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ON THE CONTINUITY OF EQUILIBRIUM AND CORE CORRESPONDENCES IN ECONOMIES WITH DIFFERENTIAL INFORMATION*

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Abstract

We study upper semi-continuity of the private core, the coarse core, and the Radner equilibrium correspondences for economies with differential information, with Boylan (1971) topology on agents' information fields.

Keywords: Economies with differential information, Radner equilibrium, rational expectations, coarse core, private core, continuity.

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1 Introduction

We study the behavior of several notions of competitive equilibrium and core in economies with differential information. Our aim is to check whether these solution concepts respond continuously to changes in agents' characteristics in the underlying economy. This is a basic problem in general equilibrium theory, that was studied by several authors in the context of economies with complete information. For instance, Kannai (1970) considered continuity properties of the core of a pure exchange economy, while Hildenbrand (1972) and Hildenbrand and Mertens (1972) investigated the continuity of the equilibrium-set correspondence. It turns out that both solution concepts are "upper semi-continuous": when the agents' characteristics converge, together with a selection from the solution of corresponding economies, then the limit of the selection belongs to the solution of the limiting economy.

In an economy with differential information every agent is characterized by his initial endowment of commodities (which is a random variable, whose value is determined by the realization of the state of nature), his state-dependent utility function, and his private information represented by a σ-field on the space of states of nature (i.e., an agent can tell whether the realized state of nature is contained in any given set from the field). In measuring "closeness" of these economies, the only non-standard part lies in evaluating the "distance" between agents' information endowments. We do so by means of Boylan (1971) metric, following Allen (1983) who was the first to apply topologies on information fields in economies with differential information. (Allen proved continuity[1] of the consumer demand and the value of information with respect to Boylan metric.)

We first consider a notion of competitive equilibrium for economies with differential information introduced in Radner (1968, 1982), that imposes measurability restrictions with respect to the private information on each agent's trades. We show that if the space of random commodity bundles is L_p^m (where m is the number of commodities, and $1 \leq p < \infty$) and the prices lie in its dual, then the Radner equilibrium correspondence is upper semi-continuous, see Theorem 1.

We do not consider another equilibrium concept, that of "rational expectations," in the study of continuity of equilibrium correspondences. The reason is simple: the rational expectations equilibrium correspondence cannot be upper semi-continuous since the information revealed to the agents by prices may have severe discontinuities; e.g., a sequence of fully revealing price systems can converge to a constant price system which reveals no information whatsoever.

In the literature there are several notions of core for economies with differential information (for a comprehensive survey see Forges, Minelli and Vohra (2002)). Among them, two notions stand out in that they are nonempty under

[1] Cotter (1986, 1987) showed that continuity properties could be obtained with a weaker topology. Stinchocombe (1990) derived further results for both topologies. Van Zandt (2002) studied continuity of solutions to constrained maximization problems with respect to the Cotter (1986) topology on the information fields.

quite general conditions on the economy. These are Wilson (1978) coarse core, where a blocking coalition considers its interim payoffs (following the revelation of private information to agents) given a common knowledge event, and Yannelis (1991) private core, where blocking is based on ex-ante payoffs, but measurability of all allocations with respect to agents' private information is required. The relation between the private core and the Radner equilibrium correspondence is analogous to that of the core and the Walrasian equilibrium correspondence in a complete information economy – see Einy, Moreno and Shitovitz (2001) for an equivalence result in economies with differential information and a continuum of traders. Here we show that the private core correspondence may not be upper semi-continuous even in simple economies (see Example 1). However, the private core correspondence *is* upper semi-continuous for every converging sequence of economies where agents' information fields approach their limiting information fields from above (which corresponds to the case of decreasing information quality; see Theorem 2).

The private core correspondence is also upper semi-continuous when the limiting economy is a complete information economy (see Theorem 3). This stands in contrast to Theorem 1 of Krasa and Shafer (2001), that shows discontinuity of this correspondence when the agents' common prior[2] over the set of states of nature changes, in a way that the information becomes complete in the end. Here we view the common prior as an invariable characteristic of the economy[3] (together with the sets of agents, commodities, and states of nature), and not as part of the agents' characteristics. Accordingly, in all our results the information fields of the agents (and their other characteristics) are allowed to vary, while the common prior stays fixed. This is precisely what allows us to obtain the positive result of Theorem 3.

Finally, we examine the coarse core correspondence, and find that it is also fragile, as it may fail to be upper semi-continuous even when the private core correspondence is (see Example 2).

2 The Model

We consider a pure exchange economy $\mathcal{E}$ with differential information. The commodity space is R^m_+. The set of *agents* is $N = \{1, 2, ..., n\}$. The uncertainty in the economy is described by a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is the space of states of nature, which we assume to be compact and metrizable, $\mathcal{F}$ is a σ-field of subsets of Ω, and μ is the *common prior* of the agents - a countably additive probability measure on $(\Omega, \mathcal{F})$. The initial information of agent $i \in N$ is given by a σ-subfield $\mathcal{F}_i$ of $\mathcal{F}$; that is, for every $A \in \mathcal{F}_i$ agent i knows whether the realized state of nature is contained in A.

[2] To interpret the model of Krasa and Shafer (2001) in our setting, one has to view this common prior as defined on the product of the set of states of nature and the signals receivable by agents, rather than the set of states of nature alone (as we do here, with information fields as a device of revealing information).

[3] Or, more precisely, of its information structure determined by the underlying economic data.

For any $1 \leq p < \infty$ we write L_p^m for the Banach space of all $\mathcal{F}$-measurable functions[4] $x : \Omega \to R^m$ such that $\|x\|_p \equiv \left(\int_\Omega \|x(\omega)\|^p \, d\mu(\omega)\right)^{\frac{1}{p}} < \infty$ (here $\|\cdot\|$ stands for the Euclidean norm on R^m). It is well known that $L_p^m \subset L_1^m$ and that convergence in L_p^m-norm implies convergence in L_1^m-norm.

Given $S \subseteq N$, an *S-assignment* in economy $\mathcal{E}$ is an S-tuple $\mathbf{x} = \left(x^i\right)_{i \in S}$ of non-negative functions (*commodity bundles*) in which $x^i \in L_1^m$ for every $i \in S$. An N-assignment is simply referred to as an *assignment.* There is a fixed initial assignment of commodities in the economy, $\mathbf{e} = (e^1, ..., e^n)$; e^i is referred to as the *initial endowment* of agent i. An *S-allocation* is an S-assignment $\mathbf{x} = \left(x^i\right)_{i \in S}$ that[5] satisfies the feasibility constraint:

$$\sum_{i \in S} x^i(\omega) \leq \sum_{i \in S} e^i(\omega) \text{ for } (\mu\text{-})\text{almost every } \omega \in \Omega. \tag{1}$$

Given a subfield $\mathcal{F}'$ of $\mathcal{F}$, we use the extended notion of $\mathcal{F}'$-measurability by calling a function, which is equal μ-almost everywhere to an $\mathcal{F}'$-measurable function, also *$\mathcal{F}'$-measurable.* A *private* S-allocation is an S-allocation $\mathbf{x} = \left(x^i\right)_{i \in S}$ such that x^i is $\mathcal{F}^i$-measurable for every $i \in S$. An N-allocation (respectively, private N-allocation) is called an *allocation* (respectively, *private allocation*).

The preferences of agent i over the commodity space are represented by a state-dependent utility function, $u^i : \Omega \times R_+^m \to R_+$, measurable with respect to the product field $\mathcal{F} \times \mathcal{B}$ (where $\mathcal{B}$ is the σ-field of Borel sets in R_+^m) and continuous[6]. We will always assume that $u^i(\omega, \cdot)$ is concave on R_+^m for every $\omega \in \Omega$, and non-decreasing. If $\mathbf{x} = (x^1, ..., x^n)$ is an assignment, we denote

$$U^i\left(x^i\right) = \int_\Omega u^i\left(\omega, x^i(\omega)\right) d\mu, \tag{2}$$

whenever the integral exists.

To sum up, an *economy with differential information,* $\mathcal{E}$, is described by the collection $\left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$.

A possible interpretation[7] of the above economy is the following. It extends over two periods of time. In the first period there is uncertainty about the state of nature. In this period, agents make contracts on redistribution of their initial endowments either before the state of nature is realized (ex-ante) or after receiving their private information (interim). In the second period agents carry out previously made agreements, and consumption takes place.

In order to define convergence of economies with differential information, we use a pseudo-metric (introduced in Boylan (1971)) on the family $\mathcal{F}^*$ of σ-

[4] Or, to be precise, their equivalence classes, where any two functions which are equal μ-almost everywhere are identified.

[5] For any two vectors $x = (x_1, ..., x_m), y = (y_1, ..., y_m) \in R^m$ we write $x \geq y$ when $x_k \geq y_k$ for every $k = 1, ..., m$, and $x > y$ when $x \geq y$ and $x \neq y$.

[6] Continuity of u implies its $\mathcal{F} \times \mathcal{B}$-measurability when $\mathcal{F}$ is the σ-field of Borel sets in Ω.

[7] This paragraph is a quotation from Allen and Yannelis (2001), p. 265, slightly modified to fit our setting.

subfields of $\mathcal{F}$, given by

$$d(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1} \inf_{B \in \mathcal{F}_2} \mu(A \triangle B) + \sup_{B \in \mathcal{F}_2} \inf_{A \in \mathcal{F}_1} \mu(A \triangle B),$$

where $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the "symmetric difference" of A and B. Consider a sequence $\{\mathcal{E}_k\}_{k=1}^{\infty}$ of economies $\mathcal{E}_k = \left(e_k^i, u_k^i, \mathcal{F}_k^i\right)_{i=1}^n$ with differential information. We say that $\{\mathcal{E}_k\}_{k=1}^{\infty}$ *converges*[8] to an economy $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$ if for every $i \in N$:

i)[9] $e_k^i \to_{k\to\infty}^{L_1^m} e^i$;

ii) $u_k^i \to_{k\to\infty} u^i$ uniformly on every compact subset of $\Omega \times R_+^m$;

iii) $\mathcal{F}_k^i \to_{k\to\infty} \mathcal{F}^i$ in Boylan pseudo-metric.

Finally, if X is a random variable on $(\Omega, \mathcal{F}, \mu)$ and $\mathcal{F}'$ is a σ-subfield of $\mathcal{F}$, denote by $E(X \mid \mathcal{F}')$ the conditional expectation[10] of X with respect to $\mathcal{F}'$.

3 Continuity of Radner Equilibrium Correspondence

Radner (1968, 1982) introduced a notion of competitive equilibrium for economies with differential information. Here we study the continuity of Radner equilibrium correspondence.

In what follows we consider commodity bundles that are members of L_p^m, for some given $p \geq 1$. Let $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$ be an economy with differential information. A *price system* π is a non-negative function in the unit sphere of L_q^m (i.e., $\|\pi\|_q = 1$), where $q \in (1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. It is elementary that any price functional φ in the economy (which is a continuous linear functional in the dual of L_p^m, restricted to the cone of commodity bundles in L_p^m) is representable by a unique price system π_φ : for any commodity bundle x, $\varphi(x) = \int_\Omega \pi_\varphi \cdot x d\mu$. Given a price system π, the budget set of $i \in N$ is given by

$$B^i(\pi) = \left\{ x \in L_p^m \mid x \text{ is non-negative and } \mathcal{F}^i\text{-measurable and } \int_\Omega \pi \cdot x d\mu \leq \int_\Omega \pi \cdot e^i d\mu \right\}.$$

A Radner equilibrium of $\mathcal{E}$ is a pair $(\mathbf{x}, \pi)$, where π is a price system and $\mathbf{x} = (x^1, ..., x^n)$ is a private allocation such that for every $i \in N$ x^i maximizes U^i on $B^i(\pi)$. Denote by $RE(\mathcal{E})$ the set of Radner equilibria in $\mathcal{E}$.

The following theorem (whose proof, together with all other proofs, is given in Section 6) establishes upper semi-continuity of the Radner equilibrium correspondence.

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[8] We use the language of "convergence of economies" only for the sake of convenience. As was mentioned in the introduction, the convergence here is actually restricted to the agents' characteristics.

[9] By "$\to_{k\to\infty}^{L_p^m}$" we denote convergence in the L_p^m-norm.

[10] Since $E(X \mid \mathcal{F}')$ is usually defined as a class of functions, we will always take a selection from it.

Theorem 1. Let $\{\mathcal{E}_k\}_{k=1}^{\infty}$ be a sequence of economies $\mathcal{E}_k = \left(e_k^i, u_k^i, \mathcal{F}_k^i\right)_{i=1}^n$ with differential information that converge to $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$, where e^i is strictly positive for every $i \in N$. If $\{(\mathbf{x}_k, \pi_k)\}_{k=1}^{\infty}$ is a sequence such that $(\mathbf{x}_k, \pi_k) \in RE(\mathcal{E}_k)$ for every k, and for every $i \in N$, $x_k^i \xrightarrow[k\to\infty]{L_p^m} x^i$, $\pi_k \xrightarrow[k\to\infty]{L_q^m} \pi$, then $(\mathbf{x},\pi) \in RE(\mathcal{E})$.

One may ask whether L_q^m-norm convergence of price systems can be replaced with weak* convergence[11], without affecting the result. The answer is negative, as will be made clear in Example 1 of the next section. In this example we will also show that confining attention only to Radner equilibrium *allocations* does not yield an upper semi-continuous correspondence.

4 Continuity of the Private Core

In this section we study the continuity of the private core of an economy with differential information, introduced in Yannelis (1991).

Let $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$ be an economy with differential information. The *private core* of $\mathcal{E}$ consists of all private allocations $\mathbf{x} = \left(x^1, ..., x^n\right)$ for which there do *not* exist a non-empty coalition $S \subseteq N$ and a private S-allocation $\mathbf{y} = \left(y^i\right)_{i \in S}$, such that

$$U^i\left(y^i\right) > U^i\left(x^i\right) \tag{3}$$

for every $i \in S$.

As was said in the introduction, in complete information economies the core correspondence is upper semi-continuous, see, e.g., Kanai (1970). Therefore, it is natural to ask whether upper semi-continuity continues to hold for economies with differential information. That is, if $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is a sequence of private core allocations in a converging sequence of economies $\{\mathcal{E}_k\}_{k=1}^{\infty}$, and $\{\mathbf{x}_k\}_{k=1}^{\infty}$ converges to $\mathbf{x}$, does this imply that $\mathbf{x}$ is a private core allocation for the limiting economy $\mathcal{E}$? The following example shows that the answer may be negative for *all* converging sequences $\{\mathbf{x}_k\}_{k=1}^{\infty}$, even in very simple economies.

Example 1. For every $\varepsilon \in [0,1)$ let $\mathcal{E}_\varepsilon$ be an economy in which $m = 1$, $n = 2$, $e_k^1 = e_k^2 \equiv \frac{1}{2}$, $\Omega = [0,1] \cup [2,3]$, μ is the restriction of the Lebesgue measure on the real line to Ω (normalized so as to satisfy $\mu(\Omega) = 1$), $u^1(\omega, x) = \begin{cases} x, & \text{if } \omega \in [0,1], \\ 0, & \text{if } \omega \in [2,3]; \end{cases}$, $u^2(\omega, x) = \begin{cases} 0, & \text{if } \omega \in [0,1], \\ x, & \text{if } \omega \in [2,3]. \end{cases}$, $\mathcal{F}^1$ is the finite field generated by $[0,1] \cup [2, 2+\varepsilon]$, $(2+\varepsilon, 3]$, and $\mathcal{F}^2$ is the finite field generated by $[0,1]$, $[2,3]$. Let $\mathbf{x}_k = \left(x_k^1, x_k^2\right)$ be a private core allocation[12] in the economy

[11] When $1 < p$, the weak* convergence is of course equivalent to the weak convergence.

[12] Existence of a private core allocation in such an economy can be established by using arguments of Glycopantis, Muir and Yannelis (2001) (where existence was established with "no free disposal" condition). Alternatively, one can show nonemptiness of the private core of $\mathcal{E}_\varepsilon$ by checking that it contains the initial endowments allocation.

$\mathcal{E}_{\frac{1}{2k}}$. Since $\mathbf{x}_k$ is private, it has the form

$$\mathbf{x}_k = \left(a_1(k)\,\chi_{[0,1]\cup[2,2+\frac{1}{2k}]} + a_2(k)\,\chi_{(2+\frac{1}{2k},3]}, b_1(k)\,\chi_{[0,1]} + b_2(k)\,\chi_{[2,3]}\right)$$

(where χ_A stands for the indicator function of the set A); the equality holds almost everywhere. The feasibility constraint (1) taken for $\omega \in [0,1]$ and $\omega \in [2, 2+\frac{1}{2k}]$ yields

$$a_1(k) + b_1(k) \leq 1, \text{ and } a_1(k) + b_2(k) \leq 1. \tag{4}$$

Note that

$$a_1(k) \geq \frac{1}{2} \tag{5}$$

since otherwise for $S = \{1\}$ an S-allocation $\mathbf{y} = \left(e_k^1\right)$ would satisfy

$$U^1\left(y^1\right) > U^1\left(x_k^1\right),$$

contrary to $\mathbf{x}_k$ being a private core allocation in the economy $\mathcal{E}_{\frac{1}{2k}}$ (that is, $\mathbf{x}_k$ would not be individually rational). Individual rationality of $\mathbf{x}_k$ also implies

$$b_2(k) \geq \frac{1}{2}. \tag{6}$$

From (5), (6), and (4) we deduce that

$$a_1(k) = b_2(k) = \frac{1}{2}.$$

Consider now a converging subsequence of $\{(a_1(k), a_2(k), b_1(k), b_2(k))\}_{k=1}^{\infty}$. It exists since $0 \leq a_1(k), a_2(k), b_1(k), b_2(k) \leq 1$ by the feasibility constraint, and w.l.o.g. we will assume that the sequence itself converges. This implies convergence of $\{\mathbf{x}_k\}_{k=1}^{\infty}$ pointwise almost everywhere (and thus in L_1^m-norm, or more generally any L_p^m-norm for $1 \leq p < \infty$ due to the boundedness of the sequence) to

$$\mathbf{x} = \left(a_1\chi_{[0,1]} + a_2\chi_{[2,3]}, b_1\chi_{[0,1]} + b_2\chi_{[2,3]}\right),$$

with $a_1 = b_2 = \frac{1}{2}$. Also, $\left\{\mathcal{E}_{\frac{1}{2k}}\right\}_{k=1}^{\infty}$ converges to $\mathcal{E}_0$. However, $\mathbf{x}$ is not a private core allocation of $\mathcal{E}_0$. Indeed, take $S = N$ and $\mathbf{y} = \left(\chi_{[0,1]}, \chi_{[2,3]}\right)$. $\mathbf{y}$ is obviously a private allocation in $\mathcal{E}_0$, and

$$U^i\left(y^i\right) > U^i\left(x^i\right)$$

is satisfied for every i (that is, $\mathbf{x}$ is not Pareto-optimal).

Note that this example would still work if the utility functions were slightly modified to become strictly concave and strictly increasing. The compact set Ω could also be made connected without affecting our claim.

Our final observation is that there is a sequence of Radner equilibria in $\left\{\mathcal{E}_{\frac{1}{2k}}\right\}_{k=1}^{\infty}$ that does not converge to a Radner equilibrium in $\mathcal{E}_0$. Indeed, the "no trade" scenario $(\mathbf{e}, \pi_k)$, where $\mathbf{e}$ consists of the initial endowments of the agents and

$$\pi_k = \begin{cases} (4k)^{\frac{1}{q}} \chi_{[2,2+\frac{1}{2k}]}, & \text{if } q < \infty, \\ \chi_{[2,2+\frac{1}{2k}]}, & \text{if } q = \infty \end{cases},$$

is a Radner equilibrium in $\mathcal{E}_{\frac{1}{2k}}$, but $\{\pi_k\}_{k=1}^{\infty}$ clearly does not converge in the L_q^m-norm. It does converge weakly*, but to the zero function. This shows that Theorem 1 cannot be strengthened by assuming that $\{\pi_k\}_{k=1}^{\infty}$ converges to π weakly* (since π can be zero despite that $\|\pi_k\|_q = 1$ for every k).

In the limiting economy $\mathcal{E}_0$, every Radner equilibrium has the form $(\mathbf{x}, \pi)$, where $\mathbf{x} = \left(\chi_{[0,1]}, \chi_{[2,3]}\right)$ (almost everywhere) and

$$\int_{[0,1]} \pi(\omega)\, d\mu(\omega) = \int_{[2,3]} \pi(\omega)\, d\mu(\omega).$$

This shows that the set of Radner equilibrium allocations REA is also not upper semi-continuous: $\mathbf{e} \in REA(\mathcal{E}_{\frac{1}{2k}})$ for every k, but $REA(\mathcal{E}_0) = \{\mathbf{x}\}$ and $\mathbf{x} \neq \mathbf{e}$. ■

In the above example the limiting information field is neither included, nor includes, the information fields that converge to it. However, when this does not occur, positive results on upper semi-continuity of the private core can be obtained.

Theorem 2. Let $\{\mathcal{E}_k\}_{k=1}^{\infty} = \left\{\left(e_k^i, u_k^i, \mathcal{F}_k^i\right)_{i=1}^n\right\}_{k=1}^{\infty}$ be a sequence of economies with differential information that converges[13] to $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$, such that $\mathcal{F}_k^i \supseteq \mathcal{F}^i$ for all $i \in N$ and k. If $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is such that $\mathbf{x}_k = \left(x_k^1, ..., x_k^n\right)$ is a private core allocation in $\mathcal{E}_k$, and for every $i \in N$ $x_k^i \xrightarrow[k\to\infty]{L_1^m} x^i$, then $\mathbf{x} = \left(x^1, ..., x^n\right)$ is a private core allocation in $\mathcal{E}$.

According to Theorem 2, if the information fields of agents *shrink* when they approach the limiting information fields (which can be the case when, e.g., the level of random noise increases), then the private core is upper semi-continuous. Theorem 3 below may be viewed as a dual result, showing that the private core is also upper semi-continuous when the information of agents *rises* to the fullest possible extent, i.e., when the economies converge to a complete information economy. It stands in contrast to the result of Krasa and Shafer

[13] Convergence of information fields in Boylan pseudo-metric (which is one of the aspects of convergence of economies) is not, in fact, necessary for this theorem.

(2001), showing generic discontinuity of the private core correspondence when complete information is approached by changing the common prior of agents (rather than by expanding their information fields).

Theorem 3. Let $\{\mathcal{E}_k\}_{k=1}^{\infty} = \left\{ \left(e_k^i, u_k^i, \mathcal{F}_k^i \right)_{i=1}^{n} \right\}_{k=1}^{\infty}$ be a sequence of economies with differential information that converges to $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i \right)_{i=1}^{n}$, where $\mathcal{F}^i = \mathcal{F}$ for every $i \in N$, $\mathcal{F}$ is the σ-field of all Borel sets in Ω, and each e^i is a continuous and strictly positive function. Assume also that for every two disjoint and closed subsets A and B of Ω there exist $A_k^i \in \mathcal{F}_k^i$ for every i and k such that $A \subseteq A_k^i$ and $A_k^i \cap B = \emptyset$ for all sufficiently large k (that is, $\mathcal{F}_k^i$ can separate disjoint closed subsets for all sufficiently large k). Then, if $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is such that $\mathbf{x}_k = \left(x_k^1, ..., x_k^n \right)$ is a private core allocation in $\mathcal{E}_k$ and for every $i \in N$ $x_k^i \rightarrow_{k \rightarrow \infty}^{L_1^m} x^i$, $\mathbf{x} = \left(x^1, ..., x^n \right)$ is a private core allocation in $\mathcal{E}$.

5 The Coarse Core

Let $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i \right)_{i=1}^{n}$ be an economy with differential information. The *coarse core* of $\mathcal{E}$ consists of all allocations $\mathbf{x} = \left(x^1, ..., x^n \right)$ for which there do *not* exist a non-empty coalition $S \subseteq N$, an event $A \in \bigwedge_{i \in S} \mathcal{F}^i$ with $\mu(A) > 0$ (where $\bigwedge_{i \in S} \mathcal{F}^i$ stands for the finest field included in all $\mathcal{F}^i, i \in S$, and represents the common knowledge of agents in S), and an S-assignment $\mathbf{y} = \left(y^i \right)_{i \in S}$, that satisfy the following conditions:

(i) $\sum_{i \in S} y^i(\omega) \leq \sum_{i \in S} e^i(\omega)$ for almost every $\omega \in A$;

(ii)[14] $E\left(u^i\left(\cdot, y^i(\cdot) \right) \mid \mathcal{F}^i \right)(\omega) > E\left(u^i\left(\cdot, x^i(\cdot) \right) \mid \mathcal{F}^i \right)(\omega)$ for almost every $\omega \in A$ and $i \in S$.

According to (i), $\mathbf{y}$ is feasible for agents in S, given the event A and their initial endowments. And (ii) means that agents in S can improve their conditional expected utility when they redistribute their endowments according to $\mathbf{y}$, given that A occurred. The concept of coarse core (introduced by Wilson (1978)) thus disallows existence of such S, A, and $\mathbf{y}$.

Wilson (1978) proved that the coarse core is non-empty under standard conditions on the economy, at least when the set of states of nature is finite. The following example shows that the coarse core may not be upper semi-continuous, even when the private core is[15].

[14] Recall that $E(X \mid \mathcal{F}')$ stands for the conditional expectation of X with respect to $\mathcal{F}'$.

[15] In this example, along the converging sequence of economies the information fields of agents 1 and 2 are fixed, while the information fields of agents 3 and 4 approximate full information. The proof of Theorem 3 can be used to show that the limit of any converging sequence of private core allocations will be a private core allocation in the limiting economy (commodity bundles y^1 and y^2, if used by a blocking coalition in $\mathcal{E}$, will not even have to be approximated by a continuous function in order to show that $E\left(y^1 \mid \mathcal{F}_k^i\right)$ and $E\left(y^2 \mid \mathcal{F}_k^i\right)$ approximate them uniformly).

Example 2. Consider a sequence of economies $\mathcal{E}_k = \left(e_k^i, u_k^i, \mathcal{F}_k^i\right)_{i=1}^n$ with $m = 2$ and $n = 4$, where:

1) u_k^i is state-independent, and $u_k^i(x_1, x_2) = u(x_1, x_2) = \left(\sqrt{x_1} + \sqrt{x_2}\right)^2$ for every i, k;

2) e_k^i is state-independent, and $e_k^1 = e_k^3 = (10, 110)$, $e_k^2 = e_k^4 = (110, 10)$, for every k;

3) the compact space of states Ω is the sequence $\{\omega_j\}_{j=0}^{\infty}$, where $\omega_0 = 0$, and $\omega_j = \frac{1}{j}$ for $j \geq 1$;

4) μ is a measure on Ω according to which $\mu(\{\omega_{2j}\}) = \mu(\{\omega_{2j+1}\}) > 0$ for every $j \geq 0$;

5) $\mathcal{F}_k^1 = \mathcal{F}_k^2 = \left\{\emptyset, \{\omega_{2j}\}_{j=0}^{\infty}, \{\omega_{2j+1}\}_{j=0}^{\infty}, \Omega\right\}$; that is, 1 and 2 can only tell whether the realized state of nature has even or odd index;

6) $\mathcal{F}_k^3 = \mathcal{F}_k^4$ is the minimal σ-field in Ω that contains $\{\omega_0\}, \{\omega_1\}, ..., \{\omega_k\}$, and $\{\omega_j\}_{j=k}^{\infty}$; that is, 3 and 4 can actually know the state of nature, provided its index is less than k.

It is clear that the sequence $\{\mathcal{E}_k\}_{k=1}^{\infty}$ converges to an economy $\mathcal{E} = \left(e^i, u^i, \mathcal{F}^i\right)_{i=1}^n$, which differs from those in $\{\mathcal{E}_k\}_{k=1}^{\infty}$ only in one respect: $\mathcal{F}^3 = \mathcal{F}^4$ is the field of all subsets of Ω.

Now consider an assignment $\mathbf{x} = \left(x^1, x^2, x^3, x^4\right)$, where $x^1(\omega_{2j}) = x^2(\omega_{2j+1}) = (50, 50)$ and $x^1(\omega_{2j+1}) = x^2(\omega_{2j}) = (70, 70)$ for every $j \geq 0$, and $x^3 = x^4 \equiv (60, 60)$. It is obvious that $\mathbf{x}$ is an allocation for all economies in $\{\mathcal{E}_k\}_{k=1}^{\infty}$, and for $\mathcal{E}$.

We show first that $\mathbf{x}$ is not a coarse core allocation for $\mathcal{E}$. Indeed, $S = \{2, 3\}$ can improve the conditional expected utility of both 2 and 3 given that the state of nature has an odd index. We simply take $A = \{\omega_{2j+1}\}_{j=0}^{\infty}$ (which is clearly in the common knowledge of 2 and 3 in $\mathcal{E}$: $A \in \mathcal{F}^2 \wedge \mathcal{F}^3$), and $\mathbf{y} = \left(y^i\right)_{i \in S}$ defined by $y^2(\omega_{2j+1}) \equiv (51, 51)$, $y^3(\omega_{2j+1}) \equiv (69, 69)$. Then (i) and (ii) are satisfied, and hence $\mathbf{x}$ is *not* a coarse core allocation for $\mathcal{E}$.

Our next step is to show that $\mathbf{x}$ *is* a coarse core allocation for every $\mathcal{E}_k$. (This will imply that the coarse core correspondence indeed lacks upper semi-continuity.) To this end suppose that there are k, S, A, and $\mathbf{y}$ that satisfy (i) and (ii) in the economy $\mathcal{E}_k$. It is easy to see that $\mathbf{x}$ is individually rational (given each trader's information), and so S must contain more than one trader.

Note that (ii) implies

$$\int_A u\left(y^i(\omega)\right) d\mu(\omega) = \int_A E\left(u\left(y^i(\cdot)\right) \mid \mathcal{F}^i\right)(\omega)\, d\mu(\omega)$$

$$> \int_A E\left(u\left(x^i(\cdot)\right) \mid \mathcal{F}^i\right)(\omega)\, d\mu(\omega) = \int_A u\left(x^i(\omega)\right) d\mu(\omega)$$

for every $i \in S$, since A is an element of every F_i. Therefore

$$\sum_{i \in S} \int_A u\left(y^i(\omega)\right) d\mu(\omega) > \sum_{i \in S} \int_A u\left(x^i(\omega)\right) d\mu(\omega). \tag{7}$$

At the same time, from concavity and homogeneity of u and (i) it follows that:

$$\sum_{i\in S}\int_A u\left(y^i(\omega)\right)d\mu(\omega)\le u\left(\sum_{i\in S}\int_A y^i(\omega)\,d\mu(\omega)\right)$$

$$=\left(\sqrt{\sum_{i\in S}\int_A y^i(\omega)_1\,d\mu(\omega)}+\sqrt{\sum_{i\in S}\int_A y^i(\omega)_2\,d\mu(\omega)}\right)^2$$

$$\le\left(2\sqrt{\frac{\sum_{i\in S}\int_A\left(y^i(\omega)_1+y^i(\omega)_2\right)d\mu(\omega)}{2}}\right)^2$$

$$=2\int_A\sum_{i\in S}\left(y^i(\omega)_1+y^i(\omega)_2\right)d\mu(\omega)$$

$$\le 2\int_A\sum_{i\in S}\left(e^i(\omega)_1+e^i(\omega)_2\right)d\mu(\omega)=2\int_A 120\,|S|\,d\mu(\omega)$$

$$=240\,|S|\,\mu(A).$$

Thus

$$\sum_{i\in S}\int_A u\left(y^i(\omega)\right)d\mu(\omega)\le 240\,|S|\,\mu(A). \tag{8}$$

If $S=\{1,2\}$ or $S=\{3,4\}$, then for each ω $\sum_{i\in S}u\left(x^i(\omega)\right)=480=240\,|S|$, and so (7) and (8) are inconsistent. Since $|S|>1$ (as was mentioned), we are left with the possibility that $S\cap\{1,2\}\neq\emptyset$ and $S\cap\{3,4\}\neq\emptyset$. Then $A=\Omega$ since the common knowledge of all the agents in S (where A belongs) contains only trivial information[16], and $\mu(A)>0$. However, clearly $\int_\Omega u\left(x^i(\omega)\right)d\mu(\omega)=240$ for every $i\in N$, and therefore

$$\sum_{i\in S}\int_\Omega u\left(x^i(\omega)\right)d\mu(\omega)=240\,|S|\,.$$

Thus, (7) and (8) lead to a contradiction again. We conclude that there exist no k, S, A, and $\mathbf{y}$ that satisfy (i) and (ii) in the economy $\mathcal{E}_k$, and therefore $\mathbf{x}$ is a coarse core allocation in every $\mathcal{E}_k$.■

[16] I.e., the field $\{\emptyset,\Omega\}$.

6 Proofs

Lemma 1. Let $\{\mathcal{F}_k\}_{k=1}^{\infty}$ be a sequence of σ-subfields of $\mathcal{F}$ that converges to $\mathcal{F}_0$ in Boylan pseudo-metric, and let $\{X_k\}_{k=1}^{\infty}$ be a sequence of functions in $L_1(\Omega, \mathcal{F}, \mu)$ that converges to X in L_1-norm. If X_k is $\mathcal{F}_k$-measurable for every k, then X is $\mathcal{F}_0$-measurable.

Proof. We will show that $X = E(X \mid \mathcal{F}_0)$ almost everywhere. Since L_1-norm convergence implies convergence in measure, $\{X_k\}_{k=1}^{\infty}$ converges to X in measure. Thus, it suffices to show that $\{X_k\}_{k=1}^{\infty}$ converges in measure also to $E(X \mid \mathcal{F}_0)$.

From $\mathcal{F}_k$-measurability of X_k, $X_k = E(X \mid \mathcal{F}_k)$ almost everywhere. Thus, almost everywhere,

$$|X_k - E(X \mid \mathcal{F}_0)| = |E(X_k \mid \mathcal{F}_k) - E(X \mid \mathcal{F}_0)| \tag{9}$$

$$\leq |E(X_k \mid \mathcal{F}_k) - E(X \mid \mathcal{F}_k)| + |E(X \mid \mathcal{F}_k) - E(X \mid \mathcal{F}_0)|. \tag{10}$$

For every k,

$$\int_{\Omega} |E(X_k \mid \mathcal{F}_k) - E(X \mid \mathcal{F}_k)| \, d\mu \leq \int_{\Omega} E(|X_k - X| \mid \mathcal{F}_k) \, d\mu$$

$$= \int_{\Omega} |X_k(\omega) - X(\omega)| \, d\mu(\omega) = \|X_k - X\|_1 .$$

Since $\|X_k - X\|_1 \to_{k\to\infty} 0$, also $\int_{\Omega} |E(X_k \mid \mathcal{F}_k) - E(X \mid \mathcal{F}_k)| \, d\mu \to_{k\to\infty} 0$, and hence the first summand in (10) converges to zero in measure. And $E(X \mid \mathcal{F}_k) \to_{k\to\infty} E(X \mid \mathcal{F}_0)$ in measure by Theorem 4 in Boylan (1971). Consequently, by (9)-(10), $\{X_k\}_{k=1}^{\infty}$ indeed converges in measure to $E(X \mid \mathcal{F}_0)$.■

Lemma 2. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of continuous functions on $\Omega \times R_+^m$ such that for every $\omega \in \Omega$ the function $u_k(\omega, \cdot)$ is concave, non-negative, and non-decreasing. Assume that $\{u_k\}_{k=1}^{\infty}$ converges to a function u uniformly on every compact subset of $\Omega \times R_+^m$. If $\{x_k\}_{k=1}^{\infty}$ is a sequence in L_1^m that converges to x in the L_1^m-norm, then

$$\lim_{k\to\infty} \int_{\Omega} u_k(\omega, x_k(\omega)) d\mu(\omega) = \int_{\Omega} u(\omega, x(\omega)) d\mu(\omega).$$

Proof. Due to the uniform convergence of $\{u_k\}_{k=1}^{\infty}$, the sequence of $C_k = \max_{\omega \in \Omega, \|x\| \leq 1} u_k(\omega, x)$ is bounded from above by some $C > 0$. Fix $\varepsilon > 0$. Since $\{x_k\}_{k=1}^{\infty}$ converges to x in L_1^m-norm, $\{x_k\}_{k=1}^{\infty}$ is uniformly integrable (see Proposition II.5.4 of Neveu (1965)) and x is integrable. Consequently, we can find $M > 1$ with:

$$\text{(i)} \ \sup_k \int_{\{\omega \in \Omega | \|x_k(\omega)\| > M\}} \|x_k(\omega)\| \, d\mu(\omega) < \frac{\varepsilon}{9C};$$

and

$$\text{(ii)} \ \int_{\{\omega \in \Omega | \|x(\omega)\| > M\}} \|x(\omega)\| \, d\mu(\omega) < \frac{\varepsilon}{9C}.$$

Note that

$$\left| \int_\Omega u_k(\omega, x_k(\omega)) d\mu(\omega) - \int_\Omega u(\omega, x(\omega)) d\mu(\omega) \right|$$

$$\leq \int_\Omega |u_k(\omega, x_k(\omega)) - u(\omega, x_k(\omega))| \, d\mu(\omega)$$

$$+ \int_\Omega |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, d\mu(\omega)$$

$$= \int_\Omega |u_k(\omega, x_k(\omega)) - u(\omega, x_k(\omega))| \, \chi_{\{\omega \in \Omega | \|x_k(\omega)\| \leq M\}} d\mu(\omega)$$

$$+ \int_{\{\omega \in \Omega | \|x_k(\omega)\| > M\}} |u_k(\omega, x_k(\omega)) - u(\omega, x_k(\omega))| \, d\mu(\omega)$$

$$+ \int_\Omega |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, \chi_{\{\omega \in \Omega | \|x_k(\omega)\| \leq M \text{ and } \|x(\omega)\| \leq M\}} d\mu(\omega)$$

$$+ \int_{\{\omega \in \Omega | \|x_k(\omega)\| > M \text{ or } \|x(\omega)\| > M\}} |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, d\mu(\omega)$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

(Here χ_S stands for the characteristic function of the set S). We will show that each I_j is less than ε for all sufficiently large k, and this will establish the claim.

Before proceeding further, note that concavity and monotonicity of each $u_k(\omega, \cdot)$ imply that

$$u_k(\omega, x) \leq C \max\{\|x\|, 1\} \tag{11}$$

for all k, $\omega \in \Omega$, and $x \in R^l_+$.

The uniform convergence of $\{u_k\}_{k=1}^{\infty}$ to u on $\Omega \times \{x \in R^m_+ \mid \|x\| \leq M\}$ immediately yields $\lim_{k\to\infty} I_1 = 0$, and thus

$$I_1 < \varepsilon$$

for all sufficiently large k. Inequality (11) implies that

$$I_2 \leq 2C \int_{\{\omega\in\Omega \mid \|x_k(\omega)\| > M\}} \|x_k(\omega)\| \, d\mu(\omega), \tag{12}$$

and by our choice of M,

$$I_2 < \varepsilon$$

for all k. Since x_k converges to x_0 in measure (as implied by the L^m_1-norm convergence), and $u(\cdot,\cdot)$ is uniformly continuous on the compact set $\Omega \times \{x \in R^m_+ \mid \|x\| \leq M\}$ (being a continuous function), $|u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \cdot \chi_{\{\omega\in\Omega \mid \|x_k(\omega)\| \leq M \text{ and } \|x(\omega)\| \leq M\}}$ converges in measure to zero, and thus $\lim_{k\to\infty} I_3 = 0$ by the generalized bounded convergence theorem. Once again,

$$I_3 < \varepsilon$$

for all sufficiently large k. Finally,

$$I_4 = \int_{\{\omega\in\Omega \mid \|x_k(\omega)\| > M \text{ or } \|x(\omega)\| > M\}} |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, d\mu(\omega)$$

$$\leq \int_{\{\omega\in\Omega \mid \|x_k(\omega)\| > M\}} |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, d\mu(\omega)$$

$$+ \int_{\{\omega\in\Omega \mid \|x(\omega)\| > M\}} |u(\omega, x_k(\omega)) - u(\omega, x(\omega))| \, d\mu(\omega)$$

$$\leq C \int_{\{\omega\in\Omega \mid \|x_k(\omega)\| > M\}} (\|x_k(\omega)\| + \max(\|x(\omega)\|, 1)) d\mu(\omega)$$

$$+ C \int_{\{\omega\in\Omega \mid \|x(\omega)\| > M\}} (\max(\|x_k(\omega)\|, 1) + \|x(\omega)\|) d\mu(\omega)$$

$$\leq C\int_{\{\omega\in\Omega|\|x_k(\omega)\|>M\}}(2\|x_k(\omega)\|+\|x(\omega)\|)d\mu(\omega)$$

$$+C\int_{\{\omega\in\Omega|\|x(\omega)\|>M\}}(\|x_k(\omega)\|+2\|x(\omega)\|)d\mu(\omega)$$

$$\leq C\int_{\{\omega\in\Omega|\|x_k(\omega)\|>M\}}(2\|x_k(\omega)\|+\|x(\omega)-x_k(\omega)\|+\|x_k(\omega)\|)d\mu(\omega)$$

$$+C\int_{\{\omega\in\Omega|\|x(\omega)\|>M\}}(\|x_k(\omega)-x(\omega)\|+\|x(\omega)\|+2\|x(\omega)\|)d\mu(\omega)$$

$$\leq 2C\int_{\Omega}\|x_k(\omega)-x_0(\omega)\|\,d\mu(\omega)$$

$$+3C\int_{\{\omega\in\Omega|\|x_k(\omega)\|>M\}}\|x_k(\omega)\|\,d\mu(\omega)+3C\int_{\{\omega\in\Omega|\|x_0(\omega)\|>M\}}\|x_0(\omega)\|\,d\mu(\omega)$$

$$\leq 2C\int_{\Omega}\|x_k(\omega)-x_0(\omega)\|\,d\mu(\omega)+\frac{2\varepsilon}{3}\text{ (by the choice of }M\text{)}.$$

The first summand in the above expression is less than $\frac{\varepsilon}{3}$ for all sufficiently large k, which yields

$$I_4<\varepsilon.$$

■

Proof of Theorem 1. We have to show that $(\mathbf{x},\pi)$ is a Radner equilibrium of $\mathcal{E}$. Note first that $\mathbf{x}=(x^1,...,x^n)$ is a private allocation in $\mathcal{E}$. Indeed, each x^i is $\mathcal{F}^i$-measurable by Lemma 1. Moreover, each L_1^m-norm convergent $\left\{x_k^i\right\}_{k=1}^{\infty}$ has a subsequence that converges pointwise almost everywhere, and thus $\mathbf{x}$ also satisfies the feasibility constraint (1).

Next, for every $i\in N$

$$\left|\int_{\Omega}\left(\pi_k\cdot x_k^i-\pi\cdot x^i\right)d\mu\right|\leq\int_{\Omega}\left|\pi_k\cdot\left(x_k^i-x^i\right)\right|d\mu+\int_{\Omega}\left|(\pi_k-\pi)\cdot x^i\right|d\mu. \quad (13)$$

By Hölder inequality,

$$\int_{\Omega}\left|\pi_k\cdot\left(x_k^i-x^i\right)\right|d\mu\leq\|\pi_k\|_q\left\|x_k^i-x^i\right\|_p=\left\|x_k^i-x^i\right\|_p$$

and

$$\int_{\Omega} \left|(\pi_k - \pi) \cdot x^i\right| d\mu \leq \|\pi_k - \pi\|_q \left\|x^i\right\|_p .$$

However, $\lim_{k\to\infty} \|\pi_k - \pi\|_q = \lim_{k\to\infty} \left\|x_k^i - x^i\right\|_p = 0$, and hence by (13)

$$\int_{\Omega} \pi_k \cdot x_k^i d\mu \to_{k\to\infty} \int_{\Omega} \pi \cdot x^i d\mu. \tag{14}$$

It is also obvious that

$$\int_{\Omega} \pi_k \cdot e^i d\mu \to_{k\to\infty} \int_{\Omega} \pi \cdot e^i d\mu. \tag{15}$$

From (14), (15) it follows that $x^i \in B^i(\pi)$ for every $i \in N$.

It remains to show that for every $i \in N$, x^i maximizes U^i on $B^i(\pi)$. Indeed, if this were not true, there would exist $i \in N$ and $y^i \in B^i(\pi)$ with $U^i(y^i) > U^i(x^i)$. Since $(1-\alpha)\min(y^i, M) \to_{\alpha\searrow 0, M\to\infty} y^i$ in L_1^m-norm, $U^i((1-\alpha)\min(y^i, M)) \to_{\alpha\searrow 0, M\to\infty} U^i(y^i)$ by Lemma 2, and thus w.l.o.g. y^i is bounded and satisfies[17]

$$\int_{\Omega} \pi \cdot y^i d\mu < \int_{\Omega} \pi \cdot e^i d\mu \tag{16}$$

(otherwise it can be replaced by some $(1-\alpha)\min(y^i, M)$).

Now define $y_k^i \equiv E\left(y^i \mid \mathcal{F}_k^i\right)$; it is $\mathcal{F}_k^i$-measurable. Since $y_k^i \to_{k\to\infty} y^i$ in measure (by Theorem 4 of Boylan (1971)), $y_k^i \to^{L_1^m}_{k\to\infty} y^i$ because of the uniform boundedness of y_k^i and y^i, and it can be established similarly to (14) that

$$\int_{\Omega} \pi_k \cdot y_k^i d\mu \to_{k\to\infty} \int_{\Omega} \pi \cdot y^i d\mu.$$

Together with (15) and (16) this implies that $y_k^i \in B^i(\pi_k)$ for all sufficiently large k. But $U^i(y^i) > U^i(x^i)$, and $x_k^i \to^{L_1^m}_{k\to\infty} x^i$, $y_k^i \to^{L_1^m}_{k\to\infty} y^i$. By Lemma 2, for all sufficiently large k

$$U_k^i(y_k^i) > U_k^i(x_k^i),$$

where U_k^i is defined by (2) for the utility function u_k. This cannot be consistent with $y_k^i \in B^i(\pi_k)$ since $(\mathbf{x}_k, \pi_k) \in RE(\mathcal{E}_k)$. We reached a contradiction, which leads to the conclusion that $(\mathbf{x}, \pi) \in RE(\mathcal{E})$.■

Proof of Theorem 2. As in the proof of Theorem 1, $\mathbf{x} = (x^1, ..., x^n)$ is a private allocation in $\mathcal{E}$. Assume, however, that $\mathbf{x}$ is not in the private core of $\mathcal{E}$.

[17] We use the fact that $\int_{\Omega} \pi \cdot e^i > 0$, which is due to strict positivity of e^i, and positivity of the non-vanishing π.

Then there exist a non-empty coalition S and a private S-allocation $\mathbf{y} = (y^i)_{i \in S}$ such that

$$U^i\left(y^i\right) > U^i\left(x^i\right) \tag{17}$$

for every $i \in S$.

We will now modify $\mathbf{y}$ to obtain an S-allocation in $\mathcal{E}_k$ for a certain large k. Let $\varepsilon > 0$ and denote by $k(\varepsilon)$ some positive integer k for which

$$\mu\left(A^i_{k,\varepsilon}\right) > 1 - \varepsilon \text{ for every } i \in S, \tag{18}$$

where

$$A^i_{k,\varepsilon} = \left\{\omega \in \Omega \mid \left\|e^i_k(\omega) - e^i(\omega)\right\| < \varepsilon\right\}.$$

Such $k(\varepsilon)$ is clearly well defined for all $\varepsilon > 0$, since $e^i_k \to e^i$ in measure. For this reason, we can also choose $\{k(\varepsilon)\}_{\varepsilon > 0}$ in a way that $k(\varepsilon)$ is non-increasing in ε and $\lim_{\varepsilon \searrow 0} k(\varepsilon) = \infty$. Clearly, for almost every $\omega \in \Omega$ and every commodity j

$$\sum_{i \in S} y^i(\omega) \cdot \chi_{A^i_{k(\varepsilon),\varepsilon}}(\omega) \leq \sum_{i \in S} e^i_{k(\varepsilon)}(\omega) + n\varepsilon\overline{1},$$

where $\overline{1} = (1, ..., 1) \in R^m$ (this inequality follows from the fact that $\mathbf{y}$ satisfies the feasibility constraint in $\mathcal{E}$ and the definition of $A^i_{k(\varepsilon),\varepsilon}$). Therefore, $\mathbf{y}_{k(\varepsilon)} = (y^i_{k(\varepsilon)})_{i \in S}$, defined by

$$y^i_{k(\varepsilon)}(\omega)_j \equiv \max(y^i(\omega)_j \cdot \chi_{A^i_{k(\varepsilon),\varepsilon}}(\omega) - n\varepsilon, 0)$$

for every $\omega \in \Omega$ and every commodity j, satisfies the feasibility constraint in $\mathcal{E}_{k(\varepsilon)}$:

$$\sum_{i \in S} y^i_{k(\varepsilon)}(\omega) \leq \sum_{i \in S} e^i_{k(\varepsilon)}(\omega) \text{ for almost every } \omega \in \Omega. \tag{19}$$

Note that each $A^i_{k,\varepsilon}$ is $\mathcal{F}^i_k$-measurable (since both e^i_k and e^i are $\mathcal{F}^i_k$-measurable due to the inclusion $\mathcal{F}^i_k \supseteq \mathcal{F}^i$), and so $y^i_{k(\varepsilon)}$ is $\mathcal{F}^i_{k(\varepsilon)}$-measurable. From this fact and (19) it follows that the sequence $\left\{\mathbf{y}_{k\left(\frac{1}{r}\right)}\right\}_{r=1}^{\infty}$ is a sequence of private S-allocations in economies $\left\{\mathcal{E}_{k\left(\frac{1}{r}\right)}\right\}_{r=1}^{\infty}$, and (18) implies that for each $i \in S$ $\left\{y^i_{k\left(\frac{1}{r}\right)}\right\}_{r=1}^{\infty}$ converges to y^i in measure.

Since for each $i \in S$ and r $y^i_{k\left(\frac{1}{r}\right)} \leq y^i$, the sequence $\left\{y^i_{k\left(\frac{1}{r}\right)}\right\}_{r=1}^{\infty}$ is bounded from above by an integrable function. Therefore, its convergence in measure to y^i implies convergence in the L^m_1-norm as well. However, according to this and Lemma 2, for every $i \in S$

$$\lim_{r \to \infty} U^i_{k\left(\frac{1}{r}\right)}\left(y^i_{k\left(\frac{1}{r}\right)}\right) = U^i\left(y^i\right).$$

Also, since $x_k^i \rightarrow_{k\rightarrow\infty}^{L_1^m} x^i$, for every $i \in S$

$$\lim_{r\to\infty} U^i_{k\left(\frac{1}{r}\right)}\left(x^i_{k\left(\frac{1}{r}\right)}\right) = U^i\left(x^i\right).$$

Due to assumption (17), these two equalities yield existence of r for which

$$U^i_{k\left(\frac{1}{r}\right)}\left(y^i_{k\left(\frac{1}{r}\right)}\right) > U^i_{k\left(\frac{1}{r}\right)}\left(x^i_{k\left(\frac{1}{r}\right)}\right)$$

for every $i \in S$. This contradicts the assumption that $\mathbf{x}_{k\left(\frac{1}{r}\right)}$ is a private core allocation in $\mathcal{E}_{k\left(\frac{1}{r}\right)}$.■

Proof of Theorem 3. As in the proof of Theorem 2, if the limit allocation $\mathbf{x} = (x^1, ..., x^n)$ is not in the private core of $\mathcal{E}$, there exist a non-empty coalition S and a private S-allocation $\mathbf{y} = (y^i)_{i\in S}$ such that

$$U^i\left(y^i\right) > U^i\left(x^i\right) \tag{20}$$

for every $i \in S$. We can assume w.l.o.g. that each y^i is bounded. (Otherwise it can be replaced by $\min(y^i, M)$, which converges to y^i in L_1^m-norm as $M \to \infty$. Indeed, according to Lemma 2, $U^i(\min(y^i, M)) \to_{M\to\infty} U^i(y^i)$, and thus this replacement will leave (20) intact for sufficiently large M. Clearly, the replacement also leads to a private allocation.) We will show next that it can also be assumed w.l.o.g. that each y^i is continuous.

By Lusin theorem, for every $\varepsilon > 0$ and $i \in S$ there exists a continuous function y^i_ε on Ω, such that $\mu\left(A^i_\varepsilon\right) < \varepsilon$ where $A^i_\varepsilon = \left\{\omega \mid y^i(\omega) \neq y^i_\varepsilon(\omega)\right\}$. Also denote $B_\varepsilon = \left\{\omega \mid \text{there exists } j \text{ with } \sum_{i\in S} y^i_\varepsilon(\omega)_j \geq \sum_{i\in S} e^i(\omega)_j + \varepsilon\right\}$, $B = \left\{\omega \mid \sum_{i\in S} y^i_\varepsilon(\omega) \leq \sum_{i\in S} e^i(\omega)\right\}$. Since these subsets of a compact metric space Ω are disjoint and closed (here we use continuity of both e^i and y^i_ε for every $i \in S$), they can be separated, i.e., there is a continuous function $c : \Omega \to [0, 1]$ such that $c \equiv 1$ on B and $c \equiv 0$ on B_ε. Thus, $\widetilde{\mathbf{y}}_\varepsilon = (\widetilde{y}^i_\varepsilon)_{i\in S}$ given by

$$\widetilde{y}^i_\varepsilon(\omega)_j \equiv \max(c(\omega) \cdot y^i_\varepsilon(\omega)_j - \varepsilon, 0)$$

for every $i \in S$, commodity j, and $\omega \in \Omega$, satisfies the feasibility constraint in $\mathcal{E}$:

$$\sum_{i\in S} \widetilde{y}^i_\varepsilon(\omega) \leq \sum_{i\in S} e^i(\omega) \text{ for every } \omega \in \Omega.$$

Since each $\widetilde{y}^i_\varepsilon$ is $\mathcal{F}^i$-measurable as a continuous function on Ω (recall that $\mathcal{F}^i = \mathcal{F}$ is the σ-field of all Borel sets in Ω), $\widetilde{\mathbf{y}}_\varepsilon$ is in fact a private S-allocation that consists of continuous functions. Moreover, for each $i \in S$ and a commodity j

$$y^i_\varepsilon(\omega)_j - \varepsilon \leq \widetilde{y}^i_\varepsilon(\omega)_j \leq y^i_\varepsilon(\omega)_j \tag{21}$$

for almost every $\omega \in \bigcap_{i\in S}\left(A_{\varepsilon}^{i}\right)^{c}$, since $\mathbf{y}$ satisfies the feasibility constraint $\sum_{i\in S} y^{i}(\omega) \leq \sum_{i\in S} e^{i}(\omega)$ for almost every $\omega \in \Omega$ and hence $\mu\left(\bigcap_{i\in S}\left(A_{\varepsilon}^{i}\right)^{c}\backslash B\right) = 0$. Clearly $\mu\left(\bigcap_{i\in S}\left(A_{\varepsilon}^{i}\right)^{c}\right) > 1 - n\varepsilon$, and together with (21) this implies that functions in $\widetilde{\mathbf{y}}_{\varepsilon}$ converge in measure to those in $\mathbf{y}$ as $\varepsilon \searrow 0$. From the (obvious) uniform boundedness[18] of $\widetilde{\mathbf{y}}_{\varepsilon}$ and $\mathbf{y}$ it follows that the convergence is in the L_{1}^{m}-norm as well. By Lemma 2 and (20),

$$U^{i}\left(\widetilde{y}_{\varepsilon_0}^{i}\right) > U^{i}\left(x^{i}\right)$$

for every $i \in S$ and some sufficiently small ε_0. Thus, by replacing $\mathbf{y}$ in (20) by $\widetilde{\mathbf{y}}_{\varepsilon_0}$ if necessary, we can w.l.o.g. assume that (20) holds for $\mathbf{y}$ that consists of continuous functions.

For every $k = 1, 2, ...$ and $i \in S$ define $z_k^i = E(y^i \mid \mathcal{F}_k^i)$; z_k^i is clearly $\mathcal{F}_k^i$-measurable and is (or can be chosen to be) bounded by the same constant as y^i. We will show that a subsequence of $\left\{z_k^i\right\}_{k=1}^{\infty}$ converges to y^i uniformly (almost everywhere). For every commodity j and two positive integers K, l with $0 \leq l \leq K$, consider a pair of closed subsets of Ω :

$$C_K^l(j) = \left\{\omega \mid \frac{l-1}{K}\max_{\omega\in\Omega} y_j^i \leq y^i(\omega)_j \leq \frac{l}{K}\max_{\omega\in\Omega} y_j^i\right\}$$

and

$$\widetilde{C}_K^l(j) = \left\{\omega \mid \frac{l-2}{K}\max_{\omega\in\Omega} y_j^i \geq y^i(\omega)_j \text{ or } \frac{l+1}{K}\max_{\omega\in\Omega} y_j^i \leq y^i(\omega)_j\right\}.$$

Since $\mathcal{F}_k^i$ can separate disjoint closed sets for all sufficiently large k, for every K there exists $k = k(K)$ independent of i and j such that $\mathcal{F}_{k(K)}^i$ separates $C_K^l(j)$ from $\widetilde{C}_K^l(j)$ for every j and $1 \leq l \leq K$; it can also be assumed that $\lim_{K\to\infty} k(K) = \infty$. Thus for every j and $1 \leq l \leq K$ there is a set $D_K^l(j) \in \mathcal{F}_{k(K)}^i$ such that

$$\left\{\omega \mid \frac{l-1}{K}\max_{\omega\in\Omega} y_j^i \leq y^i(\omega)_j \leq \frac{l}{K}\max_{\omega\in\Omega} y_j^i\right\} \subset D_K^l(j)$$

(and hence $D_K^1(j), D_K^2(j), ..., D_K^K(j)$ cover Ω), and

$$D_K^l(j) \subset \left\{\omega \mid \frac{l-2}{K}\max_{\omega\in\Omega} y_j^i < y^i(\omega)_j < \frac{l+1}{K}\max_{\omega\in\Omega} y_j^i\right\}.$$

Consequently, for every j and $1 \leq l \leq K$, and almost every $\omega \in D_K^l(j)$,

$$z_{k(K)}^i(\omega)_j - y^i(\omega)_j = E(y_j^i \mid \mathcal{F}_{k(K)}^i)(\omega) - y^i(\omega)_j$$

(here we use the fact that $D_K^l(j) \in \mathcal{F}_{k(K)}^i$)

$$= E(y_j^i \cdot \chi_{D_K^l(j)} \mid \mathcal{F}_{k(K)}^i)(\omega) - y^i(\omega)_j$$

[18] Here we use the (w.l.o.g.) assumption that the commodity bundles in $\mathbf{y}$ are bounded functions.

$$\leq E(\frac{l+1}{K}\max_{\omega\in\Omega} y_j^i \cdot \chi_{D_K^l(j)} \mid \digamma^i_{k(K)})(\omega) - \frac{l-2}{K}\max_{\omega\in\Omega} y_j^i$$

$$= \frac{l+1}{K}\max_{\omega\in\Omega} y_j^i - \frac{l-2}{K}\max_{\omega\in\Omega} y_j^i = \frac{3}{K}\max_{\omega\in\Omega} y_j^i.$$

Similarly,

$$z^i_{k(K)}(\omega)_j - y^i(\omega)_j \geq -\frac{3}{K}\max_{\omega\in\Omega} y_j^i$$

for almost every $\omega \in D_K^l(j)$. We conclude that

$$\left| z^i_{k(K)}(\omega)_j - y^i(\omega)_j \right| \leq \frac{3}{K}\max_{\omega\in\Omega} y_j^i \equiv \delta_K(j) \text{ for almost every } \omega \in \Omega.$$

(This means that $\left\{z^i_{k(K)}\right\}_{K=1}^{\infty}$ converges to y^i uniformly (almost everywhere).) Thus

$$\sum_{i\in S} z^i_{k(K)}(\omega)_j \leq \sum_{i\in S} y^i(\omega)_j + n\delta_{k(K)}(j) \leq \sum_{i\in S} e^i(\omega)_j + n\delta_{k(K)}(j) \qquad (22)$$

for almost every $\omega \in \Omega$.

Now denote

$$y^i_{k(K)}(\omega)_j \equiv \max(z^i_{k(K)}(\omega)_j - n\delta_K(j), 0)$$

for every $i \in S$, commodity j, and $\omega \in \Omega$. It is clear that $\left\{y^i_{k(K)}\right\}_{K=1}^{\infty}$ converges to y^i uniformly, and that (from (22)) the feasibility constraint

$$\sum_{i\in S} y^i_{k(K)}(\omega)_j \leq \sum_{i\in S} e^i(\omega)_j$$

is satisfied for every j and almost every $\omega \in \Omega$. Each $y^i_{k(K)}$ is also $\digamma^i_{k(K)}$-measurable, since so is $z^i_{k(K)}$. We conclude that $\left\{\mathbf{y}_{k(K)}\right\}_{K=1}^{\infty}$, where $\mathbf{y}_{k(K)} = \left(y^i_{k(K)}\right)_{i\in S}$, is a sequence of private S-allocations in economies $\left\{\mathcal{E}_{k(K)}\right\}_{K=1}^{\infty}$. Uniform convergence of functions implies convergence in L_1^m-norm, and hence, according to Lemma 2, for every $i \in S$

$$\lim_{K\to\infty} U^i_{k(K)}\left(y^i_{k(K)}\right) = U^i\left(y^i\right) \text{ and } \lim_{K\to\infty} U^i_{k(K)}\left(x^i_{k(K)}\right) = U^i\left(x^i\right).$$

Due to (20), there exists K such that

$$U^i_{k(K)}\left(y^i_{k(K)}\right) > U^i_{k(K)}\left(x^i_{k(K)}\right)$$

for every $i \in S$. This contradicts the assumption that $\mathbf{x}_{k(K)}$ is a private core allocation in $\mathcal{E}_{k(K)}$. ■

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