varphi(\succsim) = \bar{\sigma}^{\Pi}(\succsim)$.

Theorem 3 in Pápai [15] shows that, given a fixed single-lapping collection of coalitions Π, if agents are restricted to prefer standing on their own to any other coalition $C \notin \Pi$, then the single-lapping rule associated to Π is the unique rule that satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency.* Note that for every single-lapping rule, for each preference profile there is always a feasible coalition such that all its members think that it is their best preferred feasible coalition. Thus, single-lapping rules clearly satisfy *strategy-proofness* in any minimally rich domain. In fact, single-lapping rules also satisfy *individual rationality*, *non-bossiness*, *flexibility*, and *Pareto efficiency.*

Theorem 1. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ is single-lapping, then φ satisfies strategy-proofness, individual rationality, non-bossiness, flexibility, and Pareto efficiency.*

Proof. Let $F^{\varphi} = \Pi$. Because φ is a single-lapping rule, Π is a single-lapping collection of coalitions. Let us check that φ satisfies *strategy-proofness.* Let $\succsim \in \bar{\mathcal{D}}$. For each $i \in T^{\Pi}(1, \succsim)$, $\varphi_i(\succsim) = \text{top}(\Pi, \succsim_i)$. Thus, agents in $T^{\Pi}(1, \succsim)$ cannot manipulate. Moreover, by the definition of single-lapping rule for each $\succsim' \in \bar{\mathcal{D}}$ such that for each $i \in T^{\Pi}(1, \succsim)$ $\succsim_i = \succsim'_i$, $\varphi_i(\succsim) = \varphi_i(\succsim')$. Now, let $j \in T^{\Pi}(2, \succsim)$. If there exists $C \in \Pi$ such that $C \succ_j \varphi_j(\succsim)$, then there is $i \in T^{\Pi}(1, \succsim))$ such that $i \in C$. Note that for each $\succsim'_j \in \bar{\mathcal{D}}_j$ and each $i \in T^{\Pi}(1, \succsim)$, $\varphi_i(\succsim_{N \setminus \{j\}}, \succsim'_j) = T^{\Pi}(1, \succsim)$. Thus, $\varphi_j(\succsim) \succsim_j \varphi_j(\succsim_{N \setminus \{j\}}, \succsim'_j)$ and j

cannot manipulate. Repeating iteratively the argument with the remaining steps of the algorithm, we obtain that no agent can manipulate.

Let us check that φ satisfies *individual rationality*. By the definition of single-lapping rule, for each $i \in N$ and each $\succsim \in \bar{\mathcal{D}}$, there is $t \leq n$ such that $\varphi_i(\succsim) \in M^{\Pi}(t, \succsim)$. Note that $\{i\} \in \Pi(t, \succsim)$. By the definition of single-lapping rule, $\varphi_i(\succsim) \equiv \text{top}(\Pi(t, \succsim), \succsim_i)$. Thus, $\varphi_i(\succsim) \succsim_i \{i\}$, which proves *individual rationality*.

Let us check that φ satisfies *non-bossiness*. Let $i \in N$, $\succsim \in \bar{\mathcal{D}}$, and $\succsim_i' \in \bar{\mathcal{D}}_i$ be such that $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim_i')$. Let $i \in T^{\Pi}(t, \succsim)$. Because φ is a single-lapping rule, for each $j \in \cup_{t' \leq t} T^{\Pi}(t', \succsim)$, $\varphi_j(\succsim) = \varphi_j(\succsim_{N\setminus\{i\}}, \succsim_i')$. Moreover, because $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim_i')$, for each $k \in \cup_{t' \geq t} T^{\Pi}(t', \succsim)$, we have $\varphi_k(\succsim) = \varphi_k(\succsim_{N\setminus\{i\}}, \succsim_i')$. Then, $\varphi(\succsim) = \varphi(\succsim')$, which proves *non-bossiness*.

Let us check that φ satisfies *flexibility*. Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, k$, $C_t \in \Pi$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $t = 1, \ldots, m$ and each $i \in C_t$, $\text{top}(\mathcal{N}, \succsim_i) = C_t$. By the definition of single-lapping rule, $\varphi(\succsim) = \sigma$ and $\sigma \in R^{\varphi}$. Thus, φ satisfies *flexibility*.

Finally, *Pareto efficiency* follows immediately from the definition of single-lapping rule. Note that for each $i \in N$ and each $\succsim \in \bar{\mathcal{D}}$ there is $t \leq n$ such that $\varphi_i(\succsim) = \text{top}(\Pi(t, \succsim), \succsim_i)$. □

Note that we only need to assume that the domain of the rule is minimally rich in proving *flexibility*. The proof of the remaining axioms is domain independent. Note also that single-lapping rules satisfy *strategy-proofness* even in the unrestricted domain of preferences over coalitions $\mathcal{D}$. By restricting the set of feasible coalitions, single-lapping rules eliminate agents' opportunities for profitable misrepresentation of preferences. Our next result is, in some way, more surprising. In every minimally rich domain, single-lapping rules are the only rules that satisfy our list of axioms. Hence, reducing the set of admissible preferences for the agents does not allow for additional rules.

Theorem 2. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility then φ is a single-lapping rule.*

We present the proof of Theorem 2 in the next section. The intuition runs as follows. For every rule that satisfies our axioms, when the members of a feasible coalition of individuals agree that this coalition is the best preferred feasible coalition, this coalition is formed. Then, it remains to check that the set of feasible coalitions satisfies the single-lapping property. This step is far from being immediate and constitutes the bulk of the proof. The analysis is relatively simple for three agents societies. We use an induction argument to extend the result to arbitrary societies.

From Theorem 1 and Theorem 2, we obtain the following characterization theorem.

Theorem 3. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. A rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ is a single-lapping rule.*

Theorem 3 shows that only rules that select the unique core-stable partition given an initial set of feasible coalition satisfy our list of axioms. Hence, Theorem 3 provides further evidence on the relation between the concepts of *strategy-proofness* and unique *core-stability.* This relation has been already presented in previous works as Sönmez [19] and Pápai [15].[7] provides several novelties with respect to previous results. We do not impose any restrictions either on preferences or on feasible coalitions that ensure the existence of core-stable partitions. Instead, we obtain that the rule selects the unique core stable partition directly from our axioms. This fact allows us to obtain a characterization result that applies to every kind of coalition formation problem instead of impossibility results. In addition, our results apply to very restricted domains of preferences as the smallest minimally rich domain. Finally, we do not use *Pareto efficiency* in the characterization, instead we use two axioms, *non-bossiness* and *flexibility*, that are not included in the definition of *core-stability.*

The domains of additively representable and separable preferences are minimally rich domains. Hence, we obtain the following corollaries to Theorem 3.

Corollary 1. *A rule $\varphi : \mathcal{A} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ is a single-lapping rule.*

Corollary 2. *A rule $\varphi : \mathcal{S} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ is a single-lapping rule.*

Corollaries 1 and 2 are in sharp contrast with the results of Barberà, *et al.* [5]. These authors analyze problems in which the founding members of a society select new

[7]The line of research that investigates the existence of *strategy-proof* rules in core selecting organizations was initiated by Ledyard [14].

members for the society and their preferences over candidates are additively representable (or separable). They show that for those coalition formation problems, *strategy-proof* rules can be decomposed in a set of yes/no rules, one for each possible candidate. There are two differences between their framework and ours. Barberà, *et al.* [5] do not consider the preferences of the candidates as relevant for the social choice. Moreover, they do not consider the restrictions imposed by *individual rationality*, that we consider indispensable for the analysis of coalition formation rules.

In the light of Theorems 2 and 3, it may seem that the main message of this article is rather negative. Single-lapping rules make compatible our four axioms, but at the cost of severely restricting the set of feasible coalitions. However, there are many real-life applications in which the restrictions implied by single-lapping are likely to arise. Indeed, Pápai [15] shows that single-lapping collections of coalitions can be associated to a non-directed graph endowed with a tree structure. Tree structures are characteristic to many hierarchical societies in which only members of adjacent levels in the hierarchy are connected and can form a coalition.[8] Example 1 describes one such possibility.

Example 1. *Let $N = \{a_1, \ldots, a_{10}\}$ be a transnational firm formed by 10 units. The firm is organized hierarchically. Unit a_1 is the central management unit. Units a_2, a_3, and a_4 represent three different area branches that are under direct supervision of a_1 and that supervise their respective local branches. Units a_5 and a_6 are under the supervision of a_2, units a_7 and a_8 are under the supervision of a_3, and units a_9 and a_{10} are under the supervision of a_4.*

Put Figure 1 about here.

The different units are involved in research activities and may act on its own or it may initiate two-unit joint research ventures, but a joint venture is admissible only if both units are directly communicated in the hierarchy. Hence, the set of feasible coalitions is

$$\Pi^* = [N] \cup \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_5\}, \{a_2, a_6\}, \{a_3, a_7\}, \{a_3, a_8\}, \{a_4, a_9\}, \{a_4, a_{10}\}\}.$$

Note that Π^* is a single-lapping collection of coalitions. Assuming that the each unit manager´s preferences over research ventures only depend on the unit that her unit actually joins, it would be natural to use the single-lapping rule associated to Π^* for decentralizing the research venture formation process.

[8] See Demange [8] and [9] for more on the relation of hierarchical structures and coalitional stability.

Theorems 2 and 3 are tight if there are at least four agents. When there are only three agents, *flexibility* is directly implied by *individual rationality*. The following examples show the independence of the axioms for any arbitrary minimally rich domain $\bar{\mathcal{D}}$.[9]

Example 2 (*Strategy-proofness*). *Let $N = \{i, j, k\}$. For each $\succsim \in \bar{\mathcal{D}}$, let*

$$IR_i(\succsim) \equiv \{C \in \mathcal{C}_i, \text{ such that for each } j \in C, C \succsim_j \{j\}\}.$$

Let φ^{-SP} be such that for each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i^{-SP}(\succsim) \equiv \text{top}(IR_i(\succsim), \succsim_i)$ and for each $j \notin \text{top}(IR_i(\succsim), \succsim_i)$, $\varphi_j^{-SP}(\succsim) \equiv \{j\}$. Note that φ^{-SP} satisfies individual rationality, non-bossiness, and flexibility. However, φ^{-SP} violates strategy-proofness.[10]

Example 3 (*Individual rationality*). *Let $N = \{i, j, k\}$. Let φ^{-IR} be such that for each $\succsim \in \bar{\mathcal{D}}$, $\varphi_i^{-IR}(\succsim) = \text{top}(\mathcal{N}, \succsim_i)$, and for each $j \notin \text{top}(\mathcal{N}, \succsim_i)$, $\varphi_j^{-IR}(\succsim) = \{j\}$. The rule φ^{-IR} is dictatorial. Note that φ^{-IR} satisfies strategy-proofness, non-bossiness, and flexibility. However, φ^{-IR} violates individual rationality.*

Example 4 (*Non-Bossiness*). *Let $N = \{i, j, k\}$. Let φ^{-NB} be such that for each $\succsim \in \bar{\mathcal{D}}$,*

$$\varphi^{-NB}(\succsim) = \begin{cases} \{i, j, k\} & \text{if for each } i' \in N, \{i, j, k\} \succsim_{i'} \{i'\}, \\ (\{i, j\}, \{k\}) & \text{if } \{i, j\} \succ_i \{i\}, \{i, j\} \succ_j \{j\} \text{ and } \text{top}(\mathcal{N}, \succsim_k) = \{k\}, \\ [N] & \text{otherwise.} \end{cases}$$

Note that φ^{-NB} satisfies individual rationality, strategy-proofness, and flexibility. However, φ^{-NB} violates non-bossiness.[11]

Example 5 (*Flexibility*). *Let $N = \{i, j, k, l\}$. Let φ^{-F} be such that for each $\succsim \in \bar{\mathcal{D}}$,*

$$\varphi^{-F}(\succsim) = \begin{cases} (\{i, j\}, \{k, l\}) & \text{if for each } m \in N, \ (\{i, j\}, \{k, l\}) \succsim_m [N], \\ [N] & \text{otherwise.} \end{cases}$$

Note that φ^{-F} satisfies individual rationality, strategy-proofness, and non-bossiness. However, φ^{-F} violates flexibility.

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[9]The following examples are stated in three and four-agent societies. These examples can be easily generalized to arbitrary societies. This point is discussed in Appendix B.

[10]In order to check that φ^{-SP} is *manipulable*, let $N = \{i, j, k\}$, $\succsim \in \mathcal{D}^*$, and $\succsim_j' \in \mathcal{D}_j^*$ be such that $\{i, j\} \succ_i \{i, j, k\} \succ_i \{i\}$, $\{i, j, k\} \succ_j \{i, j\} \succ_j \{j, k\} \succ_j \{j\}$, and $\{i, k\} \succ_k \{i, j, k\} \succ_k \{k\}$; while $\{j, k\} \succ_j' \{i, j, k\} \succ_j' \{j\}$. Note that $\varphi^{-SP}(\succsim) = (\{i, j\}, \{k\})$, while $\varphi^{-SP}(\succsim_{N \setminus \{j\}}, \succsim_j') = \{i, j, k\}$. Then, $\varphi_j^{-SP}(\succsim_{N \setminus \{j\}}, \succsim_j') \succ_j \varphi_j^{-SP}(\succsim)$.

[11]In order to check that φ^{-NB} violates *non-bossiness*, let $\succsim \in \mathcal{D}^*$, $\succsim_k' \in \mathcal{D}_k^*$ be such that $\{i, j\} \succ_i \{i\}$, $\{i, j\} \succ_j \{j\}$, $\text{top}(\mathcal{N}, \succsim_k) = \{k\}$, while $\{j, k\} \succ_k' \{k\} \succ_k' \{i, j, k\}$. Note that $\varphi(\succsim) = (\{i, j\}, \{k\})$ and $\varphi(\succsim_{N \setminus \{k\}}, \succsim_k') = (\{i\}, \{j\}, \{k\})$.

At this point, we clarify the relation between *strategy-proofness*, *non-bossiness*, and *Pareto efficiency.* In many frameworks, *strategy-proofness* and *non-bossiness* directly imply *Pareto efficiency.* However, this is not the case in our framework.[12] On the other hand, although we cannot find a general and straight-forward argument that shows that *strategy-proofness*, *individual rationality*, and *Pareto efficiency*, imply *non-bossiness*, it turns out that the arguments in the proof of Theorem 2 are also valid (with minimal modifications) if we use *Pareto efficiency* instead of *non-bossiness* and *flexibility.* Hence, we can state the following theorem that parallels Theorem 3.[13]

Theorem 4. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. A rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, and Pareto efficiency if and only if φ is a single-lapping rule.*

Proof. See Appendix B. □

Theorem 4 shows that, when applied to *strategy-proof* and *individually rational* rules, *non-bossiness* and *flexibility* are equivalent to *Pareto efficiency.* While sometimes *Pareto efficiency* may seem a more palatable axiom, we think that in coalition formation problems, *non-bossiness* is also easily justifiable. We have chosen to use *non-bossiness* instead of *Pareto efficiency* because *Pareto efficiency* is part of the definition of *core-stability.* We feel that by introducing *individual rationality*–no single agent prefers stay on her own rather than accepting the coalition proposed by the rule– together with *Pareto efficiency*

[12] Consider a society formed by four agents $N = \{i, j, k, l\}$. Define the rule $\bar{\varphi}$ in the domain of separable preferences. Let $\bar{\varphi} : \mathcal{S} \to \Sigma$. Agents i and j are the founding members of a club and they are always together. Then, for each $\succsim \in \mathcal{S}$, $\{i\} \in \bar{\varphi}_j(\succsim)$, $\{j\} \in \bar{\varphi}_i(\succsim)$. Preferences of agents k and l are irrelevant for the social choice. Agent k enters the club if i likes agent k. Thus, $\{k\} \in \bar{\varphi}_i(\succsim)$ if $\{i, k\} \succsim_i \{i\}$. Agent l enters the club if j likes l. Thus, $\{l\} \in \bar{\varphi}_j(\succsim)$ if $\{j, l\} \succsim_l \{j\}$. The rule $\bar{\varphi}$ satisfies *strategy-proofness*, *non-bossiness*, and *flexibility.* However, $\bar{\varphi}$ violates *individual rationality* and *Pareto efficiency.* Let $\succsim \in \mathcal{S}$ be such that $\{i, j, k\} \succ_i \{i, k\} \succ_i \{i, j\} \succ_i \{i\} \succ_i C$ for each $C \in \mathcal{C}_i \setminus (\{i, j, k\}, \{i, k\}, \{i, j\}, \{i\})$, $\{i, j, l\} \succ_j \{j, l\} \succ_j \{i, j\} \succ_j \{j\} \succ_j C'$ for each $C' \in \mathcal{C}_j \setminus (\{i, j, l\}, \{j, l\}, \{i, j\}, \{j\})$, $\{k\} = \text{top}(\mathcal{N}, \succsim_k)$, and $\{l\} = \text{top}(\mathcal{N}, \succsim_l)$. Basically, i likes j and k but strongly dislikes l, j likes i and l and strongly dislikes k, whereas k and l would rather stay alone. Note that $\bar{\varphi}(\succsim) = \{i, j, k, l\}$, but for each $i' \in N$, $(\{i, j\}, \{k\}, \{l\}) \succ_{i'} \bar{\varphi}(\succsim)$.

[13] Note that the examples that show the independence of the axioms in Theorem 3 are also valid to show the independence of the axioms in Theorem 4. It is easy to check that φ^{-SP}, φ^{-IR}, and φ^{-F} satisfy *Pareto efficiency.* However, φ^{-NB} violates *Pareto efficiency.*

–all the members of the society do not prefer an alternative partition to the partition proposed by the rule–, we would introduce too many ingredients of the core in our framework. By focusing on *non-bossiness*, we make a cleaner connection between the non-cooperative concept of *strategy-proofness* and the cooperative concept of core-stability.

Before moving to the proof of Theorem 2, several remarks are in order.

First, we relate our results to those by Sönmez [19] and Takamiya [20]. Sönmez [19] proves that for coalition formation problems in which there is always a core-stable partition, there is a rule that satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency* if the set of core-stable partitions is always essentially single-valued. Actually, Takamiya [20] proves that the converse result is also true for coalition formation problems as the problems we analyze here. The main difference between our framework and Sönmez's one relies on the domain of preferences over coalitions. Sönmez [19] assumes the existence of certain preferences that need not exist in a minimally rich domain. Basically, in Sönmez's framework for each $i \in N$, and each $C \in (F^{\varphi} \cap \mathcal{C}_i)$, if there is an admissible preference $\succsim_i$ such that $C \succ_i \{i\}$, then there is another admissible preference $\succsim'_i$ such that for each $C' \in (F^{\varphi} \cap \mathcal{C}_i) \setminus \{i\}$, $C' \succsim'_i C$ if and only if $C' \succsim_i C$, while $C \succsim_i C'$ if and only if $C \succsim'_i C'$ and $C \succsim'_i \{i\} \succsim'_i C'$. There are minimally rich domains, namely the domain of additively representable preferences, for which such preferences are not admissible. Let $i, j, k \in N$, and assume $\{i, j\}, \{i, k\}, \{i, j, k\} \in F^{\varphi}$. Let $\succsim_i \in \mathcal{A}_i$ be such that $\{i, j, k\} \succ_i \{i, j\} \succ \{i, k\} \succ \{i\}$, but there is no $\succsim'_i \in \mathcal{A}_i$ such that $\{i, j, k\} \succ'_i \{i\}$, $\{i\} \succ'_i \{i, j\}$, and $\{i\} \succ'_i \{i, k\}$.

Finally, we conclude addressing the issue of whether Theorems 3 and 4 hold for domains of preferences strictly contained in $\mathcal{D}^*$. As $\mathcal{D}^*$ consists of the union of the domains of top and bottom preferences, it is natural to check whether there exist non-single-lapping rules that satisfy our axioms in those domains. It turns out that new possibilities arise in both domains. The domain of top preferences is included in the domain of top-responsive preferences proposed by Alcalde and Revilla [2]. These authors provide an algorithm – the top-covering algorithm– that always select a core-stable partition of the society if agents' preferences are top-responsive. In addition, their top-covering algorithm defines a *Pareto efficient* rule that satisfies our axioms in the domain of top preferences $\times_{i\in N}\mathcal{D}_i^+$. On the other hand, the rule φ^{-SP} presented in Example 2 satisfies *strategy-proofness* when defined in the domain of bottom preferences $\times_{i\in N}\mathcal{D}_i^-$.[14] Hence, we can

[14]See Appendix B for additional details.

also interpret Theorems 3 and 4 as minimal domain results. The smallest minimally rich domain $\mathcal{D}^*$ is a minimal domain for which the single-lapping rules are the unique rules that satisfy *strategy-proofness*, *individual rationality*, and either *non-bossiness* and *flexibility*, or *Pareto efficiency*. If we want to use rules for tighter domains (properly included on $\mathcal{D}^*$), then new possibilities arise.

6 Proof of Theorem 2

We begin this section by introducing some properties that are implied by our axioms. These properties incorporate the idea that a rule cannot be against the preferences of the members of the society. When there is a partition that each agent considers at least as good as every other partition, a rule should choose that best-preferred partition. A stronger requirement would be that whenever the members of a coalition consider this coalition as the best coalition, this coalition should form, independently of the preferences of the remaining agents in society. Of course, the following axioms refer to rules defined on a minimally rich domain $\bar{\mathcal{D}}$.

Unanimity. Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, m$, $C_t \in F^\varphi$. For each $\succsim \in \bar{\mathcal{D}}$, each $t = 1, \ldots, m$, and each $i \in C_t$, $\text{top}(F^\varphi, \succsim_i) = C_t$ implies $\varphi(\succsim) = \sigma$.

Top-Coalition. Let $C \in F^\varphi$ and $\succsim \in \bar{\mathcal{D}}$. If for each $i \in C$, $\text{top}(F^\varphi, \succsim_i) = C$, then for each $i \in C$, $\varphi_i(\succsim) = C$.

It is clear that *top-coalition* and *Pareto efficiency* imply *unanimity*. However, *Pareto efficiency* and *top-coalition* are logically independent. Note that *top-coalition* is a property of rules. Banerjee *et al.* [3] use the term top-coalition to name a property of preference profiles. These authors say that a preference profile satisfies the top-coalition property if for every group of agents $V \subseteq N$ there is a coalition $C \subseteq V$ that is mutually the best coalitions for all the members of C. Basically, our *top-coalition* implies that at a preference profile that satisfies Banerjee *et al.*'s top-coalition property, then the rule selects a partition in which all the coalitions such that all their members consider as the best feasible coalition are formed.

Lemma 1. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-*

proofness, non-bossiness, and flexibility, then φ satisfies unanimity.

Proof. Let $\sigma = \{C_1, \ldots, C_m\} \in \Sigma$ be such that for each $t = 1, \ldots, m$, $C_t \in F^\varphi$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $t = 1, \ldots, m$ and each $i \in C_t$, $\text{top}(F^\varphi, \succsim_i) = C_t$. By *flexibility*, $\sigma \in R^\varphi$. Then, there is $\succsim' \in \bar{\mathcal{D}}$, such that $\varphi(\succsim') = \sigma$. Let $i \in N$. Let $\succsim'' \in \bar{\mathcal{D}}$ be such that $\succsim''_i = \succsim_i$ while for each $j \in N \setminus \{i\}$, $\succsim''_j = \succsim'_j$. By *strategy-proofness*, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) \succsim_i \varphi_i(\succsim') = \text{top}(F^\varphi, \succsim_i)$. Then, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) = \varphi_i(\succsim') = \text{top}(F^\varphi, \succsim_i)$. By *non-bossiness*, $\varphi(\succsim'_{N\setminus\{i\}}, \succsim_i) = \varphi(\succsim')$. Repeating the argument as many times as necessary, we obtain $\varphi(\succsim) = \varphi(\succsim')$. □

Lemma 2. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility, then φ satisfies top-coalition.*

Proof. Let $C \in F^\varphi$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $i \in C$, $\text{top}(F^\varphi, \succsim_i) = C$. If $\#C = 1$, then the result follows from *individual rationality*. If $C = N$, then the result is immediate by *unanimity*. Let $\succsim' \in \bar{\mathcal{D}}$ be such that for each $i \in C$, $\text{top}(F^\varphi, \succsim'_i) = C$, for each $C' \in \mathcal{C}_i$ such that there is $j \in (C' \setminus C)$, $\{i\} \succ_i C'$, and for each $k \notin C$, $\succsim_k = \succsim'_k$.[15] By *individual rationality*, for each $i \in C$, $\varphi_i(\succsim') \subseteq C$. Let $\succsim'' \in \mathcal{A}$ be such that for each $i \in C$, $\succsim'_i = \succsim''_i$ while for each $k \in (N \setminus C)$, $\varphi_k(\succsim') = \text{top}(F^\varphi, \succsim''_k)$. By *strategy-proofness*, $\varphi_k(\succsim'_{N\setminus\{k\}}, \succsim''_k) = \varphi_k(\succsim')$. By *non-bossiness*, $\varphi(\succsim'_{N\setminus\{k\}}, \succsim''_k) = \varphi(\succsim')$. Repeating the arguments for each $k \in (N \setminus C)$, $\varphi(\succsim') = \varphi(\succsim'')$. By *unanimity*, for each $i \in C$, $\varphi_i(\succsim'') = C$. Then, $\varphi_i(\succsim') = C$. Finally, let $i \in C$. By *strategy-proofness*, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) \succsim_i \varphi_i(\succsim')$. Then, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) = C$. Repeating the argument as many times as necessary, we obtain that for each $i \in C$, $\varphi_i(\succsim) = C$. □

In the following lemma we prove that agents' preferences over infeasible coalitions are irrelevant for the social choice.

Lemma 3. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness and non-bossiness, then, for each $\succsim, \succsim' \in \bar{\mathcal{D}}$ such that for each $i \in N$, and each $C, C' \in (F^\varphi \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$, $\varphi(\succsim) = \varphi(\succsim')$.*

[15]Note that $\times_{i\in N}\mathcal{D}_i^- \subset \bar{\mathcal{D}}$. Thus, $\succsim'_C \in \bar{\mathcal{D}}_C$.

Proof. Let $\succsim, \succsim' \in \bar{\mathcal{D}}$ be such that for each $i \in N$, and each $C, C' \in (F^{\varphi} \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$. Let $i \in N$. By *strategy-proofness*, $\varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i) \succsim'_i \varphi_i(\succsim)$ and $\varphi_i(\succsim) \succsim_i \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Because for each $C, C' \in (F^{\varphi} \cap \mathcal{C}_i)$, $C \succ_i C'$ if and only if $C \succ'_i C'$, we have $\varphi_i(\succsim) = \varphi_i(\succsim_{N\setminus\{i\}}, \succsim'_i)$. By *non-bossiness*, $\varphi(\succsim) = \varphi(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Repeating the argument as many times as necessary, we get $\varphi(\succsim) = \varphi(\succsim')$. □

The following lemma presents the crucial step in the proof of Theorem 2.

Lemma 4. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility, then F^{φ} satisfies the single-lapping property.*

Proof. The proof is by induction on the number of agents. We first focus on three-agent societies. Then, we extend the result to arbitrary societies. We use extensively throughout the proof the fact that $\mathcal{D}^* \subseteq \bar{\mathcal{D}}$.

Claim 1. Let $n = 3$, then F^{φ} satisfies Condition (a) of the single-lapping property.

Proof. Let $N = \{i, j, k\}$. Assume to the contrary that F^{φ} does not satisfy Condition (a). Then, there are $C, C' \in F^{\varphi}$ such that $\#(C \cap C') \geq 2$. We have two cases.

Case (1.1): $F^{\varphi} = \{\{i\}, \{j\}, \{k\}, \{i, j\}, \{i, j, k\}\}$.

Let $\bar{\succsim}_k \in \mathcal{D}^*_k$ be such that $\{i, j, k\} \bar{\succ}_k \{i, k\} \bar{\succ}_k \{j, k\} \bar{\succ}_k \{k\}$. Let the rule $\bar{\varphi}^{\{i,j\}} : \bar{\mathcal{D}}_{\{i,j\}} \to \Sigma$ be such that for each $\succsim_{\{i,j\}} \in \bar{\mathcal{D}}_{\{i,j\}}$, $\bar{\varphi}^{\{i,j\}}(\succsim_{\{i,j\}}) \equiv \varphi(\succsim_{\{i,j\}}, \bar{\succsim}_k)$. By φ's *strategy-proofness*, $\bar{\varphi}^{\{i,j\}}$ satisfies *strategy-proofness.* By φ's *top-coalition*,

$$R^{\bar{\varphi}^{\{i,j\}}} = \{(\{i\}, \{j\}, \{k\}), (\{i, j\}, \{k\}), \{i, j, k\}\}.$$

By Remark 1, agent i and agent j's preferences over the partitions in $R^{\bar{\varphi}^{\{i,j\}}}$ are unrestricted. Hence, $\bar{\varphi}^{\{i,j\}}$ satisfies *strategy-proofness*, its range contains three elements, and agents' preferences over the elements of the range are unrestricted. Then, by the Gibbard-Satterthwaite Theorem, $\bar{\varphi}^{\{i,j\}}$ is *dictatorial.* Assume that i is the dictator for $\bar{\varphi}^{\{i,j\}}$. Let $\succsim_{\{i,j\}} \in \mathcal{D}^*_{\{i,j\}}$ be such that $\{i, j, k\} \succ_i \{i, j\} \succ_i \{i\}$ and $\{j\} \succ_j \{i, j\} \succ_j \{i, j, k\}$. Then,

$\varphi(\succsim_{\{i,j\}}, \bar{\succsim}_k) = \{i,j,k\}$, but $\{j\} \succ_j \varphi_j(\succsim)$, which violates *individual rationality*, a contradiction.

Case (1.2) $\{\{i\},\{j\},\{k\},\{i,j\},\{j,k\},\{i,j,k\}\} \subseteq F^{\varphi}$.

Let $\succsim^1 \in \mathcal{D}^*$ be such that,

$\succsim^1_i$:	$\succsim^1_j$:	$\succsim^1_k$:
$\{i,j\}$	$\{i,j\}$	$\{j,k\}$
$\{i\}$	$\{j\}$	$\{i,j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{k\}$
$\{i,k\}$	$\{j,k\}$	$\{i,k\}$

By *top-coalition*, $\varphi(\succsim^1) = (\{i,j\},\{k\})$.

Let $\succsim^2 \in \mathcal{D}^*$ be such that $\succsim^2_{N\setminus\{i\}} = \succsim^1_{N\setminus\{i\}}$ and $\{i,j,k\} \succsim^2_i \{i,j\} \succsim^2_i \{i,k\} \succsim^i_2 \{i\}$. By *strategy-proofness*, $\varphi_i(\succsim^2) \succsim^2_i \varphi_i(\succsim^1)$. Then, $\varphi_i(\succsim^2)$ is either $\{i,j,k\}$ or $\{i,j\}$. Because $\{j\} \succ^2_j \{i,j,k\}$, by *individual rationality*, $\varphi_i(\succsim^2) = \{i,j\}$. Then, by *non-bossiness*, $\varphi(\succsim^2) = \varphi(\succsim^1)$.

Let $\succsim^3 \in \mathcal{D}^*$ be such that $\succsim^3_{N\setminus\{j\}} = \succsim^2_{N\setminus\{j\}}$ and $\{i,j\} \succsim^3_j \{i,j,k\} \succsim^3_j \{j\}$. By *strategy-proofness*, $\varphi_j(\succsim^3) \succsim^3_j \varphi_j(\succsim^2)$. Then, $\varphi_j(\succsim^3) = \{i,j\}$. By *non-bossiness*, $\varphi(\succsim^3) = \varphi(\succsim^2)$.

Now, let $\succsim^4 \in \mathcal{D}^*$ be such that $\succsim^4_{N\setminus\{i\}} = \succsim^3_{N\setminus\{i\}}$ and $\{i,k\} \succsim^4_i \{i,j,k\} \succsim^4_i \{i\}$. Then,

$\succsim^4_i$:	$\succsim^4_j$:	$\succsim^4_k$:
$\{i,k\}$	$\{i,j\}$	$\{j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{i,j,k\}$
$\{i\}$	$\{j\}$	$\{k\}$
$\{i,j\}$	$\{j,k\}$	$\{i,k\}$

By *individual rationality*, $\varphi_i(\succsim^4) \neq \{i,j\}$, $\varphi_k(\succsim^4) \neq \{i,k\}$, and $\varphi_j(\succsim^4) \neq \{j,k\}$. By *strategy-proofness*, $\varphi_i(\succsim^3) \succsim^3_i \varphi(\succsim^4)$. Note that, $\{i,j,k\} \succ^3_i \varphi(\succsim^3)$. Then, $\varphi(\succsim^4) = (\{i\},\{j\},\{k\})$.

Let $\succsim^5 \in \mathcal{D}^*$ be such that $\succsim^5_i = \succsim^4_i$, $\{j,k\} \succ^5_j \{j\} \succ^5_j \{i,j,k\} \succsim^5_j \{i,j\}$, and $\{i,j,k\} \succsim^5_k \{j,k\} \succsim^5_k \{i,k\} \succsim^5_k \{k\}$. By *top-coalition*, $\varphi_k(\succsim^5_{N\setminus\{k\}}, \succsim^4_k) = \{j,k\}$. By *strategy-proofness*, $\varphi_k(\succsim^5) \succsim^5_k \{j,k\}$. Because $\{j\} \succ^5_j \{i,j,k\}$, by *individual rationality*, $\varphi(\succsim^5) = (\{i\},\{j,k\})$.

Let $\succsim^6 \in \mathcal{D}^*$ be such that $\succsim^6_{N\setminus\{j\}} = \succsim^5_{N\setminus\{j\}}$ and $\{i,j,k\} \succsim^6_j \{j,k\} \succsim^6_j \{i,j\} \succsim^6_j \{j\}$. Note that, by *unanimity*, $\varphi(\succsim^6_{N\setminus\{i\}}, \succsim^3_i) = \{i,j,k\}$. Hence, by *strategy-proofness*, $\varphi_i(\succsim^6) \succsim_i \{i,j,k\}$. Then, $\varphi(\succsim^6) = \{i,j,k\}$.

Finally, let $\succsim^7 \in \mathcal{D}^*$ be such that $\succsim^7_{N\setminus\{j\}} = \succsim^6_{N\setminus\{j\}}$ and $\succsim^7_j = \succsim^4_j$. Then

$\succsim^7_i$:	$\succsim^7_j$:	$\succsim^7_k$:
$\{i,k\}$	$\{i,j\}$	$\{i,j,k\}$
$\{i,j,k\}$	$\{i,j,k\}$	$\{j,k\}$
$\{i\}$	$\{j\}$	$\{i,k\}$
$\{i,j\}$	$\{j,k\}$	$\{k\}$

Note that the only difference between $\succsim^4$ and $\succsim^7$ consists of k's preference. By *strategy-proofness*, $\varphi_j(\succsim^7) \succsim^7_j \varphi_j(\succsim^6) = \{i,j,k\}$. By *individual rationality*, if $j \in \varphi_i(\succsim^7)$, then $\varphi_i(\succsim^7) = \{i,j,k\}$. Hence, $\varphi(\succsim^7) = \{i,j,k\}$. However, $\varphi_k(\succsim^7) \succ^4_k \varphi_k(\succsim^4)$, which violates *strategy-proofness*, a contradiction.

Cases (1.1) and (1.2) exhaust (up to a relabelling the agents) all the possibilities. Then, F^φ satisfies Condition (a), which concludes the proof of Claim 1. □

Claim 2. Let $n = 3$, then F^φ satisfies Condition (b) of the single-lapping property.

Proof. Assume, to the contrary, that F^φ does not satisfy Condition (b). Then, there is a list of coalitions $\{C_1, \ldots, C_m\} \subset F^\varphi$, with $m \geq 3$ and $m+1 = 1$ such that for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ and for no $i \in N$, $(C_t \cap C_{t+1}) = \{i\}$. By Claim 1, φ satisfies Condition (a). Then, we have $F^\varphi = \{\{i\},\{j\},\{k\},\{i,j\},\{j,k\},\{i,k\}\}$. Thus, for each $\succsim \in \bar{\mathcal{D}}$, there is $i' \in \{i,j,k\}$ such that

$$\varphi_{i'}(\succsim) = \{i'\} \tag{*}$$

Let $\succsim \in \mathcal{D}^*$ be such that $\{i,j\} \succ_i \{i,k\} \succ_i \{i\}$,[16] $\{j,k\} \succ_j \{i,j\} \succ_j \{j\}$, and $\{i,k\} \succ_k \{j,k\} \succ_k \{k\}$. Let $P \in \mathcal{P}$ be such that $k \; P \; i \; P \; j$, and let $\succsim'_i \in \mathcal{D}^*_i$ be $\succsim'_i = \succsim^-_i(P)$. Because $\text{top}(F^\varphi, \succsim'_i) = \text{top}(F^\varphi, \succsim_k) = \{i,k\}$, by *top-coalition*, we have that $\varphi(\succsim_{N\setminus\{i\}}, \succsim'_i) = (\{i,k\},\{j\})$. By *strategy-proofness*, $\varphi_i(\succsim) \succ_i \varphi(\succsim_{N\setminus\{i\}}, \succsim'_i)$. Then, we

[16]Note that by Lemma 3, we only need specify agents' preferences over feasible coalitions.

have that $\varphi_i(\succsim) \neq \{i\}$. Using parallel arguments, we get $\varphi_j(\succsim) \neq \{j\}$ and $\varphi_k(\succsim) \neq \{k\}$, which contradicts (*) and concludes the proof of Claim 2. □

Now, we extend the result to arbitrary finite societies.

Induction Step. There is $m \geq 3$ such that for $n = m$, if the n-agent rule φ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*, then F^{φ} satisfies the single-lapping property. We prove that this is true for $n = m + 1$.

By Claims 1 and 2, the induction hypothesis is true for $m = 3$. Let $n = m+1$. Assume that φ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility.* First, we prove two facts.

Fact 1. For each $C, C' \in F^{\varphi}$ such that $C \cup C' \neq N$, $\#(C \cap C') = 1$.

Proof. Let $C, C' \in F^{\varphi}$ be such that $(C \cup C') \neq N$. Let $j \in N \setminus (C \cup C')$. Let $\bar{\succsim}_j \in \mathcal{D}^*_j$ be such that for each $C \in \mathcal{C}_j$, $C \neq \{j\}$, $\{j\} \bar{\succ}_j C$. Let $\Sigma_{N\setminus\{j\}}$ denote all the partitions of the reduced society $N \setminus \{j\}$. Define the rule $\bar{\varphi}^{N\setminus\{j\}} : \bar{\mathcal{D}}_{N\setminus\{j\}} \to \Sigma_{N\setminus\{j\}}$ in such a way that for each $\succsim_{N\setminus\{j\}} \in \bar{\mathcal{D}}_{N\setminus\{j\}}$, $(\bar{\varphi}^{N\setminus\{j\}}(\succsim_{N\setminus\{j\}}), \{j\}) \equiv \varphi(\succsim_{N\setminus\{j\}}, \bar{\succsim}_j)$. By φ's *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*, $\bar{\varphi}^{N\setminus\{j\}}$ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility.* By the induction hypothesis, $F^{\bar{\varphi}^{N\setminus\{j\}}}$ satisfies the *single-lapping* property. By φ's *flexibility*, $C, C' \in F^{\bar{\varphi}^{N\setminus\{j\}}}$, then $\#(C \cap C') = 1$. □

Similar arguments apply to prove the following fact.

Fact 2. For each $\{C_1, \ldots, C_m\} \subseteq \Pi$ with $m \geq 3$, $\cup_{t=1}^{m} C_t \neq N$, and for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ (where $m + 1 = 1$), there is $i \in N$ such that for each $t = 1, \ldots, m$, $C_t \cap C_{t+1} = \{i\}$.

Claim 1′. F^{φ} satisfies Condition (a).

Proof. Assume, to the contrary, that there are $C, C' \in F^{\varphi}$ such that $(C \cup C') = N$, and $\#(C \cap C') \geq 2$. We replicate the arguments of three-agent societies. There are three cases:

Case (1.0′) Let $C, C' \neq N$.
By Fact 1, either $F^\varphi = \{[N], C, C'\}$, or $F^\varphi = \{[N], C, C', N\}$. Let $\bar{\succsim}_{N\setminus(C\cap C')} \in \mathcal{D}^*_{N\setminus(C\cap C')}$ be such that for each $j \in (C \setminus C')$, $\text{top}(F^\varphi, \succsim_j) = C$, whereas for each $k \in (C' \setminus C)$, $\text{top}(F^\varphi, \succsim_k) = C'$. Define the rule $\bar{\varphi}^{C\cap C'} : \bar{\mathcal{D}}_{C\cap C'} \to \Sigma$ in such a way that for each $\succsim_{C\cap C'} \in \bar{\mathcal{D}}_{C\cap C'}$, $\bar{\varphi}^{C\cap C'}(\succsim_{C\cap C'}) \equiv \varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cap C')})$. Because φ is *strategy-proof*, $\bar{\varphi}^{C\cap C'}$ is *strategy-proof*. Moreover, by *top-coalition*, $R^{\bar{\varphi}^{C\cap C'}} = \{[N], (C, [C' \setminus C]), (C', [C \setminus C'])\}$. By Remark 1, the preferences of the agents in $(C\cap C')$ over the partitions in $R^{\bar{\varphi}^{C\cap C'}}$ are not restricted. By the Gibbard-Satterthwaite Theorem, $\varphi^{C\cap C'}$ is *dictatorial*. Let $i \in (C \cap C')$ be a dictator for $\varphi^{C\cap C'}$. Let $\succsim_{C\cap C'} \in \mathcal{D}^*_{C\cap C'}$ be such that $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_i) = C'$, and for each $j \in (C\cap C')\setminus\{i\}$, $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_j) = \{j\}$. Then, $\varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cup C')}) = (C', [C\setminus C'])$, which violates φ's *individual rationality*, a contradiction.

Case (1.1′) Let $C' = N$, and for no $j \in C$ there is $k \in N \setminus C$ and $C'' \subset N$ with $C'' \in F^\varphi$ such that $\{j, k\} \subseteq C''$.
Let $\succsim_{N\setminus C} \in \mathcal{D}^*_{N\setminus C}$ be such that for each $j \in (N\setminus C)$, $\text{top}(F^\varphi, \bar{\succ}_j) = N$. Define now the rule $\bar{\varphi}^C : \bar{\mathcal{D}}_C \to \Sigma$ in the following way. For each $\succsim_C \in \bar{\mathcal{D}}^*$, $\bar{\varphi}^C(\succ_C) \equiv \varphi(\succsim_C, \bar{\succsim}_{N\setminus C})$. Clearly, $\bar{\varphi}^C$ satisfies *strategy-proofness*. Moreover, by *top-coalition*, for each $i \in C$, $F^\varphi \cap \mathcal{C}_i = F^{\bar{\varphi}^C}$. Hence, by Remark 1, the preferences of the agents in C over partitions in $R^{\bar{\varphi}^C}$ are not restricted. By the Gibbard-Satterthwaite Theorem, $\bar{\varphi}^C$ is *dictatorial*, which, by an already familiar argument, violates φ's *individual rationality*, a contradiction.

Case (1.2′) Let $C' = N$, and for some $j \in C$ there is $k \in N \setminus C$ and $C'' \subset N$ with $C'' \in F^\varphi$ such that $\{j, k\} \subseteq C''$.

Note first that, by Fact 1, for each $C'' \in (F^\varphi \setminus N)$, $\#(C \cap C'') \leq 1$. Moreover, by Fact 2, there is no cycle of three coalitions in F^φ that does not involve the grand coalition N.

Let $C, N \in F^\varphi$, let $j \in C$ be such that for some $T \subseteq N \setminus C$, $T \cup \{j\} \in F^\varphi$. Let $\bar{C} \equiv C \setminus \{j\}$. Let $T' \in F^\varphi \setminus \{C, N\}$. By Fact 1, there is no $i \in C \setminus \{j\}$, such that $\{i, j\} \subseteq T'$. By Fact 2, there is no $k \in T$ such that $\{i, k\} \subseteq T'$.

Let $\succsim^1 \in \mathcal{D}^*$ be such that for each $i \in \bar{C}$, there is $P_i^1 \in \mathcal{P}$ with $j = \max(N, P_i^1)$, $N_i^+(P) = C$, and $\succsim_i^1 = \succsim_i^- (P_i^1)$, for j there is $P_j^1 \in \mathcal{P}$ with $N_j^+(P) = C$, and $\succsim_j^1 = \succsim_j^- (P_j^1)$, while for each $k \in N \setminus C$, there is $P_k^1 \in \mathcal{P}$ with $N_k^+(P) = \{j\} \cup \{k\}$, and $\succsim_k^1 = \succsim_k^+ (P_k^1)$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim^1) = C$.

Next, let $\succsim^2 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\bar{C}}^1 = \succsim_{N\setminus\bar{C}}^2$, while for each $i \in \bar{C}$ there is $P_i^2 \in \mathcal{P}$ such that $j = \max(N, P_i^2)$, $N_i^+(P_i^2) = N$, and $\succsim_i^2 = \succsim_i^+ (P_i^2)$. Note that for each $i \in \bar{C}$, $N = \text{top}(F^\varphi, \succsim_i^2)$ and $C = \text{top}(F^\varphi \setminus N, \succsim_i^2)$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) \succsim_i^2 \varphi_i(\succsim^1) = C$. By *individual rationality*, $\varphi_j(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) \neq N$. Then, $\varphi_i(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) = C$. By *non-bossiness*, $\varphi(\succsim_{N\setminus\{i\}}^1, \succsim_i^2) = \varphi(\succsim^1)$. Repeating the same argument iteratively with each $i \in \bar{C}$, we get $\varphi(\succsim^2) = \varphi(\succsim^1)$.

Let $\succsim^3 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\{j\}}^2 = \succsim_{N\setminus\{j\}}^3$ and $\succsim_j^3 = \succsim_j^+ (P_j^1)$. Note that $\text{top}(F^\varphi, \succsim_j^3) = C$. By *strategy-proofness*, $\varphi_j(\succsim^3) \succsim_j^3 \varphi(\succsim^2)$. Then, $\varphi_j(\succsim^3) = C$, and by *non-bossiness*, $\varphi(\succsim^3) = \varphi(\succsim^2)$.

Let $\succsim^4 \in \mathcal{D}^*$ be such that $\succsim_{N\setminus\bar{C}}^3 = \succsim_{N\setminus\bar{C}}^4$, while for each $i \in \bar{C}$ there is $P_i^4 \in \mathcal{P}$ such that for some $\bar{k} \in T$, $\max(N, P_i^4) = \bar{k}$, $N_i^+(P_i^4) = T \cup \{i\}$, and $\succsim_i^4 = \succsim_i^+ (P_i^4)$. Note that by Fact 2 and our assumptions on F^φ, for each $i \in \bar{C}$, $\text{top}(F^\varphi, P_i^4) = N$, and for each $C \in F^\varphi \cap \mathcal{C}_i$, if $C \neq N$, then $\{i\} \succsim_i^4 C$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim^3) \succsim_i^3 \varphi_i(\succsim_{N\setminus\{i\}}^3, \succsim_i^4)$. Hence, $\varphi_i(\succsim_{N\setminus\{i\}}^3, \succsim_i^4) \neq N$. Repeating the argument for each $i \in \bar{C}$, we obtain that $\varphi(\succsim^4) \neq N$. Clearly, for each $i \in \bar{C}$, N is the only coalition in F^φ that is preferred to staying on her own. On the other hand, for agent j, the coalitions that are preferred to staying alone include some member of $\bar{C}$. Finally, each agent $k \in N \setminus C$ requires the presence of agent j in order to consider a coalition better than staying on her own. Then, by *individual rationality*, we have that $\varphi(\succsim^4) = [N]$.

Consider now the profile $\succsim^5 \in \mathcal{D}^*$, such that for each $i \in \bar{C}$, $\succsim_i^5 = \succsim_i^4$, for some $P_j^5 \in \mathcal{P}$ such that there is $\bar{k} \in T$, with $\max(N, P_j^5) = \bar{k}$ and $N_j^+(P_j^5) = N$, and $\succsim_j^5 = \succsim_j^+ (P_j^5)$, while for each $k \in N \setminus C$, there is $P_k^5 \in \mathcal{P}$ such that $j = \max(N, P_k^5)$, $N = N_k^+(P_k^5)$, and $\succsim_k^5 = \succsim_k^+ (P_k^5)$. By *unanimity*, $\varphi(\succsim^5) = N$.

Finally, let $\succsim^6 \in \mathcal{D}^*$, be such that for each $\succsim_C^6 = \succsim_C^4$, while $\succsim_{N\setminus C}^6 = \succsim_{N\setminus C}^5$. That is, we only change agent j's preferences with respect to the previous profile. By *strategy-proofness*, $\varphi_j(\succsim^6) \succsim^6 \varphi_j(\succsim^5) = N$. By *individual rationality*, for each $i \in \bar{C}$, if $j \in \varphi_i(\succsim^6)$, then $\varphi_i(\succsim^6) = N$. Hence, $\varphi(\succsim^6) = N$. Clearly, $\succsim^6$ only differs from $\succsim^4$ in the preferences of the agents who belong to $N \setminus C$. Let $k \in N \setminus T$. By *strategy-proofness*, we have that $\varphi_k(\succsim_{N\setminus\{k\}}^6, \succsim_k^4) \succsim_k^4 \varphi_k(\succsim^6) = N$. Then, $j \in \varphi_k(\succsim_{N\setminus\{k\}}^6, \succsim_k^4)$. By *individual rationality*, there is $i \in \bar{C}$ such that $i \in \varphi_j(\succsim^6_{N\setminus\{k\}}, \succsim^4_k)$. By Fact 1 and our assumptions over F^φ, $\varphi(\succsim^6_{N\setminus\{k\}}, \succsim^4_k) = N$. Repeating the argument as many times as necessary, we get that $\varphi(\succsim^4) = N$, a contradiction.

Cases $(1.0')$, $(1.1')$ and $(1.2')$ exhaust all the possibilities. Then, this suffices to prove that F^φ satisfies Condition (a). □

Claim 2′. *F^φ satisfies Condition (b).*

Proof. Assume, to the contrary, that φ does not satisfy Condition (b). Then, there is a list of coalitions $\{C_1, \ldots, C_m\}$, with $m \geq 3$ such that for each $t = 1, \ldots, m$, $(m+1 = 1)$, $(C_t \cap C_{t+1}) \neq \{\varnothing\}$, and there is no $i \in N$ such that for each $t = 1, \ldots, m$, $\{i\} = (C_t \cap C_{t+1})$. Because we have just proved that F^φ satisfies Condition (a) of the single-lapping property, we have that for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) = 1$. By Fact 2, $\cup_{t=1}^m C_t = N$. Moreover, $F^\varphi = \{C_1, \ldots, C_m\} \cup [N]$.

For each $t = 1, \ldots, m$, let $i_t \equiv (C_t \cap C_{t+1})$. Note that for each $t = 1, \ldots, m$ and each $j \in (C_t \setminus \{i_{t-1}, i_t\})$, $F^\varphi \cap \mathcal{C}_j = \{C_t, \{j\}\}$. On the other hand, for each $t = 1, \ldots, m$, $F^\varphi \cap \mathcal{C}_{i_t} = \{C_t, C_{t+1}, \{i_t\}\}$. Then, by Remark 1, minimal richness of the domain of preferences does not introduce any restriction on how the agents may order the different coalitions they may belong to. From now on, we only describe agents' preferences over feasible coalitions.

For each $t = 1, \ldots, m$, let $i_t \equiv (C_t \cap C_{t+1})$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $t = 1, \ldots, m$ and each $j \in (C_t \setminus \{i_{t-1}, i_t\})$, $\text{top}(F^\varphi, \succsim_j) = C_t$, and for each $t = 1, \ldots, m$, $\text{top}(F^\varphi, \succsim_{i_t}) = C_{t+1}$, and $C_t \succ_{i_t} \{i_t\}$. Let $t \in \{1, \ldots, m\}$. Let $\succsim'_{i_t} \in \bar{\mathcal{D}}_{i_t}$ be such that $\text{top}(F^\varphi, \succsim'_{i_t}) = C_t$. By *top-coalition*, $\varphi_{i_t}(\succsim_{N\setminus\{i_t\}}, \succsim'_{i_t}) = C_t$. By *strategy-proofness*, $\varphi_{i_t}(\succsim) \succsim_{it} \varphi_{i_t}(\succsim_{N\setminus\{i_t\}}, \succsim'_{i_t})$. Thus, for each $t = 1, \ldots, m$; $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$.

Assume first that m is odd. Then, there is $t' \in \{1, \ldots, m\}$ such that $\varphi_{i_{t'}}(\succsim) = \{i_{t'}\}$, a contradiction with $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$ for each $t = 1, \ldots, m$.

Assume now that m is even. Without loss of generality, assume that for each t odd, $\varphi_{i_t}(\succsim) = C_{t+1}$ and for each t' even, $\varphi_{i_{t'}}(\succsim) = C_{t'}$. Let $\bar{t}$ be even. Let $P_{\bar{t}} \in \mathcal{P}$ be such that $N^+_{\bar{t}}(P_{\bar{t}}) = C_{\bar{t}+1}$. Let $\succsim'_{i_{\bar{t}}} = \succsim^-_{\bar{t}} (P_{\bar{t}}) \in \mathcal{D}^*_{i_{\bar{t}}}$. Note that $\text{top}(F^\varphi, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}+1}$ and for each $T \nsubseteq C_{\bar{t}+1}$, $\{i_{\bar{t}}\} \succ'_{i_{\bar{t}}} T$. By *individual rationality*, $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) \neq C_{\bar{t}}$. Let $\succsim'_{i_{\bar{t}-1}} \in \mathcal{D}^*_{i_{\bar{t}-1}}$ be such that $\text{top}(F^\varphi, \succsim'_{i_{\bar{t}-1}}) = C_{\bar{t}-1}$. By *top-coalition*, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1}, i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}-1}, i_{\bar{t}}\}}) = C_{\bar{t}-1}$.

By *strategy-proofness*, we have that $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}}\}}) \succsim_{i_{\bar{t}-1}} \varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1},i_{\bar{t}}\}}, \succsim'_{\{i_{\bar{t}-1},i_{\bar{t}}\}})$. Then, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}-1}$, and $\varphi_{i_{\bar{t}-2}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}-1}$. Repeating the argument as many times as necessary, for each t odd, $\varphi_{i_t}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_t$, while for each t' even $\varphi_{i_{t'}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{t'+1}$, and $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) = C_{\bar{t}+1}$. Then, we get $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}}, \succsim'_{i_{\bar{t}}}) \succ_{i_{\bar{t}}} \varphi_{i_{\bar{t}}}(\succsim)$, which violates *strategy-proofness*, and suffices to prove Claim $2'$ and Lemma 4. □

Proof of Theorem 2. Let φ satisfy *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. By Lemma 2, φ satisfies *top-coalition*. Let $\succsim \in \bar{\mathcal{D}}$. By Lemma 4, F^{φ} satisfies the single-lapping property. Thus, there is $C \in F^{\varphi}$ such that for each $i \in C$, $\text{top}(F^{\varphi}, \succsim_i) = C$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim) = C$. Moreover, by *top-coalition*, for each $\succsim' \in \bar{\mathcal{D}}$ such that $\succsim_C = \succsim'_C$, for each $i \in C$, $\varphi_i(\succsim') = C$. Let $\Sigma_{N\setminus C}$ denote the set of all possible partitions of the reduced society $N \setminus C$. Define now the restricted social choice function $\bar{\varphi}^{N\setminus C} : \bar{\mathcal{D}}_{N\setminus C} \to \Sigma_{N\setminus C}$, in such a way that for each $\succsim_{N\setminus C} \in \bar{\mathcal{D}}_{N\setminus C}$, $(\bar{\varphi}^{N\setminus C}(\succsim_{N\setminus C}), C) \equiv \varphi(\succsim_{N\setminus C}, \succsim_C)$. Clearly, $\bar{\varphi}^{N\setminus C}$ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. Moreover, $F^{\bar{\varphi}^{N\setminus C}} = \{C' \in F^{\varphi}, C \cap C' = \{\varnothing\}\}$, and $F^{\bar{\varphi}^{N\setminus C}}$ satisfies the single-lapping property. Repeating the same arguments as many times as necessary, we get $\varphi(\succsim) = \bar{\sigma}^{F^{\varphi}}(\succsim)$. □

7 Implications for Matching Problems

In this section we present several applications of our results for matching problems. Many interesting matching problems, as marriage problems and college admission problems are characterized by initial restrictions on the set of feasible coalitions. However, we see that these initial restrictions are not going to suffice to obtain rules that satisfy our sets of axioms. In this section, we investigate the additional restrictions that single-lapping rules introduce in matching problems.

7.1 Marriage and Roommate Problems

Marriage problems are a special class of coalition formation problems. In a marriage problem, agents are divided in two disjoint groups.[17] These two sets are usually interpreted

[17]See Roth and Sotomayor [16] for a comprehensive exposition of modeling and analysis of such problems.

as a set of men and a set of women. Each man has preferences over the couples he may form with any woman and remaining single, and each woman has preferences over the couples she may form with any man and remaining single. The set of feasible coalitions consists of all single agents and all the couples formed by a man and a woman.

Let $M, W \subset N$ be such that $M \cap W = \varnothing$, and $M \cup W = N$, and define

$$\Pi^{(M,W)} \equiv \{(m, w) \subset N, m \in M, w \in W\} \cup [N].$$

Let $\bar{\mathcal{D}} \subseteq \mathcal{D}$ be a minimally rich domain. The rule $\varphi^m : \bar{\mathcal{D}} \to \Sigma$ is a (M,W)***-marriage rule*** if $F^{\varphi^m} \subseteq \Pi^{(M,W)}$. We say that the (M,W)-marriage rule φ^m has ***full-range*** if $F^{\varphi^m} = \Pi^{(M,W)}$.

From Theorems 3 and 4, we immediately obtain the following corollary.[18]

Corollary 3. *Let $M, W \subset N$ be such that $M \cap W = \varnothing$, and $M \cup W = N$, and let $\bar{\mathcal{D}}$ be a minimally rich domain.*

(a) A (M,W)-marriage rule φ^m satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ^m is a single-lapping rule.

(b) A (M,W)-marriage rule φ^m satisfies strategy-proofness, individual rationality, and Pareto efficiency if and only if φ^m is a single-lapping rule.

(c) If $\#M \geq 2$ and $\#W \geq 2$, then there is no full-range, strategy-proof, and individually rational (M,W)-marriage rule that satisfies either non-bossiness and flexibility, or Pareto efficiency.

Note that if $\#M = 1$ or $\#W = 1$, every (M,W)-marriage rule is a single-lapping rule. Clearly, if $\#M \geq 2$ and $\#W \geq 2$, then the set of feasible coalitions of every *full-range* (M,W)-marriage rule does not satisfy the single-lapping property. In this case, we can construct (M,W)-marriage rules that satisfy our axioms if we do not allow some couples to form.

Example 6. *Let $N = \{i, j, k, l\}$. Let $M = \{i, j\}$ and $W = \{k, l\}$. Consider the collection of coalitions $\Pi^m = \{\{i, k\}, \{i, l\}, \{j, k\}\} \cup [N]$. Note that Π^m satisfies the single-lapping*

[18] Pápai [15] has proved part (*b*) of the Corollary. Alcalde and Barberà [1] and Sönmez [19] have proven the last implication of part (*c*). Note that minimal richness of the domain does not introduce any restriction in the way agents may compare couples.

property. Hence, the single-lapping rule associated to Π^m *is a* (M, W)*-marriage rule that satisfies strategy-proofness, individual rationality, non-bossiness, flexibility and Pareto efficiency. However, the couple* $\{j, l\}$ *is not feasible.*

A generalization of marriage problems is known as *roommate problems.* There is a set of agents that have to be organized in couples (or in groups of a given cardinality). For instance, there are a number of rooms available and we can assign either 1 or 2 persons to each room. (Some room may remain empty.)

Let $q \in \mathbb{N}$ be such that $2 \leq q \leq n$. We interpret the integer q as the number of beds available in each room. For each integer q, let $\Pi^q \equiv \{C \in \mathcal{N},\ \#C \leq q\}$.

Let $\bar{\mathcal{D}}$ be a minimally rich domain. We say that the rule $\varphi^q : \bar{\mathcal{D}} \to \Sigma$ is a q-roommate rule if $F^{\varphi^q} \subseteq \Pi^q$. We say that the q-roommate rule φ^q has ***full range*** if $F^{\varphi^q} = \Pi^q$.

It is not difficult to see that for every q, a q-roommate rule, its set of feasible coalitions violates Condition (b) of the single-lapping property.[19] Hence, we obtain the following corollary.

Corollary 4. *Let* $q \in \mathbb{N}$ *be such that* $2 \leq q \leq n$*, and let* $\bar{\mathcal{D}}$ *be a minimally rich domain.*

(a) *A* q*-roommate rule* φ^q *satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if* φ^q *is a single-lapping rule.*

(b) *A* q*-marriage rule* φ^q *satisfies strategy-proofness, individual rationality, and Pareto efficiency if and only if* φ^q *is a single-lapping rule.*

(c) *There is no full-range, strategy-proof, and individually rational* q*-roommate rule that satisfies either non-bossiness and flexibility, or Pareto efficiency.*

7.2 College Admission Problems when Students Care about Classmates

Another generalization of the marriage problem is known as the college admission problem. There are two disjoint sets of agents, a set of colleges C, and a set of new students S. Each college $c \in C$ has a number of free slots and may admit up to a quota of q_c new students. Colleges have preferences over cohorts of new students. New students have

[19] For instance, let $N = \{1, 2, 3\}$ and $q = 2$. Clearly, for every full-range 2-roommate rule φ^q, $\Pi^q = \{C \in \mathcal{N},\ \#C \leq 2\} = [N] \cup \{1, 2\} \cup \{1, 3\} \cup \{2, 3\}$.

preferences over colleges and classmates. A coalition is feasible if either is a singleton or it contains exactly one college and the number of students assigned to each college is not larger than its respective quota q_c.[20]

Let $C, S \subset N$ be such that $C \cap S = \varnothing$, and $C \cup S = N$. The set C represent a set of colleges, and S a cohort of new students. Let $\{q_c\}_{c \in C} \in \mathbb{N}^{\#C}$ denote the list of quotas of available slots in each college. Let $\Pi^{(C,S,\{q_C\})} \equiv \{(c, S_c),\ c \in C,\ S_c \subseteq S, \text{ and } \#S_c \leq q_c\} \cup [N]$.

Let $\bar{\mathcal{D}}$ be a minimally rich domain. The rule $\varphi^c : \bar{\mathcal{D}} \to \Sigma$ is a $(C, S, \{q_c\})$-***college admission rule*** if $F^{\varphi^c} \subseteq \Pi^{(C,S,\{q_C\})}$. We say that the $(C, S, \{q_c\})$-college admission rule φ^c has ***full-range*** if $F^{\varphi^c} = \Pi^{(C,S,\{q_C\})}$.

In a similar fashion as for marriage and roommate problems, we obtain the following corollary.

Corollary 5. *Let $C, S \subset N$ be such that $C \cap S = \varnothing$, and $C \cup S = N$ and let $\bar{\mathcal{D}}$ be a minimally rich domain.*

(a) A $(C, S, \{q_c\})$-college admission rule φ^m satisfies strategy-proofness, individual rationality, non-bossiness, and flexibility if and only if φ^m is a single-lapping rule.

(b) A $(C, S, \{q_c\})$-college admission rule φ^m satisfies strategy-proofness, individual rationality, and Pareto efficiency if and only if φ^m is a single-lapping rule.

(c) If $\#S \geq 2$ and either $\#C \geq 2$ or if $C = \{c\}$, $q_c \geq 2$, then there is no full-range, strategy-proof, and individually rational $(C, S, \{q_c\})$-college admission rule that satisfies either non-bossiness and flexibility, or Pareto efficiency.

Sönmez [18] shows that if students only care about the college they attend and do not care about their classmates, and each college has an unlimited number of available slots, then there is always a unique core-stable partition. Moreover the rule that selects that partition satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency.* Of course, his result does not contradicts our Theorem 4, since in Sönmez's framework it is assumed that students do not care about their classmates. However, the different

[20]When students care about the identity of their classmates, Dutta and Massó [11] have shown that core-stable partitions may fail to exist. This is not the case if students only care about the college they attend. See Roth and Sotomayor [16] for further details on the existence of core-stable partitions in college-admission problems.

domains of preferences that we analyze explain that we cannot obtain impossibility results as corollaries to Theorems 3 and 4 in Sönmez's framework.

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8 Appendices

8.1 Appendix A: Proofs of Remark 1 and Remark 2

Proof of Remark 1. Let $i \in N$, and $C, C' \in \mathcal{C}_i$ be such that $C \neq C'$.

1. First, we prove that there exists $\succsim \in \mathcal{D}_i^*$ such that $\{i\} \succ C \succ C'$. We consider two cases:

   (a) There exists an agent $x \in C' \setminus C$. Let $P \in \mathcal{P}$ be such that $N_i^+(P) = \{i\}$ and $x = \min(N, P)$. Then, $\{i\} \succ_i^- (P)\; C \succ_i^- (P)\; C'$.

   (b) $C' \subset C$. Let $P' \in \mathcal{P}$ be such that $N_i^+(P') = (C \setminus C') \cup \{i\}$. Then, we have that $\{i\} \succ_i^- (P')\; C \succ_i^- (P')\; C'$.

2. Now, we prove that there exists $\succsim' \in \mathcal{D}_i^*$ such that $C \succ' \{i\} \succ' C'$. We have to consider two cases:

   (a) There exists an agent $x \in C \setminus C'$. Let $P \in \mathcal{P}$ be such that $N_i^+(P) = \{i, x\}$. Then, $C \succ_i^+ (P)\; \{i\} \succ_i^+ (P)\; C'$.

   (b) $C \subset C'$. Let $P' \in \mathcal{P}$ be such that $N_i^+(P') = C$. Then $C \succ_i^- (P')\; \{i\} \succ_i^- (P')\; C'$.

3. Finally, we check there exists $\succsim'' \in \mathcal{D}_i^*$ such that $C \succ'' C' \succ'' \{i\}$. We have to consider two cases:

   (a) There exists an agent $x \in C \setminus C'$. Let $P \in \mathcal{P}$ be such that $N_i^+(P) = N$ and $x = \max(N, P)$. Then, $C \succ_i^+ (P)\; C' \succ_i^+ (P)\; \{i\}$.

   (b) $C \subset C'$. Let $P' \in \mathcal{P}$ be such that $N_i^+(P') = C$. Then, $C \succ_i^+ (P')\; C' \succ_i^+ (P')\; \{i\}$.

□

Proof of Remark 2. First, we check that additive preferences are separable. Let $i \in N$, and $\succsim_i \in \mathcal{A}_i$. Then , let the utility function u_i be such that for each $C, C' \in \mathcal{C}_i$, $C \succsim_i C'$ if and only if $\sum_{j \in C} u_i(j) \geq \sum_{j' \in C'} u_i(j')$. Let $j \in N \setminus \{i\}$, and let $C \in \mathcal{C}_i$ be such that $j \notin C$. Note that if $\{i, j\} \succ_i \{i\}$, then $u_i(j) > 0$. This implies that $\left(\sum_{k \in C} u_i(k)\right) + u_i(j) > \sum_{k \in C} u_i(k)$. On the other hand, if $\left(\sum_{k \in C} u_i(k)\right) + u_i(j) > \sum_{k \in C} u_i(k)$, then $u_i(j) > 0$. Hence, $\succsim_i \in \mathcal{S}_i$.

Next, we prove the inclusion $\mathcal{D}_i^* \subset \mathcal{A}_i$. We check that for each $i \in N$ and each $\succsim \in \mathcal{D}_i^*$, we can construct a utility function u_i that rationalizes $\succsim$ as an additively representable preference.

Let $\succsim_i \in \mathcal{D}_i$ be such that for some $P \in \mathcal{P}$, $\succsim_i = \succsim_i^+ (P)$. Let $t^* \equiv \{t \in \mathbb{N} : \ i = N_i^+(t, P)\}$, and $\bar{t} \equiv n - t^*$. For each $j \in N_i^+(P) \setminus \{i\}$, if $j = N_i^+(k, P)$, then let $u_i(j) = n^{n-k}$, $u_i(i) = 0$, and for each $j' \in N_i^-(P) \setminus \{i\}$, if $j' = N_i^-(k', P)$, then let $u_i(j') = -(n^{\bar{t}-k})$. It is clear that for each $C, C' \in \mathcal{C}_i$, $C \succsim_i^+ (P) C'$ if and only if $\sum_{c \in C} u_i(c) \geq \sum_{c' \in C'} u_i(c')$. Thus, $\succsim_i^+ (P) \in \mathcal{A}_i$.

Now, let $\succsim_i' \in \mathcal{D}_i$ be such $\succsim_i' = \succsim_i^- (P)$. For each $j \in N_i^+(P) \setminus \{i\}$, if $j = N_i^+(k, P)$, then let $u_i'(j) = n^{t^* - k - 1}$, $u_i'(i) = 0$, whereas for each $j' \in N_i^-(P) \setminus \{i\}$, if $j' = N_i^-(k', P)$, then let $u_i'(j') = -(n^{n-k'})$. It is clear that for each $C, C' \in \mathcal{C}_i$, $C \succsim_i^- (P) C'$ if and only if $\sum_{c \in C} u_i'(c) \geq \sum_{c' \in C'} u_i'(c')$. Thus, $\succsim_i^- (P) \in \mathcal{A}_i$.

In order to check that the inclusion is proper when $n \geq 4$, assume that $\{i, j, k, l\} \subseteq N$. Let the preference $\succsim_i \in \mathcal{A}_i$ be such that

$$\{i,j,k\} \succ_i \{i,j\} \succ_i \{i,k\} \succ_i \{i,j,k,l\} \succ_i \{i\} \succ_i \{i,j,l\} \succ \{i,k,l\} \succ_i \{i,l\}.$$

The preference $\succsim_i$ can be obtained from a utility function u_i such that $u_i(i) = 0$, $u_i(j) = 3$, $u_i(k) = 2$, and $u_i(l) = -4$. However, there is no $P \in \mathcal{P}$ such that the restriction of either $\succsim_i^+ (P)$ or $\succsim_i^- (P)$ to the coalitions formed by $\{i, j, k, l\}$ coincides with $\succsim_i$. Note that if for some $P \in \mathcal{P}$, $\{i, j\} \succ_i^+ (P) \{i\}$, then $\{i, j, l\} \succ_i^+ (P) \{i\}$. On the other hand, if for some $P' \in \mathcal{P}$, $\{i\} \succ_i^- (P') \{i, l\}$, then $\{i\} \succ_i^- (P') \{i, j, k, l\}$. Finally, let $\succsim_i' \in \mathcal{S}$ be such that

$$\{i,j,k\} \succ_i \{i,k\} \succ_i \{i,j\} \succ_i \{i,j,k,l\} \succ_i \{i,j,l\} \succ \{i,k,l\} \succ_i \{i\} \succ_i \{i,l\}.$$

It is not difficult to check that $\succsim_i' \notin \mathcal{A}_i$, because for each $\succsim_i'' \in \mathcal{A}_i$, if $\{i, k\} \succ_i'' \{i, j\}$, then $\{i, k, l\} \succ_i'' \{i, j, l\}$.

The proof of (b) is just a matter of checking. Let $N = \{i, j, k\}$, $\mathcal{D}_i^*$, $\mathcal{A}_i$ and $\mathcal{S}_i$ consist of the following eight preferences:

$\succsim_i^1$:	$\succsim_i^2$:	$\succsim_1^3$:	$\succsim_i^4$:	$\succsim_i^5$:	$\succsim_i^6$:	$\succsim_i^7$:	$\succsim_i^8$:
$\{i,j,k\}$	$\{i,j,k\}$	$\{i,j\}$	$\{i,j\}$	$\{i,k\}$	$\{i,k\}$	$\{i\}$	$\{i\}$
$\{i,j\}$	$\{i,k\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j\}$	$\{i,k\}$
$\{i,k\}$	$\{i,j\}$	$\{i\}$	$\{i,j,k\}$	$\{i\}$	$\{i,j,k\}$	$\{i,k\}$	$\{i,j\}$
$\{i\}$	$\{i\}$	$\{i,k\}$	$\{i,k\}$	$\{i,j\}$	$\{i,j\}$	$\{i,j,k\}$	$\{i,j,k\}$

Note that $\succsim_i^1, \succsim_i^2, \succsim_i^7, \succsim_i^8 \in \mathcal{D}_i^+ \cap \mathcal{D}_i^-$, while $\succsim_i^3, \succsim_i^5 \in \mathcal{D}_i^+ \setminus \mathcal{D}_i^-$, and $\succsim_i^4, \succsim_i^6 \in \mathcal{D}_i^- \setminus \mathcal{D}_i^+$. $\square$

8.2 Appendix B: Supplemental Material. For the Convenience of the Referee Only

In this appendix we present additional material that may help referee's work. First we present the proofs of Theorem 4 (that replicates the proof of Theorem 3) and Remark 3 (that can be found in Pápai [15]). Next, we check the robustness of the examples that show the independence of our axioms presented in section 5. Finally, we discuss the possibility of constructing rules that satisfy *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility* in the domains of top and bottom preferences.

8.2.1 Proof of Theorem 4 and Remark 3

Before proving Theorem 4, we introduce some intermediate results.

Lemma 5. *Let $N = \{i, j, k\}$ and let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies individual rationality and Pareto efficiency, then φ satisfies non-bossiness.*

Proof. Assume, to the contrary, that there is a rule φ that satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*, but violates *non-bossiness*. Then, without loss of generality, there are $\succsim \in \bar{\mathcal{D}}$, $\succsim'_i \in \bar{\mathcal{D}}_i$ such that $\varphi(\succsim) = (\{i\}, \{j, k\})$ and $\varphi(\succsim_{N\setminus\{i\}}, \succsim'_i) = [N]$. By *individual rationality*, $\{j, k\} \succ_j \{j\}$ and $\{j, k\} \succ_k \{k\}$, which contradicts *Pareto efficiency* of φ at profile $(\succsim_{N\setminus\{i\}}, \succsim'_i)$. □

Lemma 6. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, and Pareto efficiency, then φ satisfies top-coalition.*

Proof. Let $C \in F^\varphi$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $i \in C$, $\text{top}(F^\varphi, \succsim_i) = C$. If $\#C = 1$, then the result follows from *individual rationality.* If $C = N$, then the result is immediate by *Pareto efficiency.* Hence, assume that $C \subsetneq N$. Let $i \in C$, and let $P_i \in \mathcal{P}$ be such that $N_i^+(P_i) = C$. Let $\succsim \in \bar{\mathcal{D}}$ be such that for each $i \in C$, $\succsim'_i = \succsim_i^-(P_i)$, and for each $j \in N \setminus C$, $\succsim'_j = \succsim_j$. By *individual rationality*, for each $i \in C$, $\varphi_i(\succsim') \subseteq C$. Hence, by *Pareto efficiency*, for each $i \in C$, $\varphi_i(\succsim') = C$. Next, let $i \in C$. By *strategy-proofness*, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) \succsim_i \varphi_i(\succsim') = C$. Then, $\varphi_i(\succsim'_{N\setminus\{i\}}, \succsim_i) = C$. Repeating the argument, with the remaining agents $i \in C$, we obtain that for each $i \in C$, $\varphi_i(\succsim) = C$. □

The following lemma parallels Lemma 4 of the text.

Lemma 7. *Let $\bar{\mathcal{D}}$ be a minimally rich domain. If a rule $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfies strategy-proofness, individual rationality, and Pareto efficiency, then F^{φ} satisfies the single-lapping property.*

Proof. The proof replicates the arguments of Lemma 4. First, we prove the result for three-agent societies and then we use an induction argument to extend the result to arbitrary societies.

Assume that $n = 3$. Note first that *Pareto efficiency* implies *flexibility*. Because $n = 3$, by Lemma 5, *individual rationality* and *Pareto efficiency* imply *non-bossiness*. Then, φ satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility*. By the arguments in Claims 1 and 2 of the proof of Lemma 4, F^{φ} satisfies the single-lapping property.

Induction Step. There is $m \geq 3$ such that for $n = m$, if the n-agent rule φ satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*, then F^{φ} satisfies the single-lapping property. We prove that this is true for $n = m + 1$.

The induction hypothesis is true for $m = 3$. Let $n = m + 1$. Assume that φ satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. First, we prove two facts.

Fact 1′. For each $C, C' \in F^{\varphi}$ such that $C \cup C' \neq N$, $\#(C \cap C') = 1$.

Proof. Let $C, C' \in F^{\varphi}$ be such that $(C \cup C') \neq N$. Let $j \in N \setminus (C \cup C')$. Let $\bar{\succsim}_j \in \bar{\mathcal{D}}_j$ be such that for each $C \in \mathcal{C}_j$, $C \neq \{j\}$, $\{j\} \bar{\succ}_j C$. Let $\Sigma_{N \setminus \{j\}}$ denote all the partitions of the reduced society $N \setminus \{j\}$. Define the rule $\bar{\varphi}^{N \setminus \{j\}} : \bar{\mathcal{D}}_{N \setminus \{j\}} \to \Sigma_{N \setminus \{j\}}$ in such a way that for each $\succsim_{N \setminus \{j\}} \in \bar{\mathcal{D}}_{N \setminus \{j\}}$, $(\bar{\varphi}^{N \setminus \{j\}}(\succsim_{N \setminus \{j\}}), \{j\}) \equiv \varphi(\succsim_{N \setminus \{j\}}, \bar{\succsim}_j)$. By φ's *strategy-proofness*, *individual rationality*, and *Pareto efficiency*, $\bar{\varphi}^{N \setminus \{j\}}$ satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. By the induction hypothesis, $F^{\bar{\varphi}^{N \setminus \{j\}}}$ satisfies the *single-lapping* property. By φ's *flexibility*, $C, C' \in F^{\bar{\varphi}^{N \setminus \{j\}}}$, then $\#(C \cap C') = 1$. □

The same arguments apply to prove the following fact.

Fact 2′. For each $\{C_1, \ldots, C_m\} \subseteq \Pi$ with $m \geq 3$, $\cup_{t=1}^{m} C_t \neq N$, and for each $t = 1, \ldots, m$, $\#(C_t \cap C_{t+1}) \geq 1$ (where $m + 1 = 1$), there is $i \in N$ such that for each $t = 1, \ldots, m$, $C_t \cap C_{t+1} = \{i\}$.

Claim 1″. F^φ satisfies Condition (a).

Proof. Assume, to the contrary, that there are $C, C' \in F^\varphi$ such that $(C \cup C') = N$, and $\#(C \cap C') \geq 2$. There are three cases. Note that in the proof of Cases $(1.0')$ and $(1, 1')$, of Claim $1'$ of Lemma 4, we have used *top-coalition* but not *non-bossiness*. Then, the arguments there apply without any modification here. We include the whole proof for the sake of completeness.

Case (1.0″) Let $C, C' \neq N$.
By Fact $1'$, either $F^\varphi = \{[N], C, C'\}$, or $F^\varphi = \{[N], C, C', N\}$. Let $\bar{\succsim}_{N\setminus(C\cap C')} \in \bar{\mathcal{D}}_{N\setminus(C\cap C')}$ be such that for each $j \in (C \setminus C')$, $\text{top}(F^\varphi, \succsim_j) = C$, whereas for each $k \in (C' \setminus C)$, $\text{top}(F^\varphi, \succsim_k) = C'$. Define the rule $\bar{\varphi}^{C\cap C'} : \bar{\mathcal{D}}_{C\cap C'} \to \Sigma$ in such a way that for each $\succsim_{C\cap C'} \in \bar{\mathcal{D}}_{C\cap C'}$, $\bar{\varphi}^{C\cap C'}(\succsim_{C\cap C'}) \equiv \varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cap C')})$. Because φ is *strategy-proof*, $\bar{\varphi}^{C\cap C'}$ is *strategy-proof*. Moreover, by *top-coalition*, $R^{\bar{\varphi}^{C\cap C'}} = \{[N], (C, [C' \setminus C]), (C', [C \setminus C'])\}$. By Remark 1, the preferences of the agents in $(C\cap C')$ over the partitions in $R^{\bar{\varphi}^{C\cap C'}}$ are not restricted. By the Gibbard-Satterthwaite Theorem, $\varphi^{C\cap C'}$ is *dictatorial*. Let $i \in (C \cap C')$ be a dictator for $\varphi^{C\cap C'}$. Let $\succsim_{C\cap C'} \in \bar{\mathcal{D}}_{C\cap C'}$ be such that $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_i) = C'$, and for each $j \in (C\cap C')\setminus\{i\}$, $\text{top}(F^{\bar{\varphi}^{C\cap C'}}, \succsim_j) = \{j\}$. Then, $\varphi(\succsim_{C\cap C'}, \bar{\succsim}_{N\setminus(C\cup C')}) = (C', [C\setminus C'])$, which violates φ's *individual rationality*.

Case (1.1″) Let $C' = N$, and for no $j \in C$ there is $k \in N \setminus C$ and $C'' \subset N$ with $C'' \in F^\varphi$ such that $\{j, k\} \subseteq C''$.
Let $\succsim_{N\setminus C} \in \bar{\mathcal{D}}_{N\setminus C}$ be such that for each $j \in (N \setminus C)$, $\text{top}(F^\varphi, \bar{\succ}_j) = N$. Define now the rule $\bar{\varphi}^C : \bar{\mathcal{D}}_C \to \Sigma$ in the following way. For each $\succsim_C \in \bar{\mathcal{D}}_C$, $\bar{\varphi}^C(\succ_C) \equiv \varphi(\succsim_C, \bar{\succsim}_{N\setminus C})$. Clearly, $\bar{\varphi}^C$ satisfies *strategy-proofness.* Moreover, by *top-coalition*, for each $i \in C$, $F^\varphi \cap \mathcal{C}_i = F^{\bar{\varphi}^C} \cap \mathcal{C}_i$. By Remark 1, the preferences of the agents in C over partitions in $R^{\bar{\varphi}^C}$ are free. Thus, by the Gibbard-Satterthwaite Theorem, $\bar{\varphi}^C$ is *dictatorial*, which, by an already familiar argument, violates φ's *individual rationality.*

Case (1.2″) Let $C' = N$, and for some $j \in C$ there is $k \in N \setminus C$ and $C'' \subset N$ with $C'' \in F^\varphi$ such that $\{j, k\} \subseteq C''$.

Note first that, by Fact $1'$, for each $C'' \in (F^\varphi \setminus N)$, $\#(C \cap C'') \leq 1$. Moreover, by Fact $2'$, there is no cycle of three coalitions in F^φ that does not involve the grand coalition N.

Let $C, N \in F^\varphi$, let $j \in C$ be such that for some $T \subseteq N \setminus C$, $T \cup \{j\} \in F^\varphi$. Let $\bar{C} \equiv C \setminus \{j\}$. Let $T' \in F^\varphi \setminus \{C, N\}$. By Fact 1, there is no $i \in \bar{C}$, such that $\{i, j\} \subseteq T'$. By Fact 2, there is no $k \in T$ such that $\{i, k\} \subseteq T'$.

Let $\succsim^1 \in \mathcal{D}^*$ be such that for each $i \in \bar{C}$, there is $P_i^1 \in \mathcal{P}$ with $j = \max(N, P_i^1)$, $N_i^+(P) = C$, and $\succsim_i^1 = \succsim_i^- (P_i^1)$, for j there is $P_j^1 \in \mathcal{P}$ with $N_j^+(P) = C$, and $\succsim_j^1 = \succsim_j^- (P_j^1)$, while for each $k \in N \setminus C$, there is $P_k^1 \in \mathcal{P}$ with $N_k^+(P) = \{j\} \cup \{k\}$, and $\succsim_k^1 = \succsim_k^+ (P_k^1)$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim^1) = C$.

Next, let $\succsim^2 \in \mathcal{D}^*$ be such that $\succsim_{N \setminus \bar{C}}^1 = \succsim_{N \setminus \bar{C}}^2$, while for each $i \in \bar{C}$ there is $P_i^2 \in \mathcal{P}$ such that $j = \max(N, P_i^2)$, $N_i^+(P_i^2) = N$, and $\succsim_i^2 = \succsim_i^+ (P_i^2)$. Note that for each $i \in \bar{C}$, $N = \text{top}(F^\varphi, \succsim_i^2)$ and $C = \text{top}(F^\varphi \setminus N, \succsim_i^2)$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim_{N \setminus \{i\}}^1, \succsim_i^2) \succsim_i^2 \varphi_i(\succsim^1) = C$. By *individual rationality*, $\varphi_j(\succsim_{N \setminus \{i\}}^1, \succsim_i^2) \neq N$. Then, $\varphi_i(\succsim_{N \setminus \{i\}}^1, \succsim_i^2) = C$. Repeating the same argument iteratively with each $i \in \bar{C}$, we get that for each $i \in C$, $\varphi_i(\succsim^2) = \varphi_i(\succsim^1)$.

Let $\succsim^3 \in \mathcal{D}^*$ be such that $\succsim_{N \setminus \{j\}}^2 = \succsim_{N \setminus \{j\}}^3$ and $\succsim_j^3 = \succsim_j^+ (P_j^1)$. Note that $\text{top}(F^\varphi, \succsim_j^3) = C$. By *strategy-proofness*, $\varphi_j(\succsim^3) \succsim_j^3 \varphi(\succsim^2)$. Then, $\varphi_j(\succsim^3) = C$, and for each $i \in c$, $\varphi_i(\succsim^3) = \varphi_i(\succsim^2)$.

Let $\succsim^4 \in \mathcal{D}^*$ be such that $\succsim_{N \setminus \bar{C}}^3 = \succsim_{N \setminus \bar{C}}^4$, while for each $i \in \bar{C}$ there is $P_i^4 \in \mathcal{P}$ such that for some $\bar{k} \in T$, $\max(N, P_i^4) = \bar{k}$, $N_i^+(P_i^4) = T \cup \{i\}$, and $\succsim_i^4 = \succsim_i^+ (P_i^4)$. Note that by Fact 2 and our assumptions on F^φ, for each $i \in \bar{C}$, $\text{top}(F^\varphi, P_i^4) = N$, and for each $C \in F^\varphi \cap \mathcal{C}_i$, if $C \neq N$, then $\{i\} \succsim_i^4 C$. Let $i \in \bar{C}$, by *strategy-proofness*, $\varphi_i(\succsim^3) \succsim_i^3 \varphi_i(\succsim_{N \setminus \{i\}}^3, \succsim_i^4)$. Hence, $\varphi_i(\succsim_{N \setminus \{i\}}^3, \succsim_i^4) \neq N$. Repeating the argument for each $i \in \bar{C}$, we obtain that $\varphi(\succsim^4) \neq N$. Note that for each $i \in \bar{C}$, N is the only coalition in F^φ that is preferred to staying on her own. On the other hand, agent j prefers a coalition C rather than on her own only if C includes some member of $\bar{C}$. Finally, each agent $k \in N \setminus C$ requires the presence of agent j in order to consider a coalition better than staying on her own. Then, by *individual rationality*, we have that $\varphi(\succsim^4) = [N]$, which contradicts *Pareto efficiency*, because for each $i \in N$, $N \succ_i \{i\}$.

Cases $(1.0'')$, $(1.1'')$ and $(1.2'')$ exhaust all the possibilities. Then, this suffices to prove that F^φ satisfies Condition (a). □

Claim 2″. F^φ satisfies Condition (b).

Proof. Note that in the proof of Claim 2′ of Lemma 4, we do not use *non-bossiness*. Hence the arguments there apply without any modification here. We include the proof for the sake of completeness.

Assume, to the contrary, that φ does not satisfy Condition (b). Then, there is a list of coalitions $\{C_1,\ldots,C_m\}$, with $m \geq 3$ such that for each $t=1,\ldots,m$, $(m+1=1)$, $(C_t \cap C_{t+1}) \neq \{\varnothing\}$, and there is no $i \in N$ such that for each $t=1,\ldots,m$, $\{i\}=(C_t \cap C_{t+1})$. As we have just proved that F^φ satisfies Condition (a) of the single-lapping property, for each $t=1,\ldots,m$, $\#(C_t \cap C_{t+1})=1$. By Fact 2, $\cup_{t=1}^m C_t = N$. Moreover, $F^\varphi = \{C_1,\ldots,C_m\} \cup [N]$.

For each $t=1,\ldots,m$, let $i_t \equiv (C_t \cap C_{t+1})$. Note that for each $t=1,\ldots,m$ and each $j \in (C_t \setminus \{i_{t-1},i_t\})$, $F^\varphi \cap \mathcal{C}_j = \{C_t,\{j\}\}$. On the other hand, for each $t=1,\ldots,m$, $F^\varphi \cap \mathcal{C}_{i_t} = \{C_t, C_{t+1}, \{i_t\}\}$. Then, by Remark 1, minimal richness of the domain of preferences does not introduce any restriction on how the agents may order the different coalitions they may belong to.

Let $\succsim \in \mathcal{D}^*$ be such that for each $t=1,\ldots,m$ and each $j \in (C_t \setminus \{i_{t-1},i_t\})$, $\text{top}(F^\varphi,\succsim_j)=C_t$, and for each $t=1,\ldots,m$, $\text{top}(F^\varphi,\succsim_{i_t})=C_{t+1}$, and $C_t \succ_{i_t} \{i_t\}$. By *top-coalition* and the repeated application of *strategy-proofness*, for each $t=1,\ldots,m$; $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$.

Assume first that m is odd. Then, there is $t' \in \{1,\ldots,m\}$ such that $\varphi_{i_{t'}}(\succsim)=\{i_{t'}\}$, a contradiction with $\varphi_{i_t}(\succsim) \succsim_{i_t} C_t$ for each $t=1,\ldots,m$.

Assume now that m is even. Without loss of generality, assume that for each t odd, $\varphi_{i_t}(\succsim)=C_{t+1}$ and for each t' even, $\varphi_{i_{t'}}(\succsim)=C_{t'}$. Let $\bar{t}$ be even. Let $P_{\bar{t}} \in \mathcal{P}$ be such that $N_{\bar{t}}^+(P_{\bar{t}})=C_{\bar{t}+1}$. Let $\succsim'_{i_{\bar{t}}} = \succsim_{\bar{t}}^-(P_{\bar{t}}) \in \mathcal{D}^*_{i_{\bar{t}}}$. Note that $\text{top}(F^\varphi,\succsim'_{i_{\bar{t}}})=C_{\bar{t}+1}$ and for each $T \nsubseteq C_{\bar{t}+1}$, $\{i_{\bar{t}}\} \succ'_{i_{\bar{t}}} T$. By *individual rationality*, $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}}) \neq C_{\bar{t}}$. Let $\succsim'_{i_{\bar{t}-1}} \in \mathcal{D}^*_{i_{\bar{t}-1}}$ be such that $\text{top}(F^\varphi,\succsim'_{i_{\bar{t}-1}})=C_{\bar{t}-1}$. By *top-coalition*, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1},i_{\bar{t}}\}},\succsim'_{\{i_{\bar{t}-1},i_{\bar{t}}\}})=C_{\bar{t}-1}$. By *strategy-proofness*, we have that $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{\{i_{\bar{t}}\}}) \succsim_{i_{\bar{t}-1}} \varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}-1},i_{\bar{t}}\}},\succsim'_{\{i_{\bar{t}-1},i_{\bar{t}}\}})$. Then, $\varphi_{i_{\bar{t}-1}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}})=C_{\bar{t}-1}$, and $\varphi_{i_{\bar{t}-2}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}})=C_{\bar{t}-1}$. Repeating the argument as many times as necessary, for each t odd, $\varphi_{i_t}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}})=C_t$, while for each t' even $\varphi_{i_{t'}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}})=C_{t'+1}$, and $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}})=C_{\bar{t}+1}$. Then, we get $\varphi_{i_{\bar{t}}}(\succsim_{N\setminus\{i_{\bar{t}}\}},\succsim'_{i_{\bar{t}}}) \succ_{i_{\bar{t}}} \varphi_{i_{\bar{t}}}(\succsim)$, which violates *strategy-proofness*, a contradiction. This suffices to prove that F^φ satisfies Condition (b) of the single-lapping property and concludes the proof of Lemma 7. □

Proof of Theorem 4. By Theorem 1, every single-lapping rule satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. Thus, we focus on the converse result. Let $\varphi : \bar{\mathcal{D}} \to \Sigma$ satisfy *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. By Lemma 6, φ satisfies *top-coalition*. Let $\succsim \in \bar{\mathcal{D}}$. By Lemma 7, F^{φ} satisfies the single-lapping property. Then, there is $C \in F^{\varphi}$ such that for each $i \in C$, $\text{top}(F^{\varphi}, \succsim_i) = C$. By *top-coalition*, for each $i \in C$, $\varphi_i(\succsim) = C$. Moreover, again by *top-coalition*, for each $\succsim' \in \bar{\mathcal{D}}$ such that $\succsim_C = \succsim'_C$, for each $i \in C$, $\varphi_i(\succsim') = C$. Let $\Sigma_{N \setminus C}$ denote the set of all possible partitions of the reduced society $N \setminus C$. Define now the restricted social choice function $\bar{\varphi}^{N\setminus C} : \bar{\mathcal{D}}_{N\setminus C} \to \Sigma_{N\setminus C}$, in such a way that for each $\succsim_{N\setminus C} \in \bar{\mathcal{D}}_{N\setminus C}$, $(\bar{\varphi}^{N\setminus C}(\succsim_{N\setminus C}), C) \equiv \varphi(\succsim_{N\setminus C}, \succsim_C)$. Clearly, $\bar{\varphi}^{N\setminus C}$ satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency*. Moreover, $F^{\bar{\varphi}^{N\setminus C}} = \{C' \in F^{\varphi}, C \cap C' = \{\varnothing\}\}$, and $F^{\bar{\varphi}^{N\setminus C}}$ satisfies the single-lapping property. Repeating the same arguments as many times as necessary, we get $\varphi(\succsim) = \bar{\sigma}^{F^{\varphi}}(\succsim)$. □

Proof of Remark 3, Theorem 1, Pápai [15]. Let Π be a single-lapping collection of coalitions and let $\succsim \in \mathcal{D}$. Assume, by way of contradiction, that there is no $C \in \Pi$ such that for each $i \in C$, $C = \text{top}(\Pi, \succsim_i)$. Let $i \in N$. Then $[N] \subseteq \Pi$ implies that $\text{top}(\Pi, \succsim_i) \neq \{i\}$. Thus, there exists $j \in \text{top}(\Pi, \succsim_i)$ such that $\text{top}(\Pi, \succsim_j) \succ_j \text{top}(\Pi, \succsim_i)$. Then, there is $k \in \text{top}(\Pi, \succsim_j)$ such that $\text{top}(\Pi, \succsim_k) \succ_k \text{top}(\Pi, \succsim_j)$. By condition (a) of the single-lapping property, $i \neq k$. Then, there is $l \in \text{top}(\Pi, \succsim_k)$ such that $\text{top}(\Pi, \succsim_l) \succ_l \text{top}(\Pi, \succsim_k)$. By condition (a) of the single-lapping property, $j \neq l$. By condition (b) of the single-lapping property, $i \neq l$. Repeating the argument in the same fashion, we get a contradiction, since there is a finite number of agents. □

8.2.2 On the Robustness of Examples 2 to 5

Examples 2, 3, and 5 can be straight-forwardly extended to societies with 4 or more agents so we concentrate on Example 4.

We can design obtain a rule that satisfies all our axioms except *non-bossiness* and *Pareto efficiency*, by embedding three-agent rules replicating φ^{-NB} defined in Example 4 in a tree structure. The next example presents such a possibility in a four-agent society.

Example 7. *Let $N = \{i, j, k, l\}$ and let $\bar{\mathcal{D}}$ be a minimally rich domain. Consider the $\{i, j, k\}$-society rule $\varphi^{-NB} : \bar{\mathcal{D}}_{N\setminus\{l\}} \to \Sigma_{N\setminus\{l\}}$ defined in Example 4 in the main text. Next,* *let the rule* $\varphi^* : \bar{\mathcal{D}} \to \Sigma$ *be such that* $F^{\varphi^*} = F^{\varphi^{-NB}} \cup \{k,l\} \cup \{l\}$ *and for each* $\succsim \in \bar{\mathcal{D}}$

$$\varphi^*_k(\succsim) = \begin{cases} \{k,l\} & \text{if } \{k,l\} \succ_l \{l\} \text{ and } \{k,l\} \succ_k \varphi_k^{-NB}(\succsim_{N\setminus\{l\}}), \\ \varphi_k^{-NB}(\succsim_{N\setminus\{l\}}) & \text{otherwise,} \end{cases}$$

and for each $h \in \{i,j\}$,

$$\varphi^*_h(\succsim) = \begin{cases} \{h\} & \text{if } \varphi^*_k(\succsim) = \{k,l\}, \\ \varphi_h^{-NB}(\succsim_{N\setminus\{l\}}) & \text{otherwise.} \end{cases}$$

Clearly, φ^* *satisfies strategy-proofness, individual rationality, and flexibility, but it violates non-bossiness and Pareto efficiency.*

Finally, we want to clarify a point in which Example 5 may be misleading. Although φ^{-F} is not a single-lapping rule, $F^{\varphi^{-F}}$ satisfies the single-lapping property. In four-agent societies is not easy to design rules that violate *flexibility* and satisfy the remaining axioms with a non-single-lapping set of feasible coalitions. However, in five-agent societies we can find clear-cut examples showing that *flexibility* is indeed necessary for obtaining Lemma 4.

Example 8. *Let* $N = \{i,j,k,l,m\}$ *and let* $\bar{\mathcal{D}}$ *be a minimally rich domain. Let* $\varphi' : \bar{\mathcal{D}} \to \Sigma$ *be such that* $F^{\varphi'} = \{\{i,j\},\{k,l,m\},\{k,l\}\} \cup [N]$*, and for each* $\succsim \in \bar{\mathcal{D}}$*:*

$$\varphi'_i(\succsim) = \begin{cases} \{i,j\} & \text{if } \{i,j\} \succsim_i \{i\} \text{ and } \{i,j\} \succsim_j \{j\}, \\ \{i\} & \text{otherwise,} \end{cases}$$

and

$$\varphi'_k(\succsim) = \begin{cases} \{k,l,m\} & \text{if } \varphi_i(\succsim) = \{i,j\} \text{ and } \{k,l,m\} \succsim_h \{h\} \text{ for each } h \in \{k,l,m\}, \\ \{k,l\} & \text{if } \varphi_i(\succsim) = \{i\} \text{ and } \{k,l\} \succsim_k \{k\}, \{k,l\} \succsim_l \{l\}, \\ \{k\} & \text{otherwise.} \end{cases}$$

Clearly, φ' *satisfies strategy-proofness, individual rationality, non-bossiness, Pareto efficiency, but it violates flexibility.*

8.2.3 Examples of Rules for Top and Bottom Preferences

First, we analyze the relation between our work and Alcalde and Revilla [2]. More specifically, we show that top preferences are included in the domain of top responsive preferences presented by Alcalde and Revilla [2]. In addition, we present their top covering algorithm and show that it implicitly defines a rule that satisfies *strategy-proofness*, *individual rationality*, *non-bossiness*, and *flexibility* in the domain of top preferences.

First, we need a piece of notation. For each $C \in \mathcal{N}$, let $\mathcal{C}$ denote the set of all non-empty subsets of C.

Let $i \in N$ and $\succsim_i \in \mathcal{D}_i$. The preference $\succsim_i$ is ***top-responsive*** if for each $C, C' \in \mathcal{C}_i$:

- If $\text{top}(\mathcal{C}, \succsim) \succ_i \text{top}(\mathcal{C}', \succsim_i)$, then $C \succ_i C'$.
- If $\text{top}(\mathcal{C}, \succsim_i) = \text{top}(\mathcal{C}', \succsim_i)$ and $C \subset C'$, then $C \succ_i C'$.

Let $\mathcal{D}_i^{\text{top}}$ denote the domain of all top responsive preferences in $\mathcal{D}_i$.

Lemma 8. *Let $i \in N$ and $\succsim \in \mathcal{D}_i$. If $\succsim_i \in \mathcal{D}_i^+$, then $\succsim_i \in \mathcal{D}_i^{\text{top}}$*

Proof. Let $P \in \mathcal{P}$ be such that $\succsim_i = \succsim_i^+ (P)$. Note first that for each $C \in \mathcal{C}_i$, $\text{top}(\mathcal{C}, \succsim_i) = C_i^+(P)$. Next, let $C, C' \in \mathcal{C}_i$. Assume first that $C_i^+(P) \succ_i C_i'^+(P)$. As $\succsim_i = \succsim_i^+ (P)$, $C \succ_i C'$. Finally, assume that $C_i^+(P) = C_i'^+(P)$ and $C \subset C'$. Then, $C_i^-(P) \subset C_i'^-(P)$. Let t' be the smallest integer t such that $C_i^-(t, P) \neq C_i'^-(t, P)$. Note that $C_i^-(t', P) \in C_i'^-(P)$. Then, $C_i^-(t', P)\ P\ C_i'^-(t', P)$. Hence, $C \succ_i C'$. □

Top-Covering Algorithm. For each $S, C \in \mathcal{N}$ such that $S \subseteq C$ and each $\succsim \in \times_{i \in N} \mathcal{D}_i^{\text{top}}$, define $TC(S, C, \succsim) \equiv S \cup_{i \in S} \text{top}(\mathcal{C}, \succsim_i)$. Next, for each $i \in N$, each $C \in \mathcal{C}_i$, and each $\succsim \in \times_{i \in N} \mathcal{D}_i^{\text{top}}$, let $TC^1(\{i\}, C, \succsim) \equiv TC(\{i\}, C, \succsim)$. Once $TC^t(\{i\}, C, \succsim)$ is defined for some integer $t \geq 1$, let $TC^{t+1}(\{i\}, C, \succsim) \equiv TC[TC^t(\{i\}, \succsim), C, \succsim]$ It is immediate that for each $i \in N$, each $C \in \mathcal{C}_i$, and each $\succsim \in \times_{i \in N} \mathcal{D}_i^{\text{top}}$,

$$TC^{\#C}(\{i\}, C, \succsim) = TC[TC^{\#C}(\{i\}, \succsim), C, \succsim].$$

Finally, for each $i \in N$, each $C \in \mathcal{C}_i$, and each $\succsim \in \times_{i \in N} \mathcal{D}_i^{\text{top}}$, define

$$TC^*(\{i\}, C, \succsim) \equiv TC^{\#C}(\{i\}, C, \succsim).$$

Let $\succsim \in \times_{i \in N} \mathcal{D}_i^{\text{top}}$. Define

$$N(1, \succsim) \equiv N,$$
$$S(1, \succsim) = \{i \in N(1, \succsim) \text{ s.t. } \{i\} \in \cap_{j \in TC^*(\{i\}, N, \succsim)} TC^*(\{j\}, N, \succsim)\}.$$

Once, $N(t,\succsim)$ and $S(t,\succsim)$ are defined for some integer $t \geq 1$, define

$$N(t+1,\succsim) \equiv N(t,\succsim) \setminus S(t,\succsim),$$
$$S(t+1,\succsim) \equiv \{i \in N(t,\succsim) \text{ s.t. } \{i\} \in \cap_{j\in TC^*(\{i\},N(t+1,\succsim),\succsim)} TC^*(\{j\}, N(t+1,\succsim),\succsim)\}.$$

Note that for each $\succsim \in \times_{i\in N}\mathcal{D}_i^{\text{top}}$, there is an integer $t' \leq n$ such that $N(t'+1,\succsim) = \varnothing$. Moreover, the algorithm chooses a unique partition of the society for every preference profile. Then, we can define a coalition formation rule in the following fashion.

Top-Covering Rule. Let the rule $\varphi^{\text{top}} : \times_{i\in N}\mathcal{D}_i^{\text{top}} \to \Sigma$ be such that for each $i \in N$ and each $\succsim \in \times_{i\in N}\mathcal{D}_i^{\text{top}}$, if $i \in S(t,\succsim)$, then $\varphi_i^{\text{top}}(\succsim) = TC^*(\{i\}, N(t,\succsim),\succsim)$.

Alcalde and Revilla [2] have proven that the *top covering rule* satisfies *strategy-proofness*, *individual rationality*, and *Pareto efficiency* in the domain $\times_{i\in N}\mathcal{D}_i^{\text{top}}$. Moreover, by the iterative definition of φ^{top}, and using similar arguments to those in Theorem 1, we see that φ^{top} satisfies *non-bossiness*. As $\mathcal{D}^+ \subset \times_{i\in N}\mathcal{D}_i^{\text{top}}$, the *top covering rule* satisfies our axioms in $\mathcal{D}_i^+$.

In order to find a *strategy-proof* and non-single-lapping rule on the domain of bottom preferences, we recall φ^{-SP} defined on Example 2 of the text.

Let $N = \{j,k,l\}$. For each $\succsim \in \times_{i\in N}\mathcal{D}_i^-$, let

$$IR_j(\succsim) \equiv \{C \in \mathcal{C}_j, \text{ such that for each } j' \in C, C \succsim_{j'} \{j'\}\}.$$

Let φ^{-SP} be such that for each $\succsim \in \times_{i\in N}\mathcal{D}_i^-$, $\varphi_j^{-SP}(\succsim) \equiv \text{top}(IR_j(\succsim),\succsim_j)$ and for each $j' \notin \text{top}(IR_j(\succsim),\succsim_j)$, $\varphi_{j'}^{-SP}(\succsim) \equiv \{j'\}$.

Note that for every $\succsim \in \times_{i\in N}\mathcal{D}_i^-$, φ_j^{-SP} is j's preferred coalition from a set of coalitions that does not depends on j's preferences. Hence, for each $\succsim \in \times_{i\in N}\mathcal{D}_i^-$, and each $\succsim'_j \in \mathcal{D}_j^-$, $\varphi_j^{-SP} \succsim_j \varphi_j^{-SP}(\succsim_{N\setminus\{j\}},\succsim'_j)$.

Next, we check that the remaining members of the society cannot manipulate φ^{-SP} on the domain of bottom preferences. Let $\succsim \in \times_{i\in N}\mathcal{D}_i^-$ and $\succsim'_k \in \mathcal{D}_k^-$. Let $(\succsim_{N\setminus\{k\}},\succsim'_k) = \succsim'$. We consider four cases:

- $\varphi_k^{-SP}(\succsim) = \{k\}$ but $\varphi_k^{-SP}(\succsim' \neq \{j\}$. Since j's preferences do not change from $\succsim$ to $\succsim'$, necessarily $\varphi_k(\succsim') \in IR_j(\succsim')$, but $\varphi_k(\succsim') \notin IR_j(\succsim)$. Thus, $\{k\} \succ_k \varphi^{-SP}(\succsim')$, and $\varphi_k^{-SP}(\succsim) \succsim_k \varphi_k(\succsim')$.

- $\varphi_k^{-SP}(\succsim) \neq \{k\}$, but $\varphi_k^{-SP}(\succsim') = \{k\}$. Note that $\varphi_k^{-SP}(\succsim) \in IR_i(\succsim)$ implies that $\varphi_k^{-SP}(\succsim) \succ_k \{k\}$. Then, $\varphi_k^{-SP}(\succsim) \succsim_k \varphi_k(\succsim')$.

- $\varphi_k^{-SP}(\succsim) = \{j,k\}$ and $\varphi_k^{-SP}(\succsim') = \{j,k,l\}$. Note that since l preferences do not change, if $\{j,k,l\} \in IR_j(\succsim')$, then $\{j,k,l\} \succsim'_k \{k\}$. As $\succsim'_k \in \mathcal{D}_k^-$, this implies that $\{j,k\} \succsim'_k \{k\}$, and $\{j,k\} \in IR_j(\succsim')$. Thus, $\{j,k,l\} \in IR_j(\succsim')$ but $\{j,k,l\} \notin IR_j(\succsim)$. Then, $\{k\} \succ_k \{j,k,l\}$, and $\varphi_k^{-SP}(\succsim) \succsim_k \varphi_k(\succsim')$.

- $\varphi_k^{-SP}(\succsim) = \{j,k,l\}$ and $\varphi_k^{-SP}(\succsim') = \{j,k\}$. Note that $\{j,k,l\} \in IR_j(\succsim)$ implies that $\{j,k,l\} \succ_k \{k\}$. Since $\succsim_k \in \mathcal{D}_k^-$, $\{j,k,l\} \succ_j \{j\}$ implies $\{j,k,l\} \succ_k \{j,k\}$. Then, $\varphi_k^{-SP}(\succsim) \succsim_k \varphi_k(\succsim')$.

We can apply the same argument to agent l to conclude that φ^{-SP} satisfies *strategy-proofness* on the domain of bottom preferences if there are only three agents.

Unfortunately, if there are at least four agents in the society, then φ^{-SP} does not satisfy *strategy-proofness* in $\times_{i\in N}\mathcal{D}_i^-$. However, we can design *strategy-proof* rules in this domain for large societies by embedding rules like φ^{-SP} in a tree-structure as we did in Example 7.

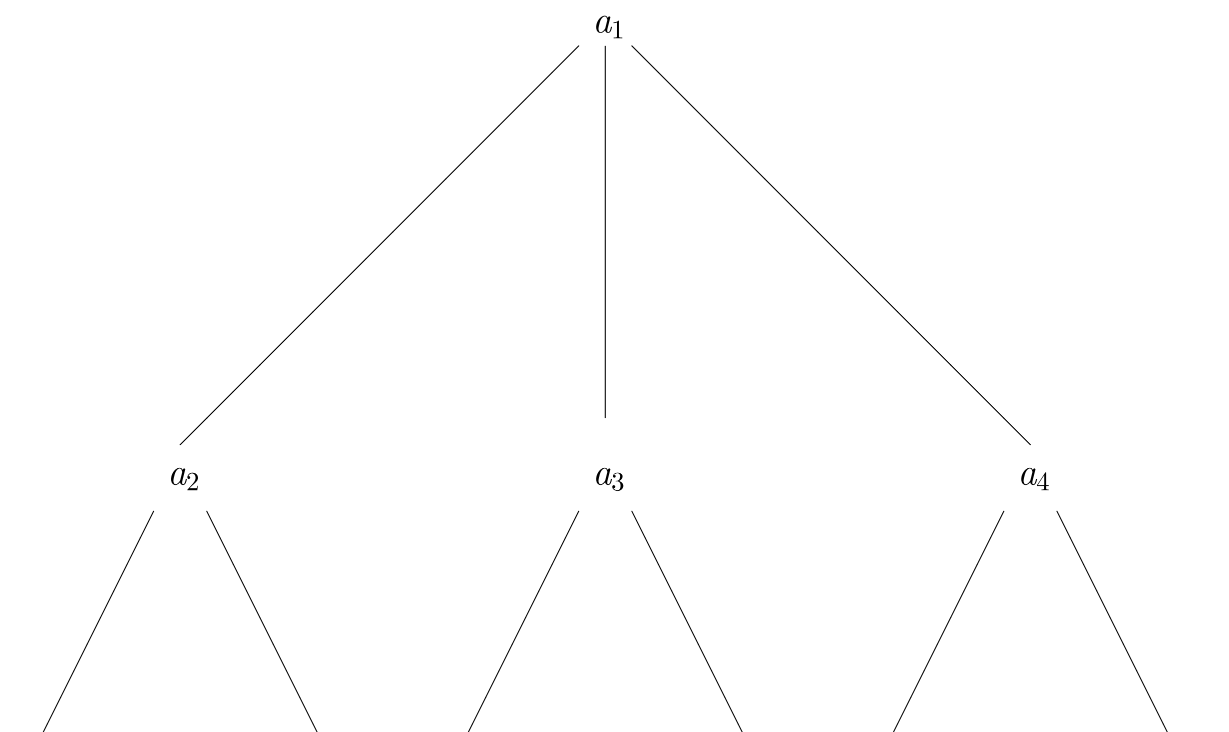

Figure 1: Example 1, A Hierarchical Organization.