UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 04-08
Statistics and Econometrics Series 01
February 2004

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

RESTLESS BANDIT MARGINAL PRODUCTIVITY INDICES I: SINGLE-PROJECT CASE AND OPTIMAL CONTROL OF A MAKE-TO-STOCK *M/G/1* QUEUE

José Niño-Mora*

Abstract

This paper develops a framework based on convex optimization and economic ideas to formulate and solve by an index policy the problem of optimal dynamic effort allocation to a generic discrete-state restless bandit (i.e. binary-action: work/rest) project, elucidating a host of issues raised by Whittle (1988)´s seminal work on the topic. Our contributions include: (i) a unifying definition of a project´s marginal productivity index (MPI), characterizing optimal policies; (ii) a complete characterization of indexability (existence of the MPI) as satisfaction by the project of the law of diminishing returns (to effort); (iii) sufficient indexability conditions based on partial conservation laws (PCLs), extending previous results of the author from the finite to the countable state case; (iv) application to a semi-Markov project, including a new MPI for a mixed long-run-average (LRA)/ bias criterion, which exists in relevant queueing control models where the index proposed by Whittle (1988) does not; and (v) optimal MPI policies for service-controlled make-to-order (MTO) and make-to-stock (MTS) *M/G/1* queues with convex back order and stock holding cost rates, under discounted and LRA criteria.

Keywords: Stochastic scheduling, restless bandit, index policy, Markov decision process, resource allocation, diminishing returns, marginal productivity, shadow wage, efficient frontier, Lagrange multiplier, convex optimization, make-to-order, make-to-stock, production-inventory control, conservation laws, achievable performance.

*Niño-Mora, Departamento de Estadística, Universidad Carlos III de Madrid, C/ Madrid 126, 28903 Getafe (Madrid), e-mail: jose.nino@uc3m.es. Supported in part by the Spanish Ministry of Science and Technology (MCyT) Grant BEC2000-1027 and a Ramón y Cajal Research Fellowship, NATO Collaborative Linkage Grant PST.CLG.976568, and the Joint Spanish-USA (Fulbright) Commission for Scientific and Technological Cooperation Grant 2000-20132.

1 Introduction

This paper develops a framework based on convex optimization and economic ideas to formulate and solve by an index policy the problem of optimal dynamic effort allocation to a generic discrete-state *restless bandit (RB)* (i.e. binary-action: work/rest) project, elucidating a host of issues raised by Whittle (1988)'s seminal work on the topic. The framework is deployed to address the solution by index policies of service-controlled make-to-order (MTO) and make-to-stock (MTS) $M/G/1$ queues with convex backorder and stock holding cost rates, under discounted and long-run-average (LRA) criteria. In the companion paper Niño-Mora (2004) (see an abridged version in Niño-Mora (2003)), the single-project results obtained here are used to address corresponding multi-project problems, yielding a heuristic hedging point and index policy, along with a lower bound on optimal cost.

Our proposed framework draws on and combines in a unifying setting ideas from relatively autonomous areas, including: (i) convex optimization in mathematical programming; (ii) the economic theory of optimal resource allocation; (iii) index policies for scheduling multiclass queues; (iv) index policies for multiarmed bandits and their RBP extension; (v) polyhedral methods in stochastic scheduling; and (vi) work conservation laws in service systems. To put our contributions in context, we discuss below the relevant background.

1.1 Solution approaches to resource allocation problems

The prevailing solution approaches in the domains of static/deterministic and of dynamic/stochastic resource allocation problems are radically distinct. In the former, the concern is to find a fixed allocation of resources optimizing a cost/reward objective. Formulation and solution methods are those of *mathematical programming (MP)*, which has proven widely successful, both in theory and practice. The concepts of *convexity* and *duality* play central roles, both as analysis tools, and as insightful bridges with economic interpretation. Convexity is the mathematical counterpart of the economic *law of diminishing returns (LDR)*, under which, as use of a resource increases, its *marginal productivity* diminishes. Duality relates to the *resource valuation problem*, which is to find a resource's *shadow price*, giving its intrinsic value in the model at hand. Two fundamental results holding under the LDR are: (i) a resource's shadow price at a given use level equals its marginal productivity; and (ii) to achieve an optimal allocation, use of a resource must be increased as long as its price is lower than its marginal productivity, until both coincide. The classic texts of Kantorovich (1965) and Koopmans (1957) provide illuminating accounts of such ideas.

In the latter domain, the concern is to design a *policy* for dynamic resource allocation to competing activities, in a *system* whose *state* evolves randomly over time. The objective is to optimize a measure of average cost/reward performance. The main modeling paradigm is furnished by *Markov decision processes (MDPs)*, especially in the *discrete-state and -action* case, to which we restrict attention. See, e.g. Puterman (1994). The leading tool is the *dynamic programming (DP)* technique, which uses the *principle of optimality* to formulate a set of *Bellman equations*, whose solution yields an *optimal policy*. Extensive research efforts have been devoted to their analytical solution in relatively simple models, by often ad hoc methods, and to their computational solution by general algorithms, such as *value* and *policy iteration*. A deep connection between the MP and DP approaches was revealed by d'Epénoux (1960) and Manne (1960), who showed that the Bellman equations for a finite MDP can be formulated and solved as a *linear programming (LP)* problem. The current status of the field remains, however, unsatisfactory. Thus, no unifying analytical solution method has emerged. Also, application of general algorithms is hindered by their computational demands (*curse of dimensionality*). Further, even when a solution is available, it is often not clear how one can gain from it insights of the kind provided by convexity and duality.

1.2 Index policies and MP approach to dynamic resource allocation

Limited research efforts have explored use of MP tools in dynamic and stochastic resource allocation problems, mostly in the area of *stochastic scheduling* (cf. Niño-Mora (2001b)). The latter concerns the optimal dynamic allocation of resources to stochastic *projects*, which can represent a variety of entities, e.g. jobs or queues.

A notorious feature of such area is the optimality, in a wide range of models, of policies characterized by *allocation indices*. In a typical result, to every project k is attached an *index* $\nu_k(i_k)$ depending only on its state i_k, such that the *index policy* which dynamically gives higher resource access priority to projects with larger index values is optimal. The optimal index often has an insightful economic interpretation, being given by, e.g., a rate of expected cost reduction per unit expected effort invested, or a critical subsidy for passivity or charge for activity.

While such results have have been obtained by ad hoc methods (e.g. *interchange arguments*), they have also been established (typically later) by LP arguments. The latter are based on formulating LP constraints on *performance measures* (e.g. mean delays). In tractable models, such constraints fully characterize the *achievable performance region* spanned by the performance vector under *admissible policies*. This is a bounded polyhedron, whose vertices are achieved by priority policies. The optimal vertex is characterized by indices, emerging in the construction of an optimal *dual* solution. In intractable models, available constraints give a tractable *relaxation*, whose dual solution may suggest a heuristic index policy. Such program has been carried out in a variety of models for scheduling a multiclass queue, and on *multiarmed bandit problems (MBPs)* and extensions.

Table 1 highlights selected results in such vein, pointing out the evolution of ideas used to obtain LP constraints, which we review next. The ground-breaking work is due to Klimov (1974), who used *aggregate flow balance* to obtain an LP formulation for the problem of scheduling a multiclass $M/G/1$ queue with feedback to minimize LRA linear holding costs. He gave an *adaptive-greedy algorithm* to construct an optimal *dual* solution in terms of a *static index* ν_k attached to classes k. He then used LP duality to prove optimality of such *Klimov index* policy.

Coffman and Mitrani (1980) introduced a different LP formulation for the *no-feedback* case of Klimov's model. The LP variables x_k represent *mean delays* for each class k, while constraints formulate *work conservation laws*, extending results of Kleinrock (1976, Ch. 3). The latter characterize the achievable performance region of mean delays as a *polymatroid*, a well-known polyhedron in polyhedral combinatorics introduced by Edmonds (1971). Optimality of the *greedy algorithm* for LP over polymatroids thus explains the *$c\mu$-index rule*'s (cf. Cox and Smith

LP constraints	Models & papers
Aggregate flow balance	Multiclass (MC) queues (feedback) Klimov (1974)
Strong conservation laws Polymatroids	MC queues (no feedback) Coffman and Mitrani (1980) Federgruen and Groenevelt (1988) Shanthikumar and Yao (1992)
Generalized conservation laws Extended polymatroids	Klimov's model, multiarmed bandits Tsoucas (1991) Bertsimas and Niño-Mora (1996)
Approximate conservation laws Extended polymatroids	MC queues (feedback & parallel servers) Dacre et al. (1999) Glazebrook and Niño-Mora (2001)
Flow balance & average activity Lagrangian relaxation	Restless bandits (RBs) Whittle (1988), Bertsimas and Niño-Mora (2000)
Partial conservation laws (PCLs) $\mathscr{F}$-extended polymatroids	RBs & MC queues (convex costs, finite state); Niño-Mora (2001a, 2002)
Diminishing returns & PCLs Efficient work-cost frontier	RBs & MC queues (convex costs, countable state); this paper, Niño-Mora (2004)

Table 1: LP formulations giving index policies for stochastic scheduling problems.

(1961)) for the scheduling problem.

The polymatroidal LP formulation was further investigated by Federgruen and Groenevelt (1988), and by Shanthikumar and Yao (1992), who explained such results through the framework of *strong (work) conservation laws*.

Coffman and Mitrani's analysis was extended to Klimov's model by Tsoucas (1991), who characterized its achievable performance region as a new type of polyhedron (*extended polymatroid*). Bertsimas and Niño-Mora (1996) furnished the theoretical foundation of such result, introducing the framework of *generalized conservation laws (GCLs)*. They further deployed GCLs to obtain a corresponding result for the *branching bandit problem*, encompassing the above models under LRA and discounted criteria, and the classic MBP.

The MBP concerns the optimal dynamic allocation of effort to a collection of projects, modeled as discounted binary-action (active/passive) discrete-state and -time MDPs which can only change state when active, and one of which must be engaged at each time. In a celebrated result, Gittins (1979) introduced an index $\nu_k(i_k)$ for each project k depending only on its state i_k, and proved optimality of the resulting *Gittins index* policy. The GCL analysis in Bertsimas and Niño-Mora (1996) yielded a new, LP-based proof of such result.

In some intractable models concerning the scheduling of a multiclass queue on parallel servers, GCLs hold only in an *approximate* sense. This yields a tractable *LP relaxation* of the achievable performance region, and explicit suboptimality bounds on heuristic index policies, which can be used to establish their asymptotic optimality in heavy traffic. See Glazebrook and Niño-Mora (2001) and Dacre et al. (1999).

1.3 RBP, indexability, and queueing control applications

The *restless bandit problem (RBP)* extension of the MBP, where passive projects can change state, is of prime concern in this paper. It provides a powerful modeling paradigm at the expense of tractability, being *P-space hard*. See Papadimitriou and Tsitsiklis (1999). Whittle (1988) introduced an index $\nu_k(i_k)$ attached to a *restless bandit (RB) project* k, proposing as a heuristic the resulting index policy. The *Whittle index* emerges in the solution of a *relaxed problem*, which further gives a *performance bound*, in terms of the *Lagrange multiplier* for an average-activity constraint. Such policy is optimal in the MBP case, and asymptotically optimal under certain conditions. See Weber and Weiss (1990).

Yet the Whittle index is *not* defined for all RB projects, only for a restricted class of so-called *indexable* projects. Whittle (1988) stated:

> "... one would very much like to have simple sufficient conditions for indexability; at the moment, none are known."

Such scope limitation prompted Bertsimas and Niño-Mora (2000) to introduce a different LP-based index policy, applying to finite-state projects, and a hierarchy of LP relaxations, giving tighter bounds at increasing computational expense. The indexability issue was taken up in Niño-Mora (2001a), where we extended GCLs to introduce the framework of *partial conservation laws (PCLs)*. Satisfaction of PCLs by the performance measures of a stochastic scheduling problem ensures optimality of index policies *with a postulated structure*, under *admissible objectives*. Use of PCLs further yielded tractable sufficient conditions for indexability (*PCL-indexability*), and an efficient algorithm for computing the Whittle index.

The polyhedral foundation of the PCL framework was developed in Niño-Mora (2002). That paper introduced extensions of the Whittle index with a significantly expanded scope, motivated by analysis of a *queueing admission control* model with *convex nondecreasing* holding cost rates. It further introduced a characterization of the index, under *PCL-indexability*, as an optimal *marginal rate* of cost decrease per unit effort increase; and a connection of PCL-indexability with the LDR.

Yet the tools in Niño-Mora (2001a, 2002), relying on polyhedral methods, apply only to *finite-state* projects. Also, they only provide *sufficient conditions* for indexability, while it would be desirable to have a complete understanding of such property. Both limitations are particularly severe in the important case of RBPs representing scheduling problems in queueing systems. Thus, the problem of scheduling a multiclass MTO/MTS $M/G/1$ queue to optimize LRA holding costs is readily formulated as an RBP, with projects corresponding to queues for each class. However, the Whittle index is *not* defined for such projects under the LRA criterion, as pointed out by Whittle (1996, Ch 14.7) himself, and by Veatch and Wein (1996). The latter authors state:

> "In contrast, the backorder problem is not indexable. $\nu(x)$ does not exist (i.e. equals $-\infty$) for all x. The difficulty is that ν is a Lagrange multiplier for the constraint on the time-average number of active arms. For the backorder problem, any stable policy must serve a time-average of ρ classes, so relaxing this constraint does not change the optimal value, and the Lagrange multiplier does not exist. In fact, no scheduling problem with a fixed utilization will be indexable."

The author first proposed (at the 2000 Madison (Wisconsin) International Conference on Stochastic Networks) to overcome such limitation by showing that the Whittle index is well defined in the MTO case, under the *discounted criterion*. Then, taking the limit of the discounted Whittle index scaled by the discount factor as this vanishes gives a convenient LRA index. Such approach is deployed in Ansell et al. (2003) and Glazebrook et al. (2003) in the MTO $M/M/1$ and $M/G/1$ cases by an ad hoc DP analysis, under the assumption that holding cost rates are *convex increasing* in the queue's state. Their approach, however, is hindered by the following limitations: (i) the convex increasing holding cost rate assumption is violated in the MTS case with backorders, where holding cost rates are V-shaped in the natural state of *net backorder* levels; (ii) it does not yield *bounds* on optimal cost under the LRA criterion, arguably more important in applications than the discounted one; (iii) it does not provide an independent concept of indexability under the LRA criterion, as it relies on establishing indexability under the discounted criterion, which is often technically cumbersome; and (iv) it does not clarify interpretation of the limiting LRA index, as it is not proven that it yields an optimal policy for the single-project LRA problem. We remark that, in complementary work, Dusonchet and Hongler (2003) have calculated the *discounted* Whittle index for an MTS $M/M/1$ queue with *linear* backorder and stock holding cost rates.

1.4 Contributions

Motivated by the issues discussed above, this paper presents the following contributions: (i) a unifying definition of a project's *marginal productivity index (MPI)*, characterizing optimal policies; (ii) a *complete characterization* of *indexability* (existence of the MPI) as satisfaction by the project of the *law of diminishing returns* (to effort); (iii) sufficient indexability conditions based on *partial conservation laws (PCLs)*, extending previous results of the author from the finite to the countable state case; (iv) application to a semi-Markov project, including a new MPI for a *mixed long-run-average (LRA)/bias criterion*, which exists in relevant queueing control models where the index proposed by Whittle (1988) does not; and (v) optimal MPI policies for service-controlled MTO and MTS $M/G/1$ queues with convex backorder and stock holding cost rates, under discounted and LRA criteria.

1.5 Structure of the paper

The rest of the paper is organized as follows. Section 2 introduces our motivating problem, concerning the scheduling of a multiclass MTO/MTS queue. Section 3 introduces the MPI policy approach to address the problem of optimal dynamic effort allocation to a generic RB project. Section 4 develops PCL-based sufficient indexability conditions for countable-state projects. Section 5 presents PCL-indexability analyses of semi-Markov projects under several criteria. Section 6 addresses the case of a service controlled MTO $M/G/1$ queue, while Section 7 investigates the corresponding MTS model.

2 Motivating problem

Consider a model for a single-product production-inventory facility. Orders of unit size arrive as a Poisson process with rate λ. A single machine, which processes all orders, makes a product unit in a *production time* distributed as a random variable with Laplace-Stieltjes transform (LST) $\psi(\cdot)$, having finite mean $1/\mu$ and variance σ^2. The arrival stream and production times are mutually independent. Denoting by $\rho = \lambda/\mu$ the traffic intensity, we assume the *stability condition* $\rho < 1$.

The facility has a *finite storage capacity* for storing up to and including $s \geq 0$ finished items in a *finished goods stock (FGS)*. Arriving orders finding the FGS empty are placed in a *backorder queue (BQ)* of unlimited size. We denote by $X^-(t)$ (resp. $X^+(t)$) the size of the FGS (resp. BQ) at time $t \geq 0$, and consider the system *state* to be the size of its *net BQ*, $X(t) = X^+(t) - X^-(t)$. The *state space* is thus $N = \{-s, \ldots, 0, 1, \ldots\}$. Notice that such setting encompasses the *pure MTO case* ($s = 0$) and the *MTS case with backorders* ($s \geq 1$).

A controller governs the system by choice of a *production-inventory policy* π, prescribing dynamically whether the machine is to be idle or working. The policy is drawn from the class Π of *admissible policies*, which are: (i) *nonpreemptive*, i.e. production of an item cannot be interrupted; thus, the *decision epoch sequence* consists of order arrival epochs to an empty system, and product completion epochs; (ii) *nonanticipative*, i.e. decisions depend on the history of the system up to and including the present epoch; and (iii) *stable*, i.e. the policy must induce an equilibrium distribution on the state process, having finite moments of the required order.

Backorder and/or stock holding costs are incurred, separably across products. Costs accrue at rate h_i per unit time while the state is $i \in N$. We will refer to the first and second-order differences $\Delta h_i \triangleq h_i - h_{i-1}$ and $\Delta^2 h_i \triangleq \Delta h_i - \Delta h_{i-1}$. We impose the following requirements on cost rates.

Assumption 2.1 Holding cost rates h_i satisfy the following:

(i) They are bounded below: $\inf\{h_i : i \in N\} > -\infty$.

(ii) They are convex: $\Delta^2 h_i \geq 0$, for $i \in N$ such that $i - 2 \in N$.

(iii) If $\psi(\cdot)$ has finite moments of up to order $m+1$, then $h_i = O(i^m)$ as $i \to +\infty$.

Our prime concern will be the *LRA production-inventory control problem*, which is to find a policy $\pi^* \in \Pi$ attaining the minimum LRA value f^* of costs incurred.

$$f^* = \inf_{\pi \in \Pi} \lim_{T \to +\infty} \frac{1}{T} \mathbb{E}^\pi \left[\int_0^T h_{X(t)}\, dt \right]. \qquad (1)$$

In the MTS case, problem (1) seems to have been addressed in the literature only in the special case where backorder and stock holding cost rates are linear. See, Buzacott and Shanthikumar (1993, Ch. 4) and the references therein. In the MTO case, Bertsekas (1987, Ch. 6.7) presents a DP-based analysis for an $M/M/1$ queue with convex *nondecreasing* holding cost rates, under the discounted criterion.

3 Optimal control of RB projects by index policies

3.1 Optimal project control problem

Consider a generic RB project, whose state $X(t)$ evolves randomly over time $t \geq 0$ across the *discrete* (finite or countable) *state space* $N \subseteq \mathbb{Z}$. Control is exercised by a *central planner*, who observes the state at a sequence of *decision epochs* $t_0 = 0 < t_1 < \cdots < t_n \to +\infty$ as $n \to +\infty$, deciding at each whether a *single operator* is allocated to work (*active* action: $a(t_n) = 1$), or is let to rest (*passive* action: $a(t_n) = 0$), in the following *period* ($a(t) = a(t_n)$ for $t \in [t_n, t_{n+1})$). We will refer to $X(t)$ and $a(t)$ as the *natural state* and *action processes*, and to $X_n = X(t_n)$ and $a_n = a(t_n)$ as the *embedded processes*. We will further refer to a period $[t_n, t_{n+1})$ where $X_n = i$ and $a_n = a$ as an (i, a)*-period*, or i*-period*, as appropriate.

Action choice results from adoption of a *policy* π, drawn from a given class Π of *nonanticipative admissible policies*: each epoch's choice must be based on the embedded processes' history. A *manager* is in charge of policy implementation.

Class Π is assumed to be *closed under randomization*. Given policies $\pi, \pi' \in \Pi$, and $q \in [0, 1]$, the policy resulting from a draw of π or π' with probabilities q and $1-q$, denoted by $q\pi + (1-q)\pi'$, is in Π. We will refer to the class $\Pi^{\mathrm{SD}} \subset \Pi$ (resp. $\Pi^{\mathrm{SR}} \subset \Pi$) of *admissible stationary deterministic* (resp. *randomized*) policies, where the chosen action is a deterministic (resp. random) function of the state.

The project accrues *holding costs* over time, whose value under a policy $\pi \in \Pi$ is evaluated by finite *cost measure* f^π. The *optimal dynamic resource allocation problem* of concern is to find an admissible policy minimizing the latter:

$$\text{Find } \pi^* \in \Pi : f^{\pi^*} = f^* \triangleq \inf \{ f^\pi : \pi \in \Pi \} . \tag{2}$$

3.2 Solution by threshold policies

Let us partition N into the *controllable state space* $N^{\{0,1\}}$, where active and passive actions differ on dynamics or costs; and the *uncontrollable state space* $N^{\{0\}}$, otherwise, where we assume that the project is rested. A policy $\pi \in \Pi^{\mathrm{SD}}$ is thus represented by the *active-state set* $S \subseteq N^{\{0,1\}}$ where it engages the project. We will then term it the S*-active policy*, writing, e.g. $S \in \Pi^{\mathrm{SD}}$, f^S. We assume both N and $N^{\{0,1\}}$ to be consecutive-integer sets *bounded below*, i.e.

$$N \triangleq \left\{ j \in \mathbb{Z} : \ell^0 \leq j \leq \ell^1 \right\} \quad \text{and} \quad N^{\{0,1\}} \triangleq \left\{ j \in \mathbb{Z} : \ell^0 < j \leq \ell^1 \right\},$$

for given $-\infty < \ell^0 < \ell^1 \leq +\infty$. Hence, $N^{\{0\}} = \{\ell^0\}$.

We aim to find an optimal policy to (2) within the class of *threshold policies*, which engage the project in states above a critical *threshold*. Writing

$$S_i \triangleq \left\{ j \in N^{\{0,1\}} : j > i \right\}, \quad i \in N,$$

threshold policies are characterized by the nested *active-state set family*

$$\mathscr{F} \triangleq \{ S_i : i \in N \} .$$

We shall henceforth refer to them as $\mathscr{F}$*-policies*, writing, e.g. f^S for $S \in \mathscr{F}$.

Our goals are: (i) elucidate conditions for existence of an optimal $\mathscr{F}$-policy; and (ii) find the latter. Our approach requires use of an appropriate *work measure* g^π, evaluating labor effort. The following properties are assumed.

Assumption 3.1

(i) $\mathscr{F} \subset \Pi^{\mathrm{SD}}$, i.e. $\mathscr{F}$-policies are admissible.

(ii) Work measure g^π is bounded above:

$$\sup\{g^\pi : \pi \in \Pi\} < +\infty.$$

(iii) Cost measure f^π is bounded below:

$$\inf\{f^\pi : \pi \in \Pi\} > -\infty.$$

(iv) Work measure g^{S_i} is decreasing in threshold state i:

$$\Delta g^{S_i} \triangleq g^{S_i} - g^{S_{i-1}} < 0, \quad i \in N^{\{0,1\}}. \tag{3}$$

(v) Achievable work performance is spanned by threshold policies:

$$\{g^\pi : \pi \in \Pi\} = \bigcup_{i \in N^{\{0,1\}}} \left[g^{S_i}, g^{S_{i-1}}\right].$$

We will further refer to the first-order differences of cost measure f^{S_i}:

$$\Delta f^{S_i} \triangleq f^{S_i} - f^{S_{i-1}}, \quad i \in N^{\{0,1\}}. \tag{4}$$

3.3 Reformulation as convex resource allocation problem

We will develop an approach grounded on convex optimization and economic resource allocation theory. Let the *achievable work-cost (performance) region* be

$$\mathbb{H} \triangleq \left\{(b,z) \in \mathbb{R}^2 : (b,z) = (g^\pi, f^\pi) \text{ for some } \pi \in \Pi\right\}.$$

Its projections give the *achievable work region*

$$\mathbb{B} \triangleq \{b \in \mathbb{R} : b = g^\pi \text{ for some } \pi \in \Pi\},$$

and the *achievable cost region*

$$\mathbb{V} \triangleq \{z \in \mathbb{R} : z = f^\pi \text{ for some } \pi \in \Pi\}.$$

Convexity of such regions follows from the requirement that Π be closed under randomization. It further extends to their *closures* $\bar{\mathbb{H}}$, $\bar{\mathbb{B}}$ and $\bar{\mathbb{V}}$.

The *efficient work-cost frontier* is given by

$$\underline{\partial}\mathbb{H} \triangleq \left\{(b,z) \in \bar{\mathbb{H}} : b \in \mathbb{B} \text{ and } z \leq f^\pi \text{ for any } \pi \in \Pi \text{ with } g^\pi = b\right\}.$$

This is characterized as the graph of *(optimal) cost function*

$$C(b) \triangleq \inf\{f^\pi : g^\pi = b, \pi \in \Pi\} = \inf\{z : (b,z) \in \mathbb{H}\}, \quad b \in \mathbb{B}, \tag{5}$$

whose convexity follows from that of region $\mathbb{H}$, so that

$$\underline{\partial}\mathbb{H} = \{(b, C(b)) : b \in \mathbb{B}\}.$$

Note that $C(b)$ is the optimal cost performance under policies using b work units.

We can now reformulate (2) as the *convex resource allocation problem*

$$\text{Find } b^* \in \mathbb{B} : C(b^*) = f^* \triangleq \inf \{C(b) : b \in \mathbb{B}\}, \tag{6}$$

which is to find an optimal amount b^* of work to be expended on the project.

To evaluate $C(b)$ we will further address the following b*-work problem*:

$$\text{Find } \pi^* \in \Pi \text{ with } g^{\pi^*} = b : f^{\pi^*} = C(b) \triangleq \inf \{f^\pi : g^\pi = b, \pi \in \Pi\}. \tag{7}$$

In what follows, a policy $\pi \in \Pi$ will be said to be b*-work feasible* if $g^\pi = b$.

3.4 Lagrangian multiplier analysis and decentralization

To address b-work problem (7) we use the *method of Lagrange multipliers*. Dualizing constraint $g^\pi = b$ by multiplier $\nu \in \mathbb{R}$ gives the *Lagrangian function*

$$\mathscr{L}_b^\pi(\nu) \triangleq f^\pi + \nu\,[g^\pi - b] = v^\pi(\nu) - \nu b, \tag{8}$$

defined for $\pi \in \Pi$ and $\nu \in \mathbb{R}$, where

$$v^\pi(\nu) \triangleq f^\pi + \nu g^\pi.$$

We interpret ν as the *wage* earned by the operator per unit work performed. Thus, $v^\pi(\nu)$ is the holding and labor costs value; and $\mathscr{L}_b^\pi(\nu)$ is the adjusted cost when work expended above (resp. below) b units is paid (resp. sold) at wage ν.

The corresponding unconstrained *Lagrangian problem* is

$$\text{Find } \pi^* \in \Pi : \mathscr{L}_b^{\pi^*}(\nu) = \mathscr{L}_b^*(\nu) \triangleq \inf \{\mathscr{L}_b^\pi(\nu) : \pi \in \Pi\}. \tag{9}$$

This is equivalent to the following ν*-wage problem*, which is to find a policy minimizing the project's holding and labor costs:

$$\text{Find } \pi^* \in \Pi : v^{\pi^*}(\nu) = v^*(\nu) \triangleq \inf \{v^\pi(\nu) : \pi \in \Pi\}. \tag{10}$$

The optimal values of problems (9) and (10) are related by

$$\mathscr{L}_b^*(\nu) = v^*(\nu) - \nu b. \tag{11}$$

Notice that problem (10) includes (2), thus recovered as the 0*-wage problem*.

Drawing on classic economic interpretation, we view problem (9) as a *decentralized planning relaxation* of *centrally planned* b-work problem (7), where: (i) the planner quotes wage ν to the manager; and (ii) this is left to autonomously solve ν-wage problem (10). This raises the possibility, discussed below, that the planner can decentralize the b-work problem's solution by wage choice.

3.5 Duality-based optimality conditions and shadow wages

To price the value of work in b-work problem (7) we will use the *dual* (or *pricing*) *problem*, which is to find a wage maximizing the (concave) objective $\mathscr{L}_b^*(\nu)$:

$$\text{Find } \nu^* \in \mathbb{R} : \mathscr{L}_b^*(\nu^*) = Q(b) \triangleq \sup \{\mathscr{L}_b^*(\nu) : \nu \in \mathbb{R}\}. \tag{12}$$

We will further use the *duality gap* for a policy π and a wage ν:

$$\Delta_b^\pi(\nu) \triangleq f^\pi - \mathscr{L}_b^*(\nu) = v^\pi(\nu) - v^*(\nu). \tag{13}$$

The next result follows immediately.

Lemma 3.2 (Weak duality)

(a) *Let policy $\pi \in \Pi$ be b-work feasible, and let $\nu \in \mathbb{R}$. Then, $\mathscr{L}_b^*(\nu) \leq f^\pi$.*

(b) $Q(b) \leq C(b)$.

Lemma 3.2 immediately yields the next result, giving a sufficient optimality condition for the b-work problem and its dual, using *strong duality* ($Q(b) = C(b)$).

Lemma 3.3 (Sufficient optimality condition) *Let $\pi^* \in \Pi$ be a b-work feasible policy for which there is a wage $\nu^* \in \mathbb{R}$ with $\Delta_b^{\pi^*}(\nu^*) = 0$. Then:*

(a) *Policy π^* is optimal for b-work problem* (7).

(b) *Wage ν^* is optimal for its dual problem* (12).

(c) *Strong duality holds: $Q(b) = C(b) = f^{\pi^*}$.*

We shall henceforth refer to a wage ν^* satisfying the optimality condition in Lemma 3.3 as a *shadow wage* for the b-work problem. In the present setting, existence of a shadow wage is also necessary for optimality.

Lemma 3.4 (Necessary optimality condition) *If π^* is an optimal policy for the b-work problem then there exists a corresponding shadow wage ν^*.*

Proof. It follows from convexity of $\mathbb{H}$ via the *separating hyperplane theorem*. □

Remark 3.5 If the sufficient optimality condition in Lemma 3.3 holds, then:

(i) The b-work problem's solution can be decentralized. The planner needs only quote wage ν^* to the manager, and let him solve the ν^*-wage problem.

(ii) Geometrically, the line $\{(b', z') \in \mathbb{R}^2 : z' + \nu^* b' = v^{\pi^*}(\nu^*)\}$ *supports* point $\left(b, f^{\pi^*}\right)$ relative to convex work-cost region $\mathbb{H}$. See, e.g. Weitzman (2000).

(iii) If there is a unique shadow wage ν^* and $C(\cdot)$ is derivable at b, then

$$\nu^* = -\frac{d}{db} C(b). \tag{14}$$

Thus, ν^* is the marginal rate of cost decrease per unit increase in work expended, or *marginal productivity of work*, in the b-work problem.

3.6 Indexability and the marginal productivity index

To relate the above analysis with threshold policies, we introduce below a tractable project class, based on structure of solutions to ν-wage problem (10) as ν varies.

Definition 3.6 ($\mathscr{F}$-indexability; MPI index) We say the project is *$\mathscr{F}$-indexable* (for f^π and g^π) if there is a nondecreasing *index* ν_i^* for $i \in N^{\{0,1\}}$ such that, for each $\ell^0 < i < \ell^1$, the S_i-active policy is optimal for the ν-wage problem iff $\nu \in [\nu_i^*, \nu_{i+1}^*]$. We say that ν_i^* is the project's *marginal productivity index (MPI).*

Notice that it is optimal (in the ν-wage problem) to work in state $i \in N^{\{0,1\}}$ iff wage ν does not exceed MPI ν_i^*. The above definition extends a particular concept of *indexability* introduced by Whittle (1988). In economic terms, indexability characterizes the *demand "curve" for work*. It gives an optimal amount of work to be expended (e.g. g^{S_i}), corresponding to a given wage (e.g. for $\nu \in [\nu_i^*, \nu_{i+1}^*]$).

The next result follows immediately.

Lemma 3.7 *If the project is $\mathscr{F}$-indexable then:*

(a) *The ν-wage problem's optimal value can be represented as*

$$v^*(\nu) = \inf \left\{ f^{S_i} + \nu g^{S_i} : i \in N \right\}, \quad \nu \in \mathbb{R}, \tag{15}$$

and is hence piecewise linear concave and nondecreasing in ν.

(b) *The project's MPI is given by*

$$\nu_i^* = -\frac{\Delta f^{S_i}}{\Delta g^{S_i}}, \quad i \in N^{\{0,1\}}. \tag{16}$$

3.7 Diminishing returns to work

We next address the issue of economic characterization of indexability. To prepare the ground, we introduce here the class of projects obeying the economic *law of diminishing returns (LDR) to work*, in a form consistent with $\mathscr{F}$-policies.

Notice that we can define index ν_i^* by (16) *without assuming $\mathscr{F}$-indexability*. We do so below. Define cost function $C^{\mathscr{F}}(\cdot) : \mathbb{B} \to \mathbb{R}$ by linear interpolation on work-cost pairs $\left(g^S, f^S\right)$, for $S \in \mathscr{F}$. Namely, for $b \in \left[g^{S_i}, g^{S_{i-1}}\right]$, let

$$\begin{aligned} C^{\mathscr{F}}(b) &\triangleq f^{S_i} + \nu_i^* \left[g^{S_i} - b\right] = q f^{S_{i-1}} + (1-q) f^{S_i} \\ &= f^{q S_{i-1} + (1-q) S_i} = f^{S^q(i)}, \end{aligned} \tag{17}$$

where

$$q = \frac{b - g^{S_i}}{-\Delta g^{S_i}} \in [0, 1]; \tag{18}$$

and $S_i^q \in \Pi^{\mathrm{SR}}$ is the policy which, at state $j \in N^{\{0,1\}}$, prescribes to: (i) engage the project if $j > i$; (ii) rest it if $j < i$; and (iii) engage it with probability (w.p.) q and rest it w.p. $1-q$ if $j = i$. Hence, for such b, $C^{\mathscr{F}}(b)$ is the cost performance achieved both by randomized policy $q S_{i-1} + (1-q) S_i$, and by policy $S^q(i)$.

Definition 3.8 ($\mathscr{F}$-diminishing returns) We say the project obeys *$\mathscr{F}$-diminishing returns to work* if (i) function $C^{\mathscr{F}}(b)$ is convex, i.e. index ν_i^* is nondecreasing; and (ii) the b-work problem's optimal cost function is $C(b) = C^{\mathscr{F}}(b)$.

The next result follows immediately.

Lemma 3.9 *If the project obeys $\mathscr{F}$-diminishing returns to work, then:*

(a) *$C(b)$ is the upper envelope of Lagrangians $\mathscr{L}_b^{S_i}(\nu_i^*)$ for $i \in N^{\{0,1\}}$, i.e.*

$$C(b) = \max \left\{ f^{S_i} + \nu_i^* \left[g^{S_i} - b\right] : j \in N^{\{0,1\}} \right\}, \ b \in \mathbb{B}, \tag{19}$$

and is hence piecewise linear convex in b.

(b) *The b-work problem, for $b \in \left[g^{S_i}, g^{S_{i-1}}\right]$, is solved by S_i^q, with q as in* (18)*.*

3.8 Characterization of indexability via diminishing returns

Economic intuition suggests that the two project classes above must be closely related. The next result establishes that they are, indeed, equivalent. See Figure 1.

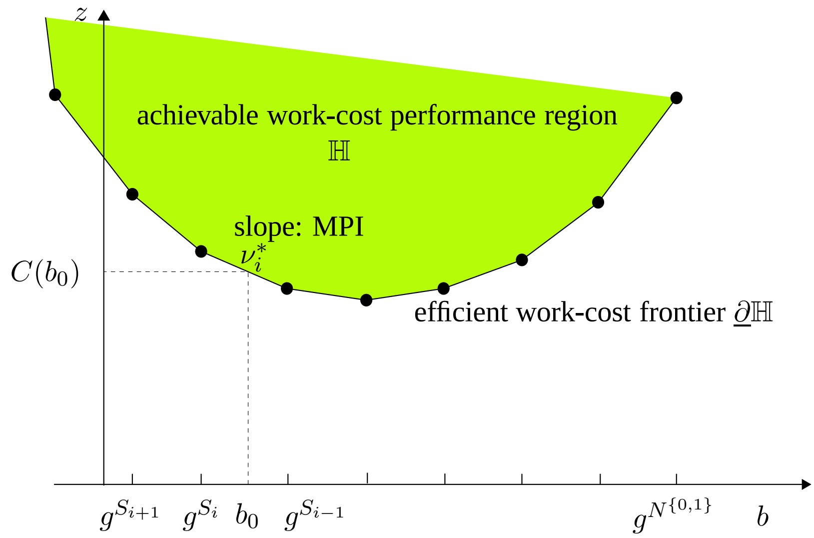

Figure 1: Indexability and diminishing returns.

Theorem 3.10 *The project is $\mathscr{F}$-indexable iff it obeys $\mathscr{F}$-diminishing returns.*

Proof. Suppose the project presents $\mathscr{F}$-diminishing returns. Then, for $\ell^0 < i < \ell^1$:

$$\begin{aligned}
\nu_i^* \le \nu \le \nu_{i+1}^* &\iff \frac{f^{S_i} - f^{S_{i-1}}}{g^{S_{i-1}} - g^{S_i}} \le \nu \le \frac{f^{S_{i+1}} - f^{S_i}}{g^{S_i} - g^{S_{i+1}}} \\
&\iff v^{S_i}(\nu) = \min\left\{v^{S_{i+k}}(\nu) : k = -1, 0, 1\right\} \\
&\iff C\left(g^{S_i}\right) + \nu g^{S_i} = \min\left\{C(g^{S_{i+k}}) + \nu g^{S_{i+k}} : k = -1, 0, 1\right\} \\
&\iff C(g^{S_i}) + \nu g^{S_i} = \min\left\{C(b) + \nu b : b \in \left[g^{S_{i+1}}, g^{S_{i-1}}\right]\right\} \\
&\iff C(g^{S_i}) + \nu g^{S_i} = \min\left\{C(b) + \nu b : b \in \mathbb{B}\right\} \\
&\iff v^{S_i}(\nu) = \min\left\{z + \nu b : (b, z) \in \mathbb{H}\right\} \\
&\iff v^*(\nu) = v^{S_i}(\nu).
\end{aligned}$$

This establishes that the project is $\mathscr{F}$-indexable.

Suppose now that the project is $\mathscr{F}$-indexable. Let $b \in \left[g^{S_{i+1}}, g^{S_i}\right]$ with $i \in N^{\{0,1\}}$. Letting q be given by (18), we can write

$$\left(g^{S_i^q}, f^{S_i^q}\right) = \left(b, f^{S_i^q}\right) = \left(b, C^{\mathscr{F}}(b)\right). \tag{20}$$

Hence, S_i^q is a feasible policy for the b-work problem. Furthermore, the duality gap (cf. (13)) associated to policy S_i^q and wage rate ν_i^* is

$$\Delta_b^{S_i^q}(\nu_i^*) = v^{S_i^q}(\nu_i^*) - v^*(\nu_i^*) = 0,$$

where the last identity follows by $\mathscr{F}$-indexability. Hence, the sufficient optimality condition in Lemma 3.3 holds, which gives, using (20), that $C(b) = C^{\mathscr{F}}(b)$. This shows that the project obeys $\mathscr{F}$-diminishing returns, and completes the proof. $\square$

4 Sufficient indexability conditions via PCLs

Suppose we aim to establish $\mathscr{F}$-indexability of an RB project model fitting in the above setting. We must then calculate index ν_i^* by (16), and show that it is nondecreasing. This a *necessary*, yet *not sufficient,* condition for $\mathscr{F}$-indexability. It remains (cf. Definition 3.6) to prove that, for each state $\ell^0 < i < \ell^1$ and wage $\nu \in [\nu_i^*, \nu_{i+1}^*]$, the S_i-active policy is optimal for ν-wage problem (10).

We present below a framework for establishing $\mathscr{F}$-indexability of an RB project, via *partial conservation laws (PCLs)*. Under the latter, it suffices to show nondecreasingness of index ν_i^*. We introduced PCLs in Niño-Mora (2001a, 2002), in a form restricted to finite-state projects, based on polyhedral methods. The approach below pursues a different course, extending the scope to countable-state projects.

4.1 Decomposition and conservation laws

The PCL framework concerns an RB project as in Section 3, whose work (g^π) and cost (f^π) measures decompose linearly in terms of *state-action occupation measures* $x_i^{a,\pi} \geq 0$. Here, $x_i^{a,\pi}$ is a measure of the time expended taking action a at i-periods under policy π. The following conditions are required to hold.

Assumption 4.1 *For any policy $\pi \in \Pi$ and state $i \in N$:*

(i) *If π takes the active action at i-periods, then $x_i^{0,\pi} = 0$.*

(ii) *If π takes the passive action at i-periods, then $x_i^{1,\pi} = 0$.*

(iii) $x_i^{0,S_i} > 0$*, and* $x_i^{1,S_{i-1}} > 0$ *(for $i > \ell^0$).*

The term "partial conservation laws (PCLs)" designates a set of properties of *performance measures* g^π and $x_i^{a,\pi}$. In what follows, we will refer to an i-period where $i \in S$, for a given $S \subseteq N^{\{0,1\}}$, as an S-*period*, and write $S^c = N^{\{0,1\}} \setminus S$.

Definition 4.2 (Partial conservation laws) We say the project's performance measures satisfy *PCLs relative to $\mathscr{F}$-policies*, or *PCL($\mathscr{F}$)* for short, if there exist coefficients $w_i^S > 0$ ($i \in N^{\{0,1\}}, S \in \mathscr{F}$) such that, for any $\pi \in \Pi$ and $S \in \mathscr{F}$:

(PCL1) $g^\pi + \sum_{i \in S} w_i^S x_i^{0,\pi} \geq g^S$, with "=" if π is passive at S^c-periods.

(PCL2) $\sum_{i \in S} w_i^S x_i^{0,\pi} \geq 0$, with "=" if π is active at S-periods.

Remark 4.3

(i) The term "partial" refers to the fact that (PCL1)–(PCL2) hold only for the family of feasible sets $S \in \mathscr{F}$. In the *strong conservation laws* (cf. Shanthikumar and Yao (1992)) and the *generalized conservation laws* (cf. Bertsimas and Niño-Mora (1996)), analogous laws hold for *all* subsets S.

(ii) Satisfaction of PCL($\mathscr{F}$) means that project control problem (2) is solved by $\mathscr{F}$-policies for a given family of linear performance objectives. Thus, $f^\pi = g^\pi + \sum_{i \in S} w_i^S x_i^{0,\pi}$ (resp. $f^\pi = \sum_{i \in S} w_i^S x_i^{0,\pi}$) is minimized by any policy prescribing to rest at S^c-periods (resp. to work at S-periods).

We will derive satisfaction of PCLs from the following requirements.

Assumption 4.4 *There exist coefficients* w_i^S, c_i^S, *for* $i \in N^{\{0,1\}}$, $S \in \mathscr{F}$, *such that:*

(i) $w_i^S > 0$, *for* $i \in N^{\{0,1\}}$ *and* $S \in \mathscr{F}$.

(ii) *Work decomposition laws: for* $\pi \in \Pi$ *and* $S \in \mathscr{F}$,

$$g^\pi + \sum_{i \in S} w_i^S x_i^{0,\pi} = g^S + \sum_{i \in S^c} w_i^S x_i^{1,\pi}.$$

(iii) *Cost decomposition laws: for* $\pi \in \Pi$ *and* $S \in \mathscr{F}$,

$$f^\pi + \sum_{i \in S^c} c_i^S x_i^{1,\pi} = f^S + \sum_{i \in S} c_i^S x_i^{0,\pi}.$$

(iv) $c_i^{S_{j-1}} - c_i^{S_j} = \dfrac{c_j^{S_j}}{w_j^{S_j}} \left[w_i^{S_{j-1}} - w_i^{S_j} \right]$, *for* $i, j \in N^{\{0,1\}}$.

We will term coefficient w_i^S (resp. c_i^S) the (i, S)-*marginal workload* (resp. (i, S)-*marginal cost*. The next result justifies such denominations.

Lemma 4.5 *For* $S \in \mathscr{F}$, $j_1 \in S$ *and* $j_2 \in S^c$, *such that* $S \setminus \{j_1\}, S \cup \{j_2\} \in \Pi^{\mathrm{SD}}$*:*

(a) $g^{S \setminus \{j_1\}} + w_{j_1}^S x_{j_1}^{0, S \setminus \{j_1\}} = g^S = g^{S \cup \{j_2\}} - w_{j_2}^S x_{j_2}^{1, S \cup \{j_2\}}$.

(b) $f^{S \setminus \{j_1\}} - c_{j_1}^S x_{j_1}^{0, S \setminus \{j_1\}} = f^S = f^{S \cup \{j_2\}} + c_{j_2}^S x_{j_2}^{1, S \cup \{j_2\}}$.

Proof. The left (resp. right) identities for g^S and for f^S follow by letting $\pi = S \setminus \{j_1\}$ (resp. $\pi = S \cup \{j_2\}$) in Assumption 4.4(ii, iii), respectively. □

We will refer in what follows to the *aggregate* marginal work and cost measures

$$\begin{aligned} W^{S,0,\pi} &\triangleq \sum_{i \in S} w_i^S x_i^{0,\pi}, \quad W^{S,1,\pi} \triangleq \sum_{i \in S^c} w_i^S x_i^{1,\pi}, \\ C^{S,0,\pi} &\triangleq \sum_{i \in S} c_i^S x_i^{0,\pi}, \quad C^{S,1,\pi} \triangleq \sum_{i \in S^c} c_i^S x_i^{1,\pi}. \end{aligned} \tag{21}$$

Remark 4.6

(i) Lemma 4.5(a) shows that Assumption 4.1(iii) is needed for consistency with Assumption 3.1(iv), since

$$\Delta g^{S_i} = -w_i^{S_i} x_i^{1,S_{i-1}} = -w_i^{S_{i-1}} x_i^{0,S_i} < 0, \quad i \in N^{\{0,1\}}.$$

(ii) By Assumption 4.4(ii), (PCL1) in Definition 4.2 can be reformulated as

(PCL1) $W^{S,1,\pi} \geq 0$, with "=" if π is passive at S^c-periods.

The next result follows immediately from the above.

Theorem 4.7 *Under Assumptions* 4.1, 4.4(i, ii), *PCL(*$\mathscr{F}$*) hold.*

4.2 PCL-indexability

Let us introduce coefficients

$$\nu_i^S \triangleq \frac{c_i^S}{w_i^S}, \quad i \in N^{\{0,1\}}, S \in \mathscr{F}, \tag{22}$$

from which we define *index* ν_i^* by

$$\nu_i^* \triangleq \nu_i^{S_i}, \quad i \in N^{\{0,1\}}. \tag{23}$$

We will term ν_i^S the (i, S)-*marginal productivity rate*. Thus, index ν_i^* is the (i, S_i)-marginal, or (i, S_{i-1})-marginal, productivity rate.

The next result ensures that if the project is $\mathscr{F}$-indexable, then index ν_i^* is indeed its MPI. See Lemma 3.7(b).

Lemma 4.8

$$\nu_i^* = \nu_i^{S_i} = \nu_i^{S_{i-1}} = \frac{\Delta f^{S_i}}{-\Delta g^{S_i}}, \quad i \in N^{\{0,1\}}.$$

Proof. The result follows from Lemma 4.5, using Assumption 4.1(iii). □

We introduce next a tractable project class based on the above.

Definition 4.9 (PCL-indexability) We say the project is *PCL($\mathscr{F}$)-indexable* if:

(i) Assumptions 4.1 and 4.4 hold, and hence performance measures satisfy PCL($\mathscr{F}$).

(ii) Index ν_i^* is nondecreasing.

The main result is that a PCL($\mathscr{F}$)-indexable project is, indeed, $\mathscr{F}$-indexable.

Theorem 4.10 (PCL indexability condition) *If the project is PCL($\mathscr{F}$)-indexable then it is $\mathscr{F}$-indexable, and ν_i^* is its MPI.*

To prove Theorem 4.10 we need several preliminary results. We start by relating marginal productivity rates to index values, which yields an index recursion.

Lemma 4.11 *For $i, j \in N^{\{0,1\}}$:*

(a) $\nu_j^{S_i} - \nu_i^* = \dfrac{w_j^{S_{i-1}}}{w_j^{S_i}} \left[\nu_j^{S_{i-1}} - \nu_i^*\right]$.

(b) $\nu_{i+1}^* = \nu_i^* + \dfrac{w_{i+1}^{S_{i-1}}}{w_{i+1}^{S_i}} \left[\nu_{i+1}^{S_{i-1}} - \nu_i^*\right]$.

Proof. (a) Using Assumption 4.4(iv) we obtain that, for $i, j \in N^{\{0,1\}}$,

$$\nu_j^{S_i} - \nu_i^* = \frac{c_j^{S_i}}{w_j^{S_i}} - \nu_i^* = \frac{c_j^{S_{i-1}} - \nu_i^* \left[w_j^{S_{i-1}} - w_j^{S_i}\right]}{w_j^{S_i}} - \nu_i^* = \frac{w_j^{S_{i-1}}}{w_j^{S_i}} \left[\nu_j^{S_{i-1}} - \nu_i^*\right].$$

(b) This part follows by letting $j = i + 1$ in part (a). □

The next result represents marginal costs as finite linear combinations of marginal workloads, involving coefficients ν_i^* and $\Delta\nu_i^* = \nu_i^* - \nu_{i-1}^*$.

Lemma 4.12 *For $i, j \in N^{\{0,1\}}$:*

(a) $c_j^{S_{i-1}} = \nu_i^* w_j^{S_{i-1}} + \sum_{k=i+1}^{j} w_j^{S_{k-1}} \Delta\nu_k^*, \quad i \le j.$

(b) $c_j^{S_i} = \nu_i^* w_j^{S_i} - \sum_{k=j}^{i-1} w_j^{S_k} \Delta\nu_{k+1}^*, \quad i \ge j.$

Proof. (a) For $i < j$, Assumption 4.4(iv) and *summation by parts* gives

$$
\begin{aligned}
c_j^{S_j} &= c_j^{S_{i-1}} + \sum_{k=i}^{j} \left[c_j^{S_k} - c_j^{S_{k-1}} \right] = c_j^{S_{i-1}} + \sum_{k=i}^{j} \nu_k^* \left[w_j^{S_k} - w_j^{S_{k-1}} \right] \\
&= c_j^{S_{i-1}} + \left[\nu_j^* w_j^{S_j} - \nu_i^* w_j^{S_{i-1}} \right] - \sum_{k=i+1}^{j} w_j^{S_{k-1}} \Delta\nu_k^*,
\end{aligned}
$$

whence the result follows (the case $i = j$ is trivial).

(b) For $j < i$ (again, the case $i = j$ is trivial), we have

$$
\begin{aligned}
c_j^{S_i} &= c_j^{S_{j-1}} + \sum_{k=j}^{i} \left[c_j^{S_k} - c_j^{S_{k-1}} \right] = c_j^{S_{j-1}} + \sum_{k=j}^{i} \nu_k^* \left[w_j^{S_k} - w_j^{S_{k-1}} \right] \\
&= c_j^{S_{j-1}} + \left[\nu_i^* w_j^{S_i} - \nu_j^* w_j^{S_{j-1}} \right] - \sum_{k=j}^{i-1} w_j^{S_k} \Delta\nu_{k+1}^*,
\end{aligned}
$$

whence the result follows. This completes the proof. □

The next result is an analog of Lemma 4.12 in terms of aggregate measures.

Lemma 4.13 *Suppose the project is PCL($\mathscr{F}$)-indexable. Then, for $i \in N^{\{0,1\}}$:*

(a) $C^{S_{i-1},0,\pi} = \nu_i^* W^{S_{i-1},0,\pi} + \sum_{k \in S_i} W^{S_{k-1},0,\pi} \Delta\nu_k^*.$

(b) $C^{S_i,1,\pi} = \nu_i^* W^{S_i,1,\pi} - \sum_{k \in S_{i-1}^c} W^{S_k,1,\pi} \Delta\nu_{k+1}^*.$

Proof. (a) Using Lemma 4.12(a), we obtain

$$
\begin{aligned}
C^{S_{i-1},0,\pi} &\triangleq \sum_{j \in S_{i-1}} c_j^{S_{i-1}} x_j^{0,\pi} = \sum_{j \in S_{i-1}} \left[\nu_i^* w_j^{S_{i-1}} + \sum_{k=i+1}^{j} w_j^{S_{k-1}} \Delta\nu_k^* \right] x_j^{0,\pi} \\
&= \nu_i^* \sum_{j \in S_{i-1}} w_j^{S_{i-1}} x_j^{0,\pi} + \sum_{k \in S_i} \sum_{j \in S_{k-1}} w_j^{S_{k-1}} x_j^{0,\pi} \Delta\nu_k^* \\
&= \nu_i^* W^{S_{i-1},0,\pi} + \sum_{k \in S_i} W^{S_{k-1},0,\pi} \Delta\nu_k^*,
\end{aligned}
$$

where the interchange on the order of summation is justified, in the countable state case, by the nonnegativity of the terms involved.

(b) Using Lemma 4.12(b) and arguing along the same lines as in part (a) gives

$$\begin{aligned} C^{S_i,1,\pi} &\triangleq \sum_{j\in S_i^c} c_j^{S_i} x_j^{1,\pi} = \sum_{j\in S_i^c}\left[\nu_i^* w_j^{S_i} - \sum_{k=j}^{i-1} w_j^{S_k}\Delta\nu_{k+1}^*\right] x_j^{1,\pi} \\ &= \nu_i^* \sum_{j\in S_i^c} w_j^{S_i} x_j^{1,\pi} - \sum_{k\in S_{i-1}^c}\sum_{j\in S_k^c} w_j^{S_k} x_j^{1,\pi}\Delta\nu_{k+1}^* \\ &= \nu_i^* W^{S_i,1,\pi} - \sum_{k\in S_{i-1}^c} W^{S_k,1,\pi}\Delta\nu_{k+1}^*, \end{aligned}$$

which completes the proof. □

4.3 Workload reformulation and indexability proof

The next result is the cornerstone of our indexability proof. It formulates the difference between ν-wage problem (10)'s objective under an arbitray policy and under an $\mathscr{F}$-policy as a linear combination of aggregate work measures.

Lemma 4.14 (Workload reformulation) *Suppose the project is PCL($\mathscr{F}$)-indexable. Then, for any state $\ell^0 < i < \ell^1$, wage $\nu \in \mathbb{R}$ and policy $\pi \in \Pi$, the objective of ν-wage problem* (10) *can be represented as*

$$\begin{aligned} v^\pi(\nu) = v^{S_i}(\nu) + W^{S_i,1,\pi}\left[\nu - \nu_i^*\right] + W^{S_i,0,\pi}\left[\nu_{i+1}^* - \nu\right] \\ + \sum_{k\in S_{i+1}} W^{S_{k-1},0,\pi}\Delta\nu_k^* + \sum_{k\in S_{i-1}^c} W^{S_k,1,\pi}\Delta\nu_{k+1}^*. \end{aligned}$$

Proof. Using in turn Assumption 4.4(iv) and Lemma 4.13 gives

$$\begin{aligned} f^\pi = f^{S_i} + C^{S_i,0,\pi} - C^{S_i,1,\pi} = f^{S_i} + \nu_{i+1}^* W^{S_i,0,\pi} - \nu_i^* W^{S_i,1,\pi} \\ + \sum_{k\in S_{i+1}} W^{S_{k-1},0,\pi}\Delta\nu_k^* + \sum_{k\in S_{i-1}^c} W^{S_k,1,\pi}\Delta\nu_{k+1}^*. \end{aligned}$$

On the other hand, by Assumption 4.4(iv) we have

$$g^\pi = g^{S_i} + W^{S_i,1,\pi} - W^{S_i,0,\pi}.$$

The required expression for $v^\pi(\nu) = f^\pi + \nu g^\pi$ follows directly by substitution of the above formulae for f^π and g^π. □

We can now prove the main result of this section.

Proof of Theorem 4.10. Let $\ell^0 < i < \ell^1$ and $\pi \in \Pi$. It follows immediately from Lemma 4.14 and Definition 4.9 that, for $\nu \in [\nu_i^*, \nu_{i+1}^*]$,

$$v^\pi(\nu) \geq v^{S_i}(\nu).$$

This (cf. Definition 3.6) completes the proof. □

The next result characterizes the MPI as a locally optimal marginal productivity rate, in a dual min/max relation.

Theorem 4.15 *Suppose the project is PCL($\mathscr{F}$)-indexable. Then,*

$$\min_{j \in S_{i-1}} \nu_j^{S_{i-1}} = \nu_i^{S_{i-1}} = \nu_i^* = \nu_i^{S_i} = \max_{j \in S_i^c} \nu_j^{S_i}, \quad i \in N^{\{0,1\}}.$$

Proof. By Lemma 4.12(a), we have that, for $i, j \in N^{\{0,1\}}$ with $i \leq j$,

$$\nu_j^{S_{i-1}} = \nu_i^* + \sum_{k=i+1}^{j} \frac{w_j^{S_{k-1}}}{w_j^{S_{i-1}}} \Delta\nu_k^*,$$

whence the "min" identity readily follows.

By Lemma 4.12(b), we have that, for $i, j \in N^{\{0,1\}}$ with $i \geq j$,

$$\nu_j^{S_i} = \nu_i^* - \sum_{k=j}^{i-1} \frac{w_j^{S_k}}{w_j^{S_i}} \Delta\nu_{k+1}^*,$$

whence the "max" identity follows. □

Remark 4.16 In Theorem 4.15:

(i) The "max" identity characterizes MPI ν_i^* as the maximum (j, S_i)-marginal productivity rate over states $j \in S_i^c$.

(ii) The "min" identity characterizes ν_i^* as the minimum (j, S_{i-1})-marginal productivity rate over states $j \in S_{i-1}$.

4.4 MPI characterization under V-shaped marginal workloads

We have found that, in a variety of applications, marginal workloads are *V-shaped*, in the following sense, which implies an alternative characterization of the MPI.

Assumption 4.17 $w_i^{S_k}$ *is V-shaped as k varies, being minimized at $k = i$, i.e.*

$$w_i^{S_{\ell 0}} \geq w_i^{S_{\ell 0+1}} \geq \cdots \geq w_i^{S_i} \leq w_i^{S_{i+1}} \leq \cdots, \quad i \in N^{\{0,1\}}.$$

We need the following preliminary result.

Lemma 4.18 *Suppose the project is PCL($\mathscr{F}$)-indexable and Assumption* 4.17 *holds. Then, for fixed $i \in N^{\{0,1\}}$, $\nu_i^{S_j}$ is nondecreasing in threshold state j, i.e.*

$$\nu_i^{S_j} \geq \nu_i^{S_{j-1}}, \quad j \in N^{\{0,1\}}, \quad \textit{with "=" if } j = i.$$

Proof. The equality part follows by Lemma 4.8(b). As for the inequality, reformulating the identity in Lemma 4.11(a) gives that, for $i, j \in N^{\{0,1\}}$,

$$\nu_i^{S_j} - \nu_i^{S_{j-1}} = \left(\frac{w_i^{S_{j-1}}}{w_i^{S_j}} - 1 \right) \left(\nu_i^{S_j} - \nu_j^* \right). \tag{24}$$

Consider the case $j \geq i + 1$. Then, by Assumption 4.17 and the "max" identity in Theorem 4.15, the two factors in the right-hand side (RHS) of (24) are nonpositive, and hence their product is nonnegative.

Consider now the case $j \leq i$. Then, drawing on index monotonicity and the "min" identity in Theorem 4.15 gives

$$\nu_j^* \leq \nu_{j+1}^* \leq \nu_i^{S_j},$$

so the second factor in the RHS of (24) is nonnegative; Assumption 4.17 shows that the first factor is nonnegative. Hence, so is their product, completing the proof. □

Theorem 4.19 (Alternative MPI characterization) *Suppose the project is PCL($\mathscr{F}$)-indexable and Assumption* 4.17 *holds. Then, the MPI has the characterization*

$$\min_{S \in \mathscr{F} \colon S^c \ni i} \nu_i^S = \nu_i^* = \max_{S \in \mathscr{F} \colon S \ni i} \nu_i^S, \quad i \in N^{\{0,1\}}. \tag{25}$$

Proof. By Lemma 4.18 we can write, for $j_1 \in S_{i-1}$ and $j_2 \in S_{i-1}^c$,

$$\nu_i^{S_{j_1}} \geq \nu_i^{S_i} = \nu_i^* = \nu_i^{S_{i-1}} \geq \nu_i^{S_{j_2}},$$

which implies

$$\min_{j \in N \colon S_j^c \ni i} \nu_i^{S_j} = \nu_i^* = \max_{j \in N \colon S_j \ni i} \nu_i^{S_j}, \quad i \in N^{\{0,1\}}.$$

The latter identities are readily reformulated into the required result. □

Remark 4.20 In Theorem 4.19:

(i) The "max" identity characterizes MPI ν_i^* as the maximum (i, S)-marginal productivity rate over $\mathscr{F}$-policies S that are active at i-periods. This extends to RBs Gittins (1979)'s index characterization for non-restless bandits.

(ii) The "min" identity characterizes MPI ν_i^* as the minimum (i, S)-marginal productivity rate over $\mathscr{F}$-policies S that are passive at i-periods.

5 PCL-indexability analysis of a semi-Markov project

This section specializes the PCL-indexability framework to a semi-Markov project, under three relevant performance criteria, including a new one. The main result is that to establish PCL($\mathscr{F}$)-indexability it suffices to check the following conditions.

Assumption 5.1

(i) Positive marginal workloads: $w_i^S > 0, \quad i \in N^{\{0,1\}}, S \in \mathscr{F}$.

(ii) Nondecreasing index: $\nu_i^* \leq \nu_{i+1}^*, \quad \ell^0 < i < \ell^1$.

Consider a semi-Markov RB project satisfying Assumption 3.1. When $X_n = i$ and $a_n = a$ is chosen, the joint distribution of the length $t_{n+1} - t_n$ of the ensuing (i, a)-period and state X_{n+1} is given by the *transition distribution*

$$Q_{ij}^a(t) \triangleq \mathbb{P}\left\{t_{n+1} - t_n \leq t, X_{n+1} = j \mid X_n = i, a_n = a\right\},$$

with associated LST

$$\beta_{ij}^{a,\alpha} \triangleq \mathbb{E}\left[1_{\{X_{n+1}=j\}} e^{-\alpha(t_{n+1}-t_n)} \mid X_n = i, a_n = a\right] = \int_0^\infty e^{-\alpha t}\, dQ_{ij}^a(t)$$

and *one-period transition probabilities*

$$p_{ij}^a \triangleq \mathbb{P}\{X_{n+1} = j \mid X_n = i, a_n = a\} = \lim_{t\to+\infty} Q_{ij}^a(t) = \lim_{\alpha\searrow 0} \beta_{ij}^{a,\alpha}.$$

From $Q_{ij}^a(t)$ we obtain the distribution of the length of an (i,a)-period,

$$F_i^a(t) \triangleq \mathbb{P}\{t_{n+1} - t_n \le t \,|\, X_n = i, a_n = a\} = \sum_{j\in N} Q_{ij}^a(t),$$

having LST

$$\beta_i^{a,\alpha} \triangleq \mathbb{E}\left[e^{-\alpha(t_{n+1}-t_n)} \mid X_n = i, a_n = a\right] = \sum_{j\in N} \beta_{ij}^{a,\alpha}$$

satisfying

$$\sup\{\beta_i^{a,\alpha} : i \in N, a \in \{0,1\}\} < 1,$$

and finite mean

$$m_i^a \triangleq \mathbb{E}[t_{n+1} - t_n \mid X_n = i, a_n = a] = \int_0^\infty t\, dF_i^a(t),$$

with

$$\inf\{m_i^a : i \in N, a \in \{0,1\}\} > 0.$$

The evolution of process $X(t)$ *within* an (i,a)-period is characterized by

$$\widehat{p}_{ij}^a(t) \triangleq \mathbb{P}\{X(t_n + t) = j \,|\, X_n = i, a_n = a, t_{n+1} - t_n > t\},$$

the probability that state j is occupied t time units after a decision epoch, given that the next epoch has not occurred yet.

The project accrues holding costs at rate h_j^a per unit time while it occupies state j and action a prevails. Holding cost rates are assumed to be *bounded below*:

$$\inf\left\{h_j^a : j \in N, a \in \{0,1\}\right\} > -\infty; \tag{26}$$

5.1 Discounted criterion

We start with the *expected total discounted (ETD)* criterion, with factor $\alpha > 0$. Letting $\mathbb{E}_i^\pi[\cdot]$ be expectation under π starting at i, the ETD value of costs accrued is

$$f_i^\pi \triangleq \mathbb{E}_i^\pi\left[\int_0^\infty h_{X(t)}^{a(t)} e^{-\alpha t}\, dt\right],$$

and the ETD amount of work expended is

$$g_i^\pi \triangleq \mathbb{E}_i^\pi\left[\int_0^\infty a(t) e^{-\alpha t}\, dt\right].$$

We consider the initial state to be drawn from a probability mass function $\mathbf{p} = (p_i)_{i \in N}$. Denoting by $\mathbb{E}_{\mathbf{p}}^{\pi}[\cdot]$ the corresponding expectation, we will use the *ETD cost measure* and the *ETD work measure* given, respectively, by

$$f^{\pi} \triangleq \mathbb{E}_{\mathbf{p}}^{\pi}\left[\int_0^{\infty} h_{X(t)}^{a(t)} e^{-\alpha t}\, dt\right] = \sum_{i \in N} p_i f_i^{\pi}.$$

and

$$g^{\pi} \triangleq \mathbb{E}_{\mathbf{p}}^{\pi}\left[\int_0^{\infty} a(t) e^{-\alpha t}\, dt\right] = \sum_{i \in N} p_i g_i^{\pi}.$$

We now reformulate the model into *discrete time*, as in Puterman (1994, Ch. 11), using coefficients β_{ij}^a and β_i^a as defined above (where now factor α is implicit), and

$$\begin{aligned} m_i^a &\triangleq \mathbb{E}\left[\int_0^{t_{n+1}-t_n} e^{-\alpha t}\, dt \mid X_n = i, a_n = a\right] = \frac{1-\beta_i^a}{\alpha}, \\ c_i^a &\triangleq \mathbb{E}\left[\int_0^{t_{n+1}-t_n} h_{X(t)}^a e^{-\alpha t}\, dt \mid X_n = i, a_n = a\right]. \end{aligned}$$

Notice that c_i^a (resp. m_i^a) is the ETD cost (resp. time) accrued over an (i,a)-period; β_i^a is the state- and action-dependent discrete-time discount factor; and β_{ij}^a is the discounted one-period transition probability from i to j under a.

We can use the above coefficients to characterize measures g_i^S and f_i^S, for given $S \in \mathscr{F}$, as the unique solutions to the linear equation systems given next.

Lemma 5.2 (Evaluation equations) *For every $S \in \mathscr{F}$:*

(a) $$\begin{cases} g_i^S = m_i^1 + \sum_{j \in N} \beta_{ij}^1 g_j^S & \text{if } i \in S \\ g_i^S = \sum_{j \in N} \beta_{ij}^0 g_j^S & \text{if } i \in N \setminus S. \end{cases}$$

(b) $$\begin{cases} f_i^S = c_i^1 + \sum_{j \in N} \beta_{ij}^1 f_j^S & \text{if } i \in S \\ f_i^S = c_i^0 + \sum_{j \in N} \beta_{ij}^0 f_j^S & \text{if } i \in N \setminus S. \end{cases}$$

We will use as *ETD state-action occupation measure* the ETD number of (j,a)-periods spanned under policy π, given by

$$x_j^{a,\pi} \triangleq \mathbb{E}_{\mathbf{p}}^{\pi}\left[\sum_{n=0}^{\infty} 1\{X_n = j, a_n = a\}\, e^{-\alpha t_n}\right] = \sum_{i \in N} p_i x_{ij}^{a,\pi},$$

where

$$x_{ij}^{a,\pi} \triangleq \mathbb{E}_i^{\pi}\left[\sum_{n=0}^{\infty} 1\{X_n = j, a_n = a\}\, e^{-\alpha t_n}\right].$$

We will write $\mathbf{x}^{a,\pi} = (x_j^{a,\pi})_{j \in N}$, $\mathbf{g}^{\pi} = (g_i^{\pi})_{i \in N}$, $\mathbf{f}^{\pi} = (f_i^{\pi})_{i \in N}$, $\mathbf{m}^a = (m_j^a)_{j \in N}$, $\mathbf{c}^a = (c_j^a)_{j \in N}$ and $\mathbf{B}^a = (\beta_{ij}^a)_{i,j \in N}$; and for $S, S' \subseteq N$, $\mathbf{B}_{SS'}^a =$

$(\beta^a_{ij})_{i \in S, j \in S'}$, $\mathbf{f}^\pi_S = (f^\pi_i)_{i \in S}$. We can thus represent work and cost measures g^π and f^π as *linear* performance measures, by

$$
\begin{aligned}
g^\pi &= \mathbf{x}^{1,\pi}\mathbf{m}^1 = \sum_{j \in N} m^1_j x^{1,\pi}_j \\
f^\pi &= \mathbf{x}^{0,\pi}\mathbf{c}^0 + \mathbf{x}^{1,\pi}\mathbf{c}^1 = \sum_{a \in \{0,1\}} \sum_{j \in N} c^a_j x^{a,\pi}_j.
\end{aligned}
\tag{27}
$$

It is well known in MDP theory that measures $x^{a,\pi}_j$ satisfy the set of linear equations given next, formulating *detailed flow balance* identities. Let $\mathbf{I} = (\delta_{ij})_{i,j \in N}$, where δ_{ij} is Kronecker's delta, be the identity matrix indexed by N.

Lemma 5.3 (ETD detailed flow balance) *For any policy $\pi \in \Pi$,*

$$\mathbf{x}^{0,\pi}\left(\mathbf{I} - \mathbf{B}^0\right) + \mathbf{x}^{1,\pi}\left(\mathbf{I} - \mathbf{B}^1\right) = \mathbf{p}.$$

Denote by $\langle a, S \rangle$ the policy taking action a in the initial period, and following the S-active policy afterwards. We define the *ETD (i, S)-marginal workload* by

$$
\begin{aligned}
w^S_i &\triangleq g^{\langle 1,S \rangle}_i - g^{\langle 0,S \rangle}_i = \begin{cases} g^S_i - g^{\langle 0,S \rangle}_i & \text{if } i \in S \\ g^{\langle 1,S \rangle}_i - g^S_i & \text{if } i \in S^c, \end{cases} \\
&= m^1_i + \sum_{j \in N} (\beta^1_{ij} - \beta^0_{ij}) g^S_j,
\end{aligned}
\tag{28}
$$

the marginal increase in ETD work expended resulting from using policy $\langle 1, S \rangle$ instead of $\langle 0, S \rangle$, starting at i. We further define the *ETD (i, S)-marginal cost* by

$$
\begin{aligned}
c^S_i &\triangleq f^{\langle 0,S \rangle}_i - f^{\langle 1,S \rangle}_i = \begin{cases} f^{\langle 0,S \rangle}_i - f^S_i & \text{if } i \in S \\ f^S_i - f^{\langle 1,S \rangle}_i & \text{if } i \in S^c, \end{cases} \\
&= c^0_i - c^1_i + \sum_{j \in N} \left(\beta^0_{ij} - \beta^1_{ij}\right) f^S_j,
\end{aligned}
\tag{29}
$$

the marginal cost decrease resulting from using policy $\langle 0, S \rangle$ instead of $\langle 1, S \rangle$.

As in (22)–(23), and under Assumption 5.1(i), we define the *ETD (i, S)-marginal productivity rate* and the *ETD index* by $\nu^S_i = c^S_i / w^S_i$ and $\nu^*_i = \nu^{S_i}_i$, respectively.

We will need the following result.

Lemma 5.4 *For every $S \in \mathscr{F}$:*

(a) $\mathbf{w}^S_S = (\mathbf{I}_{SN} - \mathbf{B}^0_{SN})\mathbf{g}^S$ *and* $\mathbf{w}^S_{S^c} = \mathbf{m}^1_{S^c} - (\mathbf{I}_{S^cN} - \mathbf{B}^1_{S^cN})\mathbf{g}^S$.

(b) $\mathbf{c}^S_S = \mathbf{c}^0_S - (\mathbf{I}_{SN} - \mathbf{B}^0_{SN})\mathbf{f}^S$ *and* $\mathbf{c}^S_{S^c} = (\mathbf{I}_{S^cN} - \mathbf{B}^1_{S^cN})\mathbf{f}^S - \mathbf{c}^1_{S^c}$.

Proof. It follows from Lemma 5.2 and (28)–(29). □

Define ETD aggregate measures $W^{S,0,\pi}$, $W^{S,1,\pi}$, $C^{S,0,\pi}$ and $C^{S,1,\pi}$ by (21). We next establish satisfaction of work and cost decomposition laws.

Lemma 5.5 *Under any policy $\pi \in \Pi$:*

(a) *Workload decomposition laws:*

$$g^{\pi} + W^{S,0,\pi} = g^{S} + W^{S,1,\pi}, \quad S \in \mathscr{F}.$$

(b) *Cost decomposition laws:*

$$f^{\pi} + C^{S,1,\pi} = f^{S} + C^{S,0,\pi}, \quad S \in \mathscr{F}.$$

Proof. (a) Using Lemma 5.3, Lemma 5.2(a), Lemma 5.4(a), and $x_{\ell^0}^{1,\pi} = 0$, gives

$$\begin{aligned}
0 &= \left[\mathbf{x}^{0,\pi}(\mathbf{I} - \mathbf{B}^0) + \mathbf{x}^{1,\pi}\,(\mathbf{I} - \mathbf{B}^1) - \mathbf{p}\right]\mathbf{g}^S \\
&= \mathbf{x}^{0,\pi}(\mathbf{I} - \mathbf{B}^0)\mathbf{g}^S + \mathbf{x}^{1,\pi}\left[(\mathbf{I} - \mathbf{B}^1)\mathbf{g}^S - \mathbf{m}^1\right] - \mathbf{p}\mathbf{g}^S + \mathbf{x}^{1,\pi}\mathbf{m}^1 \\
&= \mathbf{x}_S^{0,\pi}\mathbf{w}_S^S - \mathbf{x}_{S^c}^{1,\pi}\mathbf{w}_{S^c}^S - g^S + g^{\pi}.
\end{aligned}$$

(b) Using Lemma 5.2(b), Lemma 5.3, Lemma 5.4(b), and $x_{\ell^0}^{1,\pi} = 0$, gives

$$\begin{aligned}
0 &= \left[\mathbf{x}^{0,\pi}(\mathbf{I} - \mathbf{B}^0) + \mathbf{x}^{1,\pi}\,(\mathbf{I} - \mathbf{B}^1) - \mathbf{p}\right]\mathbf{f}^S \\
&= \mathbf{x}^{0,\pi}\left[(\mathbf{I} - \mathbf{B}^0)\mathbf{f}^S - \mathbf{c}^0\right] + \mathbf{x}^{1,\pi}\left[(\mathbf{I} - \mathbf{B}^1)\mathbf{f}^S - \mathbf{c}^1\right] \\
&\qquad - \mathbf{p}\mathbf{f}^S + \mathbf{x}^{0,\pi}\mathbf{c}^0 + \mathbf{x}^{1,\pi}\mathbf{c}^1 \\
&= -\mathbf{x}_S^{0,\pi}\mathbf{c}_S^S + \mathbf{x}_{S^c}^{1,\pi}\mathbf{c}_{S^c}^S - f^S + f^{\pi}.
\end{aligned}$$

□

The next result establishes Assumption 4.4(iv).

Lemma 5.6 *Suppose Assumption 5.1(i) holds. Then, for every $j \in N^{\{0,1\}}$:*

(a) $f_i^{S_j} - f_i^{S_{j-1}} = \nu_j^* \left[g_i^{S_{j-1}} - g_i^{S_j}\right], \quad i \in N.$

(b) $c_i^{S_{j-1}} - c_i^{S_j} = \nu_j^* \left[w_i^{S_{j-1}} - w_i^{S_j}\right], \quad i \in N^{\{0,1\}}.$

Proof. (a) We have, taking $\pi = S_{j-1}$ and $S = S_j$ in Lemma 5.5(a, b):

$$g_i^{S_{j-1}} = g_i^{S_j} + w_j^{S_j} x_{ij}^{1,S_{j-1}}, \quad i \in N,$$

$$f_i^{S_{j-1}} + c_j^{S_j} x_{ij}^{1,S_{j-1}} = f_i^{S_j}, \quad i \in N.$$

Hence,

$$f_i^{S_j} - f_i^{S_{j-1}} = c_j^{S_j} x_{ij}^{1,S_{j-1}} = \frac{c_j^{S_j}}{w_j^{S_j}} w_j^{S_j} x_{ij}^{1,S_{j-1}} = \frac{c_j^{S_j}}{w_j^{S_j}} \left[g_i^{S_{j-1}} - g_i^{S_j}\right].$$

(b) Using part (a) we have that, for $i \in N^{\{0,1\}}$:

$$\begin{aligned}
c_i^{S_{j-1}} - c_i^{S_j} &= \sum_{j \in N} \left(\beta_{ij}^1 - \beta_{ij}^0\right)\left(f_i^{S_j} - f_i^{S_{j-1}}\right) \\
&= \nu_j^* \sum_{j \in N} \left(\beta_{ij}^1 - \beta_{ij}^0\right)\left(g_i^{S_{j-1}} - g_i^{S_j}\right) = \nu_j^* \left(w_i^{S_{j-1}} - w_i^{S_j}\right).
\end{aligned}$$

This completes the proof. □

We can now give the main result for the ETD criterion.

Theorem 5.7 *Under Assumption 5.1, the project is PCL($\mathscr{F}$)-indexable relative to the ETD criterion, and ν_i^* is its discounted MPI.*

Proof. Assumption 4.1 holds. Assumption 5.1(i) and Lemmas 5.5–5.6 ensure satisfaction of Assumption 4.4. Hence, part (i) of Definition 4.9 holds, as does its part (ii) by Assumption 5.1(ii). The proof is completed by invoking Theorem 4.10. □

5.2 Long-run average criterion

We turn now to the *long-run average (LRA)* criterion, which we address by drawing on the above results, using a *vanishing discount approach*. For clarity, we include factor α in the notation above. The following *ergodicity* conditions are required.

Assumption 5.8

(i) For every state $i \in N$, the S_i-active policy induces on embedded Markov chain X_n the single positive recurrent class $S_i \cup \{i\}$.

(ii) Policies in Π are *stable*, in that there exist finite measures $x_j^{a,\pi}$, f^π and g^π, independent of the initial state i, given for any policy $\pi \in \Pi$ by

$$x_j^{a,\pi} \triangleq \lim_{\alpha \searrow 0} \alpha x_j^{a,\pi,\alpha} = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}_i^\pi \left[\sum_{k=0}^{n} 1\{X_k = j, a_k = a\} \right],$$

$$f^\pi \triangleq \lim_{\alpha \searrow 0} \alpha f_i^{\pi,\alpha} = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_i^\pi \left[\int_0^t h_{X(s)}^{a(s)} \, ds \right],$$

$$g^\pi \triangleq \lim_{\alpha \searrow 0} \alpha g_i^{\pi,\alpha} = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_i^\pi \left[\int_0^t a(s) \, ds \right].$$

(iii) For every $i \in N^{\{0,1\}}$ and $S \in \mathscr{F}$, there exist finite w_i^S and c_i^S given by

$$w_i^S \triangleq \lim_{\alpha \searrow 0} w_i^{S,\alpha} = \lim_{t \to +\infty} \left\{ \mathbb{E}_i^{\langle 1,S \rangle} \left[\int_0^t a(s) \, ds \right] - \mathbb{E}_i^{\langle 0,S \rangle} \left[\int_0^t a(s) \, ds \right] \right\},$$

$$c_i^S \triangleq \lim_{\alpha \searrow 0} c_i^{S,\alpha} = \lim_{t \to +\infty} \left\{ \mathbb{E}_i^{\langle 0,S \rangle} \left[\int_0^t h_{X(s)}^{a(s)} \, ds \right] - \mathbb{E}_i^{\langle 1,S \rangle} \left[\int_0^t h_{X(s)}^{a(s)} \, dt \right] \right\}.$$

As the above notation suggests, we will use f^π, g^π and $x_j^{a,\pi}$ as the LRA cost, work, and state-action occupation measures, respectively.

We define the *LRA* (i, S)*-marginal workload* and the *LRA* (i, S)*-marginal cost* as the limits w_i^S and c_i^S in Assumption 5.8(iii), respectively. Thus, w_i^S (resp. c_i^S) is the expected long-run cumulative marginal increase (resp. decrease) in work expended (resp. in holding cost accrued) resulting from using policy $\langle 1, S \rangle$ instead of $\langle 0, S \rangle$ (resp. $\langle 0, S \rangle$ instead of $\langle 1, S \rangle$), starting at i.

Provided Assumption 5.1(i) holds, we define the *LRA* (i, S)*-marginal productivity rate* and the *LRA index* by $\nu_i^S = c_i^S / w_i^S$ and $\nu_i^* = \nu_i^{S_i}$, respectively.

The main result for the LRA criterion is as follows.

Theorem 5.9 *Under Assumption 5.1, the project is PCL($\mathscr{F}$)-indexable relative to the LRA criterion, and ν_i^* is its LRA MPI.*

Proof. It is readily verified that Assumptions 4.1–4.4 hold. As an example, Lemmas 5.5–5.6 immediately yield LRA counterparts, by taking appropriate limits as α vanishes. Hence, Definition 4.9 holds and, by Theorem 4.10, the result follows. $\square$

5.3 Mixed LRA/bias criterion

In several models, such as that in Section 2, the LRA MPI discussed above *does not exist*. Such is the case in countable-state models where the LRA fraction of time the project must be worked on is *constant*, as often occurs in queueing systems. We propose here to overcome such problem by introducing a new, *mixed LRA/bias criterion*, where cost measures are as in the LRA case, but work measures correspond to Blackwell (1962)'s *bias* criterion. For a review on mixed criteria in MDPs see Feinberg and Shwartz (2002).

In addition to Assumption 5.8, we require the following conditions to hold.

Assumption 5.10

(i) There exists $\rho \in (0, 1)$ such that, for any policy $\pi \in \Pi$ and state $i \in N$,

$$\rho = \lim_{\alpha \searrow 0} \alpha g_i^{\pi,\alpha} = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_i^\pi \left[\int_0^t a(s)\, ds \right],$$

and there exists a finite measure g_i^π given by

$$g_i^\pi \triangleq \lim_{\alpha \searrow 0} \left\{ g_i^{\pi,\alpha} - \frac{\rho}{\alpha} \right\} = \mathbb{E}_i^\pi \left[\int_0^\infty (a(t) - \rho)\, dt \right].$$

(ii) For every $i \in N^{\{0,1\}}$ and $S \in \mathscr{F}$, there exists a finite w_i^S given by

$$w_i^S \triangleq \lim_{\alpha \searrow 0} \frac{w_i^{S,\alpha}}{\alpha} = \lim_{t \to +\infty} t \left\{ \mathbb{E}_i^{\langle 1,S \rangle} \left[\int_0^t a(s)\, ds \right] - \mathbb{E}_i^{\langle 0,S \rangle} \left[\int_0^t a(s)\, ds \right] \right\}.$$

We interpret *bias* measure g_i^π as the expected total cumulative *excess work* expended over the LRA nominal allocation ρ, under policy π, starting at i. Again, as in the ETD criterion, we will consider that the initial state $X(0)$ is drawn from a probability mass function $\mathbf{p}$, and define the *bias work measure* by

$$g^\pi \triangleq \lim_{\alpha \searrow 0} \left\{ g^{\pi,\alpha} - \frac{\rho}{\alpha} \right\} = \mathbb{E}_{\mathbf{p}}^\pi \left[\int_0^\infty (a(t) - \rho)\, dt \right].$$

Further, notice that Assumption 5.10(ii) implies that the w_i^S's defined for the LRA criterion in Assumption 5.8(iii) are zero, being hence of no use in the PCL framework. Instead, we will use as *bias (i, S)-marginal workload* the new coefficient w_i^S defined in Assumption 5.10(ii), representing the limiting (as time grows to infinity) time-scaled marginal increase in expected work expended resulting from using policy $\langle 1, S \rangle$ instead of $\langle 0, S \rangle$, starting at i.

Measures f^π and $x_j^{a,\pi}$, and marginal costs c_i^S are defined according to the LRA criterion. Provided Assumption 5.1(i) holds, we define the *LRA/bias (i, S)-marginal productivity rate* and the *LRA/bias index* by $\nu_i^S = c_i^S / w_i^S$ and $\nu_i^* = \nu_i^{S_i}$.

The main result for the mixed LRA/bias criterion is as follows.

Theorem 5.11 *Under Assumption 5.1, the project is PCL($\mathscr{F}$)-indexable relative to the LRA/bias criterion, and ν_i^* is its LRA/bias MPI.*

Proof. The result follows along the lines of Theorem 5.9, by taking appropriate limits as the discount factor vanishes in the results for the ETD criterion. □

6 Optimal MPI policy for control of an MTO queue

This section draws on the above to carry out a *PCL-indexability analysis* of the pure MTO case of the model described in Section 2. Notice that $N^{\{0\}} = \{0\}$.

In the analyses below we draw on standard results on the $M/G/1$ queue. See, e.g. (Kleinrock 1975, Ch. 5). We will refer to the number-in-system process $L(t)$ for an $M/G/1$ queue operated under the standard (S_0-active) policy. Our main concern is to establish $\mathscr{F}$-indexability under the LRA/bias criterion of Section 5.3. We will draw on preliminary results for the discounted criterion.

6.1 Preliminary results: Discounted criterion

We start with the ETD criterion of Section 5.1, under factor $\alpha > 0$. Since active and passive one-period discount factors are now constant, we will denote them by

$$\beta_1 = \psi(\alpha) \quad \text{and} \quad \beta_0 = \frac{\lambda}{\alpha + \lambda}.$$

Similarly, we will denote the ETD lengths of active and passive periods by

$$m_1 = \frac{1 - \beta_1}{\alpha} \quad \text{and} \quad m_0 = \frac{1 - \beta_0}{\alpha} = \frac{1}{\alpha + \lambda}.$$

Let a_j denote the discounted probability that j customers arrive during a service. The a_j's are characterized by their *z-transform*

$$a^*(z) \triangleq \sum_{j=0}^{\infty} a_j z^j = \psi(\alpha + \lambda - \lambda z).$$

Notice that

$$a^*(1) = \beta_1. \tag{30}$$

From the a_j's one can readily obtain the discounted transition probabilities β_{ij}^a.

We will further use the distribution of the length of a busy period starting with one customer, under the standard policy. Its LST's value $\phi = \phi(\alpha)$ is characterized as the unique solution $0 < \phi < 1$ of the fixed-point equation

$$a^*(\phi) = \phi. \tag{31}$$

For example, in the *exponential service-time* case, the latter equation becomes

$$\frac{\mu}{\alpha + \mu + \lambda(1 - \phi)} = \phi, \quad \text{i.e.} \quad \lambda\phi^2 - (\alpha + \lambda + \mu)\phi + \mu = 0.$$

Denoting the discriminant by $d = \sqrt{(\alpha + \lambda + \mu)^2 - 4\lambda\mu}$, the two solutions are

$$\phi_1 = \frac{\alpha + \lambda + \mu - d}{2\lambda} \quad \text{and} \quad \phi_2 = \frac{\alpha + \lambda + \mu + d}{2\lambda}. \tag{32}$$

Since $0 < \phi_1 < 1 < \phi_2$, the required solution is $\phi = \phi_1$.

Note:

$$\phi_1 \phi_2 = \frac{1}{\rho} \quad \text{and} \quad \phi_1 + \phi_2 = \frac{\alpha + \lambda + \mu}{\lambda}.$$

Work and marginal work measures

We address next calculation and analysis of ETD work and marginal work measures g_i^S and w_i^S. We will use the fact that, for each $k \geq 0$, $X(t)$ is a *regenerative process* under the S_k-active policy, having as *renewal epochs* those where *least recurrent state* k is hit, which marks the completion of a *cycle*.

We will make use of the *ETD time to hit state 0* starting at $i \geq 1$ under the standard policy, and of the *ETD recurrence time to 0* (starting at 0), denoted by M_{i0} and M_{00}, respectively. The M_{i0}'s are characterized by the recursion

$$M_{i0} = \begin{cases} M_{10} + \phi M_{i-1,0} & \text{if } i \geq 2 \\ \dfrac{1-\phi}{\alpha} & \text{if } i = 1. \end{cases}$$

whose solution is

$$M_{i0} = \frac{1-\phi^i}{\alpha}, \quad i \geq 1.$$

Further, we have

$$M_{00} = \frac{1 - \beta_0 \phi}{\alpha} = \frac{\alpha + \lambda - \lambda\phi}{\alpha(\alpha+\lambda)}. \tag{33}$$

We will also make use of the *ETD busy time during a cycle* (under S_0), given by

$$B_{00} = \beta_0 M_{10}.$$

Measures $g_i^{S_k}$ are represented next in terms of $g_0^{S_0}$, by standard arguments.

Lemma 6.1 *For $i \geq 0$:*

(a) $g_i^{S_0} = \dfrac{1}{\alpha} - \dfrac{(1-\beta_0)\phi^i}{\alpha(1-\phi\beta_0)} = \dfrac{1}{\alpha} - \dfrac{\phi^i}{\alpha+\lambda-\lambda\phi} = \begin{cases} M_{i0} + \phi^i g_0^{S_0} & \text{if } i \geq 1 \\ \dfrac{B_{00}}{\alpha M_{00}} & \text{if } i = 0. \end{cases}$

(b) *For $k \geq 1$,* $g_i^{S_k} = \begin{cases} \beta_0^{k-i} g_0^{S_0} & \text{if } 0 \leq i < k \\ g_{i-k}^{S_0} & \text{if } i \geq k. \end{cases}$

The next result characterizes discounted marginal work measures w_i^S.

Lemma 6.2 *For $i, k \geq 1$:*

(a) $w_i^{S_0} = (1-\beta_0)M_{i0} = \dfrac{1-\phi^i}{\alpha+\lambda} = \begin{cases} w_1^{S_0} + \phi w_{i-1}^{S_0} & \text{if } i \geq 2 \\ \dfrac{1-\phi}{\alpha+\lambda} & \text{if } i = 1. \end{cases}$

(b) $w_i^{S_k} = \begin{cases} w_{i-k}^{S_0} & \text{if } i > k \\ w_1^{S_{k-i+1}} & \text{if } 2 \leq i \leq k. \end{cases}$

(c) $w_1^{S_k} = \begin{cases} \dfrac{a_0}{\phi} w_1^{S_0} & \textit{if } k = 1 \\ (1-\beta_0)m_1 + \beta_0 w_1^{S_{k-1}} + \displaystyle\sum_{j=k+1}^{\infty} a_j w_{j-k}^{S_0} & \textit{if } k \geq 2. \end{cases}$

Proof. (a) Using (28), Lemma 6.1, and $M_{i+1,0} = M_{i0} + \phi^i M_{10}$, gives, for $i \geq 1$:

$$\begin{aligned} w_i^{S_0} &\triangleq g_i^{\langle 1,S_0 \rangle} - g_i^{\langle 0,S_0 \rangle} = g_i^{S_0} - \beta_0\, g_{i+1}^{S_0} \\ &= \left[M_{i0} + \phi^i g_0^{S_0}\right] - \beta_0 \left[M_{i+1,0} + \phi^{i+1} g_0^{S_0}\right] \\ &= \left[M_{i0} + \phi^i g_0^{S_0}\right] - \beta_0 \left[M_{i0} + \phi^i M_{10} + \phi^{i+1} g_0^{S_0}\right] = (1-\beta_0) M_{i0}. \end{aligned}$$

(b) This part follows from (28) and Lemma 6.1(b).

(c) For $k = 1$, use Lemma 6.1(b), (30) and (31) to get:

$$\begin{aligned} w_1^{S_1} &\triangleq g_1^{\langle 1,S_1 \rangle} - g_1^{\langle 0,S_1 \rangle} = m_1 + \sum_{j=0}^{\infty} a_j g_j^{S_1} - g_1^{S_1} \\ &= m_1 + a_0 \beta_0 g_0^{S_0} + \sum_{j=1}^{\infty} a_j g_{j-1}^{S_0} - g_0^{S_0} \\ &= m_1 + a_0 \beta_0 g_0^{S_0} + \sum_{j=1}^{\infty} a_j \left[\frac{1}{\alpha} - \left(\frac{1}{\alpha} - g_0^{S_0}\right) \phi^{j-1}\right] - g_0^{S_0} \\ &= m_1 + a_0 \beta_0 g_0^{S_0} + \frac{1}{\alpha} \sum_{j=1}^{\infty} a_j - \frac{1}{\phi} \left(\frac{1}{\alpha} - g_0^{S_0}\right) \sum_{j=1}^{\infty} a_j \phi^j - g_0^{S_0} \\ &= m_1 + a_0 \beta_0 g_0^{S_0} + \frac{1}{\alpha} [\beta_1 - a_0] - \frac{1}{\phi} \left(\frac{1}{\alpha} - g_0^{S_0}\right) (\phi - a_0) - g_0^{S_0} \\ &= \frac{a_0}{\phi} \left[\frac{1-\phi}{\alpha} - (1 - \beta_0 \phi) g_0^{S_0}\right] = \frac{a_0}{\phi} w_1^{S_0}. \end{aligned}$$

Let now $k \geq 2$. Using Lemma 6.1(b) we can write

$$\begin{aligned} w_1^{S_k} &\triangleq g_1^{\langle 1,S_k \rangle} - g_1^{\langle 0,S_k \rangle} = m_1 + \sum_{j=0}^{\infty} a_j g_j^{S_k} - g_1^{S_k} \\ &= m_1 + \sum_{j=0}^{k-1} a_j g_j^{S_k} + \sum_{j=k}^{\infty} a_j g_j^{S_k} - g_1^{S_k} \\ &= m_1 + \beta_0 \sum_{j=0}^{k-1} a_j g_j^{S_{k-1}} + \sum_{j=k}^{\infty} a_j g_{j-1}^{S_{k-1}} - g_0^{S_{k-1}} \\ &= m_1 + \beta_0 \left[w_1^{S_{k-1}} - m_1 - \sum_{j=k}^{\infty} a_j g_j^{S_{k-1}} + g_1^{S_{k-1}}\right] + \sum_{j=k}^{\infty} a_j g_{j-1}^{S_{k-1}} - g_0^{S_{k-1}} \\ &= (1-\beta_0) m_1 + \beta_0 w_1^{S_{k-1}} + \sum_{j=k}^{\infty} a_j \left[g_{j-1}^{S_{k-1}} - \beta_0 g_j^{S_{k-1}}\right] \\ &= (1-\beta_0) m_1 + \beta_0 w_1^{S_{k-1}} + \sum_{j=k+1}^{\infty} a_j w_{j-k}^{S_0}. \end{aligned}$$

$$
\begin{array}{ccccccccc}
\boxed{w_1^{S_0}} & \rightarrow & \boxed{w_1^{S_1}} & \rightarrow & \boxed{w_1^{S_2}} & \rightarrow & \boxed{w_1^{S_3}} & \rightarrow & \cdots \\
\downarrow & \searrow & & \searrow & & \searrow & & \searrow & \\
w_2^{S_0} & & w_2^{S_1} & & w_2^{S_2} & & w_2^{S_3} & & \ddots \\
\downarrow & \searrow & & \searrow & & \searrow & & \searrow & \\
w_3^{S_0} & & w_3^{S_1} & & w_3^{S_2} & & w_3^{S_3} & & \ddots \\
\downarrow & \searrow & & \searrow & & \searrow & & \searrow & \\
\vdots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

Figure 2: Direction of calculations for marginal workloads $w_i^{S_k}$.

This completes the proof. $\square$

Remark 6.3 Lemma 6.2 yields a recursion for calculating ETD marginal workloads $w_i^{S_k}$, for $i \geq 1$, $k \geq 0$. The calculation's backbone consists of *pivot terms* $w_1^{S_k}$, for $k \geq 0$. Calculations proceed in the order indicated by arrows in Figure 2.

The next result shows that ETD marginal workloads are positive, as required.

Proposition 6.4 *ETD marginal workloads $w_i^{S_k}$ satisfy Assumption* 5.1(a).

Proof. The result follows by induction on k using the recursions in Lemma 6.2. $\square$

Cost and marginal cost measures

We proceed to calculate the required ETD cost and marginal cost measures f_i^S and c_i^S. In the analyses below, we include in the notation the holding cost rate sequence relative to which cost measures are defined, when this is different from the original sequence $\mathbf{h} = (h_0, h_1, \ldots)$, writing $\mathbf{h}^i = (h_i, h_{i+1}, \ldots)$ for $i \geq 0$. We will further refer to sequences of first- and second-order differences $\Delta \mathbf{h}^i$, $\Delta^2 \mathbf{h}^i$.

Analogously as in the above analyses, we will make use of the *ETD cost to hit* 0 *starting at* $i > 0$ under the standard policy , denoted by $V_{i0}(\cdot)$, and of the *ETD cost accrued over a cycle*, denoted by $V_{00}(\cdot)$.

The next result characterizes ETD measures $f_i^S(\mathbf{h})$, via standard arguments.

Lemma 6.5

(a) $f_i^{S_0} = \begin{cases} V_{i0} + \phi^i f_0^{S_0} & \text{if } i \geq 1 \\ \dfrac{V_{00}}{\alpha M_{00}} & \text{if } i = 0. \end{cases}$

(b) $f_i^{S_k} = \begin{cases} f_{i-k}^{S_0}(\mathbf{h}^k) & \text{if } i \geq k \\ m_0(h_i + \cdots + \beta_0^{k-i-1} h_{k-1}) + \beta_0^{k-i} f_0^{S_0}(\mathbf{h}^k) & \text{if } 0 \leq i < k. \end{cases}$

Proof. (a) By standard arguments, we have

$$
\begin{aligned}
f_i^{S_0} &= V_{i0} + \phi^i f_0^{S_0}, \quad i \geq 1 \\
f_0^{S_0} &= \frac{1 - \beta_0}{\alpha} h_0 + \beta_0 f_1^{S_0}.
\end{aligned}
$$

Solving for $f_0^{S_0}$ and $f_1^{S_0}$ gives

$$\begin{aligned} f_0^{S_0} &= \frac{1-\beta_0}{\alpha(1-\beta_0\phi)} h_0 + \frac{\beta_0 V_{10}^{S_0}}{1-\beta_0\phi} \\ f_1^{S_0} &= \frac{(1-\beta_0)\phi}{\alpha(1-\beta_0\phi)} h_0 + \frac{V_{10}^{S_0}}{1-\beta_0\phi}, \end{aligned}$$

whence the result follows, using (33).

Part (b) is trivial. □

The next result characterizes the required ETD marginal costs c_i^S.

Lemma 6.6

(a) $c_i^{S_0} = m_0(h_i - \phi^i h_0) + \beta_0 V_{i0}(\Delta \mathbf{h}^1) - (1-\beta_0)V_{i0}$, *for* $i \geq 1$.

(b) $c_i^{S_k} = c_{i-k}^{S_0}(\mathbf{h}^k), \quad i > k$.

Proof. (a) We have, for $i \geq 1$,

$$\begin{aligned} c_i^{S_0} &\triangleq f_i^{\langle 0,S_0\rangle} - f_i^{\langle 1,S_0\rangle} = m_0 h_i + \beta_0 f_{i+1}^{S_0} - f_i^{S_0} \\ &= m_0 h_i + \beta_0 \left[V_{i+1,0} + \phi^{i+1} f_0^{S_0}\right] - \left[V_{i0} + \phi^i f_0^{S_0}\right] \\ &= m_0(h_i - \phi^i h_0) + \beta_0 V_{i0}(\Delta \mathbf{h}^1) - (1-\beta_0)V_{i0}, \end{aligned}$$

where we have used (29), Lemma 6.5(a), $V_{i+1,0} = V_{i0}(\mathbf{h}^1) + \phi^i V_{10}$, and

$$f_0^{S_0} = m_0 h_0 + \beta_0 f_1^{S_0} = m_0 h_0 + \beta_0 \left[V_{10} + \phi f_0^{S_0}\right].$$

Part (b) follows from (29) and Lemma 6.5(b). □

The next result gives representations for index $\nu_i^* = c_i^{S_{i-1}}/w_i^{S_{i-1}}$.

Proposition 6.7 *For* $i \geq 1$*:*

$$\begin{aligned} \nu_i^* &= \nu_1(\mathbf{h}^{i-1}) \\ &= \left(\lambda + \frac{\alpha}{1-\phi}\right) \mathbb{E}_0 \left[\int_0^\infty e^{-\alpha t} \left\{\Delta h_{L(t)+i} - \frac{\alpha}{\lambda}\left(h_{L(t)+i-1} - h_{i-1}\right)\right\} dt\right] \\ &= \left(\lambda + \frac{\alpha}{1-\phi}\right) \left\{\mathbb{E}_0 \left[\int_0^\infty e^{-\alpha t} h_{L(t)+i}\, dt\right] - \mathbb{E}_1 \left[\int_0^\infty e^{-\alpha t} h_{L(t)+i-1}\, dt\right]\right\} \end{aligned}$$

Proof. The first identity follows from Lemma 6.2(b) and Lemma 6.6(b). The second identity follows from Lemma 6.2(a) and Lemma 6.6(a), through

$$\begin{aligned}
\nu_1(\mathbf{h}^{i-1}) &= \frac{m_0(h_i - \phi h_{i-1}) + \beta_0 V_{10}(\Delta \mathbf{h}^i) - (1-\beta_0) V_{10}(\mathbf{h}^{i-1})}{(1-\beta_0) M_{10}} \\
&= \frac{V_{00}(\Delta \mathbf{h}^i) + m_0(1-\phi) h_{i-1} - (1-\beta_0) V_{10}(\mathbf{h}^{i-1})}{(1-\beta_0) M_{10}} \\
&= \frac{M_{00}}{m_0 M_{10}} \frac{V_{00}(\Delta \mathbf{h}^i) + m_0(1-\phi) h_{i-1} - \alpha m_0 V_{10}(\mathbf{h}^{i-1})}{\alpha M_{00}} \\
&= \frac{M_{00}}{m_0 M_{10}} \frac{V_{00}(\Delta \mathbf{h}^i) - \alpha m_0 V_{10}(\mathbf{h}^{i-1} - h_{i-1}\mathbf{1})}{\alpha M_{00}} \\
&= \frac{M_{00}}{m_0 M_{10}} \left\{ \frac{V_{00}(\Delta \mathbf{h}^i)}{\alpha M_{00}} - \frac{1-\beta_0}{\beta_0} \frac{V_{00}(\mathbf{h}^{i-1} - h_{i-1}\mathbf{1})}{\alpha M_{00}} \right\} \\
&= \frac{\alpha + \lambda(1-\phi)}{1-\phi} \mathbb{E}_0 \left[\int_0^\infty e^{-\alpha t} \left\{ \Delta h_{L(t)+i} - \frac{\alpha}{\lambda} \left(h_{L(t)+i-1} - h_{i-1} \right) \right\} dt \right]
\end{aligned}$$

where $\mathbf{1}$ denotes a sequence of 1's, and we have used

$$V_{10}(h_{i-1}\mathbf{1}) = M_{10} h_{i-1} = \frac{1-\phi}{\alpha} h_{i-1}.$$

The third identity follows from

$$\nu_i^* = \frac{f_i^{S_i} - f_i^{S_{i-1}}}{g_i^{S_{i-1}} - g_i^{S_i}} = \frac{f_0^{S_0}(\mathbf{h}^i) - f_1^{S_0}(\mathbf{h}^{i-1})}{g_1^{S_0} - g_0^{S_0}}.$$

□

Remark 6.8

1. If one can show that index ν_i^* is nondecreasing under Assumption 2.1, then Theorem 5.7 (using Proposition 6.4) shows that it is indeed the discounted MPI. We emphasize that such result is *not needed* for our analysis in Section 6.2 of indexability under the LRA/bias criterion, which is our prime concern.

2. In the *linear* holding cost case $h_j = cj$, $j \geq 0$, or $\mathbf{h} = c\mathbf{e}$, we can use Bell (1971)'s classic accounting argument to write

   $$\alpha f_i^{S_0} = c \left[i + \frac{\lambda}{\alpha} - \alpha \frac{\beta_1}{1-\beta_1} g_i^{S_0} \right], \quad i \geq 0.$$

   Substitution in the second identity in Proposition 6.7 gives the *constant* MPI

   $$\nu_i^* = c \frac{\beta_1}{1-\beta_1}, \quad i \geq 1. \tag{34}$$

   The latter agrees with the optimal *discounted $c\mu$-index rule* for scheduling a multiclass $M/G/1$ queue with linear holding costs.

3. We readily obtain the following limiting α-scaled index

   $$\nu_i^{\mathrm{LRA}} \triangleq \lim_{\alpha \searrow 0} \alpha \nu_i^* = \mu \mathbb{E}\left[\Delta h_{L+i}\right].$$

   It seems reasonable to consider ν_i^{LRA} an index corresponding to the LRA criterion. Section 6.2 will show that ν_i^{LRA} is indeed the MPI corresponding to a new, LRA-bias criterion.

In the exponential service case, we obtain a remarkably simpler evaluation of the ETD index, from which its monotonicity is apparent. Let τ be a random stopping time having an exponential distribution with rate α, independent of process $\{L(t) : t \geq 0\}$. Let $z_1 = \rho\phi_1 < \rho$, where ϕ_1 is given by (32). Recall that, if $L(0) = 0$, then $L(\tau)$ has a geometric distribution with success probability $1 - z_1$.

Theorem 6.9 *Suppose the service time distribution is exponential and Assumption* 2.1 *holds. Then, the model is $\mathscr{F}$-indexable under the ETD criterion, having MPI*

$$\begin{aligned}\nu_i^* &= \mu\mathbb{E}_0\left[\int_0^\infty e^{-\alpha t}\Delta h_{L(t)+i}\,dt\right] = \frac{\mu}{\alpha}\mathbb{E}_0\left[\Delta h_{L(\tau)+i}\right]\\ &= \frac{\mu}{\alpha}(1-z_1)\sum_{j=0}^\infty \Delta h_{j+i}z_1^j, \quad i \geq 1.\end{aligned}$$

Proof. Let $\tau_0 >$ be the first time to hit 0 for $L(t)$. Drawing on elementary properties of the $M/M/1$ queue, including the strong Markov property, we have, for $i \geq 1$:

$$\begin{aligned}\mathbb{E}_1\left[h_{L(\tau)+i-1}\right] &= \mathbb{E}_1\left[h_{L(\tau)+i-1} \mid \tau_0 > \tau\right]\mathbb{P}_1\left\{\tau_0 > \tau\right\}\\ &\quad + \mathbb{E}_1\left[h_{L(\tau)+i-1} \mid \tau_0 \leq \tau\right]\mathbb{P}_1\left\{\tau_0 \leq \tau\right\}\\ &= \mathbb{E}_0\left[h_{L(\tau)+i}\right]\mathbb{P}_1\left\{\tau_0 > \tau\right\}\\ &\quad + \mathbb{E}_0\left[h_{L(\tau)+i-1} \mid \tau_0 \leq \tau\right]\mathbb{P}_1\left\{\tau_0 \leq \tau\right\}.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\mathbb{E}_0\left[h_{L(\tau)+i-1}\right] - \mathbb{E}_1\left[h_{L(\tau)+i-1}\right] &= \mathbb{P}_1\left\{\tau_0 < \tau\right\}\mathbb{E}_0\left[\Delta h_{L(\tau)+i}\right]\\ &= \phi_1\mathbb{E}_0\left[\Delta h_{L(\tau)+i}\right].\end{aligned}$$

Now, substitution into the last identity for ν_i^* in Proposition 6.7, gives, after simplification, the required identities. It is now immediate that the index is nondecreasing under Assumption 2.1. Therefore, by Theorem 5.7, ν_i^* is the ETD MPI. □

Remark 6.10 In the exponential service case:

1. It is easily shown that $\alpha\nu_i^*$ is decreasing in discount factor α. It is insightful to consider the limiting α-scaled indices

$$\begin{aligned}\nu_i^{\mathrm{LRA}} &\triangleq \lim_{\alpha\searrow 0}\alpha\nu_i^* = \mu\mathbb{E}\left[\Delta h_{L+i}\right],\\ \nu_i^{\mathrm{myopic}} &\triangleq \lim_{\alpha\nearrow+\infty}\alpha\nu_i^* = \mu\Delta h_i.\end{aligned}$$

   Notice that it is natural to interpret ν_i^{myopic} as a *myopic index*.

2. For a quadratic cost rate $h_j = cj^2$ we obtain the MPI

$$\nu_i^* = \frac{c\mu}{\alpha}\left[2i - 1 + \frac{2z_1}{1-z_1}\right], \quad i \geq 1. \tag{35}$$

   The corresponding LRA and myopic indices are

$$\begin{aligned}\nu_i^{\mathrm{LRA}} &= c\mu\left[2i - 1 + \frac{2\rho}{1-\rho}\right],\\ \nu_i^{\mathrm{myopic}} &= c\mu\left[2i-1\right].\end{aligned}$$

   Notice that term $2\rho/(1-\rho)$ in ν_i^{LRA} accounts for long-term congestion.

6.2 LRA/bias criterion

We now address indexability under the LRA/bias criterion of Section 5.3. We will draw on the ETD measures above, which we will write making factor α explicit.

The next result characterizes *bias* (or *excess*) work measures $g_i^{S_k}$, g^{S_k}.

Lemma 6.11

(a) *For* $i \geq 0$, $g_i^{S_0} = \begin{cases} g_0^{S_0} + \dfrac{i}{\mu} & \text{if } i \geq 1 \\ -\lambda \dfrac{\sigma^2 + 1/\mu^2}{2(1-\rho)} & \text{if } i = 0. \end{cases}$

(b) *For* $k \geq 1$, $i \geq 0$, $g_i^{S_k} = g_0^{S_0} + \dfrac{i-k}{\mu}$.

(c) *For* $k \geq 0$, $g^{S_k} = g_0^{S_0} + \dfrac{\mathbb{E}[X(0)] - k}{\mu}$.

Proof. (a, b) The results follow by subtracting ρ/α from ETD measures $g_i^{S_k,\alpha}$ in identities of Lemma 6.1, and taking limits as α vanishes, using l'Hôpital's rule.

(c) This part follows easily applying (a, b) to $g^{S_k} = \sum_{i=0}^{\infty} p_i g_i^{S_k}$. □

The next result characterizes *bias marginal workloads* $w_i^{S_k}$. Note that below a_j denotes the *undiscounted* probability that j customers arrive during a service.

Lemma 6.12 *For* $i, k \geq 1$*:*

(a) $w_i^{S_0} = \dfrac{M_{i0}}{\lambda} = \dfrac{1/\lambda}{1-\rho}\dfrac{i}{\mu} = \begin{cases} \dfrac{1/\lambda}{1-\rho}\dfrac{1}{\mu} + w_{i-1}^{S_0} & \text{if } i \geq 2 \\ \dfrac{1/\lambda}{1-\rho}\dfrac{1}{\mu} & \text{if } i = 1. \end{cases}$

(b) $w_i^{S_k} = \begin{cases} w_{i-k}^{S_0} & \text{if } i > k \\ w_1^{S_{k-i+1}} & \text{if } 2 \leq i \leq k. \end{cases}$

(c) $w_1^{S_k} = \begin{cases} a_0 w_1^{S_0} & \text{if } k = 1 \\ \dfrac{1}{\lambda\mu} + w_1^{S_{k-1}} + \displaystyle\sum_{j=k+1}^{\infty} a_j w_{j-k}^{S_0} & \text{if } k \geq 2. \end{cases}$

Proof. To obtain the stated identities it suffices to divide by $\alpha > 0$ the corresponding discounted identities in Lemma 6.2, and take limits as α vanishes. □

Remark 6.13 As in the ETD case (cf. Remark 6.3), Lemma 6.2 yields a recursion for calculating bias marginal workloads $w_i^{S_k}$. See Figure 2.

The next result establishes the required properties of bias marginal workloads.

Proposition 6.14 *Bias marginal workloads* $w_i^{S_k}$ *satisfy the following:*

(a) *They are positive, i.e. Assumption* 5.1 *holds.*

(b) *They are V-shaped, satisfying Assumption* 4.17 *with strict inequalities.*

Proof. (a) The result follows by induction using Lemma 6.12. See Remark 6.13.

(b) We have, using Lemma 6.12(a, b) that, for $1 \leq k \leq i-1$:

$$w_i^{S_k} - w_i^{S_{k-1}} = w_{i-k}^{S_0} - w_{i-k+1}^{S_0} = -\frac{1}{\lambda\mu(1-\rho)} < 0.$$

Also, using Lemma 6.12(b, c) gives that, for $k \geq 1$:

$$w_k^{S_k} - w_k^{S_{k-1}} = w_1^{S_1} - w_1^{S_0} = -\left(1 - a_0\right) w_1^{S_0} < 0,$$

where the inequality follows from $w_1^{S_0} > 0$ (part (a)) and $a_0 < 1$.

Furthermore, part (a) and Lemma 6.12(b, c) give that, for $k \geq i+1$,

$$w_i^{S_k} - w_i^{S_{k-1}} = w_1^{S_{k-i+1}} - w_1^{S_{k-i}} = \frac{1}{\lambda\mu} + \sum_{j=k-i+2} a_j w_{j-k+i-1}^{S_0} > 0.$$

This completes the proof. □

We next address calculation of LRA cost measures. Let L be a random variable having the *equilibrium distribution* of the number in system for the $M/G/1$ queue of concern, under the standard (S_0) policy. The next result follows immediately.

Lemma 6.15

$$f^{S_i} = \mathbb{E}\left[h_{L+i}\right], \quad i \geq 0.$$

The next result characterizes the required LRA marginal costs c_i^S. Terms V_{10} below are undiscounted counterparts of corresponding terms in Section 6.1.

Lemma 6.16

(a) $c_i^{S_0} = \dfrac{h_i - h_0}{\lambda} + V_{i0}(\Delta\mathbf{h}^1), \quad i \geq 1.$

(b) $c_i^{S_k} = c_{i-k}^{S_0}(\mathbf{h}^k), \quad i > k.$

Proof. The result follows by letting α vanish in Lemma 6.6's identities. □

The following result gives representations for index $\nu_i^* = c_i^{S_{i-1}}/w_i^{S_{i-1}}$.

Proposition 6.17

$$\nu_i^* = \nu_1^*(\mathbf{h}^{i-1}) = \frac{\Delta h_i/\lambda + V_{10}(\Delta\mathbf{h}^i)}{w_1^{S_0}} = \mu\mathbb{E}\left[\Delta h_{L+i}\right], \quad i \geq 1.$$

Proof. Scale by α identities in Proposition 6.7 and let α vanish to get the result. □

We can finally present the main result of this section.

Theorem 6.18 *Under Assumption* 2.1:

(a) *The model is $\mathscr{F}$-indexable under the LRA/bias criterion, having MPI $\nu_i^* = \mu\mathbb{E}\left[\Delta h_{L+i}\right]$, for $i \geq 1$.*

(b) *MPI* ν_i^* *has the characterization* (25) *in Theorem* 4.19.

Proof. (a) This part follows from Theorem 5.11, using Proposition 6.14(a) to ensure positivity of bias marginal workloads, and expression $\nu_i^* = \mu \mathbb{E}[\Delta h_{L+i}]$ in Proposition 6.17 to ensure nondecreasingness of the index.

(b) This part follows from Theorem 4.19 using Proposition 6.14(b). □

Remark 6.19

(i) For a *linear cost rate* $h_j = cj$, the LRA/bias MPI is $\nu_i^* = c\mu$, consistently with the *$c\mu$-index rule* for scheduling a multiclass $M/G/1$ queue.

(ii) For a *quadratic cost rate* $h_j = cj^2$, the LRA/bias MPI has the evaluation

$$\nu_i^* = c\mu \left[2i - 1 + \frac{2\rho + \lambda^2(\sigma^2 - 1/\mu^2)}{1-\rho} \right], \quad i \geq 1.$$

(iii) Denoting by $\nu_i^{*,m}$ the LRA/bias MPI under cost rates $h_j = j^m$, $j \geq 0$, where $m \geq 1$ is an integer, we readily obtain the evaluation

$$\nu_i^{*,m} = \mu \sum_{k=0}^{m-1} \binom{m}{k} \left\{ i^{m-k} - (i-1)^{m-k} \right\} \mathbb{E}\left[L^k \right], \quad i \geq 1.$$

Notice that if, e.g., $h_j = c_0 + c_1 j + c_2 j^2$, then $\nu_i^* = c_1 \nu_i^{*,1} + c_2 \nu_i^{*,2}$.

(iv) Suppose costs are given customer-wise, by a polynomial cost rate $h(T)$ on the current delay T accrued by a customer. To use the above results, we can obtain an *equivalent holding cost rate* h_j by drawing on Marshall and Wolff (1971)'s extension of Little's law for $M/G/K$ queues.

7 Optimal MPI policy for control of an MTS queue

We next extend the above analysis to the MTS case $s \geq 1$ of Section 2's model. Notice that $N^{\{0\}} = \{-s\}$. Investigation of indexability reduces to the MTO case in Section 6. This follows by Morse (1958)'s classic argument, showing that analysis of an MTS queue reduces to that of a related MTO queue. Thus, it suffices to consider the state process $Y(t) = X(t) + s \geq 0$, corresponding to the number-in-system of an MTO $M/G/1$ queue. Process $X(t)$ under the S_i-active policy is equivalent to process $Y(t)$ under the S_{i+s} policy, for each $i \in N$. It follows that all the results derived in Section 6 for the MTO case carry over to the MTS case.

We thus obtain from Theorem 6.18 the following result under the LRA/bias criterion. Let L be as in Theorem 6.18. As before, we write $\mathbf{h}^0 = (h_0, h_1, \ldots)$.

Theorem 7.1 *Under Assumption* 2.1:

(a) *The model is $\mathscr{F}$-indexable under the LRA/bias criterion, having MPI*

$$\begin{aligned} \nu_i^* &= \mu \mathbb{E}[\Delta h_{L+i}] \\ &= \begin{cases} \nu_i^*(\mathbf{h}^0) & \text{if } i \geq 1 \\ \nu_1^*(\mathbf{h}^0) \mathbb{P}\{L > -i\} + \mu \sum_{j=0}^{-i} \Delta h_{i+j} \mathbb{P}\{L = j\} & \text{if } i \leq 0. \end{cases} \end{aligned} \tag{36}$$

(b) *MPI* ν_i^* *has the characterization* (25) *in Theorem* 4.19.

Proof. The results follow from the above discussion via Theorem 6.18. To obtain the second case in (36) we use the fact that $L \mid L \geq j \sim L + j$, for $j \geq 0$, where $\sim$ denotes equality in distribution. Hence, for $i \leq 0$:

$$\mathbb{E}\left[\Delta h_{L+i}\right] = \mathbb{E}\left[\Delta h_{L+i} \mid L > -i\right] \mathbb{P}\{L > -i\} + \sum_{j=0}^{-i} \mathbb{E}\left[\Delta h_{L+i} \mid L = j\right] \mathbb{P}\{L = j\}$$
$$= \mathbb{E}\left[\Delta h_{L+1}\right] \mathbb{P}\{L > -i\} + \sum_{j=0}^{-i} \Delta h_{i+j} \mathbb{P}\{L = j\},$$

which yields the result. $\square$

While our focus in on the LRA/bias criterion, it is insightful to consider indexability under the discounted criterion in the exponential service case. From the above discussion and Theorem 6.9 we readily obtain the next result (where we include discount factor $\alpha > 0$ in the notation). Let z_1, τ be as in Theorem 6.9.

Theorem 7.2 *Suppose service times are exponential and Assumption* 2.1 *holds. Then, the model is* $\mathscr{F}$*-indexable under the ETD criterion, having MPI*

$$\nu_i^{\alpha,*} = \frac{\mu}{\alpha} \mathbb{E}\left[\Delta h_{L(\tau)+i}\right] = \begin{cases} \nu_i^{\alpha,*}(\mathbf{h}^0) & \textit{if } i \geq 1 \\ \nu_1^{\alpha,*}(\mathbf{h}^0) z_1^{-i+1} + \dfrac{\mu(1 - z_1)}{\alpha} \displaystyle\sum_{j=0}^{-i} \Delta h_{i+j} z_1^j & \textit{if } i \leq 0. \end{cases} \tag{37}$$

Remark 7.3

(i) For $i \geq 1$, $\nu_i^*(\mathbf{h}^0)$ is the *MTO* MPI obtained in the previous Section. Thus, the first case in (36) shows that *the MTS MPI extends the MTO MPI*. This agrees with a result of Ha (1997) for the linear cost exponential service case.

(iii) Under *linear FGS holding costs* ($h_j = -c^{\mathrm{F}} j$, $j \leq -1$), we obtain the MPI

$$\nu_i^* = \begin{cases} \nu_i^*(\mathbf{h}^0) & \text{if } i \geq 1 \\ \left\{\nu_1^*(\mathbf{h}^0) + c^{\mathrm{F}}\mu\right\} \mathbb{P}\left\{L \geq -i + 1\right\} - c^{\mathrm{F}}\mu & \text{otherwise.} \end{cases} \tag{38}$$

(iii) Consider the *linear costs* case, $h_j = c^{\mathrm{B}} j$ for $j \geq 0$, and $h_j = -c^{\mathrm{F}} j$ for $j \leq -1$, where $c^{\mathrm{B}}, c^{\mathrm{F}} > 0$. We readily obtain from (36) the MPI evaluation

$$\nu_i^* = \begin{cases} c^{\mathrm{B}}\mu & \text{if } i \geq 1 \\ \left[\left(c^{\mathrm{B}} + c^{\mathrm{F}}\right) \mathbb{P}\left\{L \geq -i + 1\right\} - c^{\mathrm{F}}\right] \mu & \text{otherwise.} \end{cases} \tag{39}$$

In the $M/M/1$ case, substituting $\mathbb{P}\{L \geq j\} = \rho^j$ above gives the index

$$\nu_i^* = \begin{cases} c^{\mathrm{B}}\mu & \text{if } i \geq 1 \\ \left[\left(c^{\mathrm{B}} + c^{\mathrm{F}}\right) \rho^{-i+1} - c^{\mathrm{F}}\right] \mu & \text{otherwise.} \end{cases}$$

Remarkably, the latter is the *myopic(T) index* of Peña-Pérez and Zipkin (1997, p. 926), which they obtain by a look-ahead argument. It has been shown in de Véricourt et al. (2000) that such index characterizes the optimal policy for a multiclass make-to-stock queue in a limited region of the state space. It is insightful to further evaluate the myopic index as in Remark 6.10:

$$\nu_i^{\text{myopic}} \triangleq \lim_{\alpha \to +\infty} \alpha \nu_i^{\alpha,*} = \mu \Delta h_i = \begin{cases} c^{\text{B}}\mu & \text{if } i \geq 1 \\ -c^{\text{F}}\mu & \text{otherwise,} \end{cases}$$

i.e. it recovers the index derived by Wein (1992) by a heavy-traffic analysis.

(iv) The MPI characterizes the condition under which the MTS capability can improve performance relative to MTO operation: Such will be the case iff $\nu_0^* > 0$. Using (36), and assuming $\Delta h_0 < 0 < \nu_1^*(\mathbf{h}^0)$, we obtain

$$\nu_0^* > 0 \iff \rho > \frac{-\mu \Delta h_0}{\nu_1^*(\mathbf{h}^0) - \mu \Delta h_0}. \tag{40}$$

In the linear cost case, the latter condition reduces to

$$\rho > \frac{c^{\text{F}}}{c^{\text{B}} + c^{\text{F}}}. \tag{41}$$

The MTO vs. MTS issue is thus resolved by *traffic load condition* (40): in *light traffic* ($\rho \approx 0$), MTO suffices; in *heavy traffic* ($\rho \approx 1$), MTS is better.

(v) The MPI characterizes the optimal threshold level(s). For $i \leq -1$, the S_i-active policy (produce when $X^-(t) < -i$) is optimal if $\nu_{i+1}^* > 0$ and $\nu_i^* \leq 0$.

References

Ansell, P. S., Glazebrook, K. D., Niño-Mora, J., and O'Keeffe, M. (2003). Whittle's index policy for a multi-class queueing system with convex holding costs. *Math. Methods Oper. Res.*, 57(1):21–39.

Bell, C. E. (1971). Characterization and computation of optimal policies for operating an $M/G/1$ queue with removable server. *Oper. Res.*, 19:208–218.

Bertsekas, D. P. (1987). *Dynamic Programming: Deterministic and Stochastic Models*. Prentice-Hall, Englewood Cliffs, New Jersey.

Bertsimas, D. and Niño-Mora, J. (1996). Conservation laws, extended polymatroids and multiarmed bandit problems; a polyhedral approach to indexable systems. *Math. Oper. Res.*, 21(2):257–306.

Bertsimas, D. and Niño-Mora, J. (2000). Restless bandits, linear programming relaxations, and a primal-dual index heuristic. *Oper. Res.*, 48(1):80–90.

Blackwell, D. (1962). Discrete dynamic programming. *Ann. Math. Statist.*, 33:719–726.

Buzacott, J. A. and Shanthikumar, J. G. (1993). *Stochastic Models of Manufacturing Systems*. Prentice-Hall, Englewood Cliffs, New Jersey.

Coffman, Jr., E. G. and Mitrani, I. (1980). A characterization of waiting time performance realizable by single-server queues. *Oper. Res.*, 28(3, part 2):810–821.

Cox, D. R. and Smith, W. L. (1961). *Queues*. Methuen, London.

Dacre, M., Glazebrook, K., and Niño-Mora, J. (1999). The achievable region approach to the optimal control of stochastic systems. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 61(4):747–791. With discussion.

de Véricourt, F., Karaesmen, F., and Dallery, Y. (2000). Dynamic scheduling in a make-to-stock system: A partial characterization of optimal policies. *Oper. Res.*, 48:811–819.

d'Epénoux, F. (1960). Sur un problème de production et de stockage dans l'aléatoire. *RAIRO Rech. Opér.*, 14(4):3–16. (Engl. trans.) A probabilistic production and inventory problem. *Management Sci.* **10**, 98–108, 1963.

Dusonchet, F. and Hongler, M. O. (2003). Continuous-time restless bandit and dynamic scheduling for make-to-stock production. *IEEE Trans. Robotics and Automat.*, 19(6):977–990.

Edmonds, J. (1971). Matroids and the greedy algorithm. *Math. Program.*, 1:127–136.

Federgruen, A. and Groenevelt, H. (1988). Characterization and optimization of achievable performance in general queueing systems. *Oper. Res.*, 36(5):733–741.

Feinberg, E. A. and Shwartz, A. (2002). Mixed criteria. In Feinberg, E. A. and Shwartz, A., editors, *Handbook of Markov Decision Processes: Methods and Applications*, pages 209–230. Kluwer.

Gittins, J. C. (1979). Bandit processes and dynamic allocation indices. *J. Roy. Statist. Soc. Ser. B*, 41(2):148–177. With discussion.

Glazebrook, K. D., Lumley, R. R., and Ansell, P. S. (2003). Index heuristics for multiclass $M/G/1$ systems with nonpreemptive service and convex holding costs. *Queueing Systems Theory Appl.*, 45:81–111.

Glazebrook, K. D. and Niño-Mora, J. (2001). Parallel scheduling of multiclass $M/M/m$ queues: Approximate and heavy-traffic optimization of achievable performance. *Oper. Res.*, 49(4):609–623.

Ha, A. Y. (1997). Optimal dynamic scheduling policy for a make-to-stock production system. *Oper. Res.*, 45:42–53.

Kantorovich, L. V. (1965). *The Best Use of Economic Resources*. Harvard University Press, Cambridge, Massachusetts. English translation.

Kleinrock, L. (1975). *Queueing Systems. Vol. I: Theory*. Wiley, New York.

Kleinrock, L. (1976). *Queueing Systems. Vol. II: Computer Applications*. Wiley, New York.

Klimov, G. P. (1974). Time-sharing queueing systems. I. *Theory Probab. Appl.*, 19:532–551.

Koopmans, T. C. (1957). *Three Essays on the State of Economic Science*, chapter Allocation of resources and the price system. McGraw-Hill.

Manne, A. S. (1960). Linear programming and sequential decisions. *Management Sci.*, 6:259–267.

Marshall, K. T. and Wolff, R. W. (1971). Customer average and time average queue lengths and waiting times. *J. Appl. Probab.*, 8:535–542.

Morse, P. M. (1958). *Queues, Inventories and Maintenance: The Analysis of Operational Systems with Variable Demand and Supply*. Wiley, New York.

Niño-Mora, J. (2001a). Restless bandits, partial conservation laws and indexability. *Adv. in Appl. Probab.*, 33(1):76–98.

Niño-Mora, J. (2001b). Stochastic scheduling. In Floudas, C. A. and Pardalos, P. M., editors, *Encyclopedia of Optimization. Vol. 5*, pages 367–372. Kluwer, Dordrecht.

Niño-Mora, J. (2002). Dynamic allocation indices for restless projects and queueing admission control: a polyhedral approach. *Math. Program.*, 93(3, Ser. A):361–413.

Niño-Mora, J. (2003). Restless bandit marginal productivity indices, diminishing returns, and scheduling a multiclass make-to-order/-stock queue. In *Proceedings of the 41st Annual Allerton Conference on Communications, Control and Computing*, pages 100–109.

Niño-Mora, J. (2004). Restless bandit dynamic allocation indices II: Multi-project case and dynamic scheduling of a multiclass make-to-order/-stock $M/G/1$ queue. Technical report, Department of Statistics, Universidad Carlos III de Madrid. Submitted.

Papadimitriou, C. H. and Tsitsiklis, J. N. (1999). The complexity of optimal queuing network control. *Math. Oper. Res.*, 24(2):293–305.

Peña-Pérez, A. and Zipkin, P. (1997). Dynamic scheduling rules for a multiproduct make-to-stock queue. *Oper. Res.*, 45:919–930.

Puterman, M. L. (1994). *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley, New York.

Shanthikumar, J. G. and Yao, D. D. (1992). Multiclass queueing systems: Polymatroidal structure and optimal scheduling control. *Oper. Res.*, 40(suppl. 2):S293–S299.

Tsoucas, P. (1991). The region of achievable performance in a model of Klimov. Technical Report RC16543, IBM T. J. Watson Research Center, Yorktown Heights, New York.

Veatch, M. H. and Wein, L. M. (1996). Scheduling a multiclass make-to-stock queue: Index policies and hedging points. *Oper. Res.*, 44:634–647.

Weber, R. R. and Weiss, G. (1990). On an index policy for restless bandits. *J. Appl. Probab.*, 27:637–648.

Wein, L. M. (1992). Dynamic scheduling of a multiclass make-to-stock queue. *Oper. Res.*, 40:724–735.

Weitzman, M. L. (2000). An 'economics proof' of the supporting hyperplane theorem. *Econ. Letters*, 68:1–6.

Whittle, P. (1988). Restless bandits: Activity allocation in a changing world. In Gani, J., editor, *A Celebration of Applied Probability*, *J. Appl. Probab. Special Vol. 25A*, pages 287–298. Applied Probability Trust.

Whittle, P. (1996). *Optimal Control: Basics and Beyond*. Wiley, Chichester.