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ON THE EQUIVALENCE BETWEEN SUBGAME PERFECTION AND SEQUENTIALITY *

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Abstract

We identify the maximal set of finite extensive forms for which the sets of subgame perfect and sequential equilibrium strategies coincide for any possible assignment of the payoff function. We also identify the maximal set of finite extensive forms for which the outcomes induced by the two solution concepts coincide.

Keywords: extensive form, subgame perfect equilibrium, sequential equilibrium.

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1 Introduction

Analysis of backwards induction in finite extensive form games provides useful insights for a wide range of economic problems. The basic idea of backwards induction is that each player uses a best reply to the other players' strategies, not only at the initial node of the tree, but also at any other information set.

An attempt to capture this type of rationality is due to Selten [14], who defined the subgame perfect equilibrium concept. While subgame perfection has some important applications, it has the drawback that does not always eliminate irrational behavior at information sets reached with zero probability. In order to solve this problem, Selten [15] introduced the more restrictive notion of "trembling-hand" perfection.

The sequential equilibrium concept, due to Kreps and Wilson [9], requires that every player maximizes her expected payoff at every information set, according to some consistent beliefs. They also showed that "trembling-hand" perfection implies sequentiality, which in turn implies subgame perfection. Blume and Zame [3] established that for each fixed extensive form, sequential equilibrium coincide with "trembling-hand" perfect equilibrium for generic payoffs.

Although a weaker concept than "trembling-hand" perfection, sequential equilibrium seems to be the direct generalization of the idea of backwards induction to games of imperfect information. It satisfies in such games all the properties that characterize subgame perfection (backwards induction) in games of perfect information.[1]

However, because of difficulties encountered when characterizing the set of sequential equilibria, the implications of using this more appealing concept as compared to subgame perfection are seldom discussed in economic models in which agents are supposed to act according to the backwards induction principle.

[1] Perfect and proper equilibrium do not satisfy all these properties, see Kohlberg and Mertens [8] for details.

In this paper we find the maximal set of finite extensive forms for which sequential and subgame perfect equilibrium yield the same equilibrium strategies, for every possible payoff function. This set can be characterized as follows: in a given extensive form, if for any behavior strategy combination every information set is reached with positive probability inside its minimal subform, then subgame perfection implies sequential rationality for any possible payoffs.

Whenever the extensive form does not have the structure above, payoffs can be assigned such that the set of subgame perfect equilibria does not coincide with the set of sequential equilibria. However, it may still happen that the set of equilibrium payoffs of both concepts coincides for any possible assignment of the payoff function. Thus, we also identify the maximal set of finite extensive forms for which subgame perfect and sequential equilibrium always yield the same equilibrium payoffs.[2] This completes the description of the information structures where applying sequential rationality does not make a relevant difference with respect to subgame perfection.

The paper is organized as follows: in Section 2 we briefly introduce the main notation and terminology of extensive form games. This closely follows Van Damme [16]. Section 3 contains definitions. Results are formally stated and proved in Section 4. In Section 5 we give some examples where our results can be applied. Section 6 concludes.

2 Notation and Terminology

An $n-$player extensive form is a sextuple $\Xi = (T, \leq, P, U, C, p)$, where T is the finite set of nodes and $\leq$ is a partial order on T, representing precedence. Furthermore, $(T, \leq)$ forms an arborescence with a unique root α. We use the

[2] We actually prove something stronger. For that set of extensive forms, subgame and sequential equilibrium always induce the same set of equilibrium outcomes.

notation $x < y$ to say that node y comes after node x. The immediate predecessor of $x \neq \alpha$ is $P(x) = \max\{y; y < x\}$, and the set of immediate successors of x is $S(x) = \{y : x \in P(y)\}$. The set of endpoints of the tree is $Z = \{x : S(x) = \emptyset\}$ and $X = T \setminus Z$ is the set of decision points. We write $Z(x) = \{y \in Z : x < y\}$ to denote the set of terminal successors of x, and if A is an arbitrary set of nodes we write $Z(A) = \{z \in Z(x) : x \in A\}$.

The player partition, P, is a partition of X into sets $P_0, P_1, ..., P_n$, where P_i is the set of decision points of player i and P_0 stands for the set of nodes where chance moves.

The information partition U is an n-tuple $(U_1, ..., U_n)$, where U_i is a partition of P_i into information sets of player i, such that (i) if $u \in U_i$, $x, y \in u$ and $x < z$ for $z \in X$, then we cannot have $z < y$, and (ii) if $u \in U_i$, $x, y \in u$, then $|S(x)| = |S(y)|$. Therefore, if u is an information set and $x \in X$, it makes sense to write $u < x$. Also, if $u \in U_i$, we often refer to player i as the owner of the information set u.

If $u \in U_i$, the set C_u is the set of choices available for i at u. A generic choice $c \in C_u$ is a collection of $|u|$ nodes with one, and only one, element of $S(x)$ for each $x \in u$. If player i chooses $c \in C_u$ at the information set $u \in U_i$ when she is actually at $x \in u$, then the next node reached by the game is the element of $S(x)$ contained in c. The entire collection $C = \{C_u : u \in \bigcup_{i=1}^{n} U_i\}$ is called the choice partition.

The probability assignment p specifies for every $x \in P_0$ a completely mixed probability distribution p_x on $S(x)$.

We define a finite n-person extensive form game as a pair $\Gamma = (\Xi, r)$, where Ξ is an n-player extensive form and r, the payoff function, is an n-tuple $(r_1, ..., r_n)$, where r_i is a real valued function with domain Z.

We assume throughout that the extensive form Ξ meets perfect recall, i.e.

for all $i \in \{1, ..., n\}$, $u, v \in U_i$, $c \in C_u$ and $x, y \in v$, we have $c < x$ if and only if $c < y$. Therefore, we can say that choice c comes before the information set v (to be denoted $c < v$) and that the information set u comes before the information set v (to be denoted $u < v$).

A behavior strategy b_i of player i is a sequence of functions $(b_i^u)_{u \in U_i}$ such that $b_i^u : C_u \to \mathbb{R}_+$ and $\sum_{c \in C_u} b_i^u(c) = 1, \forall u$. The set B_i represents the set of behavior strategies available to player i. A behavior strategy combination is an element of $B = \prod_{i=1}^n B_i$. As common in extensive form games, we restrict attention to behavior strategies.[3] From now on, we simply refer to them as strategies.

If $b_i \in B_i$ and $c \in C_u$ with $u \in U_i$, then $b_i \backslash c$ denotes the strategy b_i changed so that c is taken with probability one at u. If $b \in B$ and $b_i' \in B_i$ then $b \backslash b_i'$ is the strategy combination $(b_1, ..., b_{i-1}, b_i', b_{i+1}, ..., b_n)$. If c is a choice of player i then $b \backslash c = b \backslash b_i'$, where $b_i' = b_i \backslash c$.

A strategy combination $b \in B$ induces a probability distribution $\mathbb{P}^b$ on the set of terminal nodes Z. If A is an arbitrary set of nodes, we write $\mathbb{P}^b(A)$ for $\mathbb{P}^b(Z(A))$. If $x \in X$, let $\mathbb{P}_x^b$ denote the probability distribution on Z if the game is started at x and the players play according to the strategy combination b.

A system of beliefs μ is a function $\mu : X \setminus P_0 \to [0,1]$ such that $\sum_{x \in u} \mu(x) = 1, \forall u$. Given a system of beliefs μ, a strategy combination b and an information set u, we define the probability distribution $\mathbb{P}_u^{b,\mu}$ on Z as $\mathbb{P}_u^{b,\mu} = \sum_{x \in u} \mu(x) \mathbb{P}_x^b$. An assessment (b, μ) is a strategy combination together with a system of beliefs.

If players play according to the strategy profile b, $R_i(b) = \sum_{z \in Z} \mathbb{P}^b(z) r_i(z)$ denotes player i's expected payoff at the beginning of the game and $R_{ix}(b) = \sum_{z \in Z} \mathbb{P}_x^b(z) r_i(z)$ denotes player i's expected payoff at node x. In a similar way, $R_{iu}(b) = \sum_{z \in Z} \mathbb{P}^b(z \mid u) r_i(z) = \sum_{x \in u} \mathbb{P}^b(x \mid u) R_{ix}(b)$ is player i's expected

[3] We can do this without loss of generality due to perfect recall and Kuhn's theorem, see Kuhn [10].

payoff at the information set u with $\mathbb{P}^b(u) > 0$. Furthermore, under the system of beliefs μ, $R^{\mu}_{iu}(b) = \sum_{z \in Z} \mathbb{P}^{b,\mu}_u(z) r_i(z)$ denotes player i's expected payoff at the information set u. Denote by $R(b)$ the n-tuple $(R_1(b), R_2(b), ..., R_n(b))$.

The strategy b_i is said to be a best reply against b if $b_i \in \arg\max_{b'_i \in B_i} R_i(b \backslash b'_i)$. Likewise, the strategy b_i is a best reply against b at the information set $u \in U_i$ if it maximizes $R_{iu}(b \backslash b'_i)$.

The strategy b_i is a best reply against (b, μ) at the information set $u \in U_i$ if $b_i \in \arg\max_{b'_i \in B_i} R^{\mu}_{iu}(b \backslash b'_i)$. If b_i prescribes a best reply against (b, μ) at every information set $u \in U_i$, we say that b_i is a sequential best reply against (b, μ). The strategy combination b is a sequential best reply against (b, μ) if it prescribes a sequential best reply against (b, μ) for every player.

Let $\hat{T} \subset T$ be a subset of nodes such that (i) $\exists y \in \hat{T}$ with $y < x, \forall x \in \hat{T}, x \neq y$, (ii) if $x \in \hat{T}$ then $S(x) \subset \hat{T}$, and (iii) if $x \in \hat{T}$ and $x \in u$ then $u \subset \hat{T}$. Then we say that $\Xi_y = (\hat{T}, \hat{\leq}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ is a subform of Ξ starting at y, where $(\hat{\leq}, \hat{P}, \hat{U}, \hat{C}, \hat{p})$ are defined from Ξ in $\hat{T}$ by restriction. A subgame is a pair $\Gamma_y = (\Xi_y, \hat{r})$, where $\hat{r}$ is the restriction of r to the endpoints of Ξ_y. We denote by b_y the restriction of $b \in B$ to the subform Ξ_y (to the subgame Γ_y). The restriction of a system of beliefs μ to the subform Ξ_y (to the subgame Γ_y) is denoted by μ_y.

3 Definitions

Let us first review the concept of Nash equilibrium.

Definition 1 (Nash Equilibrium) *A strategy combination $b \in B$ is a Nash equilibrium of Γ if every player is playing a best reply against b.*

We denote by $NE(\Gamma)$ the set of Nash equilibria of Γ. If b is a Nash equilibrium, every player is playing a best reply for the entire game. However, this does not need to be true for every subgame. Subgame perfect equilibrium accounts for this fact.

Definition 2 (Subgame perfect equilibrium) *A strategy combination b is a subgame perfect equilibrium of Γ if, for every subgame Γ_y of Γ, the restriction b_y constitutes a Nash equilibrium of Γ_y.*

We denote by $SPE(\Gamma)$ the set of subgame perfect equilibria of Γ and by $SPEP(\Gamma) = \{R(b) : b \in SPE(\Gamma)\}$ the set of subgame perfect equilibrium payoffs.

Sequential rationality is a refinement of subgame perfection. Every player must maximize at every information set according to her beliefs about how the game has evolved so far. If b is a completely mixed strategy profile, beliefs are perfectly defined by Bayes' rule. Otherwise, such beliefs should meet a consistency requirement.

Definition 3 (Consistent assessment) *An assessment (b, μ) is consistent if there exists a sequence $\{(b_t, \mu_t)\}_t$, where b_t is a completely mixed strategy combination and $\mu_t(x) = \mathbb{P}^{b_t}(x \mid u)$ if $x \in u$, such that*

$$\lim_{t \to \infty} (b_t, \mu_t) = (b, \mu). \tag{1}$$

Therefore, a sequential equilibrium is not a strategy profile but a consistent assessment.

Definition 4 (Sequential equilibrium) *A sequential equilibrium of Γ is a consistent assessment (b, μ) such that b is a sequential best reply against (b, μ).*

If Γ is an extensive game, we denote by $SQE(\Gamma)$ the set of strategies b such that (b, μ) is a sequential equilibrium of Γ, for some μ. Moreover, $SQEP(\Gamma) =$

$\{R(b) : b \in SQE(\Gamma)\}$ denotes the set of sequential equilibrium payoffs. Clearly, $SQE(\Gamma) \subseteq SPE(\Gamma)$, for any game Γ.

We now introduce some new definitions that are needed for the results.

Definition 5 (Minimal Subform of an Information Set) *Given an information set u, the minimal subform that contains u is the subform Ξ_y that contains u and does not include any other proper subform that contains u.*

We say that $\Gamma_y = (\Xi_y, \hat{r})$ is the minimal subgame that contains u if Ξ_y is the minimal subform that contains u.

In a given extensive form there are information sets that are always reached with positive probability. The next two definitions formalize this idea.

Definition 6 (Surely Relevance) *An information set u is surely relevant in the extensive form Ξ if $|C_u| > 1$ implies that $\mathbb{P}^b(u) > 0, \forall b \in B$.*

As will be seen in Proposition 1, the surely relevance property is strictly related with the maximizing behavior of the player who moves at that information set. If there is only one choice available at an information set, the player is obviously maximizing. Hence, we also consider such an information set as surely relevant.

See Figure 1 for a game form where all information sets are surely relevant.

Definition 7 (Subform Surely Relevance) *An information set u is subform surely relevant in the extensive form Ξ if it is surely relevant in its minimal subform.*

See figures 1, 2 and 3 for some game forms where all information sets are subform surely relevant. Conversely, see figures from 4 to 9 for some game forms with information sets that are not subform surely relevant.

4 Results

The three "best reply" concepts introduced in Section 2 relate each other, as it is shown in (i) and (ii) in the next lemma. The third assertion of the same lemma shows that the maximizing behavior at an information set is independent of the subgame of reference.

Lemma 1 *Fix a game $\Gamma = (\Xi, r)$. The following assertions are true:*

(i) Given a strategy combination b, if $u \in U_i$ is such that $\mathbb{P}^b(u) > 0$ and b_i is a best reply against b, then b_i is a best reply against b at the information set u.

(ii) Given a consistent assessment (b, μ), if $u \in U_i$ is such that $\mathbb{P}^b(u) > 0$ and b_i is a best reply against b at the information set u, then b_i is a best reply against (b, μ) at the information set u.

(iii) If Γ_y is the minimal subgame that contains u and (b_y, μ_y) is the restriction of some assessment (b, μ) to Γ_y, then b_i is a best reply against (b, μ) at the information set u in the game Γ if and only if $b_{y,i}$ is a best reply against (b_y, μ_y) at the information set u in the game Γ_y.

Proof. Part (i) is a known result.[4] Proofs for (ii) and (iii) are trivial. ■

The next proposition confines the set of extensive forms for which the sequential equilibrium solution concept actually refines subgame perfection.

Proposition 1 *If Ξ is an extensive form such that every information set is subform surely relevant, then for any possible payoff vector r, the game $\Gamma = (\Xi, r)$ is such that $SPE(\Gamma) = SQE(\Gamma)$. If Ξ is an extensive form such that there exists an information set that is not subform surely relevant, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $SPE(\Gamma) \neq SQE(\Gamma)$.*

[4] See Van Damme [16], Theorem 6.2.1.

Proof. Let us prove the first part of the proposition. We only have to show that $SPE(\Gamma) \subseteq SQE(\Gamma)$. Consider $b \in SPE(\Gamma)$ and build a consistent assessment (b,μ).[5] Construct the set $\tilde{U}(b,\mu) = \bigcup_{i=1}^{n} \left\{ u \in U_i : b_i \notin \arg\max_{\tilde{b}_i \in B_i} R^{\mu}_{iu}(b \backslash \tilde{b}_i) \right\}$; we have to prove that it is empty. Suppose by contradiction that $\tilde{U}(b,\mu) \neq \emptyset$, and take $u \in \tilde{U}(b,\mu)$, obviously $|C_u| > 1$. Let Γ_y be the minimal subgame that contains u. By Lemma 1, $\mathbb{P}^b_y(u) = 0$. However, u must be subform surely relevant. This provides the contradiction.

Let us now prove the second part of the proposition. Suppose $u \in U_i$ is an information set that is not subform surely relevant and let $c \in C_u$ be an arbitrary choice available at u. Assign the following payoffs:

$$\begin{aligned} r_i(z) &= 0 \quad \forall i \qquad \text{if} \quad z \in Z(c) \\ r_i(z) &= 1 \quad \forall i \qquad \text{elsewhere.} \end{aligned} \tag{2}$$

Clearly any strategy $b_i = b_i \backslash c$ cannot be part of a sequential equilibrium since playing a different choice at u gives player i strictly higher expected payoff at that information set.

We now have to show that there exists a subgame perfect equilibrium b such that $b_i = b_i \backslash c$. By assumption there exists b' such that $\mathbb{P}^{b'}_y(u) = 0$ in the minimal subgame Γ_y that contains u. Consider the strategy combination $b = b' \backslash c$, it is also true that $\mathbb{P}^b_y(u) = 0$. Note that b_y is a Nash equilibrium of Γ_y since nobody can obtain a payoff larger than one. By the same argument, b is inducing a Nash equilibrium in every subgame, hence it is a subgame perfect equilibrium. This completes the proof. ■

[5] A general method to define consistent assessments (b,μ) for any given $b \in B$, in an extensive form, is the following: take a sequence of completely mixed strategy combination $\{b_t\}_t \to b$ and for each t, construct $\mu^t(x) = \mathbb{P}^{b_t}(x \mid u) \in [0,1]$, $\forall x \in u$, for all information sets u. Call $k = |X \setminus P_0|$. The set $[0,1]^k$ is compact and since $\mu^t \in [0,1]^k, \forall t$, there exists a subsequence of $\{t\}$, call it $\{t_j\}$, such that $\left\{\mu^{t_j}\right\}_{t_j}$ converges in $[0,1]^k$. Define beliefs as $\mu = \lim_{j\to\infty} \mu^{t_j}$.

Figures 1, 2 and 3 contain some examples of extensive forms where the first part of Proposition 1 applies. The extensive forms in figures 4, 5 and 6 illustrate the second part.

Notice that the payoff assignment in the previous proof provides a difference in equilibrium strategies but not in equilibrium payoffs. The reason is that this difference in equilibrium payoffs cannot always be achieved. In figures 7, 8 and 9 there are examples of extensive forms with information sets that are not subgame surely relevant and, however, the sets of sequential and subgame perfect equilibrium payoffs always coincide. Proposition 2 provides a sufficient and necessary condition for equilibrium payoffs sets to be equal, for any conceivable payoff function. Before that, an additional piece of notation is needed.

Given a node $x \in T$, we call path to x the set of choices $Path(x) = \{c \in \bigcup_u C_u : c < x\}$. Let Ξ_y be the minimal subform that contains the information set u. Construct the set $B(u) = \left\{b \in B : \mathbb{P}_y^b(u) > 0\right\}$. We say that player i can avoid the information set u if there exists a strategy combination $b \in B(u)$, and a choice $c \in C_v$, with $v \in U_i$, such that $\mathbb{P}_y^{b\backslash c}(u) = 0$.[6] Therefore, associated with any information set, we can construct a list (possibly empty) of players who can avoid it. The following lemma is useful for the proof of Proposition 2.

Lemma 2 *Let Ξ be an extensive form such that every information may only be avoided by its owner. Let (b, μ) and (b', μ') be two consistent assessments. If b and b' are such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y, then $\mu = \mu'$.*

Proof. Let (b, μ) and (b', μ') be two consistent assessments such that $\mathbb{P}_y^b = \mathbb{P}_y^{b'}$ for every subform Ξ_y. Note that b' can be obtained from b by changing behavior at information sets that are reached with zero probability within its minimal subform. Hence, without loss of generality, let b and b' differ only at one of such

[6] We are restricting to the minimal subform. This is embedded in the notation. For the sake of clarification: assigning probability zero to an entire subgame is not the same as avoiding an information set contained in that subgame.

information sets, say $u \in U_i$, and let Ξ_y be its minimal subform. The shift from b to b' may cause a change in beliefs only at information sets that come after u and have the same minimal subform Ξ_y. Let $v \in U_j$ be one of those information sets.

If $j = i$, perfect recall implies that there is no change in beliefs at the information set v. If $j \neq i$ there are two possible cases, either $\mathbb{P}_y^b(v) > 0$ or $\mathbb{P}_y^b(v) = 0$. In the first case the beliefs at v are uniquely defined, therefore, $\mu(x) = \mu'(x), \forall x \in v$ and moreover, $\mu(x) = \mu'(x) = 0, \forall x \in v$ such that $u < x$. In the second case, since the information set v can only be avoided by player j there exists a choice $c \in C_w$ of player j such that $\mathbb{P}_y^{b\backslash c}(v) > 0$, otherwise player i would be able to avoid the information set u. Let $b'' = b\backslash c$ and $b''' = b'\backslash c$, then by the discussion of the first case, $\mu''(x) = \mu'''(x), \forall x \in v$, furthermore, perfect recall implies $\mu''(x) = \mu(x)$ and $\mu'''(x) = \mu'(x), \forall x \in v$, which in turn implies $\mu(x) = \mu'(x), \forall x \in v$. ■

Remark 1 *If in an extensive form every information set may only be avoided by its owner, then beliefs are always uniquely defined (consider $b' = b$ in Lemma 2).*[7] *If we consider extensive forms such that only one choice is available at information sets that can be avoided by a different player than its owner, then beliefs are still uniquely defined whenever they are useful, that is, whenever an actual choice is faced. However, this does not need to be true at information sets where only one choice can be taken. This enlarged set of extensive forms is the one under study in the next proposition.*

Proposition 2 *If Ξ is an extensive form such that every information set that is not subform surely relevant can only be avoided by its owner, then for any possible payoff vector r, the game $\Gamma = (\Xi, r)$ is such that $SPEP(\Gamma) = SQEP(\Gamma)$.*

[7] A complete characterization is the following: in an extensive form Ξ, beliefs are always uniquely defined if and only if $|u| > 1$ implies that u may only be avoided by its owner.

If Ξ is an extensive form with an information set that is not subform surely relevant and that can be avoided by a different player than its owner, then we can find a payoff vector r such that for the game $\Gamma = (\Xi, r)$, $SPEP(\Gamma) \neq SQEP(\Gamma)$.

Proof. Let us prove the first part of the proposition. We need to prove that $\forall b \in SPE(\Gamma)$, $R(b) \in SQEP(\Gamma)$. Take an arbitrary $b \in SPE(\Gamma)$ and construct some consistent beliefs μ.

If the set $\tilde{U}(b,\mu) = \bigcup_{i=1}^{n} \left\{ u \in U_i : b_i \notin \arg\max_{\tilde{b}_i \in B_i} R^{\mu}_{iu}(b \backslash \tilde{b}_i) \right\}$ is empty, then $b \in SQE(\Gamma)$ and $R(b) \in SQEP(\Gamma)$. Otherwise, we need to find a sequential equilibrium (b^*, μ^*) such that $R(b^*) = R(b)$.

Step 1: Take an information set $u \in \tilde{U}(b,\mu)$, obviously $|C_u| > 1$. Let i be the player that moves at this information set, and let Γ_y be the minimal subgame that contains u. Notice that by Lemma 1, u should be such that $\mathbb{P}^b_y(u) = 0$, hence it is not subform surely relevant. By assumption, u can only be avoided by player i.

Step 2: Let b' be the strategy profile b modified so that player i plays a best reply against (b, μ) at the information set u. Construct a consistent assessment (b', μ'). Notice that $\mathbb{P}^{b'} = \mathbb{P}^b$ and, in particular, $\mathbb{P}^{b'}_y = \mathbb{P}^b_y$. By Lemma 2, μ and μ' assign the same probability distribution to information sets where more than one choice is available.

Step 3: We now prove that $b' \in SPE(\Gamma)$. For this we need $b'_y \in NE(\Gamma_y)$. Given the strategy profile b'_y in the subgame Γ_y, player i cannot profitably deviate because this would mean that she was also able to profitably deviate when b_y was played in the subgame Γ_y, which contradicts $b_y \in NE(\Gamma_y)$.

Suppose now that there exists a player $j \neq i$ who has a profitable deviation $b''_{y,j}$ from $b'_{y,j}$ in the subgame Γ_y. The hypothesis on the extensive form Ξ implies $\mathbb{P}_y^{b \backslash b''_{y,j}} = \mathbb{P}_y^{b' \backslash b''_{y,j}}$, which further implies that $b''_{y,j}$ should also have been a profitable deviation from b_y. However, this is impossible since $b_y \in NE(\Gamma_y)$.

Step 4: By step 2, $\left|\tilde{U}(b', \mu')\right| = \left|\tilde{U}(b, \mu)\right| - 1$. If $\left|\tilde{U}(b', \mu')\right| \neq \emptyset$, apply the same type of transformation to b'. Suppose that the cardinality of $\tilde{U}(b, \mu)$ is q, then in the qth transformation we will obtain a consistent assessment $(b^{(q)}, \mu^{(q)})$ such that $b^{(q)} \in SPE(\Gamma)$, $\mathbb{P}^b = \mathbb{P}^{b^{(q)}}$, and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$. Observe that, $b^{(q)} \in SPE(\Gamma)$ and $\tilde{U}(b^{(q)}, \mu^{(q)}) = \emptyset$ imply $b^{(q)} \in SQE(\Gamma)$, and that $\mathbb{P}^b = \mathbb{P}^{b^{(q)}}$ implies $R(b) = R(b^{(q)})$. Therefore $(b^{(q)}, \mu^{(q)})$ is the sequential equilibrium (b^*, μ^*) we were looking for.

Let us now prove the second part of the proposition. For notational convenience, it is proved for games without proper subgames, however, the argument extends immediately to the general case. Suppose that u is an information set that is not subform surely relevant. Suppose also that it can be avoided by a player, say player j, different from the player moving at it, say player i. Note that there must exist an $x \in u$ and a choice $c \in C_v$, where $v \in U_j$, such that if $b = b \backslash Path(x)$, then $\mathbb{P}^{b \backslash c}(u) = 0$ is true.

Let $f \in C_u$ be an arbitrary choice available to player i at u. Assign the following payoffs:

$$\begin{array}{rcl} r_j(z) = 0 & \text{if} & z \in Z(c) \\ r_i(z) = r_j(z) = 0 & \text{if} & z \in Z(f) \\ r_i(z) = r_j(z) = 1 & \text{if} & z \in Z(u) \setminus Z(f). \end{array} \tag{3}$$

Let $d \in Path(x)$ with $d \notin C_v$, assign payoffs to the terminal nodes, whenever allowed by 3, in the following fashion:

$$r_k(z) > r_k(z') \text{ where } z \in Z(d) \text{ and } z' \in Z(C_w \setminus \{d\}). \tag{4}$$

Player k above is the player who has choice d available at the information set w. Give zero to every player everywhere else.

In words, player j moves with positive probability in the game. She has two choices, either moving towards the information set u and letting player i decide, or moving away from the information set u. If she moves away she gets zero for sure. If she lets player i decide, player i can either make both get zero by choosing f, or make both get one by choosing something else. Due to 4, no player will disturb this description of the playing of the game.

This game has a Nash equilibrium in which player i moves f and player j obtains a payoff equal to zero by moving c. However, in every sequential equilibrium of this game, player i does not choose f and, as a consequence, player j takes the action contained in $Path(x) \cap C_v$. Therefore, in every sequential equilibrium, players i and j obtain a payoff strictly larger than zero.[8] This completes the proof. ■

5 Examples

These results can be applied to many games considered in different branches of the economic literature. It allows to identify in a straightforward way, the finite extensive form games of imperfect information for which subgame perfect equilibria are still conforming with backwards induction, when following the minimal interpretation of the principle given by Kohlberg [7].

Besley and Coate [2] proposed an economic model of representative democracy. The political process is a three-stage game. In stage 1, each citizen decides whether or not to become a candidate for public office. At the second stage, voting takes place over the list of candidates. At stage 3 the candidate with the most votes chooses the policy. Besley and Coate solved this model using subgame perfection and found multiple subgame perfect equilibria with very different outcomes in terms of number of candidates. This may suggest that

[8] Equilibrium payoffs are not necessarily equal to one due to eventual moves of nature.

some refinement might give sharper predictions. However, given the structure of the game that they considered, it follows immediately from the results of the previous section that all subgame perfect equilibria in their model are also sequential. Thus, no additional insights would be obtained by requiring this particular refinement.

The information structure of Besley and Coate's model is a particular case of the more general framework offered by Fudenberg and Levine [5]. They characterized the information structure of finite-horizon multistage games as "almost" perfect, since each period players simultaneously choose actions, no Nature moves are allowed and there is no uncertainty at the end of each stage. As they noticed, sequential equilibrium does not refine subgame perfection in this class of games. This claim can also be obtained as an implication of Proposition 1 in the present paper.

In their version of the Diamond-Dybvig [4] model, Adão and Temzelides [1] discussed both the issue of potential banking instability as well as that of the decentralization of the optimal deposit contract. They addressed the first question in a model with a "social planner" bank. The bank offers the efficient contract as a deposit contract in the initial period. In the first stage agents sequentially choose whether to deposit in the bank or to remain in autarky. In the second stage, those agents who were selected by Nature to be patient, simultaneously choose whether to misrepresent their preferences and withdraw, or report truthfully and wait. The reduced normal form of the game has two symmetric Nash equilibria in pure strategies. The first one has all agents choosing depositing in the bank and reporting faithfully, the second one has all agents choosing autarky. The fact that both equilibria are sequential is captured by their Proposition 2. As in the other examples, because of the game form they used, our Proposition 1 also covers their claim.

In the implementation theory framework, Moore and Repullo [11], present the strength of subgame perfect implementation. If a choice function is implementable in subgame perfect equilibria by a given mechanism and the strategy space is finite, by analyzing the extensive form of the mechanism game, our work perfectly demarcates whether this very mechanism also implements in sequential equilibria. Namely, if in the extensive form of the mechanism game all information sets are subform surely relevant, then the mechanism also implements in sequential equilibria. If not every information set is subform surely relevant, then there exist economies for which the mechanism does not double implement in subgame perfect and sequential equilibria.

More examples can be found in Game Theory textbooks. Good references are Fudenberg and Tirole [6], Myerson [12] and Osborne and Rubinstein [13]. Notice that whenever are presented extensive games where subgame perfect and sequential equilibrium differ, there are information sets that are not subform surely relevant in the game form of such extensive games. As examples consider figures 8.4 and 8.5 in Fudenberg and Tirole, figures from 4.8 to 4.11 in Myerson and figures 225.1 and 230.1 in Osborne and Rubinstein.

6 Conclusion

For the class of extensive form games sharing the property that all information sets are subform surely relevant, the set of sequential equilibrium strategies is equal to the set of subgame perfect equilibrium strategies. However, for different information structures the equality of these sets crucially depends on the payoffs of the players.

If we are only concerned with equilibrium payoffs, we can have information sets that are not subform surely relevant, as long as the player who can avoid it is the same as its owner. In this case, the solution concepts of subgame perfection and sequential equilibrium yield the same set of equilibrium payoffs, for any possible assignment of the payoff function.

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A Figures

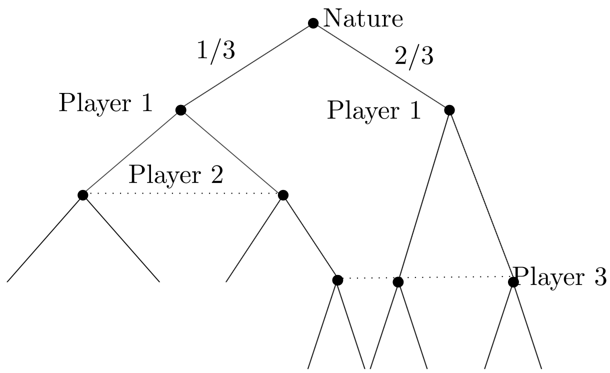

Figure 1

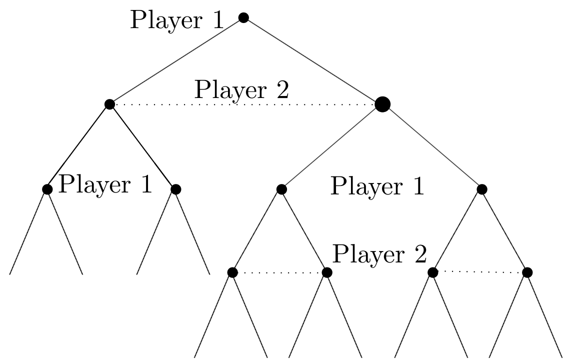

Figure 2

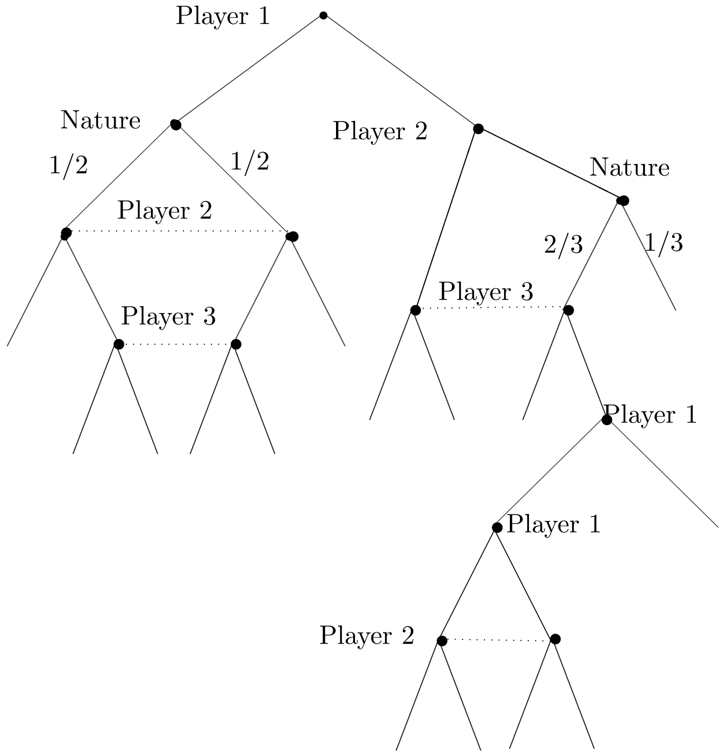
Player 1
Nature
1/2
1/2
Player 2
Player 2
Player 3
Nature
2/3
1/3
Player 3
Player 1
Player 1
Player 2

Figure 3

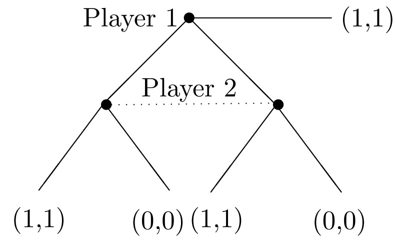

Figure 4

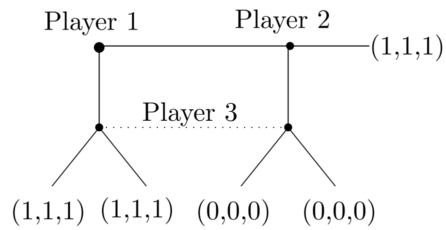

Figure 5

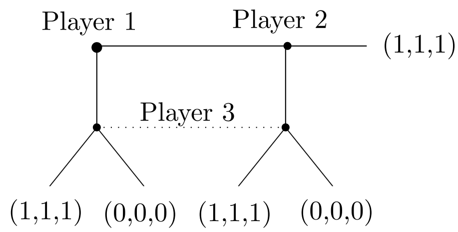

Figure 6

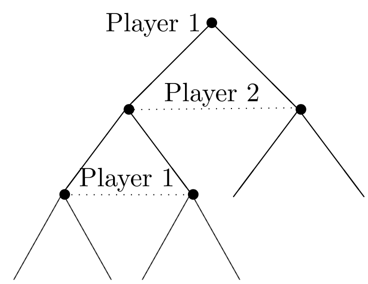

Figure 7

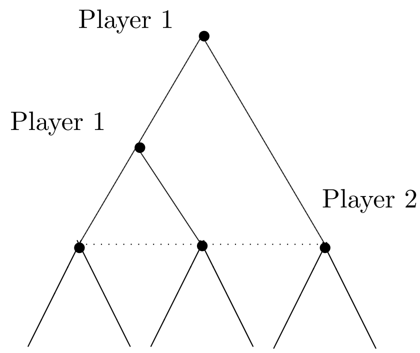

Figure 8

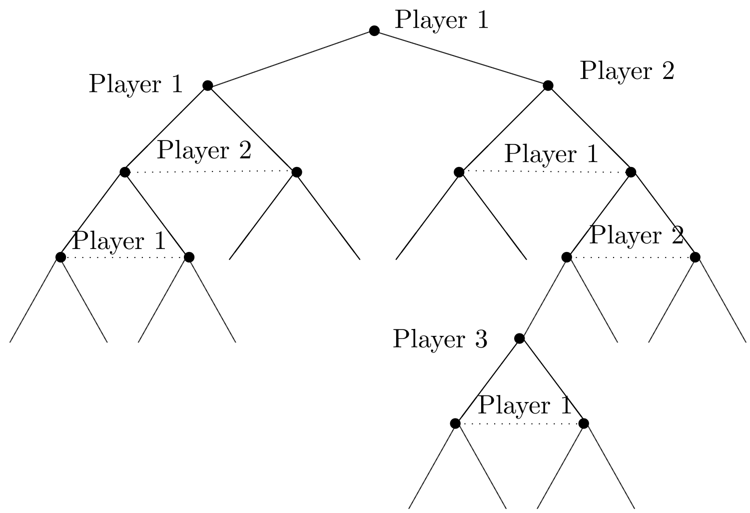
Player 1
Player 1
Player 2
Player 2
Player 1
Player 1
Player 2
Player 3
Player 1

Figure 9