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RANGE UNIT ROOT TESTS

Felipe M. Aparicio, Alvaro Escribano and Ana García*

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Keywords: Unit roots tests, Strong serial dependence; Structural breaks; Nonlinearity; Additive outliers, Near-unit root time series; Invariance; Robustness; Running ranges.

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Since the seminal paper by Dickey and Fuller in 1979, unit-root tests have conditioned the standard approaches to analyse time series with strong serial dependence, the focus being placed in the detection of eventual unit roots in an autorregressive model fitted to the series. In this paper we propose a completely different method to test for the type of "long-wave" patterns observed not only in unit root time series but also in series following more complex data generating mechanisms. To this end, our testing device analyses the trend exhibited by the data, without imposing any constraint on the generating mechanism. We call our device the *Range Unit Root* (RUR) *Test* since it is constructed from the running *ranges* of the series. These statistics allow a more general characterization of *strong serial dependence* in the mean behavior, thus endowing our test with a number of desirable properties. Among these properties are the invariance to nonlinear monotonic transformations of the series and the robustness to the presence of level shifts and additive outliers. In addition, the RUR test outperforms the power of standard unit root tests on *near-unit-root* stationary time series.

Key Words: Unit Roots Tests, Strong Serial Dependence, Structural Breaks, Nonlinearity, Additive Outliers, Near-Unit Root Time Series, Invariance, Robustness, Running Ranges.

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1 Introduction

Many overwhelming low-frequency non-periodic components in time series are associated with the presence of unit roots in their data generating process (DGP). Such time series are said to be *integrated.* The pioneering work of Nelson and Plosser (1982) [36] led to the belief that many economic time series were best described in this way. This promted a large amount of research on unit root time series, covering both theoretical and empirical aspects. The unit root paradigm has important practical implications since it entails that shocks have a permanent effect on a variable, or equivalently that the fluctuations they cause are not transitory. As an example, one such implication is the *Purchasing Power Parity (PPP) hypothesis*, which asserts that fluctuations in the real exchange rates of any countries are stationary (Dornbush, 1988 [13]).

The existence of unit roots in time series is investigated by means of unit root tests. The application of standard unit root tests, such as the Dickey-Fuller (DF hereafter) test (Dickey and Fuller, 1979 [12]), has been an important step in the construction of a useful parametric model for many economic time series. In a one-sided DF test, the null hypothesis of a unit root in a series x_t, say H_0 : $(1-B)(x_t-\mu_t) = \xi_t$, is tested against the alternative H_1 : $(1-\rho B)(x_t-\mu_t) = \xi_t$ with $|\rho| < 1$, where μ_t denotes the mean of x_t. If the alternative is rejected then x_t is supposed to follow a unit root time series model.

Unit root time series models impose, however, severe restrictions on the DGP's of the data. For example, when the errors are negatively correlated, the DF test exhibits important size distortions (Schwert, 1989 [57]). Many real world time series exhibit also nonlinearities, outliers and structural breaks. All these features, which cannot be properly captured with random-walk-like models, fool standard unit root tests (see for instance, Granger and Hallman 1991 [18], and Ermini and Granger, 1993 [14]).

Alternative procedures for testing unit roots were proposed by Lo (1991 [28]), Kwiatowski et al. (1992 [25]), Stock (1994 [55]) and the recent contributions of Bai and Perron (1998 [7]). Yet, all these tests rely on assumptions which are too restrictive in practice. In fact, they were reported to have poor power performances when confronted to deviations from the standard linear context (see for example, Sims, 1988 [52]; Perron, 1989 [38]; Perron, 1990 [39]; Schotman and Van Dijk, 1991 [50]; Aparicio, 1995 [2]).

The appropriate handling of such departures as level shifts, trend breaks and nonlinearities calls for the development of robust unit root tests. Certainly, the rejection of the unit root hypothesis by standard tests together with the acceptance of a wider null by a robust testing procedure will lead us to seek alternative models for the time series structure.

In this paper, we propose a nonparametric device called the *Range Unit Root* (RUR hereafter) test, since its test statistic is constructed from the *running ranges* of the series. The RUR test is a natural follow-up of the methodology proposed in Aparicio (1995 [2]) and in Aparicio and Granger (1995 [4]) for robustizing cointegration tests. The key idea behind it the fact that the average number of level crossings for a unit root time series is smaller than for a stationary one (Burridge and Guerre, 1996 [9]).

The RUR test outperforms standard unit root tests in several aspects. First, because of its invariance to monotonic transformations of the series and its robustness to the presence of additive outliers, the size is not distorted under the latter deviations. This amounts at widening the null hypothesis. Second, because of its robustness to structural breaks or level shifts in the DGP of the series, the new test approaches the power of standard unit root tests when such features are not present. And third, because it does not depend on the variance of any stationary alternative, it improves considerably the power of standard tests on near-unit-root stationary time series.

The structure of the paper is as follows. In Section 2 we explain the heuristics which motivate the proposed methodology. This will lead us to define the RUR test in section 3, and study both its small sample and asymptotic properties under the null hypothesis of unit root. In section 4 we establish the invariance properties of the test, and analyse its small-sample power performances and robustness against a number of departures from the standard assumptions. Section 5 introduces a modification of the former RUR test that improves both its small-sample power in the presence of level shifts, and its size when additive outliers corrupts early the series. In section 6 we apply our testing methodology to a set of four real time series and compare the results with those obtained by means of standard unit root tests.Finally, after the concluding remarks in Section 7, an appendix is devoted to the proofs of the theoretical results.

2 A Characterization of Integrated Time Series based on Ranges

Many time series not generated by unit-root models exhibit similar mean behavior as those which are. The objective of this section is to present a nonparametric procedure for testing unit-root like features not necessarily caused by unit roots in time series. What we want is a procedure that is invariant, or at least robust, to certain departures from the standard unit root model. To achieve this, we first propose a characterization of this "long wave" in terms of what we call *Low-Frequency Features* (LFF hereafter). Following Granger and Terasvirta (1993 [19]) and Anderson and Vahid (1998 [1]), a *feature* is essentially any dominating statistical property exhibited by a time series. Features may refer to either the mean behavior or to higher-order moments of the series, such as heteroskedasticity. Here we are interested in the former; in particular in.those features that are potentially useful in revealing the presence of stochastic trends. These include the autocorrelation structure of the series or of nonlinear transformations of the latter, any existing growth rate, and whatever measure of mean reversion. Features are endowed with some algebraic properties. For instance, if x_t has a feature while y_t has not, then both λx_t and $y_t + x_t$ as well as any delayed replica of x_t, say x_{t-p} (where p is a positive integer), will have that feature. Roughly speaking, we could say that a time series has strong dependence in the mean if it exhibits a LFF.

Here we will consider a particular class of LFF's that are obtained by taking the difference of the *extremes* in an evergrowing sample of the series. This results in a sequence of *running ranges*. Formally, for a given time series x_t, the terms $x_{1,i} = \min\{x_1, \cdots, x_i\}$ and $x_{i,i} = \max\{x_1, \cdots, x_i\}$ are called the *i-th extremes* (see for instance Galambos, 1984 [17]). The sequence of ranges for x_t is then defined as $R_i^{(x)} = x_{i,i} - x_{1,i}$, for $i = 1, 2, 3, \cdots, n$, with n denoting the sample size. Basically, a process defined by a sequence of ranges is an *integrated jump process*, where the jumps $\Delta R_i^{(x)} = R_i^{(x)} - R_{i-1}^{(x)}$ are nonnegative quantities that will be different from zero each time i that a new maximum or a minimum is reached.

The behavior of the running ranges can be used to assess serial dependence in a single time series as well as the relationship between two time series (Aparicio and Escribano, 1998 [3]). One important finding is that the range sequence $R_i^{(x)}$ for stationary time series is *stochastically bounded* [1], whereas it is not for null-recurrent time series such as integrated time series or those having monotonic trends.

Figures 1, 2, 3 and 4 show respectively the sequences of running ranges corresponding to a realization of a random walk process $y_t = y_{t-1} + e_t$, where $e_t \sim Nid(0, 1)$ (Figure 1), a stationary Gaussian AR(1) process $y_t = 0.5y_{t-1} + e_t$ (Figure 2), the AR(1) process $y_t = 0.5y_{t-1} + \xi_t$, where the model errors ξ_t follow a student t distribution with 5 degrees of freedom (Figure 3), and finally the same model with ξ_t following a Cauchy distribution (Figure 4).

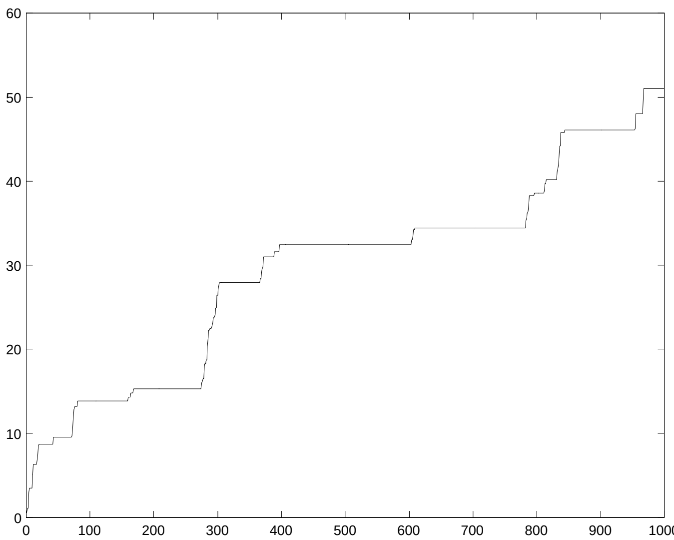

Figure 1. Sequence of running ranges for a Gaussian random walk process $y_t = y_{t-1} + e_t$, where $e_t \sim Nid(0, 1)$.

[1] A nonnegative sequence s_t is said to be *stochastically bounded* if for every positive real number ϵ,there exists a finite positive constant δ_ϵ such that $sup_t P(s_t \leq \delta_\epsilon) \geq 1 - \epsilon$.

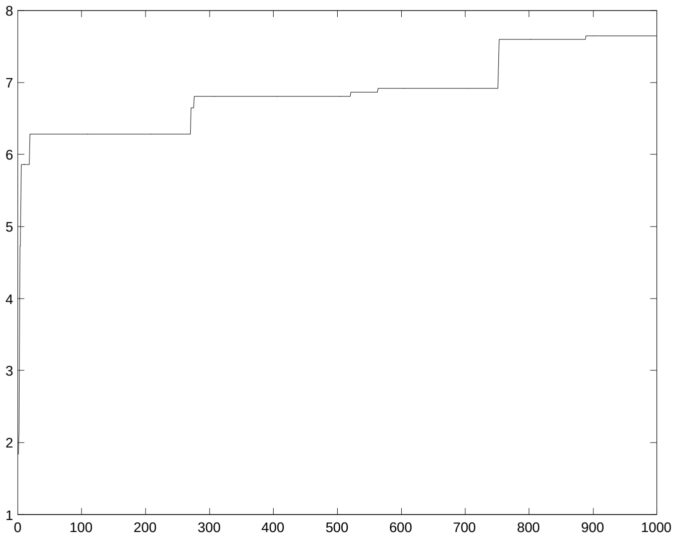

Figure 2. Sequence of running ranges for the Gaussian stationary AR(1) process $y_t = 0.5y_{t-1} + e_t$, with $e_t \sim Nid(0, 1)$.

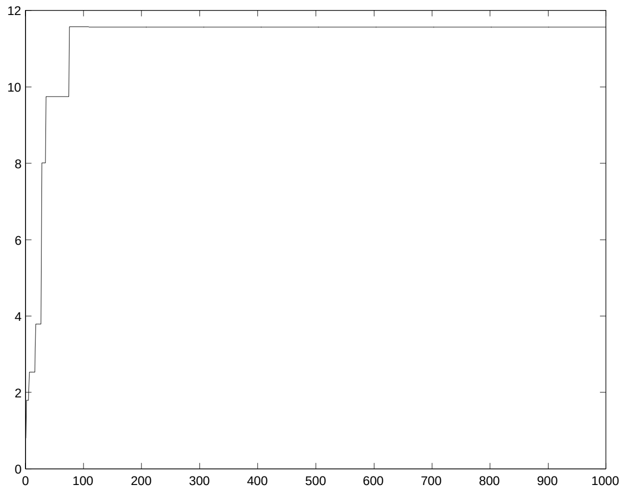

Figure 3. Sequence of running ranges for the AR(1) process $y_t = 0.5y_{t-1} + \xi_t$, where ξ_t is an *i.i.d.* sequence of random variables with a Student-t distribution with 5 degrees of freedom.

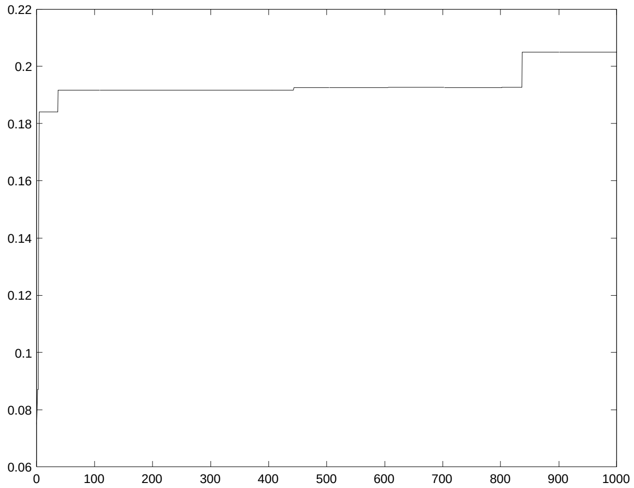

Figure 4. Sequence of running ranges for the AR(1) process $y_t = 0.5y_{t-1} + \xi_t$, where ξ_t is an *i.i.d.* sequence of random variables with a Cauchy distribution.

Figures 5 and 6 illustrate the same fact by showing an estimate of the probability of jumps larger than a small real number δ. The probability was estimated from 1000 replications and for a sample size of $n = 1000$. We took $\delta = 0.01$, although the rates of convergence to zero of the plotted frequencies were not significantly different for other small values of δ. It can be seen from these figures that when x_t is a stationary AR(1) series the frequency of new jump arrivals goes to zero for increasing t, while it seems to reach a floor when x_t is a random walk. Apparently, the existence of a "long-wave" is related to the persistence of new jumps. Put in other words, the asymptotic behavior of the jump sequence $\Delta R_t^{(x)}$ conveys useful information on the presence of an ongoing LFF or trending pattern. The graphs suggest therefore that the long-run frequency of the event $\Delta R_t^{(x)} > 0$ could be used as a measure of the *persistence* in a time series.

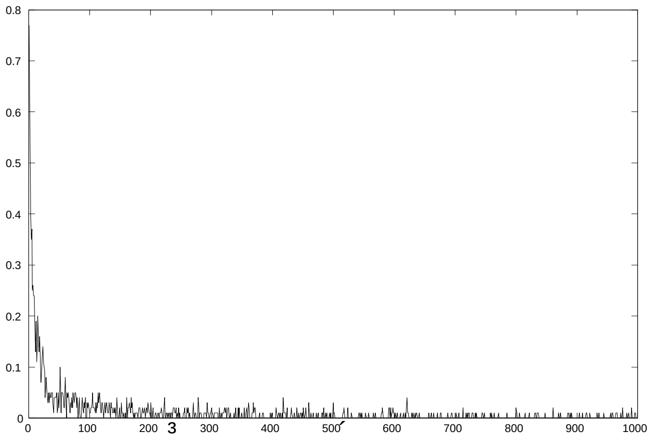

Figure 5. Estimate of $P \quad \Delta R_t^{(x)} > 0.01 \quad$ versus t when $x_t = 0.6x_{t-1} + \epsilon_t$ and $\epsilon_t \sim Nid(0, 1)$.

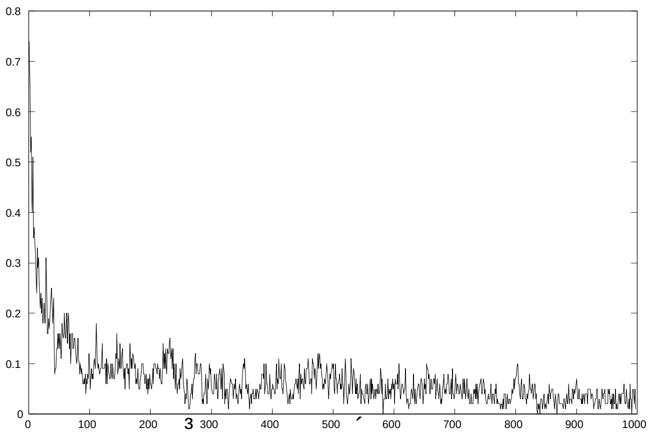

Figure 6. Estimate of $P \quad \Delta R_t^{(x)} > 0.01 \quad$ versus t when $x_t = x_{t-1} + \epsilon_t$ and $\epsilon_t \sim Nid(0, 1)$.

Another equivalent aspect of the previously illustrated property is captured by the mean interarrival times between consecutive maxima (or, equivalently, minima). These are also the "jump interarrival times" for the sequences of running range. Figures 7 and 8 show respectively the mean interarrival times between the first 50 consecutive maxima for a Normal random walk $y_t = 0.5y_{t-1} + e_t$, and for a stationary AR(1) Normal time series from the model $y_t = 0.5y_{t-1} + e_t$, where $e_t \sim Nid(0, 1)$. These mean interarrival times were estimated from 1000 replications of the models, each with a sample size of $n = 1000$.

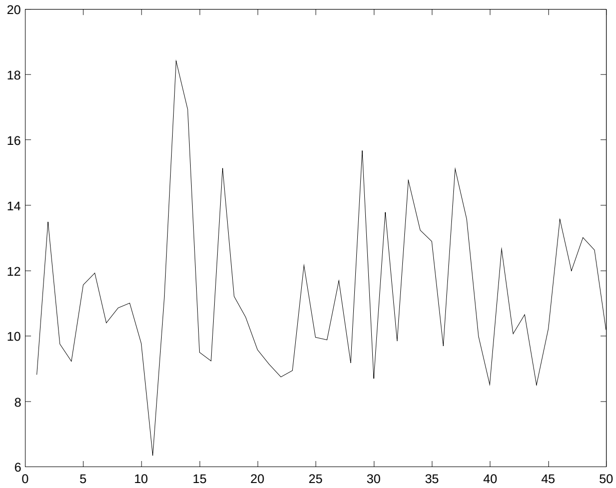

Figure 7. Mean interarrival times between the first 50 consecutive maxima of the Gaussian random walk process $y_t = y_{t-1} + e_t$, where $e_t \sim Nid(0, 1)$.

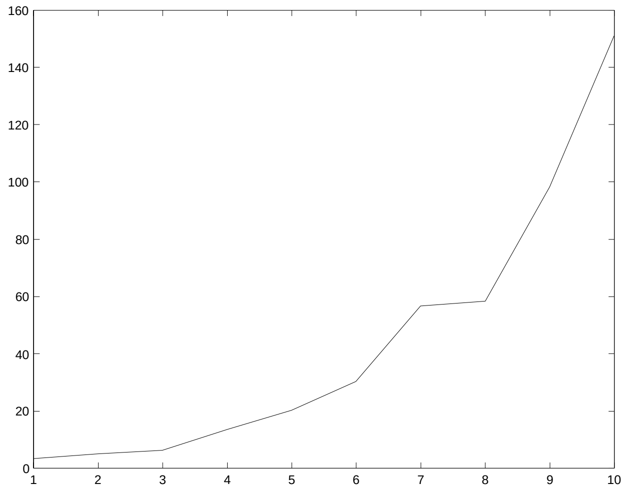

Figure 8. Mean interarrival times between the first 50 consecutive maxima of the Gaussian AR(1) process $y_t = 0.5y_{t-1} + e_t$, where $e_t \sim Nid(0, 1)$.

The figures show clearly that the sequence of interarrival times is stable for a random walk, but exploding for the stationary AR(1) process.

3 The Range Unit Root (RUR) test

In this section we first present the test statistic upon which the proposed unit root testing methodology is based. Then we analyse its small-sample behavior under the null hypothesis of a single unit root, provide some asymptotic results concerning this behavior, and finally, study its small-sample power performances.

3.1 The test statistic

In the sequel we will consider the statistic $J_0^{(n)}$ defined below for testing the null hypothesis of a single unit root in a time series. We will refer to this testing device as the *Range Unit Root* (RUR hearafter) test. Farther in the paper we will show that this test is either robust or invariant to a number of departures from this null hypothesis.

$$J_0^{(n)} = \frac{1}{\sqrt{n}} \sum_{t=1} \mathbf{1}(\Delta R_t^{(x)} > 0). \tag{1}$$

The following set of assumptions from Phillips (1987 [43]) on the model errors ϵ_t will be needed in the subsequent results. These assumptions allows trading an increasing degree of temporal dependence against a decreasing degree of heteroskedasticity (and viceversa) in the process.

A1. $E(\epsilon_t) = 0$.

A2. $\sup_t E(|\epsilon_t|^p) < C < \infty$ for some $p > 2$.

A3. $0 < \lim_{n\to\infty} E\left[n^{-1} \sum_{t=1}^{n} \epsilon_t^2\right] < \infty$.

A4. $\{\epsilon_t\}_{t=1}^{\infty}$ is strong mixing with mixing coefficients $\{\alpha_m\}_{m=1}^{\infty}$ satisfying $\sum_{m=1}^{\infty} \alpha_m^{1-2/p} < \infty$.

Our first result refers to the rate of divergence of the series of partial sums $\sum_{t=1}^{n} \mathbf{1}(\Delta R_t^{(x)} > 0)$.

Theorem 1 *Under the null hypothesis $H_0 : x_t = x_{t-1} + \epsilon_t$, where ϵ_t satisfies assumptions A1-A4, $J_0^{(n)}$ converges to a non-degenerate random variable as n grows to infinity.*

Proof: see Appendix 1.

The statistic $J_0^{(n)}$ can be interpreted as a measure of the errors in predicting the range of x_t at t, $R_t^{(x)}$, by means of its value at time $t-1$, $R_{t-1}^{(x)}$. More exactly, $n^{-1/2} J_0^{(n)}$ represents the proportion of these prediction errors in a sample of size n. Given the non-ergodic nature of x_t when $x_t = x_{t-1} + \epsilon_t$, the "re-scaled" sample proportion $J_0^{(n)}$ does not converge to a constant value but to a random variable, as it will be shown later. On the contrary, when $x_t \sim I(0)$ the range sequence $R_t^{(x)}$ is stochastically bounded. Therefore $J_0^{(n)}$ should be expected to converge in probability to zero. This means that when $x_t \sim I(0)$, ${}_t^{(x)} = R_{t-1}^{(x)}$ is a consistant predictor of $R_t^{(x)}$, while it is not when $x_t \sim I(1)$. As the sample size approaches infinity, $J_0^{(n)}$ measures the persistence of the one-step-ahead range prediction errors, or equivalently, the divergence rate of the range of x_t. Essentially, the test statistic $J_0^{(n)}$ will be expected to take comparatively large values for $I(1)$ time series while small for $I(0)$ time series.

3.2 Small-sample behavior under the null

Summary statistics for $J_0^{(n)}$ under the null hypothesis are given in Table 1 for a sample size of n =1000, and for $Nid(0,1)$ errors.

summary statistics	*minimum*	*maximum*	*mean*	*median*	*std. dev.*
estimates	0.80	4.65	2.11	2.02	0.63

Table 1: Summary statistics for $J_0^{(n)}$ estimated from samples of random walks with $Nid(0,1)$ errors, for a sample size of $n = 1000$.

The critical values of $J_0^{(n)}$ were estimated under the model $x_t = x_{t-1} + \epsilon_t$, where $\epsilon_t \sim Nid(0,1)$. The critical values, estimated from 10000 replications, for eigth different sample sizes and six significance levels ($\alpha = 0.01, 0.025, 0.05, 0.10, 0.90, 0.95$),.are shown in Table 2. Figure 9 shows the convergence towards their asymptotic values as the sample size n is increased.

$\alpha \mid n$	100	250	500	1000	2000	3000	4000	5000
0.01	0.9	0.9391	1.0119	1.0435	1.1180	1.1502	1.0594	1.0465
0.025	1.0	1.0752	1.1180	1.2333	1.1404	1.1502	1.2491	1.1031
0.05	1.1	1.2017	1.2075	1.2649	1.2746	1.3145	1.3123	1.3152
0.10	1.3	1.3282	1.3416	1.3598	1.2969	1.3510	1.3756	1.4425
0.90	2.8	2.9725	2.9963	3.0990	3.2870	2.9212	3.0042	2.7577
0.95	3.1	3.2888	3.3541	3.3520	3.6001	3.2498	3.3046	3.1396

Table 2: Empirical critical values of the RUR test for different sample sizes and for different significance levels.

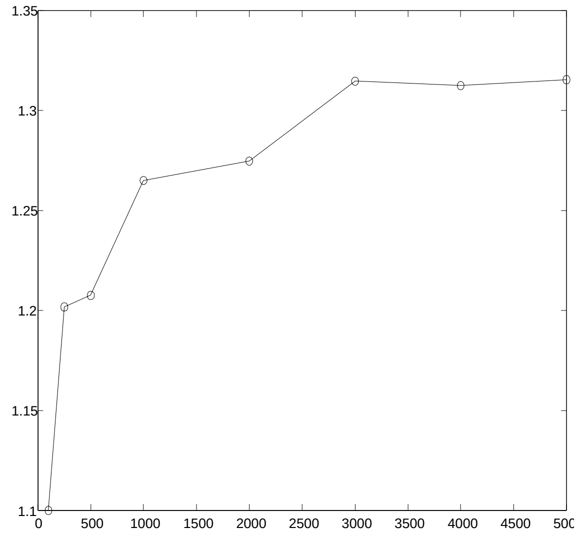

Figure 9. Convergence of the 5%-level empirical critical values of $J_0^{(n)}$ (vertical axis) towards the asymptotic values with increasing sample size (horizontal axis).

Figure 10 shows the empirical density of $J_0^{(n)}$ estimated by kernel smoothing, again under the null hypothesis of a random walk with $Nid(0,1)$ errors. The estimates were obtained from 1000 replications and for different sample sizes using the *Epanechnikov* kernel (see for example Silverman, 1986 [51] and Hardle, 1990 [21]), which is optimal in the the mean-square error (MSE) sense[2].

[2] The density of $J_0^{(n)}$ was estimated as $\hat{f}_h(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)$, with $K(.)$ given by $K(u) = \frac{3}{4}(1-u^2)1(|u| \leq 1)$.

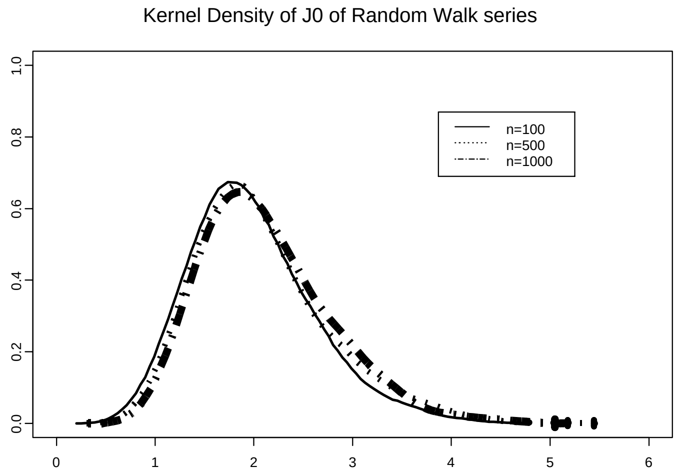

Figure 10. Plot of the empirical density of $J_0^{(n)}$ under the null hypothesis $H_0:\ x_t = x_{t-1} + \epsilon_t$, where $\epsilon_t \sim Nid(0,1)$. The density was estimated using the Epanechnikov kernel on different sample sizes.

3.3 Asymptotics

In order to obtain asymptotic results for our test statistic $J_0^{(n)}$, it is useful to split it into two terms:

$$J_0^{(n)} = \frac{1}{\sqrt{N}} \sum_{n=1} \mathbb{1}(\Delta R_n^{(x)} > 0)$$

$$= \frac{1}{\sqrt{N}} \sum_{n=1} \mathbb{1}(x_{n,n} = x_n) + \frac{1}{\sqrt{N}} \sum_{n=1} \mathbb{1}(x_{1,n} = x_n).$$

Now let

$$J_1^{(n)} = \frac{1}{\sqrt{N}} \sum_{n=1} \mathbb{1}(x_{n,n} = x_n),$$

$$J_2^{(n)} = \frac{1}{\sqrt{N}} \sum_{n=1} \mathbb{1}(x_{1,n} = x_n).$$

The theorem below establishes the asymptotic null distribution of $J_1^{(n)}$ and $J_2^{(n)}$ in terms of the *local time* of a Brownian motion process W_t.on the interval $[0,1]$

The local time $L_t(x)$ of a Brownian motion measures the amount of time spent by this process in the neighborhood of the point x up to time t (see Revuz and Yor, 1991 [47]).

$$L_t(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}(x - \epsilon < W_s < x + \epsilon) ds.$$

More details can be found in the Appendix.

Theorem 2 *Let $L_t(x)$ denote the local time defined as above. Under the null hypothesis $H_0 : x_t = x_{t-1} + \epsilon_t$ with ϵ_t satisfying assumptions A1-A4, we have: (i) $J_1^{(n)} \to 2L_1(0)$, (ii) $J_2^{(n)} \to 2L_1(0)$.*

Proof: see Appendix 2.

Our next theorem establishes the asymptotic independence of $J_1^{(n)}$ and $J_2^{(n)}$, which allows us to obtain the density of $J_0^{(n)}$ as the convolution of the densities of $J_1^{(n)}$ and $J_2^{(n)}$.

Theorem 3 *Under the null hypothesis H_0, $J_1^{(n)}$ and $J_2^{(n)}$ are asymptotically independent .*

Proof: see Appendix 3.

Corollary 1. Let f_{J1+J2} represent the asymptotic probability density function of $J_1^{(n)} + J_2^{(n)}$. Then

$f_{J1+J2}(z) = (2\pi)^{-1/2} \exp\left\{ -\frac{z^2+2}{4} \right\} (1 - \Psi(z))$ $0 < z < \infty$, where Ψ denotes the distribution function of a standard Normal random variable.

Proof: see Appendix 4.

3.4 Small-sample power performances

Here we compute the small-sample power performances of the RUR test using the estimated critical values at the 5% significance level, and against the alternative of a stationary $AR(1)$ time series with standard Normal errors that is against the model $x_t = b\ x_{t-1} + \epsilon_t$.where $\epsilon_t \sim Nid(0, 1)$. This was done for different values of the autoregressive parameter ($b = 0.5, 0.8, 0.9$) and for three different sample sizes ($n = 100, 250, 500$). The power estimates, obtained from 10000 replications of the previous model, are shown in Table 3 together with the DF results (given in brackets).

$n \mid b$	0.5	0.8	0.9	0.95	0.99
100	0.8 (1)	0.6 (0.99)	0.5 (0.5)	0.4 (0.18)	0.12 (0.0375)
250	1 (1)	1 (1)	1 (1)	0.8 (0.7)	0.47 (0.0760)
500	1 (1)	1 (1)	1 (1)	1 (0.99)	0.72 (0.39)

Table 3: Power of the RUR test at the 5% significance level against the model $x_t = b\, x_{t-1} + \epsilon_t$ for different sample sizes n $(100, 250, 500)$, and for different values of the parameter b $(0.5, 0.8, 0.9)$.

These results show the DF test outperforms the RUR test in only two cases: (i) when the sample size is comparatively small ($n = 100$), and (ii) when the autoregression parameter b is not close to 1. Otherwise, it is noteworthy the improved power performances of the RUR test over the DF test against stationary alternatives close the nonstationary border.

The power curves are plotted in Figure 11 below for three different sample sizes ($n = 100$, 250, 500). The continuous and dotted lines correspond respectively to the DF and the RUR tests.

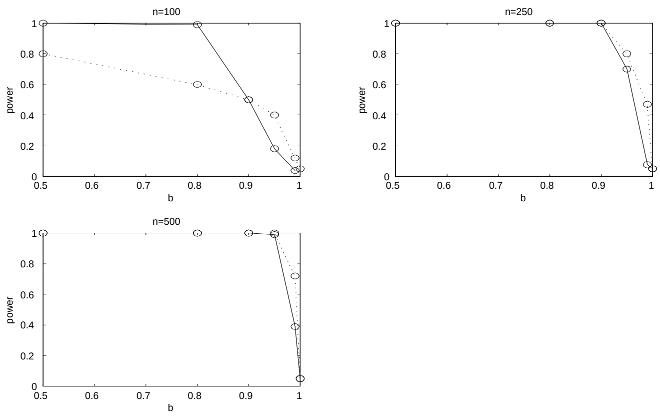

Figure 11. Plots of the power estimates of DF and RUR tests against stationary Gaussian AR(1) series.

Thus as compared to the DF test, the RUR test establishes a sharper frontier between the null hypothesis of unit root and the stationary AR(1) alternatives. This is partly a consequence of the invariance of the RUR test statistic $J_0^{(n)}$ with respect to the finite variance of the stationary alternative, which we state in the next proposition.

Proposition 1. *Let x_t be a stationary time series with finite variance σ_x^2, and let a be any nonzero real number. Let $J_0^{(n)}(x)$ be the RUR test statistic applied to x_t. Then we have:*

$$J_0^{(n)}(ax) = J_0^{(n)}(x) \tag{2}$$

Proof: see Appendix 5.

4 The RUR Test Statistic under Departures from the Standard Assumptions

Another important property of the RUR test is its robustness to departures from the standard assumptions. In this paper, we consider three types of departures: a) when a stationary time series undergoes structural breaks; b) when $I(1)$ time series are corrupted by additive outliers; and c) when $I(1)$ time series are nonlinearly transformed. In the sequel we study the small sample behavior of the RUR test in the presence of each of the above-mentioned departures from the standard unit-root tests assumptions.

4.1 Stationary time series with level shifts

Many economic and financial time series such as inflation, nominal and real interest rates can be trend-stationary with a structural break in the unconditional mean which affects the standard inferential procedures and often makes constant coefficient models to perform poorly in practice (see for instance Perron, 1990 [39], and Malliaropulos, 2000 [33]). The literature on testing for unit roots in the presence of both known and unknown break points is large (see Maddala and Kim, 1998 [32] for a review). Perron (1989 [38]) and Perron and Vogelsang (1992 [41]) reported evidence that structural breaks can make an $I(0)$ time series behave locally as $I(1)$ and, as a result, these breaks are able to fool standard unit root tests (this is shown by the simulations below). More precisely, Perron (1989 [38]) and Rappoport and Reichlin (1989 [46]) showed via Monte Carlo experiments that time series such as GNP (previously modelled as $I(1)$) appear as $I(0)$ if we allow for a segmented trend in the model during the oil crisis. In brief, if the permanent break is not explicitly taken into account standard unit root tests tend to find too many unit roots. However, as shown by Leybourne, Mills and Newbold (1998 [27]), it is also possible to reach the opposite conclusion when the break's location appears at the beginning of the sample, that is that an $I(0)$ time series be interpreted as $I(1)$. Moreover, the critical values of standard unit root tests depend on the new unknown nuisance parameters such as the number of breaks and their timing, which has led several authors (see Zivot and Andrews, 1992 [59]; Perron and Vogelsang, 1992 [41]; Banerjee, Lumsdaine and Stock , 1992 [8]; and Stock, 1994 [55]) to propose recursive and sequential testing procedures in order to estimate these parameters. In the light of the previous difficulties, it may be interesting to analyse the power of our RUR test when confronted to the alternative of a stationary $AR(1)$ time series corrupted by such breaks, that is $x_t = 0.5\ x_{t-1} + s\ D_t + \epsilon_t$, where D_t represents a dummy variable defined by $D_t = 0$ for $t \leq n/2$ and $D_t = 1$ for $t > n/2$. Table 4 provides power estimates from 10000 replications for different values of the local break size s. The power figures corresponding to the Dickey-Fuller test applied to the same replication of the model are given in brackets.

$n \mid s$	4	8	12
100	0.2 (0.00)	0.08 (0.00)	0.07 (0.00)
250	0.7 (0.00)	0.6 (0.00)	0.6 (0.00)
500	1 (0.86)	1 (0.00)	1 (0.00)

Table 4: Power of the RUR test at the 5% significance level against model $x_t = 0.5\ x_{t-1} + s\ D_t + \epsilon_t$, where $D_t = 0$ for $t \leq n/2$ and $D_t = 1$ for $t > n/2$. The power estimates are given for different values of the sample size $n(100, 250, 500)$, and of the break size s ($s = 4, 8, 12$). The performance of the DF test accompany these figures in brakets.

We remark that except for the more favorable case of $s = 4$ and $n = 500$, the Dickey-Fuller (DF) test has a much stronger bias towards nonstationarity, thus suggesting that the RUR test is less prone to misinterpret structural breaks as permanent stochastic disturbances.

In a scenario allowing for multiple breaks, we should expect an even larger decrease in power for both the RUR and the DF tests. In order to assess these power losses, we performed another experiment which included two breaks at different locations in time. Our model was therefore $x_t = 0.5\ x_{t-1} + s_1\ D_{t,1} + s_2 D_{t,2} + \epsilon_t$, where $D_{t,i}$ $(i = 1, 2)$ represents a dummy variable defined by $D_{t,i} = 0$ for $t \leq in/4$ and $D_{t,i} = 1$ for $in/4 < t \leq in/2$. Table 5 shows the power results obtained from 10000 replications of this model, for both the RUR and the DF tests (the DF figures given in brackets). Once again, the RUR test outperfoms the DF results in all cases, and is still remarkably powerful for sample sizes as small as $n = 500$, as far as the break size is not too large.

n |$\mathsf{s}_{1,2}$	(2,4)	(4,8)	(8,12)
100	0.07 (0.000)	0.005 (0.000)	0.000 (0.000)
250	0.5 (0.000)	0.200 (0.000)	0.05 (0.000)
500	1 (0.453)	0.7 (0.000)	0.6 (0.000)

Table 5: Power of the RUR test at the 5% significance level against the model $x_t = 0.5\ x_{t-1} + s_1\ D_{t,1} + s_2 D_{t,2} + \epsilon_t$, where $D_{t,i}$ $(i = 1, 2)$ represents a dummy variable defined by $D_{t,i} = 0$ for $t \leq in/4$ and $D_{t,i} = 1$ for $in/4 < t \leq in/2$. Here $\mathsf{s}_{1,2} = (s_1, s_2)'$. The power estimates are given for different sample sizes (n=100,250,500), and for different values of s_1 and s_2 ($s_1 = 2, 4, 8$, and $s_2 = 4, 8, 12$,respectively). The performance of the DF test is given by the figures in brakets

4.2 Nonlinearly transformed $I(1)$ time series

In practice, it is difficult (or even impossible) to know whether a time series exhibiting "unit-root-like" mean behavior is really $I(1)$, or rather a monotonically nonlinear transformation of an $I(1)$ series. With standard unit-root tests such as the DF test, misspecification of the true time series model may affect the rate of divergence of the test statistic, causing it to be inconsistent. It is therefore desirable that a unit-root test could avoid such ambiguities.

Granger and Hallman (1991 [18]) looked at the autocorrelation function of several nonlinear transformations of the original series and proposed a test invariant to monotonic transformations based on ranks. Ermini and Granger (1993 [14]) worked with the Hermite polynomial expansion of different nonlinear transformations of random walks, possibly with drift, and showed that the autocorrelation function is not always a reliable indicator of the degree of memory of nonlinear time series.

In this section we analyse the small sample behavior of the RUR test in the face of several nonlinear transformations of random walks. Table 6 shows the results estimated at the 5% significance level from 10000 replications of the different models and for three different sample sizes ($n = 100, 250, 500$).

Transformation	100	250	500
1) x_t^2	0.079 (0.397)	0.170 (0.406)	0.178 (0.420)
2) x_t^2, with $x_t > 0$, $\forall t$	0.03 (0.397)	0.059 (0.406)	0.048 (0.420)
3) x_t^3	0.038 (0.456)	0.057 (0.532)	0.049 (0.533)
4) $exp(x_t)$	0.03 (0.92)	0.05 (1)	0.0469 (1)
5) $exp(\frac{x_t}{75})$	0.054 (0.271)	0.0526 (0.271)	0.05 (0.301)
6) $\log(x_t + 100)$	0.043 (0.275)	0.064 (0.331)	0.051 (0.354)
7) $\log(\frac{x_t+2\sqrt{T}}{4\sqrt{T}})$, $\frac{x_t+2\sqrt{T}}{4\sqrt{T}} \in (0,1)$	0.072 (0.347)	0.054 (0.349)	0.051 (0.354)
8) $\sin(x_t)$	0.8828 (1)	0.9986 (1)	1 (1)

Table 6: Size of the RUR test against differents forms of nonlinearity applied to a random walk $x_t = x_{t-1} + \epsilon_t$, where $\epsilon_t \sim Nid(0,1)$. The proportion of model replications for which the null hypothesis was rejected at the 5% significance level are given for different sample sizes ($n = 100, 250, 500$). The total number of replications was 10000. As usual, DF performances are shown in brackets.

It is worth observing that the size tends towards its correct value in all cases except when the transformation is non-monotonic (case 1) or when it stationarizes the series (case 8). To study more precisely the effect of the logarithmic nonlinearity, in case 7), we forced the variable to take most of its values in the interval $(0,1)$. This was done by transforming linearly the series prior to applying the logarithmic transformation. Since in this interval the function is not so well approximated by a straight line, one would expect a more noticeable size distortion for the smaller sample size of $n = 100$. Overall, however, all the empirical sizes for the purley monotonic transformations seem to converge to the nominal size of 0.05 as the sample size grows. In fact, it can be shown that the RUR is invariant to monotonic nonlinear transformations of $I(1)$ time series. We state this property in the following proposition.

Proposition 2. *Let f be a monotonic transformation applied to an $I(1)$ time series x_t, and let $J_0^{(n)}(x)$ be the RUR test statistic applied to x_t. Then we have:*

$$J_0^{(n)}(f(x)) = J_0^{(n)}(x) \tag{3}$$

Proof: see Appendix 6.

This result is a natural consequence of the invariance of the number of level crossings to monotonic transformations of a time series.

4.3 Integrated time series corrupted by additive outliers

Outlying observations form another class of non-repetitive events which may occur for different reasons, including eventual measurement errors, and recordings of unusual events such as wars, disasters and dramatic policy changes. Some commonplace outlier-inducing events in economic time series are union strikes, hoarding consumer behavior in response to a policy announcement, and computer breakdown effects on unemployment or sales data collection and processing, to name a few. Outliers can also appear as a result of misspecified estimated relationships or omitted variables (see Peña, 2001 [42]).

Outliers are often classified into two groups: additive outliers (AO) and innovation outliers (IO), of which the former ones have the most insidious effects on classical inference. In both cases, standard unit root tests are biased towards the rejection of the unit root hypothesis. An AO corresponds to an external error or exogenous change in the observed value of the time series at a particular instant, but with no effect on the subsequent observations in the series. Formally, instead of observing the original series x_t, we observe a corrupted series y_t, which in the case of a single AO is given by:

$$y_t = \begin{cases} x_t & t \neq T \\ x_t + s & t = T \end{cases}$$

where s represents the outlier magnitude.

There is a sort of duality between the effects of AO's and those of structural breaks on time series. Indeed while $I(0)$ time series subject to level shifts could be misinterpreted as $I(1)$, $I(1)$ time series corrupted by AO's might look like $I(0)$ provided that the outliers are sufficiently important in magnitude or in frequence. In particular, it is known that the presence of AO's leads to a downward bias of the OLS parameter estimates in a stationary AR(1) process (Bustos and Yohai, 1986 [10]; Martin and Yohai, 1986 [34]), and that unit-root inferences are sensitive to the presence of extreme observations. For example, the DF test will have an actual size in excess of the nominal size and thus will reject the unit-root hypothesis too often. The size distortion of the DF test in the presence of this type of outliers was quantified by Haldrup and Hanses (1994 [20]), who also demonstrated that the distribution of the parameter estimates changed dramatically when both the magnitude of the outliers and their frequency become large.

Traditionally, the presence of AO's is dealt with either by attaching less weight to the extreme observations in the sample, by removing them with the inclusion of a dummy variable in the model, or by treating them as missing observations. This two-stage approach was followed in Arranz and Escribano (1998 [5]), who proposed filtering the contaminated series prior to standard unit-root testing. Single-stage robust unit-root tests were first proposed by Lucas (1995 a,b [29,30]) and by Franses and Lucas (1997 [15]) using M-estimators with high breakdown point and efficiency, instead of OLS estimators. However, these tests were really conceived for dealing with fat-tailed distributions of the model errors and, as a result, were less powerful than standard unit root tests on Normally distributed errors. Alternatively, some authors have followed a likelihood-based approach where inference is made about a particular fat-tailed distribution rather than on the Gaussian distribution (Hoek et al., 1995 [23]; Rothenberg and Stock, 1997 [49]). The use of nonparametric statistics is another avenue of research in robust unit root testing. Hasan and Kroenker (1997 [22]) applied rank-based methods.to this problem and reported improved power performances on time series corrupted by a few large observations. The RUR procedure that we propose also belongs to the nonparametric category.

The results in Table 7 and Table 8 show that the size distortions caused by the presence of an AO in the middle of the series and beyond are much smaller for the RUR test than for the DF test (shown in brackets). When the AO appears near the end of the series (Table 8) the RUR test have even lower than nominal sizes. In any of these scenarios, our alternative hypothesis is embodied in the model $y_t = x_t + s\delta_{t,\tau}$ where $x_t = x_{t-1} + \epsilon_t$, τ denotes an integer no larger than the sample size, and $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes were estimated at the 5% significance level, for different values of both τ ($\tau = n/25, n/10, n/5$) and the sample size n $(100, 250, 500)$.

$n \mid \tau$	$n/2$	$n/2+1$	$n/2+2$
100	0.0826 (0.2978)	0.0830 (0.2964)	0.0812 (0.2958)
250	0.0800 (0.1682)	0.0800 (0.1688)	0.0798 (0.1670)
500	0.0644 (0.1130)	0.0640 (0.1102)	0.0642 (0.1096)

Table 7. Size of the RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$ where $x_t = x_{t-1} + \epsilon_t$and τ is an integer no larger than the sample size n, with $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different values of the sample size $(100, 250, 500)$ and for $s = 10$. The performance of the DF test is shown in brackets.

$n \mid \tau$	$n - n/20$	$n - n/10$	$n - n/5$
100	0.0212 (0.2964)	0.0244 (0.2990)	0.0352 (0.2980)
250	0.0392 (0.1704)	0.0422 (0.1660)	0.0484 (0.1656)
500	0.0446 (0.1106)	0.0472 (0.1104)	0.0510 (0.1118)

Table 8. Size of the RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$ where $x_t = x_{t-1} + \epsilon_t$, and τ is an integer no larger than the sample size n, with $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different sample sizes $(100, 250, 500)$ and for $s = 10$. The performance of the DF test is given by the figures in brackets.

Unfortunately, an early AO will produce a jump in the sequence of ranges which may prevent other jumps from appearing, thereby biasing the RUR test towards the rejection of the null hypothesis of unit root. The bias will be larger the sooner the outlier appears in the series. In order to grasp more closely this problem, we performed another Monte Carlo experiments in which the single AO is introduced near the origin (Table 9).

The results show that when the AO appears within the first quarter of the sample, the RUR test seems to offer no real improvement over the DF test.

n |τ	$n/25$	$n/10$	$n/5$
100	0.3778 (0.2956)	0.3192 (0.2964)	0.2432 (0.3002)
250	0.2746 (0.1672)	0.2230 (0.1668)	0.1700 (0.1676)
500	0.1930 (0.1114)	0.1588 (0.1112)	0.1188 (0.1110)

Table 9. Size of the RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$ where $x_t = x_{t-1} + \epsilon_t$, and τ is an integer no larger than the sample size n, with $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different values of the sample size $(100, 250, 500)$ and for $s = 10$. The performance of the DF test is given

5 The Forward-Backward Range Unit Root (FB-RUR) Test

Unless we know the outlier locations, the amount of size distortion or bias of the RUR test, based on the statistic $J_0^{(n)}$, when confonted to time series with AO's will be uncertain. By means of a simple resampling technique, we obtain, in this section, an extension of the RUR test, called the *Forward-Backward RUR (FB-RUR) Test*, based on the statistic here noted as $J_{0,*}^{(n)}$, which remarkably reduces the size distortion when the AO occurs at the beginning of the sample, and which turns out to be smaller than with the DF test. The FB-RUR test also improves the power performances of the former RUR test.

The key idea of this extension consists in running the RUR test first forwards (from the beginning to the end of the sample) and then backwards (from the end to the beginning). The total jump counts corresponds therefore to a sample size twice the original one, thus leading to improved power performances, in general. The FB-RUR test statistic $J_{0,*}^{(n)}$ can be formulated as follows:

$$J_{0,*}^{(n)} = \frac{1}{\sqrt{2n}} \sum_{t=1}^{n} \left(\mathbf{1}(\Delta R_t^{(x)} > 0) + \mathbf{1}(\Delta R_t^{(x')} > 0) \right), \qquad (4)$$

where $x'_t = x_{n-t+1}$ denotes the time-reversed series.

Table 10 shows that the critical values at the 5% significance level for the test based on $J_{0,*}^{(n)}$ are almost undistinguishable from those of $J_0^{(n)}$, under the null hypothesis of a random walk with $Nid(0,1)$ errors.

α/n	100	250	500	1000	2000	3000	4000	5000
0.01	1.2728	1.4311	1.4863	1.5428	1.5337	1.5233	1.3416	1.6400
0.025	1.3435	1.7889	1.7076	1.7441	1.6601	1.6395	1.8000	1.7300
0.05	1.6971	1.7889	1.7709	1.8112	1.7866	1.7686	1.9230	1.7700
0.10	1.9092	1.9677	1.9606	1.9230	1.9922	1.8590	2.0571	2.0800
0.90	3.9598	4.1591	4.0477	4.2933	4.5853	4.1699	4.1144	3.9400
0.95	4.6669	4.6957	4.6802	4.5839	5.0280	4.5572	4.9193	4.3500

Table 10: Empirical critical values of the FB-RUR test based on $J_{0,*}^{(n)}$ for different sample sizes

Figure 12 shows the empirical density of $J_{0,*}^{(n)}$ estimated by kernel smoothing under the null hypothesis of a random walk with *i.i.d.* Normally distributed errors having zero mean and unit variance. The estimates were obtained again from 1000 replications of the null model and for three different sample sizes, using the *Epanechnikov* kernel.

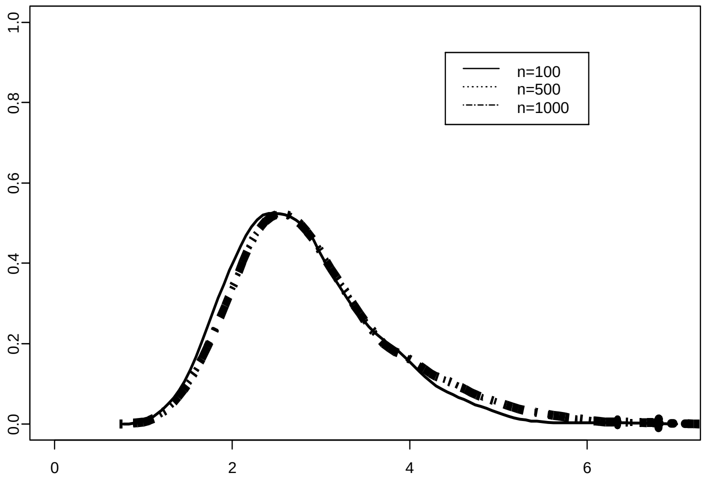

Figure 12. Plot of the empirical density of $J_{0,*}^{(n)}$ for $H_0 : x_t = x_{t-1} + \epsilon_t$ with $\epsilon_t \sim Nid(0,1)$. The density was estimated using the kernel and for different sample sizes.

The power of the FB-RUR test against the alternative of a stationary $AR(1)$ time series is shown in Table 11, using the same experimental set of parameters as was used in Table 3. We remark some improvements in power performances, especially for the smaller sample sizes, where now DF (whose results are again shown in brackets) is outperformed, or at least equated, in all the cases except when the value of the autoregression parameter b is 0.8. The corresponding power curves are plotted in Figure 13. The continuous and dotted lines correspond respectively to the DF and FB-RUR tests.

n |b	0.5	0.8	0.9	0.95	0.99
100	1.00 (1.00)	0.80 (0.99)	0.60 (0.5)	0.5 (0.18)	0.2 (0.0375)
250	1.00 (1.00)	1.00 (1.00)	1.00 (1)	0.9 (0.7)	0.52 (0.0760)
500	1.00 (1.00)	1.00 (1.00)	1.00 (1)	1 (0.99)	0.8 (0.39)

Table 11: Power of the FB-RUR test at the 5% significance level against the model $x_t = b\,x_{t-1} + \epsilon_t$ for different sample sizes ($n = 100, 250, 500$), and for different values of the autoregression parameter b in the previous model ($b = 0.5, 0.8, 0.9$)

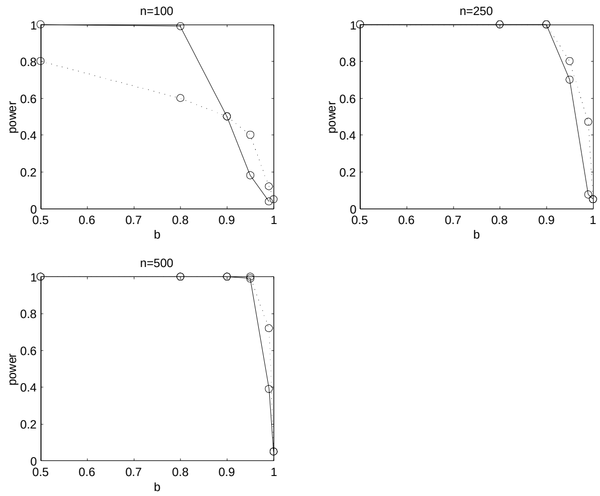

Figure 13. Power of FB-RUR and DF tests against the stationary series of Table 11.

In order to quantify the size distortion of the FB-RUR test in the presence of additive outliers, we used the same experimental framework as for the original RUR test. Tables 12 through 14 show the Monte Carlo results, depending on whether the single outlier's location is at the beginning, in the middle, or at the end of the sample. For comparison, we let the DF test results appear in brackets.

n | τ	$n/25$	$n/10$	$n/5$
100	0.1206 (0.2956)	0.0880 (0.2964)	0.0550 (0.3002)
250	0.1156 (0.1672)	0.0950 (0.1668)	0.0678 (0.1676)
500	0.0918 (0.1114)	0.0726 (0.1112)	0.0638 (0.1110)

Table 12: Size of the FB-RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$, where $x_t = x_{t-1} + \epsilon_t$, τ is a positive integer no larger than the sample size, and $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different values of the sample sizes $n(100, 250, 500)$, and for $s = 10$. The performance of the DF test is given in brackets.

n | τ	$n/2$	$n/2+1$	$n/2+2$
100	0.0826 (0.2978)	0.0830 (0.2964)	0.0812 (0.2958)
250	0.0800 (0.1682)	0.0800 (0.1688)	0.0798 (0.1670)
500	0.0644 (0.1130)	0.0640 (0.1102)	0.0642 (0.1096)

Table 13: Size of the FB-RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$ where $x_t = x_{t-1} + \epsilon_t$, τ is a positive integer no larger than the sample size, and $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different values of the sample size n $(100, 250, 500)$, and for $s = 10$. The performance of the DF test is given in brackets.

n|τ	$n - n/20$	$n - n/10$	$n - n/5$
100	0.0468 (0.2964)	0.0650 (0.2990)	0.0726 (0.2980)
250	0.0704 (0.1704)	0.0750 (0.1660)	0.0768 (0.1656)
500	0.0626 (0.1106)	0.0638 (0.1104)	0.0656 (0.1118)

Table 14: Size of the FB-RUR test against model $y_t = x_t + s\ \delta_{t,\tau}$, where $x_t = x_{t-1} + \epsilon_t$, τ is a positive integer no larger than the sample size, and $\delta_{t,\tau} = 1$ if $t = \tau$ and zero elsewhere. The sizes at the 5% significance level, estimated from 10000 replications of the model, are given for different values of the sample size n $(100, 250, 500)$, and for $s = 10$. The performance of the DF test is given by in brackets.

On the one hand, notice that even in cases where the outlying observations appear at the beginning of the data sample, the FB-RUR test robustises the DF test. Indeed, since in this case the last jump occurs at the largest outlier's location, $J_0^{(n)}$ tends to be very small (and, asymptotically, zero), whereas $J_{0,*}^{(n)}$ will only be approximately reduced by a factor of 1/2 with respect to the case of no outliers. Notice also that we should not expect any improvement in performances of the FB-RUR test over the RUR test when the AO's occur at the middle of the sample. Finally, this competitive edge of the FB-RUR test disappears when outliers occur at both the beginning and the end of the sample. However, this situation is more unlikely.

On the other hand, when considering the alternative of a stationary $AR(1)$ time series corrupted by a single structural break, we obtain an outstanding improvement in power performance over the former RUR test, for a sample size of $n = 500$, as shown in Table 15. As we already pointed, this result is not surprising since it is as though we were working on twice the original number of observations.

n |s	4	8	12
100	0.200	0.050	0.050
250	1.00	0.900	0.800
500	1.000	1.000	1.000

Table 15: Power of the FB-RUR test at the 5% significance level against model $x_t = 0.5\ x_{t-1} + s\ D_t + \epsilon_t$, where $D_t = 0$ for $t \le n/2$ and $D_t = 1$ for $t > n/2$. The power estimates are given for different values of the sample size $n(100, 250, 500)$, and for different values of s ($s = 4, 8, 12$).

Obviously, the results deteriorate when two breaks are present in the DGP of the time series, and thereby a larger sample size ($n = 500$) is required in order to notice these improvements. This is shown in Table 16. As can be seen, when $n = 500$ the simple RUR test is outperformed by the FB-RUR test for the larger level shifts.

n | $\mathsf{s}_{1,2}$	$(2, 4)$	$(4, 8)$	$(8, 12)$
100	0.080	0.000	0.000
250	0.500	0.090	0.059
500	1.000	0.900	0.800

Table 16: Power of the FB-RUR test at the 5% significance level against the model $x_t = 0.5\ x_{t-1} + s_1\ D_{t,1} + s_2 D_{t,2} + \epsilon_t$, where $D_{t,i}$ ($i = 1, 2$) represents a dummy variable defined by $D_{t,i} = 0$ for $t \le in/4$ and $D_{t,i} = 1$ for $in/4 < t \le in/2$. The power estimates are given for different values of the sample size n $(100, 250, 500)$, and for different values of s_1 and s_2 ($2, 4, 8$ and $4, 8, 12$, respectively).

Finally, as regards the robustness of the FB-RUR test to monotonic nonlinearities, no significant differences are obtained with respect to the former RUR test. It is also straightforward to show that the FB-RUR test, based on $J_{0,*}^{(n)}$, has the same invariance properties and asymptotics as the one based on $J_0^{(n)}$.

6 Empirical applications

In this section we illustrate the performances of our robust unit root testing methodology on four real time series. Our first example focusses on a series affected by structural changes, that is the monthly Brazilian inflation rate series from January 1994 to June 1996. The second study case is the annual US/Finland real exchange rates series from 1900 to 1987, which is contaminated with both additive and innovation outliers. The third and four examples deal with a couple of near-unit root stationary time series. The first of these represents the quarterly Montevideo unemployment rate, which covers the period ranging from the third quarter of 1981 to the second quarter of 2001. The second one corresponds to the annual US unemployment rate from 1955 to 1999.

6.1 Analysis of the monthly Brazilian inflation rate series: January, 1974 – June,1993

Here we analyse the monthly Brazilian inflation rate for the period covering from January 1974 to June 1993, and which yields a sample size of $n = 234$ observations. The choice of June 1993 as the end of the sample was to avoid incorporating the so-called "Real Plan", which is currently in effect.

Figure 14 shows a plot of the series exhibiting several sudden drops in the 1980's. These abrupt changes are the outcome of the various shock plans instituted by the government in an attempt to stop the process of soaring inflation.

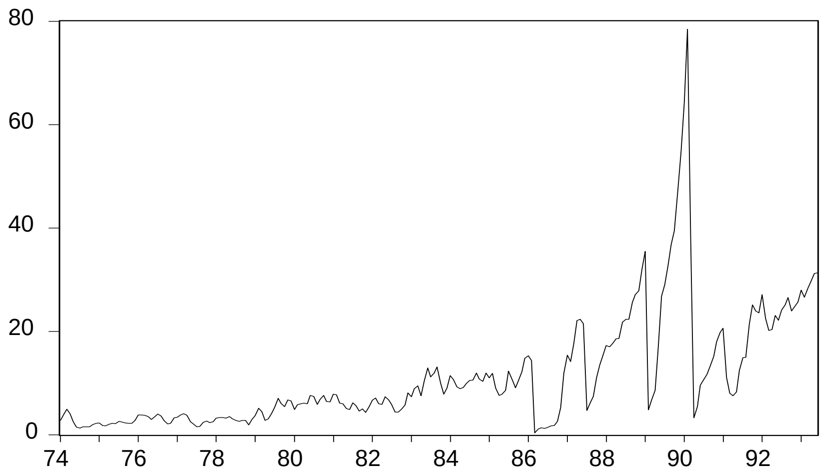

Figure 14. Monthly Brazilian inflation rate from January 1974 to June 1993.

Cati, Garcia and Perron (1999 [11]) have reported empirical results obtained with the application of standard unit root tests, such as the Augmented Dickey-Fuller (ADF) test (1979 [12]), and the Phillips-Perron test (1988 [44]). They also have reported results obtained with a modification of the Phillips-Perron test (hereafter MPP), suggested by Stock (1990 [56]). Such a modified test is less prone to size distorsions in the presence of serial correlation in the first differences of the series. These three unit roots tests concur for an overwhelming rejection of the null hypothesis of a unit root in favour of a stationary alternative. All the statistics are significant at the 1% level (the critical values, from Fuller (1976 [16]) are -29.5, for both PP and MPP test statistics, and -3.96 for the t-ratio ADF test statistic), while the values obtained for test statistics were -41.55, -37.57 and -6.61, respectively). In all cases, the truncation lag was selected using the BIC criterion (Maddala,1998 [32]). The results suggest that "shock plans" induce a strong bias in unit root tests towards stationarity, whether the true model has a unit root or not.

When applying the RUR test we obtained, for the analysed sample (January, 1974 - June, 1993), the value $J_0 = 2.2226$. Using the approximate critical values for $n = 250$ from Table 2, the null hipothesis of unit root is maintained at the 1%, 2.5%, 5% and 10% significance levels. When the FB-RUR test statistic was used instead, we obtained the value $J_{0,*} = 3.7905$, which, using the estimated critical values in Table 10 for $n = 250$, also leads to the non-rejection of the null hypothesis of a unit root at the same significance levels.

6.2 Analysis of the annual US/Finland real exchange rates: 1900-1987

In this section, the RUR and FB-RUR tests were applied to the annual series of US/Finland real exchange rates, whose logarithm is plotted in Figure 15. This series, which contains a total of $n = 88$ observations (from 1900 to 1987), was constructed using the Gross Domestic Product (GDP) deflator. Previous analyses on this series done by Vogelsang (1999 [58]), Franses and Haldrup (1994 [20]), Perron and Vogelsang (1992 [41]), and Perron and Rodriguez (2000 [40]), point to the presence of an AO at date 1918 together with IO's that produce temporary changes at dates 1917, 1932, 1949 and 1957.

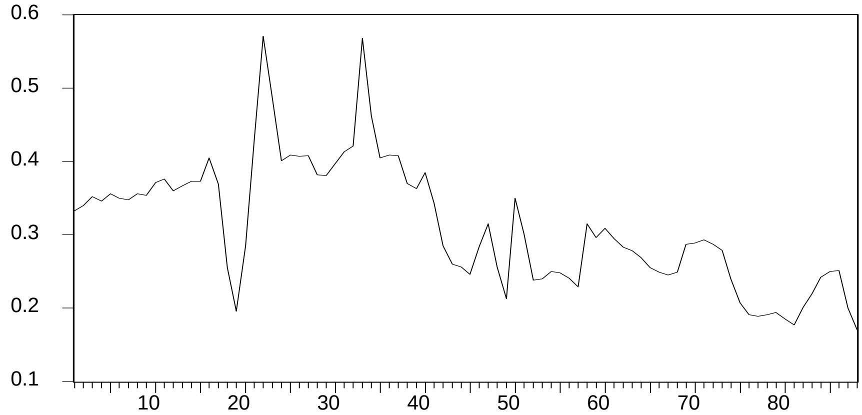

Figure 15. Logarithm of the US/Finland real exchange rates deflated annual series from 1900 to 1987.

Using the Mackinnon's critical values for the ADF test (Mackinnon, 1994 [31]), the null hypothesis of a unit root is rejected at the 5% significance level (the ADF test statistic took the value -3.732041 while the 5% critical values was -3.4614).

Alternatively, with a value of $J_0 = 1.4924$ obtained for the RUR test statistic, and the corresponding estimated critical value of 1.1726 at the 5% significance level and for $n = 88$, the null is not rejected. Similarly, for the FB-RUR test we obtained a value for its test statistic of $J_{0,*} = 2.11$, which is also larger than the corresponding estimated critical value, that is 1.7337.

6.3 Analysis of the quarterly Montevideo unemployment rate series: 1981-2001

Back as far as 1982 Nelson and Plosser [36] argued that most macroeconomic time series have unit roots, a finding which is very important in the design of macroeconomic policies and which has been corroborated ever since by many authors. For example, a great bulk of econometric research has supported the evidence of the existence of unit roots in the unemployment rate series of most countries (see for instance Mitchell, 1993 [35]; Arrufat et al.,1999 [6]; Papell et al., 2000 [37]). Yet the low power of standard unit root tests on stationary alternatives close to the nonstationary border contradicts such reports. One among many controversial study cases is the quarterly Montevideo unemployment rate series spanning the period that goes from the third quarter of 1981 to the second quarter of 2001. The 88 observations of this series are plotted in Figure 16 below.

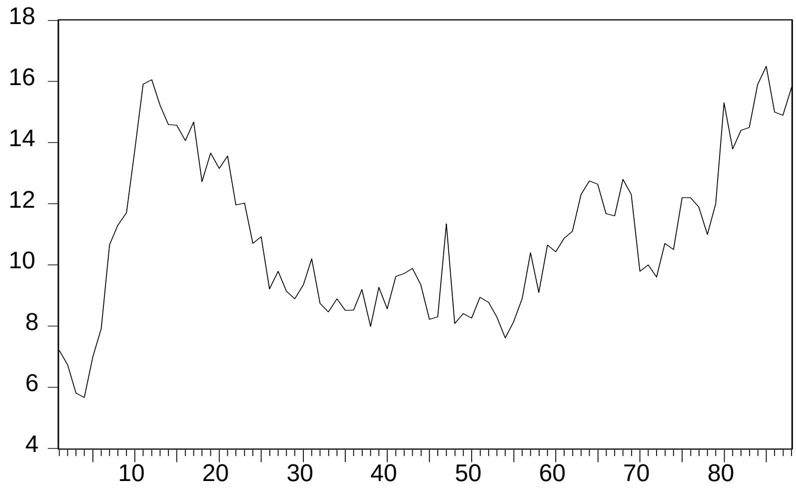

Figure 16. Quarterly Montevideo unemployment rate series, from 1981 to 2001.

Spremolla (1998 [54]) found a fractional root in this series, which suggested a stationary ARFIMA model, thus opposing a previous finding by Rodriguez (1998 [48]) of a single unit root, using the DF test. It is then natural to ask whether our RUR test is powerful enough to reject the null hypothesis of unit root in this case. We found $J_0 = 1.0660$, a value smaller than the corresponding estimated critical value at the 5% significance level (1.1726), and which leads to the rejection of the unit root hypothesis at this level.

We have to bear in mind, however, that when dealing with such small sample sizes, the RUR is not very reliable (even though it always outperforms DF, as shown in Table 3). Yet any significant discrepancy between the p-values of these tests may alert us on the inconsistency of the DF test outcome.

6.4 Analysis of the annual US unemployment rate series: 1955-1999

This series contains just $n = 55$ observations. Its plot is shown in Figure 17 below.

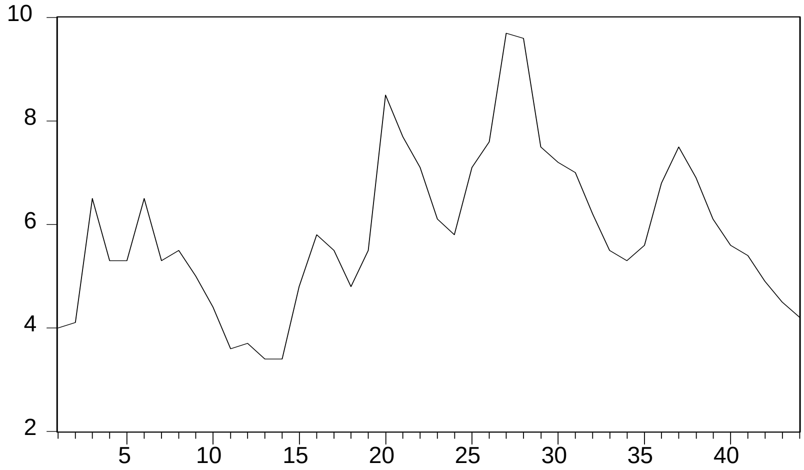

Figure 17. Annual US unemployment rate series from 1955 to 1999.

The application of an ADF unit root test to this series obtains a score of -0.510664, which when compared to MacKinnon's critical values for $n = 55$ (-4.2605, -3.5514 and -3.2081 at the 1%, 5% and 10% significance levels, respectively) leads to the non-rejection of the null hypothesis of a unit root. On the contrary, the score for the RUR test was found to be $J_0 = 0.7538$, which is smaller than the 5% estimated critical value for the given sample size (0.9899), thus rejecting the null at this level.

7 Concluding Remarks

It is important to be very careful interpreting standard unit root test results since, apart from having low power on stationary near-unit root time series and having size distortions in the presence of negatively correlated moving averages (Schwert, 1989 [57]), they are also seriously affected by other aspects of real data such as level shifts, outliers and nonlinearities. In this paper we have proposed a nonparametric methodology for testing unit roots in time series which is either robust or invariant to such departures. The new method, called the *Range Unit Root* (RUR) *Test* also outperforms the Dickey-Fuller test in terms of power on stationary near-unit root alternatives. A major drawback of our test is its sensitivity to early additive outliers in the series, which leads to considerable size distortions in such cases. However, by simply running the test forwards and backwards (FB-RUR) on the series it is possible to circumvent this problem and even improve its small-sample power performances. A few real time series were selected to illustrate the performances of our test and compare it to DF. In all the cases, we found discrepancies which question the validity of the standard test outcome.

Appendix

Proof of Theorem 1

The proof of Theorem 1 will be established by using the methodology of subsampling (see Politis, Romano and Wolf, 1999 [45], for further details). Let

$$\Psi_0^{(n)} = \frac{1}{n} \sum_{t=1} 1(R_t^{(x)} > 0) \tag{5}$$

denotes the frequency of jumps, corresponding to "arrivals" of new maxima or minima, in a sample $\{x_1,...,x_n\}$, where $1(.)$ represents the indicator function, and x_t is a random walk with $Nid(0,1)$ errors.

First we show that $\Psi_0^{(n)}$ converges to zero in probability as n grows to infinity. Indeed, noting that

$$1(\Delta R_t^{(x)} > 0) = 1(x_t > x_{t-1,t-1}) + 1(x_t < x_{1,t-1}) \tag{6}$$
$$= 1 - 1(x_{1,t-1} \leq x_t \leq x_{t-1,t-1}) \tag{7}$$

we can rewrite $\Psi_0^{(n)}$ as

$$\Psi_0^{(n)} = 1 - \sum_{t=1} 1(\frac{n^{-1/2}x_{1,t-1}}{\sigma} \leq \frac{n^{-1/2}x_t}{\sigma} \leq \frac{n^{-1/2}x_{t-1,t-1}}{\sigma}) \left(\frac{t}{n} - \frac{t-1}{n}\right) \tag{8}$$

where $\sigma^2 = \lim_{n\to\infty} E(n^{-1}x_n^2)$ represents the long-run variance of x_t. Now letting $t = [rn]$, with $r \in [0,1]$ and $[.]$ denoting the integer part, we obtain in the limit of $n \to \infty$:

$$\Psi_0^{(n)} \Rightarrow 1 - \int_0^1 1(min\{W(s)\}_{s=0}^r \leq W(r) \leq max\{W(s)\}_{s=0}^r dr = 1 - 1 = 0. \tag{9}$$

On the other hand it is clear that, under the null hypothesis of a unit root, $n\Psi_0^{(n)}$ diverges as n grows to infinity. Therefore we can let

$$\Psi_0^{(n)} = O\left(n^{-\alpha}\right), \text{with } \alpha \in (0,1).$$

To estimate α, we analysed the variation of $\Psi_0^{(n)}$ with the sample size. Here we took

$$n \in \{5000, 10000, 15000, 20000\}.$$

For each value of n, $Q = 100$ independant random walks with $Nid(0,1)$ errors were generated. Let $\hat{\Psi}_0^{(n,q)}$ be the empirical frequency of jumps corresponding to the q-th sample of size n. Then for each n, α could be estimated as the slope parameter in the regression of $\log(\hat{\Psi}_0^{(n,q)})$ against $\log(n^{-\alpha})$, or as:

$$\hat{\alpha}_n = \frac{1}{Q} \sum_{q=1} \frac{\log(\hat{\Psi}_0^{(n,q)})}{\log n}, \quad \text{for each } n. \tag{10}$$

.

The subsampling estimates in the table below show the convergence of $\hat{b}_n$ towards 0.5 as n grows to infinity.

n	5000	10000	15000	20000
$\hat{b}_n$	0.47	0.48	0.499	0.5

Proof of Theorem 2

The proof of Theorem 2 uses the concept of local time associated with a Brownian motion W_t. The local time is a measure of the time spent by a Brownian motion process in the vicinity of the point x, $(x-\epsilon, x+\epsilon)$.

For every fixed Borel set $B \in \mathcal{B}(\mathcal{R})$, let

$$I_B(\zeta) = \begin{cases} 1 & \text{if } \zeta \in B \\ 0 & \text{if } \zeta \notin B \end{cases}$$

and let λ denote the Lebesgue measure. We define the *occupation time* of the set B by the Brownian path up to time t as

$$\Gamma_t(B) = \int_0^t I_B(W_s)ds = \lambda\{0 \le s \le t;\, W_s \in B\};\, 0 \le t < \infty \qquad (11)$$

Note that because the Brownian process is continuous, this random variable is well defined as an ordinary integral along each process realization.

Now consider the open interval $B = B(x,\epsilon) = (x-\epsilon, x+\epsilon)$. From the previous definition, we have:

$$\Gamma_t[B(x,\epsilon)] = \lambda\{0 \le s \le t;\; |W_s - x| \le \epsilon\};\; 0 \le t < \infty,\;\; x \in \mathcal{R}.$$

Define the limit random variable

$$L(t,x) = L_t(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \Gamma_t[B(x,\epsilon)] \qquad (12)$$

For each real value x, $\{L(t,x)\}_{t>0}$ represents a nondecreasing (in t) and almost surely jointly continuous family of random variables, called the *local time process* of the Brownian motion. The random function $L(t,x)$ can be construed as the "spatial" density of the occupation time $\Gamma_t(B)$, since

$$\Gamma_t(B) = \int_B L(t,x)dx. \qquad (13)$$

In particular, the "local time at the origin", $x = 0$, that is,

$$L(t,0) = L_t(0) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\Gamma_t[B(0,\epsilon)]. \tag{14}$$

plays an important role in our results. If for example, $L_t(0)$ is large then this means that the Brownian path spends a lot of time close to 0, thus changing the sign relatively often.

For the final proof, we require the following lemmas:
Let M_t^w, $\max_{0\le s\le t} W_t$; *and* $Y_t = M_t^w - W_t$.

Lemma 1 (Lévy , 1948 [26]).
The processes $\{|W_t|;\, 0 \le t < \infty\}$ and $\{Y_t;\, 0 \le t < \infty\}$ have the same law.
Proof: See Karatzas and Shreve (1988 p.210 [24]).

Lemma 2 (Skorohod, 1961; Lévy, 1948[26]).
The processes $\{(M_t^w,\, 0 \le t < \infty\}$ and $\{2L_t(0);\, 0 \le t < \infty\}$ have the same law, given by:

$$P\{M_t^w < x\} = \sqrt{\frac{2}{\pi t}}\int_0^x e^{-\frac{\zeta^2}{2t}}\,d\zeta, \qquad x > 0. \tag{15}$$

That is,

$$M_t^w \sim |N(0,t)|$$

Proof: See Karatzas and Shreve (1988, p.210 [24]).

Lemma 3.
Let X_1, X_2,... , X_i, ... be a sequence of random variables with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$. If $\mu_i \xrightarrow{p} a$ and $\sigma_i^2 \to 0$, then $X_i \xrightarrow{p} a$.
Proof. See Arnold (1990, p. 239).

Proof of Theorem 2.
Let

$$J_0^{(n)} = J_1^{(n)} + J_2^{(n)},. \tag{16}$$

with

$$J_1^{(n)} = n^{-1/2}\sum_{i=1} 1(x_{i,i} = x_i), \tag{17}$$

$$J_2^{(n)} = n^{-1/2}\sum_{i=1} 1(x_{1,i} = x_i).$$

Notice that, by symmetry, $J_2^{(n)}$ is the same as $J_1^{(n)}$ when the sign of the time series is reversed. So we may restrict our analysis to proving the asymptotic distribution of $J_1^{(n)}$. Define the variable $\zeta_i = M_i - x_i$, where $M_i = \max\{x_1, ..., x_i\}$. Thus we have:

$$J_1^{(n)} = n^{-1/2} \sum_{i=1} 1(\zeta_i = 0) \tag{18}$$

Letting $\omega_i = 1(\zeta_i = 0) - 1(\zeta_i \leq 1)$ we can write:

$$J_1^{(n)} = n^{-1/2} \left(\sum_{i=1} 1(\zeta_i \leq 1) + \sum_{i=1} \omega_i \right) \tag{19}$$

We have for the first term on the right-hand side:

$$n^{-1/2} \sum_{i=1} 1(\zeta_i \leq 1) = n^{1/2} \sum_{i=1} 1\left(\frac{\zeta_i}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \right) \frac{1}{n} \tag{20}$$

With $h = \frac{1}{\sqrt{n}}$, $t = \frac{i}{n} \in (0,1)$ and "$\Rightarrow$" denoting weak convergence as $N \to \infty$, we have:

$$\begin{aligned} n^{1/2} \sum_{i=1} 1\left(\frac{\zeta_i}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \right) \frac{1}{n} &= n^{1/2} \sum_{i=1} 1\left(\frac{\zeta_{[tn]}}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \right) \left(\frac{i}{n} - \frac{i-1}{n} \right) \\ &\Rightarrow \lim_{h \to 0} \frac{1}{h} \int_0^1 1(Y_t \leq h) dt \\ &= \lim_{h \to 0} \frac{1}{h} \int_0^1 1(|W_t| \leq h) dt, \text{ from lemma 1,} \\ &= 2L_1(0), \text{ by definition of local time} \\ &\sim |N(0,1)|, \text{from lemma 2.} \end{aligned} \tag{21}$$

Therefore the term

$$n^{1/2} \sum_{i=1} 1\left(\frac{\zeta_i}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \right) \frac{1}{n}$$

converges, as $n \to \infty$, to a random variable with *pdf* :

$$f(y) = \frac{2}{\pi} \exp(-\frac{y^2}{2}), \quad y \geq 0.$$

Now for the second component of $J_1^{(n)}$, namely $V_n = n^{-1/2} \sum_{i=1} \omega_i$, we will show by means of Monte Carlo simulations that

$$V_n \xrightarrow{p} 0.$$

Figure 18 below shows the shrinking support of the *pdf* traces of V_n under the null hypothesis as the sample size n is increased from $n = 1000$ to $n = 15000$. The traces were both estimated from 10000 Monte Carlo replications.

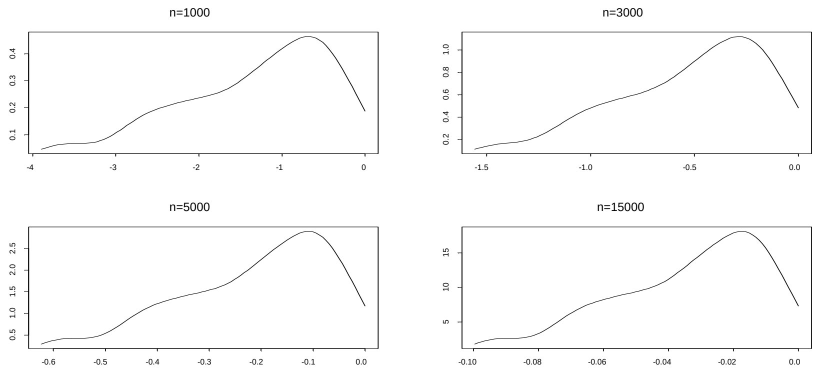

Figure 18. Density traces of the residual term for different sample sizes.

Figures 19 and 20 represent the outcome of another Monte Carlo experiment using 10000 replications of the null model, which shows that as the number of observations in the sample is increased, both the mean and variance (respectively) of V_n converge exponentially fast to zero. Formally,

$$\lim_{n\to\infty} E(V_n) = 0, \qquad \lim_{n\to\infty} Var(V_n) = 0$$

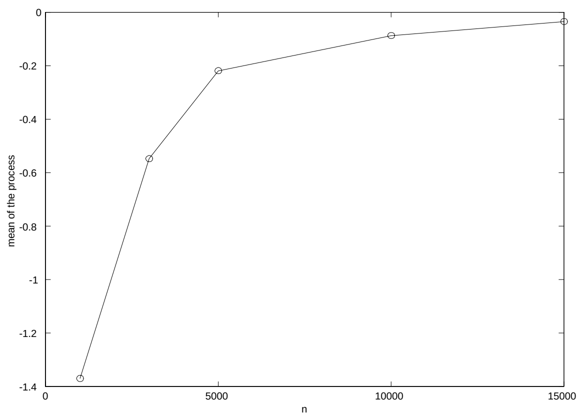

Figure 19. Convergence of $E(V_n)$ to zero as n increases to infinity.

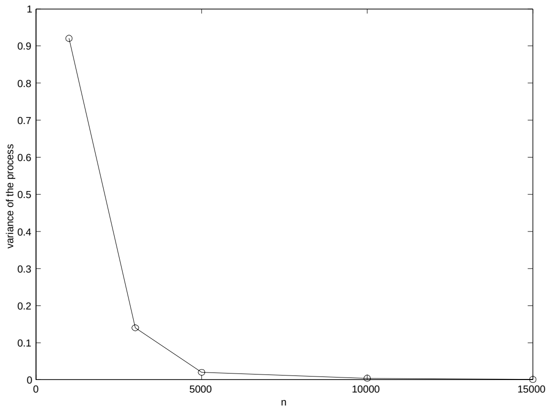

Figure 20. Convergence of $Var(V_n)$ to zero as n increases to infinity.

Using this experimental evidence, we may conclude from lemma 3 that $V_n \xrightarrow{p} 0$, and consequently that $J_1^{(n)} \Rightarrow |N(0,1)|$. By symmetry, the proof for $J_2^{(n)}$ follows similarly, since this statistic is equivalent to $J_1^{(n)}$when the sign of the time series is reversed.

Figure 21 shows the empirical density of $J_i^{(n)}$ $(i = 1, 2)$estimated by kernel smoothing , for 1000 Monte Carlo replications and a sample size of $n = 500$.

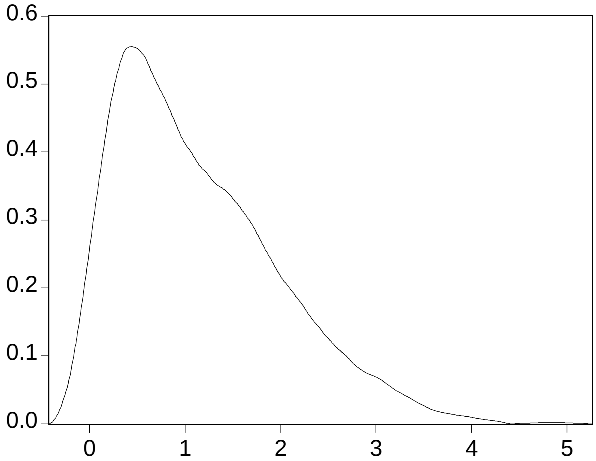

Figure 21. Kernel density estimate of $J_i^{(n)}$, for $i = 1, 2$.

Proof of Theorem 3

The proof of Theorem 3 proceeds by showing the asymptotic independence of $J_1^{(n)}$ and $J_2^{(n)}$. More precisely, we show by means of Monte Carlo experiments that the distribution of $J_0^{(n)}$ converges to the distribution of the sum of two independant random variables, each with *pdf* $|N(0,1)|$.

Figure 22 shows the kernel estimate of the *pdf* of $J_0^{(n)}$ for $n = 1000$, obtained from 1000 Monte Carlo replications of the model., This kernel estimate appears superimposed on the *pdf* estimate of the probabilistic model, that is the sum of two independent and identically distributed random variables, with distribution $|N(0,1)|$.

Figure 23 shows the estimated quantiles for both the test statistic $J_0^{(n)}$ and its probabilistic model, again for $n = 1000$. The close alignment of the quantiles is tantamount to the similarity of the distributions On the other hand, Figure 24 shows the quantiles of $J_0^{(n)}$ and its probabilistic model versus each other. The idea is that if the two samples come from the same distribution, the points should be close to the diagonal line, as it seems to be the case here. The hypothesis of an identical distribution for $J_0^{(n)}$ and its probabilistuc model is further supported by a Kolmogorov-Smirnov test. The p-values of such a test for different sample sizes are given in the table below. The results suggests clearly that, as the number of observations in the sample is increased, the p-value converges to 1.

n	1000	2000	3000	4000	5000
p-value	0.4	0.6	0.8	0.96	0.99

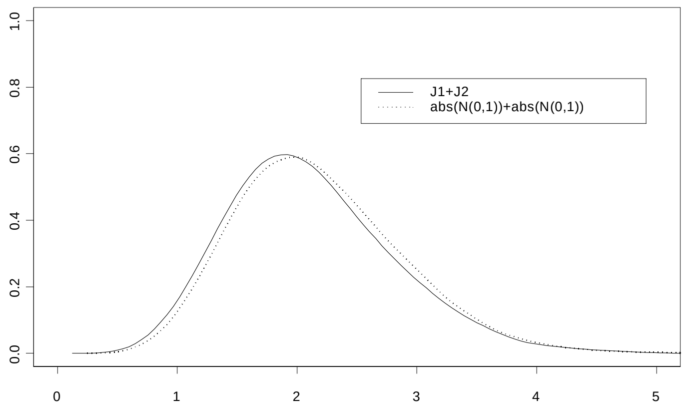

Figure 22. Comparison of the kernel density estimate of $J_0^{(n)}$ for $n = 1000$ (continuous line) with the theoretical asymptotic density (dotted line).

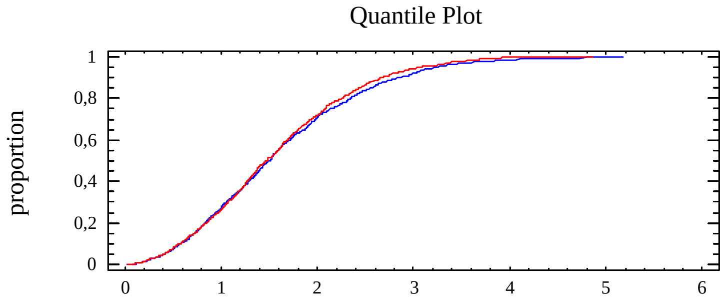

Figure 23. Estimated quantiles for both the test statistic $J_0^{(n)}$ and its probabilistic model, for a sample size of $n = 1000$.

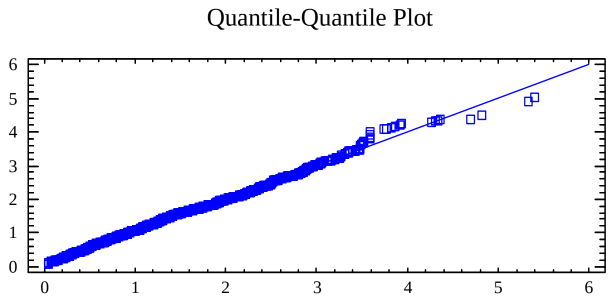

Figure 24. Estimated quantiles of the test statistic $J_0^{(n)}$ versus the quantiles of its probabilistic model, for a sample size of $n = 1000$.

Proof of Corollary 3

We have shown previously that the asymptotic density of both $J_1^{(n)}$ and $J_2^{(n)}$ is given $f(x) = \left(\frac{2}{\pi}\right)^{1/2} \exp\left(-x^2/2\right)$

Under the hypothesis of asymptotic independence of these statistics, the asymptotic density of their sum, that is of $J_0^{(n)}$, is given by the following convolution product:

$$
\begin{aligned}
f_{J1+J2}(z) &= \int_0^{\infty} f_{J1}(t) f_{J2}(z-t)dt \\
&= \frac{2}{\pi}\exp(-z^2/2)\int_0^{\infty}\exp\left(-[t-z/2]^2\right)dt \\
&= \pi^{-1}\exp\left(-\frac{z^2+2}{4}\right)\int_{-z}^{\infty}\exp\left(-\frac{u^2}{2}\right)du, \text{ where we let } 1ptu2 = t - 1ptz2 \\
&= \sqrt{\frac{2}{\pi}}\exp\left(-\frac{z^2+2}{4}\right)[1-\Psi(z)]
\end{aligned}
$$

Proof of Proposition 1

The invariance of $J_0^{(n)}$ with respect to the variance σ_x^2 of a stationary alternative x_t follows trivially from the fact that

$$
\mathbb{1}\left(\Delta R_t^{(x)} > 0\right) = \mathbb{1}(\sigma_x^{-1}\Delta R_t^{(x)} > 0) \tag{22}
$$

Proof of Proposition 2

The invariance of $J_0^{(n)}$ to monotonic nonlinear transformations $f(.)$ applied to x_t follows inmediately from:

$$
\mathbb{1}\left(f(x_t) > f(x_{t-1,t-1})\right) = \mathbb{1}(x_t > x_{t-1,t-1}) \tag{23}
$$

$$
\mathbb{1}\left(f(x_t) < f(x_{1,t-1})\right) = \mathbb{1}(x_t < x_{1,t-1}). \tag{24}
$$

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