Working Paper 06-24
Statistics and Econometrics Series 08
April 2006

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

OPTIMAL POLICIES FOR DISCRETE TIME RISK PROCESSES WITH A MARKOV CHAIN INVESTMENT MODEL

Maikol Diasparra* , Rosario Romera**

Abstract

We consider a discrete risk process modelled by a Markov Decision Process. The surplus could be invested in stock market assets. We adopt a realistic point of view and we let the investment return process to be statistically dependent over time. We assume that follows a Markov Chain model. To minimize the risk there is a possibility to reinsure a part or the whole reserve. We consider proportional reinsurance. Recursive and integral equations for the ruin probability are given. Generalized Lundberg inequalities for the ruin probabilities are derived. Stochastic optimal control theory is used to determine the optimal stationary policy which minimizes the ruin probability. To illustrate these results numerical examples are included.

Keywords: risk process; ruin probability; Markov decision processes; proportional reinsurance.

*, mdiaspar@est-econ.uc3m.es, Departamento de Estadística, Universidad Carlos III de Madrid, Avda. de la Universidad s/n, 28911 Leganés (Madrid).

**, rosario.romera@uc3m.es, Departamento de Estadística, Universidad Carlos III de Madrid, C/ Madrid 126, 28903 Getafe (Madrid).

.

OPTIMAL POLICIES FOR DISCRETE TIME RISK PROCESSES WITH A MARKOV CHAIN INVESTMENT MODEL

Maikol Diasparra[1] and Rosario Romera[2]
mdiaspar@est-econ.uc3m.es[1], mrromera@est-econ.uc3m.es[2]
Department of Statistics, Universidad Carlos III de Madrid.
AMS 2000 Subject Classification: Primary: 60K10; 62N05.
Secondary: 60J05; 93E20.
Keywords: risk process, ruin probability, Markov decision processes, proportional reinsurance
April, 2006.

Abstract

We consider a discrete risk process modelled by a Markov Decision Process. The surplus could be invested in stock market assets. We adopt a realistic point of view and we let the investment return process to be statistically dependent over time. We assume that follows a Markov Chain model. To minimize the risk there is a possibility to reinsure a part or the whole reserve. We consider proportional reinsurance. Recursive and integral equations for the ruin probability are given. Generalized Lundberg inequalities for the ruin probabilities are derived. Stochastic optimal control theory is used to determine the optimal stationary policy which minimizes the ruin probability. To illustrate these results numerical examples are included.

1 Introduction

Recently, a large body of work has been devoted to extend risk models to allow for decisor's actions thus continuing the classical works from the seventies by e.g., H.U. Gerber, H. Bühlman and A. Martin-Löf. Controlled risk processes are considered by Hipp and Taksar (2000), Hipp and Plum (2000), Schmidli (2001, 2002), Hipp and Schmidli (2004), among others. Most of the papers consider continuous-time models and some specific intervention, which allow them to explicitly obtain the optimal policy. Schäl (2004) discusses discrete-time risk processes controlled by reinsurance and non-statistically-dependent over time investment. The effect of stochastic investment returns minimizing the ruin probability in continuous-time models is considered by Browne (1995), Hipp & Plum (2000) and Gaier et al. (2003). Several models

of interes rate on ruin probabilities has been discussed by Sundt and Teugels (1995, 1997), Yang (1999), Norbert (1997) and more recently for discrete time risk processes see Cai (2002) and Cai and Dickson (2004) and references therein.

At first view, the ruin probability is not a classical performance criterion for control problems. As is pointed out by Schäl (2004) one can write the ruin probability as some total cost without discounting where one has to pay one unit of cost when entering a ruin state. After this simple observation, the results from discret-time dynamic programming apply. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. An analytic method commonly used in ruin theory is to derive inequalities for ruin probabilities (see Grandell (1991), Willmot, et al. (2000) and Willmot and Lin (2001)).

In this paper we consider the problem of minimization of the ruin probability in a discrete time risk process with proportional reinsurance and investment in a financial market. For the surplus process we consider the usual formulation in Markov decision theory (see Schäl (2004) and Hernández-Lerma and Laserre (1996)). We assume statistical dependency over time for the investment process and following a realistic point of view as is sugested in Cai (2002) and in Cai and Dickson (2004), we modelize the return process as a finite or countable state markov chain. This model could be in fact a discrete conterpart of the most frequently ocurring effect observed in continuous total return processes, e.g. mean-reverting effect. Our aim is to choose the reinsurance-investment control strategies in order to minimize the ruin probability and for sake of simplicity we restrict control policies to be Markovian and stationary. Thus we garantee optimal control strategies among the admissible control class. First, for this purpose we develop generalized Lundberg inequalities for the ruin probability that depend on the decision or control strategy. Previously we derive recursive and integral equations for the ruin probability. Second, optimality over the admissible control set can be achieved by the monotonic property of the upper bounds that we obtain.

The outline of the paper is as follows. In Section 2 the risk model is formulated. In Section 3 we derive recursive equations for the finite ruin probability and integral equations for the ultimate ruin probability. In Section 4 we obtain probability inequalities for the ultimate probability of ruin. An analysis of the new inequalities and the comparison with the Lundberg's inequality is also included. In Section 5 we report the outcomes of an illus-

trative application to the ruin probability in a risk process with a heavy tail (PH-type) claims distribution under proportional reinsurance and Markov dependency of the return process. Finally, we studying in Section 6 the optimality of our approximations.

2 Model

A general X-valued discrete-time stochastic process $\{X_n\}_{n\geq 0}$ is considered which can be observed and controlled at the beginning of each period. The stochastic development is determined by a sequence of random variables $\{W_n\}_{n\geq 1}$ on some probability space $(\Omega, \mathcal{F}, P)$ with $W_n = (Y_n, Z_n, R_n)$. Let $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ be two independent sequences of independent and identically distributed (i.i.d.) nonnegative random variables with probability distribution functions F and G respectively. We assume also independency with respect to the investment return process $\{R_n\}$. In risk theory, $X_n \in X$ (usually $X = \mathbb{R}$) describes the *surplus* (size of the fund of reserves) of an insurance company after n periods, Y_n denotes the total claims during the n-th period, i.e. from time *n-1* to time n, Z_n represents the random of period n, and R_n denotes the total retribution of the financial market during the n-th period.

A policy is a sequence $a = \{a_n\}_{n\geq 1}$ of decision functions $\varphi_n : X \to A$, with $A = [\underline{b}, 1] \times [0, 1]^d$ and $0 < \underline{b} \leq 1$. For the sake of simplicity we can consider a subset $D \subset [0, 1]^d$ for the portfolio values, for example a grid of $[0, 1]^d$. Then $\varphi_n(X_n) = a_n = (b_n, \delta_n)$ will represent the action chosen at the beginning of the period *n+1*. We denote by $\delta_n = (\delta_1^d, \ldots, \delta_n^d)$ the portfolio invest in d risky assets, and b the proportion of the claim to be paid by the insurer. We consider Markovian control policies, which depend only on the current state, and stationary, that is, $\varphi_n = \varphi$. Abusing notation, we will identify functions $\varphi : X \to A$ with stationary strategies. The (measurable) function $h(b, y)$ specifies the part of the claim y paid by the insurer. Then, $h(b, y)$ depends on the retention level b at the beginning of the respective period where $0 \leq h(b, y) \leq y$. In this article, we consider the case of *proportional reinsurance*, where

$$h(b, y) = b \cdot y \text{ with retention level } 0 < \underline{b} \leq b \leq \overline{b} \leq 1. \tag{1}$$

The premium (income) rate c is fixed. Because the insurer pays a premium rate to the reinsurer which depends on the retention level b, we consider the

net income rate $c(b)$ where

$$0 \leq c(b) \leq c = c(\bar{b}),\ 0 < \underline{b} \leq b \leq \bar{b} \leq 1$$

and $c(b)$ is an increasing function and may be calculated according to the *expected value principle* with and added safety loading θ from the reinsurer:

$$c(b) = c - (1+\theta) \cdot E[Y - h(b,Y)]/E[Z]. \tag{2}$$

For an initial state $X_0 = x$ and a control policy a, the value of the surplus process at the beginning of the period *n+1*, say $X_n^{x,a} = X_n$, is given according to

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R_n \rangle) + c(b_{n-1})Z_n - h(b_{n-1}, Y_n) \tag{3}$$

Note that (3) can be rewritten in the usual form in Markov control theory

$$X_n = f(X_{n-1}, a_{n-1}, W_n) \tag{4}$$

with the random perturbation $W_n = (Y_n, Z_n, R_n)$. In general the process X_n is not homogeneous because its transition probabilities change in time, however we get homogeneity if we consider stationary strategies of the form $\varphi_n = \varphi$.

From this fact we can assume that the transition probabilities of X_n are of the form

$$P(x, a, [x-u, \infty)) = \widehat{F}(a, u) \tag{5}$$

where $\widehat{F}(a,u)$ is the probability distribution of the amount of net losses in a single period, provided that a is the action taken by the decisor at the beginning of the period.

We adopt a realistic point of view, and we let the return process $\{R_n\}_{n\geq 1}$ where $R_n = (R_n^1, \ldots, R_n^d)$ to be statistically dependent whith the price process given by $I_n^a = \langle \delta_{n-1}, R_n \rangle$ for $n \geq 1$ provided the action a is take. Note that I_n^a depends on the decision variable a only through the portfolio $\delta_n = (\delta_1^d, \ldots, \delta_n^d)$. Moreover $\{I_n^a\}_{n\geq 0}$ is assumed to follow a Markov chain. In addition we assume that for all $n \geq 0$, I_n^a takes a finite or countable number of possible values on I. Note that I_0^a is the state of the price process before to make the investment in the financial market and the value i_0 in

fact drives the evolution of the process.
It is not hard to check that (3) is equivalent to

$$\begin{array}{rcl} X_n & = & x\prod_{l=1}^{n}\left(1+\langle\delta_{l-1}^{d},R_l^d\rangle\right) \\ & & +\sum_{l=1}^{n}\left(c(b_{l-1})Z_l-h(b_{l-1},Y_l)\prod_{m=l+1}^{n}\left(1+\langle\delta_{m-1}^{d},R_m^d\rangle\right)\right) \end{array} \tag{6}$$

where throughout this article we consider

$$\prod_{i=n_1}^{n_2} X_i = 1 \text{ and } \sum_{i=n_1}^{n_2} X_i = 0 \text{ if } n_1 > n_2$$

That is,

$$\begin{array}{rcl} X_n & = & x\prod_{l=1}^{n}\left(1+I_l^a\right) \\ & & +\sum_{l=1}^{n}\left(c(b_{l-1})Z_l-h(b_{l-1},Y_l)\prod_{m=l+1}^{n}\left(1+I_m^a\right)\right) \end{array} \tag{7}$$

we define for all $a \in A$

$$\begin{array}{c} Pr\{I_{n+1}^a = j | I_n^a = i, I_{n-1}^a = i_{n-1}, \ldots, I_0^a = i_0, a\} \\ = Pr\{I_{n+1}^a = j | I_n^a = i, a\} = p_{ij}^a \geq 0 \end{array} \tag{8}$$

where $\sum_{j=0}^{\infty} p_{ij}^a = 1$ for $i = 0, 1, 2, \ldots,$. Note that we consider in fact a collection of transition probability matrices $\{[p_{ij}^a]\}_{a\in A}$ describing the behavior of I_n^a. The ruin probabilities with infinite and finite horizon, initial surplus x, action a and given $I_0^a = i$ are defined as

$$\psi^a(x,i) = Pr\{\bigcup_{k=1}^{\infty}(X_k^{x,a} < 0)\} = Pr\{\bigcup_{k=1}^{\infty}(X_k < 0) | I_0^a = i, X_0 = x, a\} \tag{9}$$

and

$$\psi_n^a(x,i) = Pr\{\bigcup_{k=1}^{n}(X_k^{x,a} < 0)\} = Pr\{\bigcup_{k=1}^{n}(X_k < 0) | I_0^a = i, X_0 = x, a\} \tag{10}$$

where X_k is given by (7). Thus,

$$\psi_1^a(x,i) \leq \psi_2^a(x,i) \leq \cdots \leq \psi_n^a(x,i) \leq \cdots$$

and

$$\lim_{n\to\infty} \psi_n^a(x,i) = \psi^a(x,i). \tag{11}$$

Note that if $\delta_k = 0$ and $b_k = 1$ for $k = 1, \ldots$ the risk model (7) is reduced to the classical discrete time risk model without investment and reinsurance

$$X_k = x - \sum_{t=1}^{k} (Y_t - cZ_t)$$

If $\delta_k = 0$ and $b_k \in (0, 1]$ for $k = 0, 1, \ldots$ the risk model is

$$X_k = x - \sum_{t=1}^{k} (b_{t-1}Y_t - c(b_{t-1})Z_t) \tag{12}$$

and $a_k \equiv b_k$, $k = 0, 1, \ldots$.
Let $\psi^b(x)$ denote the infinite time ruin probability in the risk model given by (12), namely

$$\psi^b(x) = Pr\{\bigcup_{k=1}^{\infty} \sum_{t=1}^{k} (b_{t-1}Y_t - c(b_{t-1})Z_t) > x\}$$

If we assume stationary strategies $b_t = b_0$, $t \geq 1$ and in addition $E[b_0Y_1 - c(b_0)Z_1] < 0$, if there is a constant $R_0 > 0$ satisfying

$$E[e^{R_0(b_0Y_1 - c(b_0)Z_1)}] = 1 \tag{13}$$

we get the classical well-Known Lundberg inequality for the ruin probability

$$\psi^b(x) \leq e^{-R_0 x}. \tag{14}$$

Furthermore, if all interest rates are non-negative, i.e. $I_n^a \geq 0$ for $n = 0, 1, \ldots$, then it is easy to see that

$$\psi^a(x, i) \leq \psi^b(x), \ x \geq 0 \tag{15}$$

Finally, note that for a model with dividends (constant case), if d_n denotes the short term dividend rate in the $n-th$ period the discrete-time risk model with stochastic investment return and dividends is given by

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R_n \rangle) + c(b_{n-1})Z_n - h(b_{n-1}, Y_n) - d_n X_n$$

and rearranging X_n one obtain

$$X_n = X_{n-1}\left(\frac{(1 + \langle \delta_{n-1}, R_n \rangle)}{(1 + d_n)}\right) + \frac{c(b_{n-1})Z_n}{(1 + d_n)} - \frac{h(b_{n-1}, Y_n)}{(1 + d_n)}$$

For $1 + R'_n = \frac{1+R_n}{(1+d_n)}$, $Z'_n = \frac{Z_n}{(1+d_n)}$, and $Y'_n = \frac{Y_n}{(1+d_n)}$, and with conditions that guarantee that $\{R'_n\}$, $\{Z'_n\}$ and $\{Y'_n\}$ are independent processes with similar distributional characteristics as the original processes, then the model becomes

$$X_n = X_{n-1}(1 + \langle \delta_{n-1}, R'_n \rangle) + c(b_{n-1})Z'_n - h(b_{n-1}, Y'_n)$$

which is the same as the model without dividends (3) and it can be analyzed in the same way.

3 Recursive and integral equations for ruin probabilities

Throughout this article, we denote the tail of a distribution function F by $\overline{F}(x) = 1 - F(x)$. We first give a recursive equation for $\psi^a_n(x, i)$ in (10) and an integral equation for $\psi^a(x, i)$ in (9). These equations hold for any interest rate.

Lemma 1. *For $n = 1, 2, \ldots$ and any $X_0 = x \geq 0$,*

$$\psi^a_{n+1}(x, i) = \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \int_0^{\tau} \psi^a_n(x(1 + j) - u, j) dF(y) dG(z) + \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \overline{F}(\tau) dG(z) \quad (16)$$

with $u = b_0 y - c(b_0)z$, b_0 is the initial retention level, $\tau = \frac{x(1+j)+c(b_0)z}{b_0}$ and p^a_{ij} as in (8),

$$\psi^a_1(x, i) = \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \overline{F}(\tau) dG(z)$$

then

$$\psi^a(x, i) = \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \int_0^{\tau} \psi^a(x(1 + j) - u, j) dF(y) dG(z) + \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \overline{F}(\tau) dG(z) \quad (17)$$

Proof. Let $U_k = b_{k-1}Y_k - c(b_{k-1})Z_k$, given $Y_1 = y$, $Z_1 = z$, the control strategy a, and $I_1^a = j$, from (7), we have $U_1 = b_0Y_1 - c(b_0)Z_1 = b_0y - c(b_0)z = u$, then $X_1 = x(1 + I_1^a) - U_1 = h_1 - u$, where $h_1 = x(1 + j)$.
Thus, if $u > h_1$ then

$$Pr\{X_1 < 0 | Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\} = 1,$$

which implies that for $u > h_1$

$$Pr\{\bigcup_{k=1}^{n+1}(X_k < 0) | Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\} = 1$$

while if $0 \leq u \leq h_1$, then

$$Pr\{X_1 < 0 | Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\} = 0 \tag{18}$$

Let $\{\widetilde{Y}_n\}_{n\geq 1}$, $\{\widetilde{Z}_n\}_{n\geq 1}$, and $\{\widetilde{I}_n^a\}_{n\geq 0}$ be independent copies of $\{Y_n\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$, and $\{I_n^a\}_{n\geq 0}$, respectively. Suppose that $\widetilde{U}_k = b_{k-1}\widetilde{Y}_k - c(b_{k-1})\widetilde{Z}_k$.
Thus, (18) and (7) imply that for $0 \leq u \leq h_1$,

$$\begin{aligned}
& Pr\{\bigcup_{k=1}^{n+1}(X_k < 0) | Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\} \\
= \; & Pr\{\bigcup_{k=2}^{n+1}(X_k < 0) | Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\} \\
= \; & Pr\{\bigcup_{k=2}^{n+1}((h_1 - u)\prod_{l=1}^{k}(1 + I_l^a) - \sum_{l=1}^{k} U_l \prod_{m=l+1}^{k}(1 + I_m^a) < 0) | I_1^a = j, a\} \qquad (19) \\
= \; & Pr\{\bigcup_{k=1}^{n}(h_1 - u)\prod_{l=1}^{k}(1 + \widetilde{I}_l^a) - \sum_{l=1}^{k} \widetilde{U}_l \prod_{m=l+1}^{k}(1 + \widetilde{I}_m^a) < 0) | \widetilde{I}_0^a = j, a\} \\
= \; & \psi_n^a(h_1 - u, j) = \psi_n^a(x(1 + j) - u, j)
\end{aligned}$$

where the second equality follows from the Markov property of $\{I_n^a\}_{n\geq 0}$, and independence of $\{Y_n\}_{n\geq 1}$, $\{Z_n\}_{n\geq 1}$ and $\{I_n^a\}_{n\geq 0}$.

Let consider *condition* $A = \{Z_1 = z, Y_1 = y, I_1^a = j, I_0^a = i, a\}$. Suppose

that $F(y) = Pr(Y \leq y)$ and $G(z) = Pr(Z \leq z)$. From equation (7) we obtain

$$\begin{aligned}
\psi^a_{n+1}(x,i) &= Pr\{\bigcup_{k=1}^{n+1}(X_k < 0)|I^a_0 = i, a\} \\
&= \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty}\int_0^{\infty} Pr\{\bigcup_{k=1}^{n+1}(X_k < 0)|\ condition\ A\ \}F(y)dG(z).
\end{aligned}$$

Then

$$\begin{aligned}
\psi^a_{n+1}(x,i) &= \sum_{j=0}^{\infty} p^a_{ij} \left\{ \int_0^{\infty}\int_0^{\tau} Pr\{\bigcup_{k=1}^{n+1}(X_k < 0)|condition\ A\}dF(y)dG(z) \right. \\
&+ \left. \int_0^{\infty}\int_{\tau}^{\infty} Pr\{\bigcup_{k=1}^{n+1}(X_k < 0)|condition\ A\}dF(y)dG(z) \right\} \\
&= \sum_{j=0}^{\infty} p^a_{ij} \left\{ \int_0^{\infty}\int_0^{\tau} \psi^a_n(x(1+j)-u,j)dF(y)dG(z) + \int_0^{\infty}\int_{\tau}^{\infty} dF(y)dG(z) \right\} \\
&= \sum_{j=0}^{\infty} p^a_{ij} \left\{ \int_0^{\infty}\int_0^{\tau} \psi^a_n(x(1+j)-u,j)dF(y)dG(z) + \int_0^{\infty} \overline{F}(\tau)dG(z) \right\}
\end{aligned} \tag{20}$$

where $\tau = \frac{x(1+j)+c(b_0)z}{b_0}$. Next by letting $n \to \infty$ in (20), and by the Lebesgue dominated convergence theorem, we obtain $\lim_{n\to\infty} \psi^a_{n+1}(x,i) = \psi^a(x,i)$, i.e.

$$\psi^a(x,i) = \sum_{j=0}^{\infty} p^a_{ij} \left\{ \int_0^{\infty}\int_0^{\tau} \psi^a(x(1+j)-u,j)dF(y)dG(z) + \int_0^{\infty} \overline{F}(\tau)dG(z) \right\}$$

In particular,

$$\psi^a_1(x,i) = \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} \overline{F}((x(1+j)+c(b_0)z)/b_0)dG(z).$$

□

Note that if we consider the classical risk model without investment and reinsurance, i.e. $\delta_n = 0$ and $b = 1$ for $n \geq 0$, we obtain similar results as in Cai & Dickson [2].

4 Inequalities for ruin probabilities

In this section we assume that the return process is nonnegative, i.e. $I_n^a \geq 0$ for $n = 0, 1, \ldots$ and all $a \in A$. We will use the results obtained in the last section to find upper bounds for the ruin probability taking into account the information contributed by the Markov chain of the price process.

4.1 Inequalities for ruin probabilities by the inductive approach

Theorem 1. *Suppose that R_0 is a constant satisfying (13) then*

$$\begin{aligned} \psi^a(x,i) &\leq \beta \sum_{j=0}^{\infty} p_{ij}^a E[e^{-R_0 x(1+I_1^a)} | I_0^a = i, a],\ x \geq 0,\ i \in I \\ &= \beta E[e^{-R_0[x(1+I_1^a)]} | I_0^a = i, a] \end{aligned} \tag{21}$$

where

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^{\infty} e^{R_0 by} dF(y)}{e^{R_0 bt} \overline{F}(t)} \ \textit{with}\ b \in (0,1].$$

Proof. We apply induction method. For $n = 1$ we can write

$$\begin{aligned} \overline{F}(\theta) &= \left(\frac{\int_\theta^{\infty} e^{R_0 by} dF(y)}{e^{R_0 b\theta} \overline{F}(\theta)} \right)^{-1} e^{-R_0 b\theta} \int_\theta^{\infty} e^{R_0 by} dF(y) \\ &\leq \beta e^{-R_0 b\theta} \int_\theta^{\infty} e^{R_0 by} dF(y) \leq \beta e^{-R_0 b\theta} E[e^{R_0 bY_1} | a], \end{aligned} \tag{22}$$

for any $\theta \geq 0$. Which implies that for any $x \geq 0$, $i \geq 0$ and $b_0 \in [\underline{b}, 1]$

$$\begin{aligned} \psi_1^a(x,i) &= Pr\{X_1 < 0 | I_0^a = i, a\} = Pr\{Y_1 > \tfrac{x(1+j)+c(b_0)Z_1}{b_0} | I_0^a = i, a\} \\ &= \sum_{j=0}^{\infty} p_{ij}^a \int_0^{\infty} \overline{F}(\tfrac{x(1+j)+c(b_0)z}{b_0}) dG(z), \end{aligned}$$

which, in turn, implies by (22) that

$$\begin{aligned}
\psi_1^a(x,i) &\leq \sum_{j=0}^{\infty} p_{ij}^a \left(\beta E[e^{R_0 b Y_1}|a] \cdot \int_0^{\infty} e^{-R_0 b_0 \left(\frac{x(1+j)+c(b_0)z}{b_0}\right)} dG(z) \right) \\
&= \beta E[e^{R_0 b Y_1}|a] \sum_{j=0}^{\infty} p_{ij}^a \int_0^{\infty} e^{-R_0[x(1+j)+c(b_0)z]} dG(z) \\
&= \beta E[e^{R_0 b Y_1}|a] \cdot \sum_{j=0}^{\infty} p_{ij}^a E[e^{-R_0[x(1+j)+c(b)Z_1]}|I_0^a = i, a] \\
&= \beta E[e^{R_0 b Y_1}|a] E[e^{-R_0 c(b) Z_1}|a] E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a] \\
&= \beta E[e^{R_0[bY_1 - c(b)Z_1]}|a] \cdot E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a] \\
&= \beta E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a].
\end{aligned}$$

Under an inductive hypotheses, we assume for any $x \geq 0$ and any $i \in I$,

$$\psi_n^a(x,i) \leq \beta E[e^{-R_0 x(1+I_1^a)}|I_0^a = i, a] \tag{23}$$

thus, for $0 \leq y \leq \frac{x(1+j)+c(b_0)z}{b_0}$, with x and i replaced by $x(1+j)+c(b_0)z-b_0 y$ and j respectively in (23), and $I_1^a \geq 0$, we have

$$\begin{aligned}
\psi_n^a(x(1+j)+c(b_0)z-b_0 y, j) &\leq \beta E[e^{-R_0[x(1+j)+c(b)z-by](1+I_1^a)}|I_0^a = j, a] \\
&\leq \beta e^{-R_0[x(1+j)+c(b_0)z-b_0 y]}
\end{aligned} \tag{24}$$

therefore, replacing (24) in (16), we get

$$\begin{aligned}
\psi^a_{n+1}(x,i) &\leq \sum_{j=0}^{\infty} p^a_{ij} \left(\beta \int_0^{\infty} e^{-R_0[x(1+j)+c(b_0)z]} \int_{\tau}^{\infty} e^{R_0 b_0 y} dF(y) dG(z)\right) \\
&+ \sum_{j=0}^{\infty} p^a_{ij} \left(\beta \int_0^{\infty} e^{-R_0[x(1+j)+c(b_0)z]} \int_0^{\tau} e^{R_0 b_0 y} dF(y) dG(z)\right) \\
&= \sum_{j=0}^{\infty} p^a_{ij} \left(\beta \int_0^{\infty} e^{-R_0[x(1+j)+c(b_0)z]} \int_0^{\infty} e^{R_0 b_0 y} dF(y) dG(z)\right) \\
&= \beta E[e^{R_0 b Y_1}|a] \sum_{j=0}^{\infty} p^a_{ij} \int_0^{\infty} e^{-R_0[x(1+j)+c(b_0)z]} dG(z) \\
&= \beta E[e^{R_0 b Y_1}|a] \cdot E[e^{-R_0 c(b) Z_1}|a] \cdot E[e^{-R_0 x(1+I^a_1)}|I^a_0 = i, a] \\
&= \beta E[e^{-R_0 x(1+I^a_1)}|I^a_0 = i, a].
\end{aligned}$$

Hence, for any $n = 1, 2, \ldots$, (23) holds. Therefore, (21) follows by letting $n \to \infty$ in (23). □

Next we consider the case when the claim distribution belongs to the particular class of NWUC distributions (see (35) in appendix A for details).

Corollary 1. *Under the hypothesis of Theorem* 1*, and assuming that* $E[e^{R_0 b Y_1}|a] < \infty$ *for all* $b \in (0, 1]$ *and in addition if* F *is a NWUC distributions, then*

$$\psi^a(x,i) \leq (E[e^{R_0 b Y_1}|a])^{-1} E[e^{-R_0 x(1+I^a_1)}|I^a_0 = i, a], \ x \geq 0. \tag{25}$$

Proof. Following Willmot & Lin [19] in pp. 96–97, and defining $r = R_0 b_0 > 0$, we have

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^{\infty} e^{ry} dF(y)}{e^{rt} \overline{F}(t)} = \int_0^{\infty} e^{ry} dF(y) \tag{26}$$

that is, $\beta^{-1} = E[e^{R_0 b_0 Y_1}] = E[e^{R_0 b Y_1}|a]$. Finally replacing this equality in (21), then (25) holds. □

4.2 Inequalities for ruin probabilities by the martingale approach

Another tool for deriving inequalities for ruin probabilities is the martingale approach. The ruin probabilities associated with the discounted risk process $\{V_n, n = 1, 2 \ldots\}$, i.e.

$$\psi_n^a(x,i) \;=\; Pr\{\bigcup_{k=1}^{n}(X_k < 0)|I_0^a = i, a\} = Pr\{\bigcup_{k=1}^{n}(V_k < 0)|I_0^a = i, a\}$$

where $V_n = X_n \prod_{l=1}^{n}(1 + I_l^a)^{-1}$, $n = 1, 2, \ldots$.
In the classical risk model, $\{e^{-R_0 X_n}, n = 1, 2, \ldots\}$ is a martingale. However, for model (7), there is no constant $r > 0$ such that $\{e^{-rX_n}, n = 1, 2, \ldots\}$ is a martingale. Still, there exists a constant $r > 0$ such that $\{e^{-rV_n}, n = 1, 2, \ldots\}$ is a supermartingale, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

Proposition 1. *Assume that* $E[bY_1 - c(b)Z_1] < 0$. *In addition we suppose that for* $i \in I$, *there exists* $\rho_i > 0$ *satisfying*

$$E\left[e^{\rho_i[bY_1 - c(b)Z_1](1+I_1^a)^{-1}}|I_0^a = i, a\right] = 1. \tag{27}$$

Then,

$$R_1 = \min_{i \in I} \rho_i \geq R_0 \tag{28}$$

and for all $i \in I$

$$E\left[e^{R_1[bY_1 - c(b)Z_1](1+I_1^a)^{-1}}|I_0^a = i, a\right] \leq 1. \tag{29}$$

Proof. Note that the function

$$l_i(r) = E\left[e^{r[bY - c(b)Z](1+I_1^a)^{-1}}|I_0^a = i, a\right] - 1$$

$r > 0$, for each $i \in I$ is a convex function with $l_i(0) = 0$

$$l_i^{'}(0) = E[(bY - c(b)Z)]\, E\left[(1 + I_1^a)^{-1}|I_0^a = i, a\right] < 0.$$

Therefore, ρ_i is the unique positive root ρ of the equation $l_i(r) = 0$ on $(0, \infty)$. Further,

$$E\left[e^{R_0[bY-c(b)Z](1+I_1^a)^{-1}}|I_0^a = i, a\right] = \sum_{j=0}^{\infty} p_{ij}^a E\left[e^{R_0[b_0Y-c(b_0)Z](1+j)^{-1}}\right]$$

by Jensen's inequality

$$= \sum_{j=0}^{\infty} p_{ij}^a \cdot E\left[e^{\left[R_0[b_0Y_1-c(b_0)Z_1](1+j)^{-1}\right]}\right] \leq \sum_{j=0}^{\infty} p_{ij}^a E\left[e^{R_0[b_0Y_1-c(b_0)Z_1]}\right]^{(1+j)^{-1}}$$

$$= \sum_{j=0}^{\infty} p_{ij}^a = 1 \Longrightarrow E\left[e^{R_0[bY-c(b)Z](1+I_1^a)^{-1}}|I_0^a = i, a\right] \leq 1$$

which implies that $l_i(R_0) \leq 0$. Here $R_0 \leq \rho_i$ and

$$R_1 = \min_{i \in I} \rho_i \geq R_0.$$

Thus, (28) holds. In addition $R_i \leq \rho_i$, for all $i \in I$, which implies that $l_i(R_1) \leq 0$, i.e. (29) holds. □

Theorem 2. *Under the hypothesis of Proposition* 1*, for all* $i \in I$*,*

$$\psi^a(x, i) \leq e^{-R_1 x}, x \geq 0. \tag{30}$$

Proof. For the surplus process $\{X_k\}$ given by (7), we have

$$V_k = X_k \prod_{l=1}^{k}(1 + I_l^a)^{-1} = x - \sum_{l=1}^{k}\left((b_{l-1}Y_l - c(b_{l-1})Z_l)\prod_{t=1}^{l}(1 + I_t^a)^{-1}\right) \tag{31}$$

and $S_n = e^{-R_1 V_n}$. Then $S_{n+1} = S_n e^{-R_1(b_nY_{n+1}-c(b_n)Z_{n+1})\prod_{t=1}^{n+1}(1+I_t^a)^{-1}}$. Thus, for

any $n \geq 1$,

$$E[S_{n+1} \mid Y_1, \ldots Y_n, Z_1, \ldots, Z_n, I_1^a, \ldots I_n^a]$$

$$= S_n E\left[e^{-R_1(b_n Y_{n+1} - c(b_n) Z_{n+1}) \prod_{t=1}^{n+1} (1+I_t^a)^{-1}} \mid Y_1, \ldots Y_n, Z_1, \ldots, Z_n, I_1^a, \ldots I_n^a \right]$$

$$= S_n E\left[e^{-R_1(b_n Y_{n+1} - c(b_n) Z_{n+1})(1+I_{n+1}^a)^{-1} \prod_{t=1}^{n} (1+I_t^a)^{-1}} \mid I_1^a, \ldots I_n^a \right]$$

$$\leq S_n E\left(\left[e^{-R_1(b_n Y_{n+1} - c(b_n) Z_{n+1})(1+I_{n+1}^a)^{-1}} \mid I_1^a, \ldots I_n^a \right] \right)^{\prod_{t=1}^{n} (1+I_t^a)^{-1}}$$

$$S_n E\left(\left[e^{-R_1(b_n Y_{n+1} - c(b_n) Z_{n+1})(1+I_{n+1}^a)^{-1}} \mid I_n^a \right] \right)^{\prod_{t=1}^{n} (1+I_t^a)^{-1}} \leq S_n$$

which implies that $\{S_n, n = 1, 2, \ldots\}$ is a supermartingale.
Let $T_i = \min\{n : V_n < 0 \mid I_0^a = i\}$ where V_n is given by (31). Then T_i is a stopping time and $n \wedge T_i = \min\{n, T_i\}$ is a finite stopping time. Thus, by the optimal stopping theorem for martingales, we get

$$E(S_n \wedge T_i) \leq E(S_0) = e^{-R_1 x}.$$

Hence,

$$\begin{aligned} e^{-R_1 x} &\geq E(S_{n \wedge T_i}) \geq E((S_{n \wedge T_i}) I_{(T_i \leq n)}) \geq E((S_{T_i}) I_{(T_i \leq n)}) \\ &= E(e^{-R_1 V_{T_i}} I_{(T_i \leq n)}) \geq E(I_{(T_i \leq n)}) = \psi_n^a(x, i), \end{aligned} \tag{32}$$

where (32) follows from $V_{T_i} < 0$. Thus, (30) follows by letting $n \to \infty$ in (32). □

5 Numerical results

We present a numerical example to illustrate the bounds given by Theorems 1 and 2 and for the purpose Matlab and Maple implementations are developed.

We consider claim distribution of the PH type because they and their moments can be written in a closed form, various quantities of interest can be evaluated with relative ease, and finally, the set of PH distributions is dense in the set of all distributions with support in $[0, \infty)$.

Suppose that the claim size Y has a phase-type density ($PH(\alpha, T)$) with $\alpha = (1/2, 1/2)$, $e = (1, 1)$,

$$T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } t = -Te = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

then

$$M_Y(s) = E[e^{s \cdot Y}] = \alpha \left(-sI - T\right)^{-1} t$$

thus, $E[Y] = \frac{d}{ds} M_Y(s) \mid_{s=0} = \alpha(T)^{-2} t = 0.75$, and Y is NWUC.

We denote the length of periods by Z_k. Let $Z \sim Exp(1)$, $E[Z] = 1$, thus $M_Z(s) = E[e^{s \cdot Z}] = (1 - s)^{-1}$.

We consider an interest model with three posible interest rates: $I = \{6\%, 8\%, 10\%\}$. In addition we suppose two portfolio investment $\delta \in \{1, 2\}$. Thus the corresponding transition probability matrices of the price process I_n^a for $a_1 = (b, 1)$, $a_2 = (b, 2)$ are

$$P_{\mathbf{1}} = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.8 & 0.2 & 0 \\ 0.9 & 0.1 & 0 \end{pmatrix} \text{ and } P_{\mathbf{2}} = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0 & 0.2 & 0.8 \\ 0 & 0.1 & 0.9 \end{pmatrix}.$$

We fixed the premium income rate (2) $c = 0.975$ and the safety loading of the reinsurer $\theta = 0.1$.

Lundberg bound: In this example we can guarantee that (14) holds for each $b \in (0, 1]$. First, it is easy to see that for b fixed $E[bY_1 - c(b)Z_1] < 0$. Second, for each $b \in (0, 1]$ there exist a constant R_0 such that (13) is achieved. Note that solving

$$E[e^{R_0 b Y_1}] \cdot E[e^{-R_0 c(b) Z_1}] = 1$$

is equivalent to solve

$$(1 + c(b)R_0) = \alpha(-bR_0I - T)^{-1}t$$

The Lundberg bound for the ruin probability is

$$\psi^b(x) \leq e^{-R_0 x}, \text{ for } x \geq 0.$$

Figure 1 shows the non-linear relation between R_0 and b which is the solution of the preceding equation. Table 1 present numerical values of the bounds obtained for several admissible decision policies

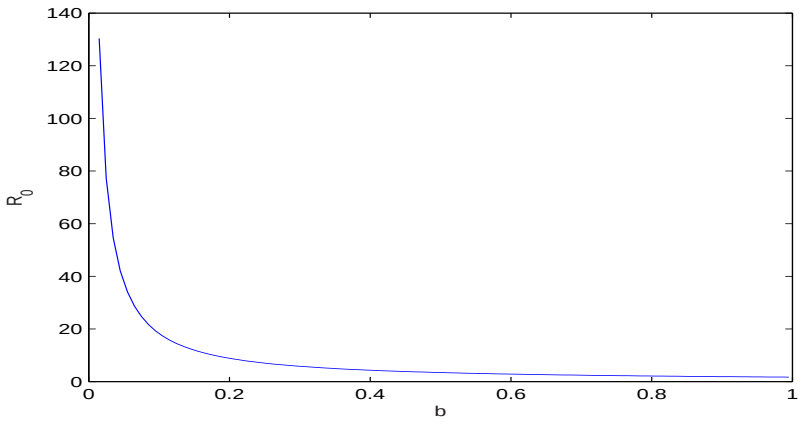

Figure 1: Relation between R_0 and b.

Induction bound: In this example the claim distribution is a NWUC (see [19], page 24) and such that $E[e^{R_0 bY_1}|a] = M_Y(R_0 b) < \infty$ for each $b \in (0, 1]$. Then Corollary 1 applies and for each $i \in I$

$$\begin{aligned}\psi^a(x,i) &\leq (E[e^{R_0 bY_1}|a])^{-1} E[e^{-R_0 x(1+I_1^a)}|I_0^a = i,a] \\ &\leq \left[\alpha\left(-bR_0I - T\right)^{-1} t\right]^{-1} \sum_{k\in\{6,8,10\}} p_{ik} e^{-R_0 x(1+k)} \text{ for } x \geq 0.\end{aligned}$$

Numerical values of this bound obtained for several admissible decision policies are presented in Table 1. As it is expected we get induction bounds

smaller than the Lundberg bounds for the same decision policies.

Martingale bound: In this example we can guarantee that hypotheses of Proposition 1 follow. Then Theorem 2 holds and we get bound (30). Condition (27) of Proposition 1

$$E\left[e^{\rho_i(bY_1-c(b)Z_1)(1+I_1^a)^{-1}}|I_0^a=i,a\right]=1,$$

is equivalent to the following condition for each $i \in I$

$$\sum_{k\in\{6,8,10\}} p_{ik}e^{\rho_i(1+k)^{-1}}M_Y\left(\frac{b\rho_i}{1+k}\right)M_Z\left(-\frac{c(b)\rho_i}{1+k}\right)=1$$

or

$$\sum_{k\in\{6,8,10\}} p_{ik}e^{\rho_i(1+k)^{-1}}\alpha\left(-b\rho_i(1+k)^{-1}I-T\right)^{-1}t\cdot\left(1+\frac{c(b)\rho_i}{1+k}\right)^{-1}=1.$$

In our example we solve $R_1=\min_{i\in I}\rho_i \geq R_0$ and then we obtain $\psi^a(x,i_1)\leq e^{-R_1x}$, $x\geq 0$. Numerical results of this bound are reported in Table 1. And it is obvious that this martingale bound improves the results for the induction bound.

Finally, we find of special interest the case of small reinsurer for which the retention level could be restricted by economic considerations. Thus we run numerical experiments in order to compare for a fixed retention level b the ruin probability bounds that could be achieved, or conversely depending on the type of the claim distribution for a fixed ruin probability level we can evaluate the admissible investment/reinsurance policies.
Figure 2 shows the bounds for different approaches with $x=5$ and $i=8\%$ while $b\in(0,1]$.

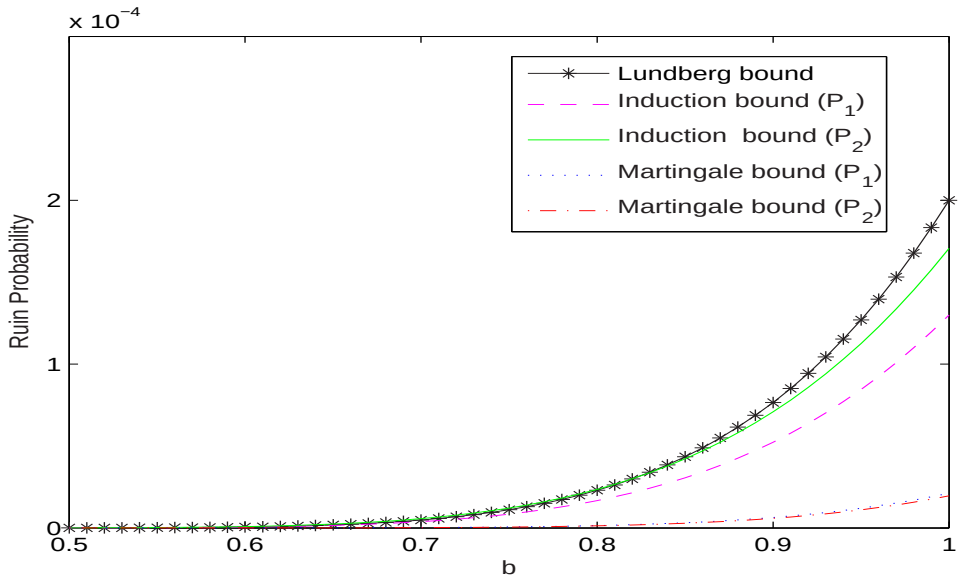

Figure 2: Bounds of ruin probability by different approaches vs b.

P_δ	b	*Lundberg bound*	*Induction bound*	*Martingale bound*	R_0	R_1
P_1	0.01	$0.223e-427$	$0.934e-453$	$\to 0$	196.94	$\to \infty$
P_1	0.25	$0.516e-15$	$0.125e-15$	$0.199e-18$	7.039	8.611
P_1	0.5	$0.323e-7$	$0.239e-7$	$0.448e-9$	3.449	4.304
P_1	0.75	$0.111e-4$	$0.118e-4$	$0.586e-6$	2.280	2.869
P_1	0.85	$0.434e-4$	$0.504e-4$	$0.317e-5$	2.008	2.532
P_1	0.95	$0.1261e-3$	$0.157e-3$	$0.120e-4$	1.794	2.265
P_1	1	$0.2e-3$	$0.255e-3$	$0.212e-4$	1.703	2.152
P_2	0.01	$0.22e-427$	$0.653e-462$	$\to 0$	196.94	$\to \infty$
P_2	0.25	$0.5166e-15$	$0.411e-16$	$0.144e-18$	7.039	8.675
P_2	0.5	$0.323e-7$	$0.138e-7$	$0.382e-9$	3.449	4.336
P_2	0.75	$0.111e-4$	$0.824e-5$	$0.527e-6$	2.280	2.891
P_2	0.85	$0.434e-4$	$0.366e-4$	$0.288e-5$	2.008	2.550
P_2	0.95	$0.126e-3$	$0.118e-3$	$0.110e-4$	1.794	2.282
P_2	1	$0.2e-3$	$0.194e-3$	$0.195e-4$	1.703	2.168

Table 1: Numerical bounds of ruin probability.

6 Optimality

In the preceding Section probability inequalities for the ultimate probability of ruin are derived. Our goal is to minimize the ruin probability $\psi_*(x) = \inf_a \psi^a(x)$, that is hitting $\Delta = (-\infty, 0]$. Following Gajek (2005) the idea is to use our Lundberg-type upper bounds as initial approximations and then iteratively apply an integral operator. Groniowska and Niemiro (2005) use a similar approximation provided that the risk model follows a random walk. Our approximations are closely related to the well-known in stochastic control theory *Policy Iteration Algorithm*, (see Hernandez-Lerma and Laserre (1996), Section 4.4), which provides decreasing approximations to the minimum, over all possible decision strategies, of the ultimate probability of ruin.

Assume that the transition probabilities of X_n are given by (5).

We consider the *Bellman operator* defined as follows. For $v \in V$, the set of all measurable functions $v : X \setminus \Delta \to [0, 1]$,

$$Bv(x) = \inf_{a \in A} [P(x, a, \Delta) + \int_{X \setminus \Delta} P(x, a, dx')v(x')]$$

If $\varphi : X \to A$, we consider the following operator

$$T^{\varphi}v(x) = P(\varphi(x), x, \Delta) + \int_{X \setminus \Delta} P(\varphi(x), x, dx')v(x')$$

We state now some results essentially standard but necessary for the sequel. They are known as the *measurable selection conditions* for the *Bellman's Principle of Optimality*.

Proposition 2. *Let A be a compact set and let assume that the stochastic kernel P is strongly continuous with respect to a*

$$u(x, a) = \int_X P(x, a, dx')w(x')$$

i.e.the measurable function $u : X \times A \to [0, 1]$ is continuous in a for every fixed $x \in X$ and for every measurable function $w : X \to [0, 1]$.

Then the Bellman operator has the following properties:

- *i) Attainability: for every $v \in V$ there exist a measurable $\varphi : X \to A$ such that $Bv = T^{\varphi}v$. Therefore, $Bv \in V$.*

- *ii) Monotonicity: if $v_1 \leq v_2$ then $Bv_1 \leq Bv_2$.*

- *iii) Monotone continuity: if $v_k \nearrow v$ then $Bv_k \nearrow Bv$.*

Proof:

Note that in the definition of operator B, the function under the infimum is continuous with respect to a. Because the kernel is assumed to be *strongly continuous*. Attainability of the infimum over the compact set A holds, and consequently the infimum is also measurable. Result *ii)* is trivial. The Lebesgue monotone convergence theorem and interchange results of limits and minima on continuous functions over compact sets imply *iii)*.

For further details see Hernandez-Lerma and Laserre (1996).

□

We turn now to the Lundberg-type inequality probabilities for the ruin probability developed in Section 4. The existence of this bounds means that for at least one strategy the probability distribution $\widehat{F}(a, .)$ has the adjustment coefficient as we assume in the following.

Proposition 3. *Let consider the upper bound $v(x) = \exp\{-R_1 x\}$ given by* (30) *in Theorem 2. Assuming that there exist $a \in A$ such that $\int_X e^{R_1 x}$, $\widehat{F}(a, dy) = 1$, where the distribution function $\widehat{F}$ is given by* (5) *then the function $v(x)$ satisfies $T^a v \leq v$.*

Proof:

Because the function $v(x) \in V$ and satisfies Proposition 2, we can state

$$\begin{aligned}
T^a v(x) &= P(x,a,\Delta) + \textstyle\int_{X\setminus\Delta} P(x,a,dx')v(x') \\
&= \textstyle\int_{(x,\infty]} \widehat{F}(a,dx') + \int_{(-\infty,x]} \widehat{F}(a,dx')v(x-x') \\
&= \widehat{F}(a,dx') + \textstyle\int_{(-\infty,x]} \exp(-R_1(x-x'))\cdot \widehat{F}(a,dx') \\
&\leq \underset{(x,\infty]}{\int} \exp(-R_1(x-x'))\cdot \widehat{F}(a,dx') + \underset{(-\infty,x]}{\int} \exp(-R_1(x-x'))\cdot \widehat{F}(a,dx') \\
&\quad \textstyle\int_R \exp(-R_1(x-x'))\cdot \widehat{F}(a,dx') \\
&= \exp(-R_1(x)) \textstyle\int_R \exp(R_1 x')\cdot \widehat{F}(a,dx') \\
&= \exp(-R_1(x)) = v(x)
\end{aligned}$$

Theorem 3. *Let assume that Proposition 3 holds. Let the distribution function $\widehat{F}$ verify that there exist $h > 0$ and $\zeta > 0$ such that $(1 - \widehat{F}(a,h)) \geq \xi$, for every $a \in A$. Then*

- *i) If we define recursively $\bar{v}_k = B\bar{v}_{k-1}$, starting with $\bar{v}_0 = v$, we obtain the monotone-decreasing convergence $\bar{v}_k \searrow \psi_*$.*

- *ii) Moreover we have that $\bar{v}_k \to \psi_*$ where the convergence may be non-monotonic.*

***Remark**: Note that the condition for $\widehat{F}$ seems to be satisfied for all reasonable risk models. It is worth noting that if the conditions is fulfilled, then for every strategy a and every initial state x the surplus process verify $P_x^a(T < \infty \text{ or } X_n \to \infty) = 1$ and consequently almost every trajectory either goes infinity or falls below zero at least once.*

Proof:

- i) By induction it can be proved that $\bar{v}_{k-1} \geq \bar{v}_k$, starting with $\bar{v}_0 \geq T^a\bar{v}_0 \geq B\bar{v}_0 = \bar{v}_1$. Because $B\psi_* = \psi_*$ by induction we obtain $\bar{v}_k \geq \psi_*$.

- ii) Note that $\psi_* \leq \psi^a \leq \bar{v}_0$ implies $\psi_* \leq \bar{v}_k$ because B is a monotonic operator and has a fixed point which is ψ_*. Because $P_x^a(T < \infty \text{ or } X_n \to \infty) = 1$ holds, then provided that $v(x) = \exp\{-R_1 x\}$ verifies $\lim_{x\to\infty} v(x) = 0$ and the dominated convergence theorem we obtain

$$\begin{aligned}\int_{X\setminus\Delta} P(x,a,dx')v(x') &= \int_{(-\infty,x]} \widehat{F}(a,dx')v(x-x') \\ &= E_x^a(I(T>k)\bar{v}_0(X_x) \to 0\end{aligned}$$

  This is true for every strategy a, in particular for a_*. Then we have

$$\bar{v}_k(x) \leq P_x^{a^*}(T<\infty) + E_x^{a^*}(I(T>k)\bar{v}_0(X_x)$$

  The first term tends to ψ_{a^*} and the second term goes to zero. It follows that $\lim \bar{v}_k \leq \psi_{a^*} = \psi_*$. Combining with i) then the convergence $\bar{v}_k \to \psi_*$ is obtained.

Remark: By the attainability property of the Bellman operator there is a stationary strategy a such that $B\bar{v}_{k-1} = T^a\bar{v}_{k-1}$. It is not difficult to prove that considering this stationary strategy $T^a\bar{v}_{k-1} \leq \bar{v}_{k-1}$ implies $\psi^a \leq \bar{v}_k$. If we compute the stationary strategy simultaneously with $\bar{v}_k$, we obtain a stationary strategy for which at each stage the ruin probability is at most $\bar{v}_k$.

References

[1] Browne, S. (1995). Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Math. Operat. Res.* **20,** 4, pp. 937–958.

[2] Cai, J. and Dickson, D. (2004). Ruin probabilities with a Markov chain interest model. *Insur.: Math. Econo.* **35,** pp. 513–525.

[3] Cai, J. (2002). Ruin probabilities with dependent rates of interest. *J. Appl. Prob.* **39,** pp. 312–323.

[4] Gajek, L. (2005). On the deficit distribution when ruin occurs discrete time model. *Insur.: Math. Econo.* **36,** pp. 13–24.

[5] Gaier, J., et al. (2003). Asymptotic ruin probabilities and optimal investment.*Ann. Appl. Probab.* **13,** no. 3, pp. 1054–1076.

[6] Grandell, J.(1991). *Aspects of risk theory.* Springer Series in Statistics. Probability and its Applications. Springer-Verlag, New York.

[7] Groniowska, A. and Niemiro, W. (2005). Controlled risk processes in discrete time: Lower and upper approximations to the optimal probability of ruin. *Insur.: Math. Econo.* **36,** pp. 433–440.

[8] Hernandez-Lerma, O. and Lasserre, J.B. (1996). *Discrete-Time Markov Control Processes: Basic Optimality Criteria.* Springer-Verlang, New York.

[9] Hipp, C. and Plum, M. (2000). Optimal investment for insurers.*Insur.: Math. Econo.* **27,** pp. 215–228.

[10] Hipp, C. and Schmidli, H. (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claim case. *Scand. Actuarial J.* **5,** pp. 321–335.

[11] Hipp, C. and Taksar, M. (2000). Stochastic control for optimal new business. *Insur.: Math. Econo.* **26,** pp. 185–192.

[12] Norberg, R. (1997). Discussion of Stochastic analysis of the interaction between investment and insurance risks. *North American Actuarial Journal.* **1,** pp. 75–76.

[13] Schäl, M.(2004). On Dicrete-Time Dynamic Programming in Insurance: Exponential Utility and Minimizing the Ruin Probability. *Scand. Actuarial J.* **3,** pp. 189–210.

[14] Schmidli, H. (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J.* **1,** pp. 55–68.

[15] Schmidli, H.(2002). On minimising the ruin probability by investment and reinsurance. *Ann. Appl. Prob.* **12,** pp. 890–907.

[16] Sundt, B. and Teugels, J.L. (1995). Ruin Estimates Under Interest Force. *Insur.: Math. Econo.* **16,** pp. 7–22.

[17] Sundt, B. and Teugels, J.L. (1997). The Adjustment Function in Ruin Estimates Under Interest Force. *Insur.: Math. Econo.* **19,** pp. 85–94.

[18] Willmot, G. E., et al. (2000). Lundberg inequalities for renewal equations. *Advances in Applied Probability.* **33,** pp. 674–689.

[19] Willmot, G. E. and Lin, X. S. (2001). *Lundberg Approximations for Compound Distributions with Insurance Applications.* Springer-Verlag, New York.

[20] Yang, H. (1999). Non-exponential bounds for ruin probability with interest effect included. *Scand. Actuarial J.* **99,** pp. 66–79.

A Stochastic orders and phase-type distributions

$$\begin{array}{ccccc} \text{DFR} & & \Rightarrow & & \text{NWU} \\ \Downarrow & & & & \Downarrow \\ \text{IMRL} & \Rightarrow & \text{2-NWU} & \Rightarrow & \text{NWUC} \end{array}$$

$\diamond$ **The df F is DFR (Decreasing Failure Rate):** if $\frac{\overline{F}(x+y)}{\overline{F}(y)}$ is nondecreasing in y for fixed $x \geq 0$, i.e. $\overline{F}(y)$ is log-convex. Then $\mu(y) = \frac{f(y)}{\overline{F}(y)}$ is nonincreasing in y.

Let $F(y)$ phase-type distribution with parameters (α, T), we want to see under what condition the phase-type distributions are DFR. That is equivalent to satisface that $\frac{\partial}{\partial y}\frac{\overline{F}(x+y)}{\overline{F}(y)}$ is non-negative in y for fixed $x \geq 0$. In this case $\frac{\overline{F}(x+y)}{\overline{F}(y)} = \frac{\alpha e^{T(x+y)}e}{\alpha e^{Ty}e}$, then

$$\frac{\partial}{\partial y}\frac{\overline{F}(x+y)}{\overline{F}(y)} = \frac{\alpha T e^{T(x+y)}\widehat{\alpha}e^{Ty}e - \alpha e^{T(x+y)}\widehat{\alpha}Te^{Ty}e}{(\alpha e^{Ty}e)^2}$$

is non-negative, if only if the numerator is non-negative, i.e,

$$\begin{array}{rl} & \alpha T e^{T(x+y)}\widehat{\alpha}e^{Ty}e - \alpha e^{T(x+y)}\widehat{\alpha}Te^{Ty}e \\ = & \alpha\{Te^{T(x+y)}\widehat{\alpha} - e^{T(x+y)}\widehat{\alpha}T\}e^{Ty}e = \alpha\{TA - AT\}e^{Ty}e \geq 0 \end{array} \tag{33}$$

where $A = e^{T(x+y)}\widehat{\alpha}$ and $\widehat{\alpha} = e \cdot \alpha = \overrightarrow{\mathbf{1}} \cdot (\alpha_1, \ldots, \alpha_n) = (\alpha_1 \cdot \overrightarrow{\mathbf{1}}, \ldots, \alpha_n \cdot \overrightarrow{\mathbf{1}})$. The previous condition (33) is only posible if $TA - AT$ is non-negative definite. The equality is when T commute with $A = e^{T(x+y)}e\alpha$.
So that, The phase-type distribution are DFR if only if $TA - AT$ is non-negative definite.
$\diamond$ **The df F is IMRL (Increasing Mean Residual Life)**: if

$$\begin{array}{rcl} r(y) & = & E[T_y] = \frac{\int_y^\infty (t-y)dF(t)}{\overline{F}(y)}dt \\ & = & \frac{\int_0^\infty \overline{F}(t+y)dt}{\overline{F}(y)} = \frac{E(Y)\overline{F}_1(y)}{\overline{F}(y)} \end{array}$$

is increasing. Where $F_1(y) = \frac{\int_0^y \overline{F}(x)dx}{E[Y]}$ with $y \geq 0$ and $E[Y] = \int_0^\infty \overline{F}(y)dy$.
Let $F(x) = PH(\alpha, T)$, always T^{-1} exist. En this case $E[Y] = -\alpha T^{-1}e$, then

$F_1(y) = \frac{\int_0^y \alpha e^{Tx} e dx}{E[Y]} = \frac{\alpha T^{-1}\{e^{Ty} - I\}e}{E[Y]}$ and $\overline{F}_1(y) = \frac{\alpha T^{-1} e^{Ty} e}{\alpha T^{-1} e}$. So $r(y) = \frac{-\alpha T^{-1} e^{Ty} e}{\alpha e^{Ty} e}$ and $\frac{\partial}{\partial y} r(y) = \frac{\alpha(-e^{Ty}\widehat{\alpha} + T^{-1} e^{Ty}\widehat{\alpha} T) e^{Ty} e}{(\alpha e^{Ty} e)^2}$.
$F(y)$ is IMRL if only if $\frac{\partial}{\partial y} r(y)$ is non-negative, i.e, $F(y)$ is IMRL iff $-e^{Ty}\widehat{\alpha} + T^{-1} e^{Ty}\widehat{\alpha} T$ is non-negative definite.
$\diamond$ **The df F is NWU (New Worse than Used):** if

$$\overline{F}(x+y) \geq \overline{F}(y) \cdot \overline{F}(x), \ \forall x \geq 0, \ \forall y \geq 0. \tag{34}$$

Let $F(x) = PH(\alpha, T)$, we say that F is NWU if only if $\overline{F}(x+y) - \overline{F}(y) \cdot \overline{F}(x) = \alpha e^{Tx}(I - e\alpha) e^{Ty} e \geq 0$, i.e, the phase-type distribution is NWU iff $I - \widehat{\alpha}$ is non-negative definite.
$\diamond$ **The df F is 2-NWU (Second New Worse than Used):** if

$$\overline{F}_1(x+y) \geq \overline{F}_1(y) \cdot \overline{F}_1(x), \ \forall x \geq 0, \ \forall y \geq 0.$$

Let $F(x) = PH(\alpha, T)$, in this case $\overline{F}_1(x) = 1 - F_1(x) = \frac{\alpha T^{-1} e^{Tx} e}{\alpha T^{-1} e}$. We say that F is 2-NWU iff

$$\overline{F}_1(x+y) - \overline{F}_1(y) \cdot \overline{F}_1(x) = \frac{\alpha T^{-1}(e\alpha T^{-1} e^{Ty} - e^{Ty} e \alpha T^{-1}) e^{Tx} e}{(\alpha T^{-1} e)^2} \geq 0.$$

The phase-type distribution is 2-NWU iff $T^{-1}(Be^{Ty} - e^{Ty} B)$ is non-negative definite, where $B = e\alpha T^{-1}$.
$\diamond$ **The df F_1 is NWUC (New Worse than Used in Convex ordering)**: if

$$\overline{F}_1(x+y) \geq \overline{F}_1(y) \cdot \overline{F}(x), \ \forall x \geq 0, \ \forall y \geq 0. \tag{35}$$

Let $F(x) = PH(\alpha, T)$, we say that F is NWUC iff

$$\overline{F}_1(x+y) - \overline{F}_1(y) \cdot \overline{F}(x) = \frac{\alpha T^{-1} e^{Ty}(I - e\alpha) e^{Tx} e}{\alpha T^{-1} e} \geq 0.$$

The phase-type distribution is NWUC iff T^{-1} and $T^{-1} e^{Ty}(I - e\alpha)$ are non-negative or non-positive definite both simultaneously.