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VALUATION OF BOUNDARY-LINKED ASSETS^{*}

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Abstract

This article studies the valuation of boundary-linked assets and their derivatives in continuous-time markets. Valuing boundary-linked assets requires the solution of a stochastic differential equation with boundary conditions, which, often, is not Markovian. We propose a wavelet-collocation algorithm for solving a Milstein approximation to the stochastic boundary problem. Its convergence properties are studied. Furthermore, we value boundary-linked derivatives using Malliavin calculus and Monte Carlo methods. We apply these ideas to value European call options of boundary-linked assets.

Key words: Boundary-linked assets, Stochastic Differential Equations with Boundary Conditions, Wavelets.

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1 Introduction

In this paper we consider pricing boundary-linked assets and their derivatives in continuous-time markets. The value of these assets are contractually linked at several dates by means of boundary constraints. Therefore, valuing boundary-linked assets requires the solution of boundary value stochastic differential equations.

A stochastic Boundary Value Problem (BVP) is defined as

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \text{ for } t \in [0, T], \quad (1)$$

with a boundary condition

$$\alpha(X) = c, \quad (2)$$

where W_t is a d -dimensional Brownian motion, X_t a continuous time d -dimensional stochastic process, α a continuous operator from the trajectories' space to \mathbb{R}^d and $c \in \mathbb{R}^d$ constant. For example, a boundary condition can be a terminal condition with $\alpha(X) = X_T$, $\alpha(X) = A_0 X_0 + A_T X_T$ where A_1 and A_T are real matrices, or a more involved condition such as $\alpha(X) = \int_0^T dA_t X_t$, where A_t is a $d \times d$ matrix which components are functions of bounded variation in $[0, T]$. The theory of stochastic BVPs has also considered some cases of non linear operators α , see e.g. Nualart and Pardoux (1991a). Other references on boundary value stochastic differential equations are Huang (1984), Ocone and Pardoux (1998) and Nualart and Pardoux (1991a).

Stochastic BVPs typically arise from the application of the Pontryagin's maximum principle to stochastic control optimum problems with finite time horizon T , where the boundary condition is given by the transversality condition (see, e.g., Malliaris and Brock, 1982, Prop. 10.1, pp. 112-113). These systems cannot usually be analytically solved, and algorithmic tools are required to cope with these problems. Despite recent contributions in stochastic BVP literature (see e.g. Ferrante et al., 1996, and Kohatsu-Higa, 2001, that focus on Stratonovich integrals), much can be done to enlarge the catalogue of techniques for solving BVP.

In this paper, we propose a projection-based method for solving stochastic BVPs. Its main idea consists of using a wavelet-collocation method to solve a finite-difference Milstein approximation to the stochastic differential problem. We prove that this procedure provides a strong approximation for the solution to (1) and (2). We study the numerical performance of the algorithm in several examples.

We apply these ideas to study the valuation of boundary-linked assets and their derivatives. The analysis of boundary linked assets is not only a theoretical problem, but can also be applied to the increasingly exotic assets traded in actual economies. With the growing sophistication of financial markets, investors are demanding new, more complex options products, tailored to their needs. In particular, there is an increasing number of financial assets which values are contractually linked at certain periods of time, such as leases and rental agreements. An illustrative example is the English real estate lease market. In English Law, two legal estates exist in buildings and land: freehold (absolute ownership which does not expire) and leasehold (temporal possession for a specified time period). Leasehold enables liability on covenants to pass from tenant to tenant and indeed from landlord to landlord. In this context, the lessor bears the risk associated with the residual market of the asset at the maturity date of the contract, and the buyer bears the short term lease risk, where the value fluctuation of the lease randomly fluctuates subject to some boundary constraints, e.g. a zero value of the leasing contract at the maturity date. The value of lease

assets can be formulated by a second order boundary value stochastic differential equation

$$\begin{aligned}\ddot{X}(t) &= b(t) \dot{X}(t) + \dot{W}(t), \quad t \in [0, T], \\ X(0) &= \rho, \quad X(T) = 0;\end{aligned}$$

that is, the acceleration of lease assets prices is proportional to their growth rate and affected by a white noise shock. Note that any second order problem can be reduced to a first order system of stochastic differential equations with boundary value conditions in the space of states, see e.g. Nualart and Pardoux (1991b). Hence, to value boundary-linked assets, we are faced with the problem of solving stochastic differential equations with boundary conditions.

Often, the solution of stochastic BVPs does not satisfy the Markovian conditional independence property, see e.g. Alabert et al. (1995) and Alabert and Ferrante (2002). Therefore, standard Black-Scholes arguments cannot sometimes be applied to value derivatives of boundary-linked assets. We propose the use of Malliavin calculus to value these derivatives. In particular, we consider the generalized Clark-Ocone formula and present a procedure for its computation based on the Monte Carlo method and wavelets approximations. To illustrate this methodology, we consider an European call option of boundary-linked assets.

The rest of the paper is organized as follows. After some preliminaries, Section 2 provides a brief introduction to stochastic BVPs. In Section 3 we present an algorithm for solving boundary value stochastic differential problems. Its numerical performance is illustrated by means of some examples. Next we study the properties of the solution approximation. In Section 4 we apply these ideas to value boundary-linked assets, of which prices are determined by a stochastic differential equation with boundary conditions. Also, we consider the valuation of boundary-linked derivatives and their numerical computation. Finally, proofs are placed in Appendix A.

2 Stochastic BVPs: Preliminaries

Now we introduce some basic notation and tools that will be used through the paper.

White Noise Process. Let \mathcal{S} be the Schwartz space in \mathbb{R} and let \mathcal{S}' be its dual (the space of tempered distributions) endowed with the weak-* topology and Borel subsets \mathcal{B} . By the Minlos Theorem, there is a probability measure μ on \mathcal{S}' such that $\int_{\mathcal{S}'} \exp \{i \langle \omega, \phi \rangle\} d\mu(\omega) = \exp \left\{ - (1/2) \|\phi\|_{L_2(\mathbb{R})}^2 \right\}$ for all $\phi \in \mathcal{S}$, where $\langle \omega, \phi \rangle$ is the evaluation of $\omega \in \mathcal{S}'$ on ϕ . The space $(\mathcal{S}', \mathcal{B}, \mu)$ is the white noise probability measure, satisfying the Itô isometry $E_\mu [\langle \omega, \phi \rangle] = \|\phi\|_{L_2(\mathbb{R})}^2$ for all $\phi \in \mathcal{S}$. Consider d independent realizations from μ , then we construct a d -dimensional Wiener process $W_t = (\langle \omega_1, I_{[0,t]} \rangle, \dots, \langle \omega_d, I_{[0,t]} \rangle)'$, which has a continuous modification in $C(\mathbb{R}; \mathbb{R}^d)$ with $W_t = 0$ for $t \leq 0$, and such that $\langle \omega, \phi \rangle = \int_{\mathbb{R}} \phi(t) dW_t$ for all $\phi \in \mathcal{S}$, in the sense of Itô's integral. Consider the Gelfand triple $\mathcal{S} \subset L_2(\mu) \subset \mathcal{S}'$, where

$$L_2(\mu) = \left\{ X : \mathcal{S} \rightarrow \mathbb{R} : \|X\|_{L_2(\mu)}^2 = \int_{\mathcal{S}'} \langle X, \omega \rangle^2 d\mu(\omega) < \infty \right\}.$$

An orthogonal basis for $L_2(\mu)$ is given by the family $\{H_k\}$, indexed by all vectors $k = (k_1, \dots, k_m)$, with $\{k_j\}_{j=1}^m \subset \mathbb{N}$, for all $m = 1, 2, \dots$ where $H_k(\omega) = \prod_{j=1}^m h_{k_j}(\langle \omega, e_j \rangle)$ and $\{h_n\}, \{e_n\}$ are Hermite polynomials and Hermite functions, respectively. Then, we define a singular white noise generalized process as follows,

$$\dot{W}_t = \left(\dot{W}_t(\omega_1), \dots, \dot{W}_t(\omega_d) \right)',$$

with $\dot{W}_t(\omega) = \sum_k e_k(t) H_k(\omega)$. A detailed review of this topic can be found, e.g. in Hida et al. (1993) and Holden et al. (1996).

Let $\Omega = C_0([0, T]; \mathbb{R}^d)$ be the space of all the continuous functions in $[0, T]$ which vanish at zero, with $T > 0$ deterministic. The restriction of the Wiener process W_t to $[0, T]$ induces a Borel probability space, which completion is denoted by (Ω, \mathcal{A}, P) . Let $\{\mathcal{A}_t\}$ denote the filtration generated by W_t , completed and made right continuous.

Malliavin calculus. Next we introduce some tools from Malliavin calculus. For all $h \in L_2([0, T]; \mathbb{R}^d)$, consider $W(h) = \int_0^T h_s dW_s$. Let $C^\infty(\mathbb{R}^n)$ be the set of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ infinitely continuously differentiable such that f and all its derivatives are bounded. We denote by \mathcal{D} the set of real random variables of the form $F = f(W(h_1), \dots, W(h_n))$, with $f \in C^\infty(\mathbb{R}^n)$ for any $n \in \mathbb{N}$ and $h_1, \dots, h_n \in L_2([0, T]; \mathbb{R}^d)$. For all $F \in \mathcal{D}$ we can define the differential operator DF as the stochastic process

$$D_t F = \sum_{j=1}^n \frac{\partial}{\partial x_j} f(W(h_1), \dots, W(h_n)) h_j(t), \quad \forall t \in [0, T],$$

and the iterated differential $D_{t_1, \dots, t_n}^n F = D_{t_1} \dots D_{t_n} F$, and $D^0 F = F$. Let $\mathbb{D}^{q,p}$ be the closure of \mathcal{D} with respect to the Sobolev norm

$$\begin{aligned} \|F\|_{q,p} &= \left(\|F\|_{L_p(\Omega)}^p + \sum_{j=1}^n \left\| \left\| D_{t_1, \dots, t_j}^j F \right\|_{L_2([0, T]^j)} \right\|_{L_p(\Omega)}^p \right)^{1/p} \\ &= \left(E[|F|^p] + \sum_{j=1}^n E \left[\left(\int_{[0, T]^j} |D_{t_1, \dots, t_j}^j F|^2 dt_1 \dots dt_j \right)^{p/2} \right] \right)^{1/p}, \end{aligned}$$

with $p \in (0, \infty)$, and $\mathbb{D}^{q,\infty}$ as the elements F in $\mathbb{D}^{q,2}$ with finite norm

$$\|F\|_{q,\infty} = \|F\|_{L_\infty(\Omega)} + \sum_{j=1}^n \left\| \left\| D_{t_1, \dots, t_j}^j F \right\|_{L_2([0, T]^j)} \right\|_{L_\infty(\Omega)}.$$

We also define $\mathbb{D}^{\infty,p} = \cap_{q \geq 1} \mathbb{D}^{q,p}$ and $\mathbb{D}^\infty = \cap_{q,p \geq 1} \mathbb{D}^{q,p}$.

The operator $D : \mathbb{D}^{1,2} \subset L_2(\Omega) \rightarrow L_2([0, T] \times \Omega)$ is known as the Malliavin derivative of $F \in \mathbb{D}^{1,2}$, with first order derivatives and second order moments. The adjoint operator of D , denoted by δ , is defined for all processes u such that $E \left[\int_{[0, T]} D_t F u_t dt \right] \leq c \|F\|_{L_2(\Omega)}$. If $u \in \text{Dom}(\delta)$, then $E[\delta(u) F] = E \left[\int_{[0, T]} D_t F u_t dt \right]$ for all $F \in \mathbb{D}^{1,2}$. Often, the operator δ is expressed as $\delta(u) = \int_0^T u_t dW_t$, and is known as the Skorohod integral. It can be proved that $\delta(u)$ is equal to the Itô integral if the process u_t is adapted. The duality can be used to establish the Clark-Ocone formula, see e.g. Karatzas and Ocone (1991); that is for all $F \in \mathbb{D}^{1,2}$, $F = E[F] + \int_0^T E[D_t F | \mathcal{A}_t] dW_t$. For an introduction to the Malliavin calculus and its properties, see e.g. Bell (1987), Ustunel (1995), Nualart (1995) and Øksendal (1997). Malliavin derivatives can be also considered as Frechet derivatives, see e.g. Øksendal (1997). Malliavin calculus can be also introduced using Wiener-Itô chaos expansions, see e.g. Houdré et al. (1994).

Stochastic BVP solutions. Consider the stochastic BVP (1) and (2). Although the study of existence of solutions for these problems is beyond the scope of this paper, we will sketch a proof using a scheme similar to that of the deterministic case. Notice that there exists a unique solution associated to $Dx(t) = 0$

with $\alpha(x) = c$ since α are linearly independent (at least over $\text{Ker}\{D\}$). Consider a Green's matrix of functions $G(t, s)$, such that any $g \in C_0([0, T]; \mathbb{R}^d)$ with Dg integrable can be expressed as follows:

$$g(t) = P_0(g)(t) + \int_0^T G(t, s) Dg(s) ds,$$

where $P_0(g)$ is the unique element in $\text{Ker}\{D\}$ which agrees with $\alpha(g)$. Furthermore,

$$Dg(t) = D_t P_0(g)(t) + \int_0^T D_t G(t, s) g(s) ds.$$

Let $\mathcal{H}_{1,2} = \left\{ h \in L_2([0, T] \times \Omega; \mathbb{R}^d) : \exists o \in (\mathbb{D}^{1,2})^d, Do = h \right\}$. We will express the stochastic BVP (1) and (2) in a more convenient way, using the following property: Define $Dx = u$, thus $u = G[x]$ and $G^{-1}[u] = x$, reciprocally, with

$$\begin{aligned} G[x](t) &= P_0(x)(t) + \int_0^T G(t, s) x(s) ds, \\ G^{-1}[u](t) &= D_t \{P_0(x)(t)\} + \int_0^T D_t G(t, s) x(s) ds. \end{aligned}$$

Therefore, defining $U = G(X)$, and the nonlinear operator

$$T[U](t) := b(t, G^{-1}[U](t)) + \sigma(t, G^{-1}[U](t)) \dot{W}_t,$$

we can express the stochastic BVP as $U = T[U]$. Then, we can guarantee the existence of solution in BVP by proving the existence of a fixed point U^0 for T , for which it suffices that T is a continuous retraction in a space isometric to $\mathcal{H}_{1,2}$, and a pathwise unique solution U^0 exists. Also, $X^0 = G^{-1}(U^0)$ is the almost sure (a.s.) unique solution of BVP, with $U_t^0 = G(U^0)(t)$ the Malliavin derivative of X_t^0 . Since

$$\begin{aligned} E \left[\|T[U]\|_{L_2[0,T]}^2 \right] &= E \left[\int_0^T \left\| b(t, G^{-1}[U](t)) + \sigma(t, G^{-1}[U](t)) \dot{W}_t \right\|^2 dt \right] \\ &= \int_0^T E \left[\|b(t, G^{-1}[U](t))\|^2 + \|\sigma(t, G^{-1}[U](t))\|^2 \right] dt, \end{aligned}$$

T is contractive if $\|b(t, x)\| \leq k \|x\|$, $\|\sigma(t, x)\| \leq c \|x\|$, for all t, x and $(k + c) \|G^{-1}\| < 1$. Note that the solution X^0 satisfies

$$X_t^0 = X_0^0 + \int_0^t b(s, X_s^0) ds + \int_0^t \sigma(s, X_s^0) dW_s, \quad (3)$$

for some initial condition X_0^0 . Given X^0 , we can define a sequence $\{X_t^n\}$ as follows

$$X_t^{n+1} = X_0^0 + \int_0^t b(s, X_s^n) ds + \int_0^t \sigma(s, X_s^n) dW_s, \quad (4)$$

where $E \left[\int_0^T |X_t^n - X_t^0|^2 dt \right] \rightarrow 0$ under appropriate Lipschitz conditions on b and σ . Hence, using (4), the existence of a unique continuous version of X^0 on $[0, T]$ can be proved using arguments similar to the case of ordinary stochastic differential equations.

Adaptativeness. The solutions of stochastic BVPs are anticipative in nature due to the boundary condition. But given an appropriate (anticipating) initial condition, the dynamics of the process is driven by an ordinary stochastic differential equation. This is the logic underlying shooting numerical methods. These methods are widely used to solve deterministic BVP (see Ascher et al (1995) for a review), and have been recently extended by Ferrante et al. (1996) to solve Stratonovich stochastic BVPs. A shooting method is a successive substitution method based on the idea of guessing the initial condition until its associate solution satisfies the boundary condition.

We use the shooting argument to define a conditional adaptativeness for solutions of BVPs. Let \mathcal{A}_0 be the completion of the smallest σ -algebra such that $\{\alpha(X^0)\}$ is measurable, and consider the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t = \mathcal{A}_0 \cap \mathcal{A}_t$. Note that conditioning on $\{\alpha(X^0) = c\}$, the unique solution X_t^0 satisfies (3), where X_0^0 is \mathcal{A}_0 measurable, and as a consequence X^0 is adapted respect to $\{\mathcal{F}_t\}_{t \geq 0}$. Therefore, conditioning on the boundary condition the expression (1) can be considered as an Itô integral or, alternatively, a generalized process (see Holden et al., 1996). Otherwise, Equation (1) should be interpreted in terms of Skorohod stochastic integrals.

3 An algorithm to solve stochastic BVPs

The numerical resolution of stochastic BVPs is the aim of this section. We propose a wavelet projection-based algorithm for solving stochastic differential equations with boundary conditions. Its main idea consists of using a wavelet-collocation method to solve a finite-difference approximation to the stochastic BVP. With this end in view, we first introduce some concepts of wavelet approximation.

Within the last decades, wavelet multiresolution methods have proved to be a flexible method for approximating relatively irregular functions with a parsimonious number of parameters. The first wavelet basis can be at least traced to the Haar (1910) work, but the theoretical foundations of wavelets have been established by physicians and mathematicians from the early 30's to the 80's. The interest on wavelets has increased since Mallat (1989) and Meyer (1992) introduced the use of multiresolution as a framework to study wavelets expansions. A historical perspective can be found in Daubechies (1992) and Meyer (1993). Excellent monographs in wavelets are Chui (1992), Daubechies (1992), Meyer (1992, 1993) and Walnut (2001).

Given the Hilbert space $L_2(\mathbb{R})$, let consider a sequence of closed subspaces $\{V_n\}_{n \in \mathbb{Z}}$ such that: i) $V_n \subset V_{n+1}$, $\forall n \in \mathbb{Z}$, ii) $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$, and iii) $\bigcup_{n \in \mathbb{Z}} V_n$ is dense in $L_2(\mathbb{R})$. In particular we say that $\{V_n\}_{n \in \mathbb{Z}}$ is a multiresolution if each subspace V_n is the span of an orthonormal basis $\{\phi_{n,k}\}_{k \in \mathbb{Z}}$, with $\phi_{n,k}(t) = 2^{n/2} \phi(2^n t - k)$ and $\phi \in L_2(\mathbb{R})$, is known as the father wavelet. This concept was introduced by Mallat (1989).

As $\{\phi_{n,k}\}_{k \in \mathbb{Z}}$ are orthonormal, if $\Pi_{V_n}(x)$ denotes the orthogonal projection of an arbitrary $x \in L_2(\mathbb{R})$ into V_n , then

$$x(t) = \lim_{n \rightarrow \infty} \Pi_{V_n}(x)(t) = \lim_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}} \langle x, \phi_{n,k} \rangle_{L_2} \phi_{n,k}(t), \quad (5)$$

in the sense of L_2 . Whenever ϕ has compact support, for each $t \in \mathbb{R}$ the summation in (5) contains a finite number of non null terms. Otherwise it should be truncated for practical applications. In practice, one of the most popular wavelets systems is the compact wavelet proposed by Daubechies, for a detailed exposition see Daubechies (1992). Let $W_2^r(\mathbb{R})$ be the Sobolev space of functions (a.s. identical) with L_2 -integrable weak derivatives up to order r . If $x \in W_2^r(\mathbb{R})$, under appropriate conditions, wavelets derivatives can also approximate the weak derivatives of x . The multiresolution ideas can be specialized to the space $L_2([0, T])$

taking a multiresolution $\{V_n\}_{n \geq 0}$. In this context, it can be proved that $\Pi_{V_n}(x) \rightarrow x$ uniformly for all $x \in C([0, T])$, see e.g. Daubechies (1994).

The first basic step of our algorithm is to consider a real wavelet multiresolution $\{V_n\}_{n=1}^\infty$ in $L_2([0, T])$. To simplify notation throughout the remainder of the paper, given a vector of d functions $x(t) = (x_1(t), \dots, x_d(t))'$, we will denote the wavelet approximation of any $x \in L_2(\mathbb{R})^d$ by

$$\Pi_{V_n}(x)(t) = \sum_{k \in \mathbb{Z}} \theta_{n,k} \phi_{n,k}(t),$$

where $\theta_{n,k} \in \mathbb{R}^d$ is a vector of coefficients.

The next step is to consider a finite-difference approximation to the stochastic differential equation. For the sake of simplicity, we first consider the problem of solving an autonomous stochastic system $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, with $\alpha(X) = c$. In particular, we consider the Milstein (1974) finite-difference approach,

$$\begin{aligned} X_n(t_{i,n}) - X_n(t_{i-1,n}) &= h_n b(X_n(t_{i-1,n})) + \sigma(X_n(t_{i-1,n}))(W_{t_{i,n}} - W_{t_{i-1,n}}) \\ &+ \sigma(X_n(t_{i-1,n})) \frac{\partial \sigma}{\partial x}(X_n(t_{i-1,n})) \left[\int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s_1} dW_{s_1} dW_{s_2} \right], \\ X_n(0) &= c. \end{aligned}$$

The double stochastic integral can be readily computed, e.g. in the scalar case

$$\int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^{s_1} dW_{s_1} dW_{s_2} = \frac{1}{2} \left((W_{t_{i,n}} - W_{t_{i-1,n}})^2 - h_n \right).$$

For the multivariate case see Kloeden and Platen (1999, Ch. 5, Sec. 8).

Thus, the third and final step of the algorithm consists of applying the wavelet-collocation method to the Milstein approximation and solving the following system of equations in $\theta_{n,k} \in \mathbb{R}$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \theta_{n,k} (\phi_{n,k}(t_{i,n}) - \phi_{n,k}(t_{i-1,n})) &= \\ h_n b(X_n(t_{i-1,n})) + \sigma \left(\sum_{k \in \mathbb{Z}} \theta_{n,k} \phi_{n,k}(t_{i-1,n}) \right) (W_{t_{i,n}} - W_{t_{i-1,n}}) \\ + \frac{1}{2} \sigma \left(\sum_{k \in \mathbb{Z}} \theta_{n,k} \phi_{n,k}(t_{i-1,n}) \right) \frac{\partial \sigma}{\partial x} \left(\sum_{k \in \mathbb{Z}} \theta_{n,k} \phi_{n,k}(t_{i-1,n}) \right) \left((W_{t_{i,n}} - W_{t_{i-1,n}})^2 - h_n \right), \\ \sum_{k \in \mathbb{Z}} \theta_{n,k} \alpha(\phi_{n,k}) &= c, \end{aligned} \tag{6}$$

where $t_{i,n} = 2^{-n}i \in [0, T]$ with $i \in \mathbb{Z}$, and $h_n = 2^{-n}$. The solution coefficients $\{\theta_{k,n}^*\}$ determine $X_n \in V_n$ as

$$X_{\theta^*,n}(t) = \sum_{k \in \mathbb{Z}^d} \theta_{n,k}^* \phi_{n,k}(t).$$

Often, system (6) has to be solved by numerical methods. There are numerous methods for solving nonlinear equations (see e.g. Rheinboldt, 1998). However, we consider Newton's method as we are faced with the problem of solving small-size systems of smooth non linear equations.

Solving BVPs with $\sigma(t) = \sigma$ for all t is particularly easy. In this case, Milstein's equations are reduced to the Euler-Maruyama approximation (see Maruyama, 1955) and the system of equations in $\theta_{n,k} \in \mathbb{R}$ to be solved is,

$$\begin{aligned} X_n(t_{i,n}) - X_n(t_{i-1,n}) &= (t_{i,n} - t_{i-1,n}) b(X_n(t_{i-1,n})) + (W_{t_{i,n}} - W_{t_{i-1,n}}), \\ \alpha(X_n) &= c, \end{aligned}$$

where $t_{i,n} = 2^{-n}i \in [0, T]$ with $i \in \mathbb{Z}$, and $h_n = 2^{-n}$.

Also, this method can be applied to the non autonomous stochastic systems, $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$, with $\alpha(X) = c$. However, instead of the Milstein equation, we should consider an expansion for non homogeneous stochastic differential equations, see e.g. Kloeden and Platen (1999, Chap 5, Sect. 5).

In order to illustrate the accuracy of the method, we compute several examples of stochastic BVPs with analytical solution. All the examples consider the compactly supported wavelets of Daubechies, with parameter N . The algorithm has been implemented and the tests have been carried out on MATLAB 6.5 on an Intel Centrino Pentium M 1.6 GHz with machine precision 10^{-16} . First we consider a very simple stochastic BVP to show how to set up parameters to compute its approximate solution.

- Example I. Consider the problem

$$\begin{aligned} dX_t &= dW_t, \quad t \in [0, 1], \\ X_{1/2} + X_1 &= 0. \end{aligned}$$

This problem has a solution of the form $X_t = -\frac{1}{2}(W_{1/2} + W_1) + W_t$. We compute the numerical approximation of its solution for a sample path of $\{W_t\}$ using Daubechies wavelets with $N = 3$ and the step length $h = 2^{-2}$ (i.e. $n = 2$ and the number of dyadic points used is 9). In order to illustrate the accuracy of the numerical solution, we perform a Monte Carlo with $N = 400$ realizations. The mean of the maximum error of the approximation $\|X^*(t_{i,n}) - X_{\theta^*,n}(t_{i,n})\|_\infty$ is 2.7717×10^{-15} and its standard deviation 1.5896×10^{-15} .

In case of being interested in a higher accuracy, we can consider a larger number n . For $n = 6$, Figure 1 shows the values of the actual solution and its approximation over the dyadic points $t_i = 2^{-6}i \in [0, 2]$ with $i \in \mathbb{Z}$.

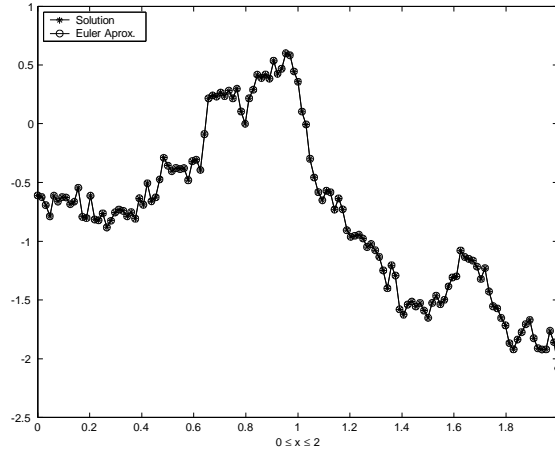


Figure 1. Numerical and exact solution of Example I with $N = 3$ and $n = 6$.

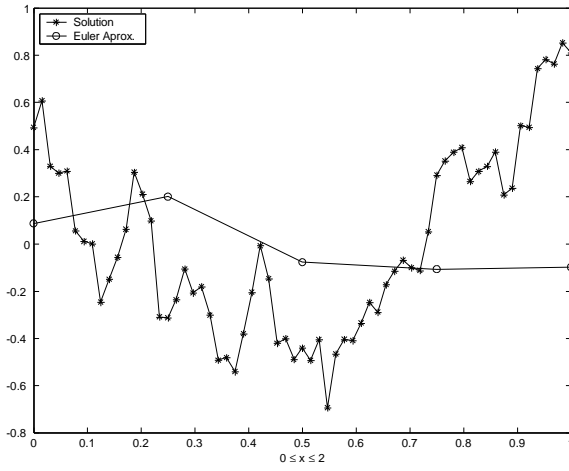
Next example is intended to demonstrate that the algorithm also behaves well in more complicated problems. However, higher number of dyadic points (in other words, higher parameter n) should be considered to get accuracy.

- Example II. Consider the problem

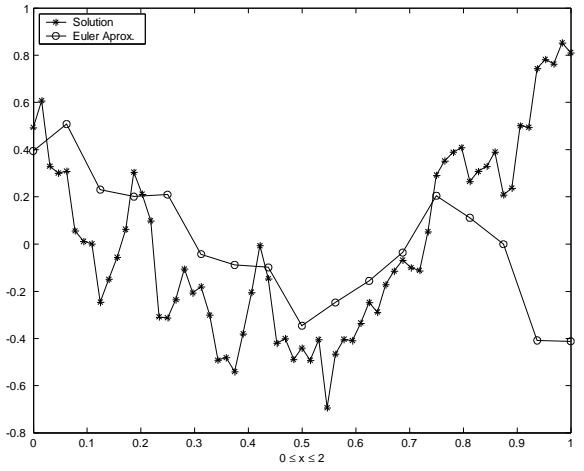
$$\begin{aligned} dX_t &= dW_t, \quad t \in [0, 1], \\ \int_0^1 X_t &= 0. \end{aligned}$$

The solution of this problem has the general form $X_t = -\int_0^1 W_t dt + W_t$.

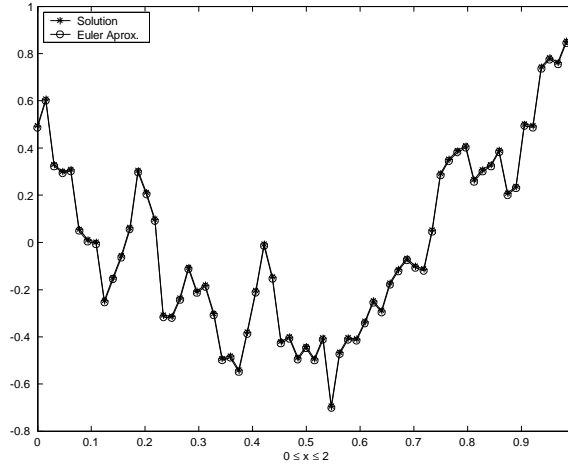
For a given sample path of $\{W_t\}$, using Daubechies wavelets with $N = 3$, Figure 2 shows the exact and the computed approximation for $n = 2, 4, 6$.



For $n = 2$



For $n = 4$



For $n = 6$

Figure 2. For $n = 2, 4, 6$, numerical and exact solutions of Example II with $N = 3$.

Table 1 reports the approximation errors to the solution and the computational cost for solving the stochastic BVP for $n = 2, 4, 6$. In this case, the approximation solution with $n = 6$ gives the smallest residual $\|X^*(t_{i,n}) - X_{\theta^*,n}(t_{i,n})\|_\infty$ as illustrated in Figure 2.

	$\ X^*(t_{i,n}) - X_{\theta^*,n}(t_{i,n})\ _\infty$	CPU (seconds)
$n = 2$	0.2058	0.02
$n = 4$	0.0997	0.03
$n = 6$	0.0075	4.66

Table1: Approximation errors and running times for computing Example 2 with $n = 2, 5, 6$.

Similarly to stochastic BVPs, most stochastic differential equations arising in real-world applications cannot be solved exactly. Numerical methods to get accurate solutions are Euler-Maruyama and Milstein schemes, among others. A review of the literature can be found, e.g. Kloeden and Platen, 1991. The proposed algorithm can also be used to solve ordinary stochastic differential equations as the following example illustrates.

- Example III. Consider the problem $dX_t = bX_t dt + \sigma X_t dW_t$ for $t \in [0, 1]$, with $X_0 = \xi$. The solution of this problem has the general form

$$X_t = \xi \exp\left(\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

Assume that $b = 2$, $\sigma = 1$ and $\xi = 1$. We compute the numerical of this problem using Daubechies wavelets with $N = 3$. Figures 3 shows that the approximation error is satisfactory for $n = 6$, although there is room for improvement in the right hand side of the time interval.

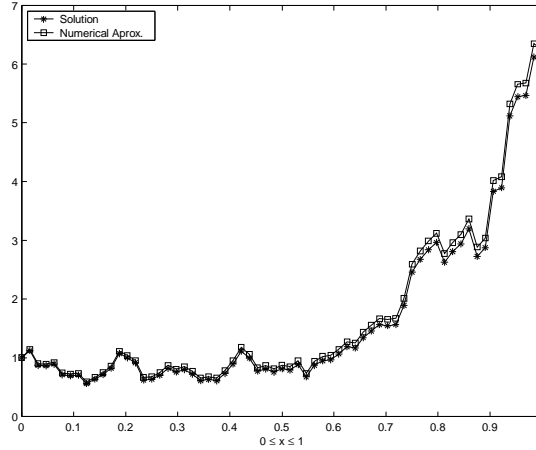


Figure 3. Numerical and exact solutions of Example III with $N = 3$ and $n = 6$.

Although we have focused on solving stochastic BVPs with linear boundary conditions, we can also apply this method to problems in which the boundary conditions are given by non linear continuous operators. When α is a non linear continuous operator, the proposed method can be applied replacing the last equation in (6) by $\alpha(\sum_{k \in \mathbb{Z}} \theta_{n,k} \phi_{n,k}) = c$. However, the convergence theory for this type of problem is beyond the scope of this paper.

3.1 Convergence analysis

In this section, we study the convergence properties of the proposed method. Proofs are placed in Appendix A. Assume

A.1. Let $\{V_n\}$ be a multiresolution in $L_2(\mathbb{R})$, with compactly supported father wavelet ϕ and assume for all $x \in W_2^r(\mathbb{R})$, with $1 \leq r \leq q$, $q \geq 1$, and all integer vector ν , $0 \leq \|\nu\|_1 \leq r-1$, it is satisfied

$$\|D^\nu x - D^\nu \Pi_{V_n}(x)\|_{L_2} = O\left(2^{-(r-\|\nu\|_1)n}\right),$$

Whenever $x \in C^r(\mathbb{R})$ with compact support the same rates are satisfied replacing the L_2 norm by the supremum norm. In spaces $L_2([a, b])$, an analogous behavior is assumed.

There are several sufficient conditions for this result that can be found in the literature, often based on the regularity of order q assumption. The father wavelet ϕ is said to be *regular of order $q \in \mathbb{N}$* , if ϕ has a version q times continuously differentiable and for $0 \leq \|\nu\|_1 \leq q$, and any positive integer $p \in \mathbb{N}$, there exists a constant $C_p > 0$ such that $|D^\nu \phi(t)| < (1 + \|t\|)^{-p} C_p$, $\forall t \in \mathbb{R}$. See Meyer (1992) for further details.

A.2. Let $X^0(t)$ be a solution of the stochastic BVP and define the curve $\mathcal{C} = \{(t, X^0(t))' : t \in [0, T]\}$.

Also, assume that, $b, \sigma \in C^2(\mathcal{N})$ where $\mathcal{N} \subset \mathbb{R}^{R+1}$ is an ε -neighborhood of \mathcal{C} in the L_∞ norm, and for some $\eta > 0$, it is satisfied

$$\Pr \left\{ \inf_{t \in [0, T]} \left| \det \left(I - D_x b(t, X^0(t)) - D_x \sigma(t, X^0(t)) \dot{W}_t \right) \right| > \eta \right\} = 1.$$

Notice that the last condition is satisfied whenever $\det(I - D_x b(t, x) - D_x \sigma(t, x) g(t))$ is non null for all (t, x) and for all $g \in L_1[0, T]$. For example, when $\sigma(t)$ does not depend on X it suffices $\det(I - D_x b(t, x)) \neq 0$ for all (t, x) . In particular, for linear stochastic BVP, $b(t, x) = bx$ and $\sigma(t, x) = \sigma$, it suffices that $\det(I - b) \neq 0$.

We start with an auxiliary result on the rate of approximation of the wavelet-Galerkin method. Given the multiresolution $\{V_n\}$, let $x_n \in V_n$ be the wavelet-Galerkin solution to the stochastic BVP; i.e. x_n satisfies

$$\Pi_{V_n} \left\{ Dx_n - b(t, x_n) - \sigma(t, x_n) \dot{W}_t \right\} = 0, \quad \alpha(x_n) = c.$$

Theorem 1 *Let consider the problem BVP with solution $X^0(t)$, and a multiresolution sequence $\{V_n\}$ in $L_2([0, T])$. Assume that A.1., A.2. are satisfied, then there exist $\delta > 0$ and an integer M such that X^0 is unique a.s. in $B(X^0, \delta) = \{X : \|X - X^0\|_\infty \leq \delta\}$, and the projected system*

$$\Pi_{V_n} \left\{ DX_n - b(t, X_n) - \sigma(t, X_n) \dot{W}_t \right\} = 0,$$

has an a.s. unique solution $X_n \in V_n \cap B(X^0, \delta)$. Furthermore, with probability one,

$$\max \{ \|X_n - X^0\|_\infty, \|DX_n - DX^0\|_\infty \} = O(2^{-n}).$$

In order to prove the convergence of the wavelet-collocation method we will use an interpolation result,

Theorem 2 Consider a multiresolution $\{V_n\}$ in $L_2(\mathbb{R})$ satisfying A.1. For each $x \in L_2(\mathbb{R})$ with an almost everywhere (a.e.) continuous version with compact support, we define $\Gamma_{V_n}(x)$ as any function $g_n \in V_n$ such that $g_n(t_{n,i}) = x(t_{n,i})$, for all $\{t_{n,i} = 2^{-n}i\}_{i \in \mathbb{Z}}$. Then, there exists a unique element in $\Gamma_{V_n}(x)$. Furthermore, assuming

1. ϕ is regular of order $q \geq 1$, and
2. the Poisson summa $\sum_{k \in \mathbb{Z}} \Phi(\omega + 2\pi k) > 0$, for almost every $\omega \in [0, 2\pi]$, being $\Phi(\omega) = \int_{\mathbb{R}} \phi(t) e^{it\omega} dt$ the Fourier transformed of ϕ ;

for all $x \in W_2^q(\mathbb{R})$ with compact support, there exist $K > 0$ and n_0 such that, $\forall n > n_0$,

$$\|\Gamma_{V_n}(x) - x\|_{L_2} \leq K \|\Pi_{V_n}(x) - x\|_{W_2^q}.$$

Given the multiresolution $\{V_n\}$, let $x_n \in V_n$ denote the wavelet-collocation solution to the stochastic BVP, and therefore

$$\Gamma_{V_n} \left\{ Dx_n - b(t, x_n) - \sigma(t, x_n) \dot{W}_t \right\} = 0, \quad \alpha(x_n) = c. \quad (7)$$

The following result is a consequence of Theorems 2 and 1.

Corollary 3 Under the assumptions of Theorems 1 and 2, the wavelet-collocation method satisfies the approximation property at rate $O(2^{-n})$.

Therefore, it remains to prove the consistence of the proposed method based on the Milstein scheme:

$$\begin{aligned} \tilde{X}_n(t_{i,n}) - \tilde{X}_n(t_{i-1,n}) &= h_n b(\tilde{X}_n(t_{i-1,n})) + \sigma(\tilde{X}_n(t_{i-1,n})) (W_{t_{i,n}} - W_{t_{i-1,n}}) \\ &\quad + \sigma(\tilde{X}_n(t_{i-1,n})) \frac{\partial \sigma}{\partial x}(\tilde{X}_n(t_{i-1,n})) \frac{1}{2} \left((W_{t_{i,n}} - W_{t_{i-1,n}})^2 - h_n \right), \end{aligned} \quad (8)$$

for $\tilde{X}_n \in V_n$.

Proposition 4 Under the assumptions of Theorems 1 and 2. Let $\tilde{x}_n \in V_n$ be the approximation generated by the proposed method and x_n the solution of the wavelet-collocation method. Then, it is satisfied $\|\tilde{x}_n - x_n\|_\infty = O_p(2^{-n})$.

4 Boundary-linked financial markets

Consider a monetary bond and d boundary-linked assets. Let assume that the bond has a continuous positive price per share $X_0(t)$ solving the stochastic differential equation

$$dX_0(t) = r(t) X_0(t) dt, \quad X_0(0) = 1, \quad (9)$$

where $r(t)$ is a progressively measurable process satisfying $\int_0^T |r(t)| dt < \infty$. Therefore $X_0(t) = \exp \left\{ \int_0^t r(s) ds \right\}$ for $t \in [0, T]$. Let $X_d(t)$ denote the price per share of each d -th boundary-linked asset and $X(t) = (X_1(t), \dots, X_d(t))'$. Assume that the initial values of the boundary-linked assets $X_1(0), \dots, X_d(0)$ are positive constants almost surely. For each $t \in [0, T]$, suppose also that these prices are governed by

$$dX(t) = b(t, X(t)) dt + \sigma(t) dW(t) \quad (10)$$

and the boundary conditions $\beta(X) = \rho$, where $\beta(X)$ is a set of d linear continuous real functionals and $\rho \in \mathbb{R}^d$. A particularly relevant example is the linear boundary value stochastic differential equations defined as,

$$dX(t) = b(t) X(t) dt + \sigma(t) dW(t) \quad (11)$$

with $\beta(X) = \rho$. Following the arguments given in Alabert and Ferrante (2002), this problem possesses a unique solution in $C^1([0, T])$ if and only if $\det \{\beta(x^s)\} \neq 0$ for some $s \in [0, T]$ (equivalently for all $s \in [0, T]$), where $x^s(t)$ is the solution of the homogeneous system $dx(t) = x(t) b(t) dt$ with $x(s) = I_d$; (i.e., problem $dx(t) = x(t) b(t) dt$ with $\beta(x) = 0$ has only the trivial solution). In this case, we can express

$$X(t) = J^{-1}(t) \rho + \int_0^t G(t, s) \sigma(s) dW(s),$$

with Green function

$$G(t, s) = J^{-1}(t) \left[\int_0^s J^{-1}(u) \nu(du) - 1_{[0, s]}(t) I_d \right] J(s),$$

being $1_{[0, s]}(t)$ the characteristic function of the set $[0, s]$.

In the linear context, we define a portfolio as a progressively measurable process $(\theta_0(t), \theta_1(t), \dots, \theta_d(t))'$ that represents the number of units of the assets for each $t \in [0, T]$. The value of a portfolio is given by $V_\theta(t) = \theta_0(t) X_0(t) + \theta(t)' X(t)$. The portfolio $\theta_0(t), \theta(t) = (\theta_1(t), \dots, \theta_d(t))'$ is called self-financing (respect to $X_0(t), X(t)$) if

$$\int_0^T \left(r(s) \theta_0(s)' X_0(s) + \theta(s)' b(s) + \sum_{k=1}^d |\theta(s)' \sigma_k(s)|^2 \right) ds < \infty,$$

$$dV_\theta(t) = \theta_0(t) dX_0(t) + \theta(t)' dX(t).$$

Then, $V_\theta(t) = V_\theta(0) + \int_0^t \theta_0(s) dX_0(s) + \int_0^t \theta(s)' dX(s)$. Notice that given an appropriate $\theta(t)'$, there exists $\theta_0(t)$ such that $(\theta_0(t), \theta(t)')$ is self-financing. A self-financing portfolio $(\theta_0(t), \theta(t)')$ is called admissible if its corresponding process $V_\theta(t)$ is a.e. lower bounded; i.e. $\exists K_\theta > 0$ such that $V_\theta(t) \geq -K_\theta$ a.e. for all $t \in [0, T]$. This is a natural constraint in real life as debts cannot infinitely increase. An admissible portfolio is called an arbitrage in the considered market, if the associated value process satisfies $V_\theta(0)$ and $V_\theta(T) \geq 0$ a.e. with $P(V_\theta(T) > 0) > 0$.

In practice, the value of a portfolio is often discounted at the non-risk rate. This means that we can normalize prices by defining $\tilde{X}_0(t) = 1$ and $\tilde{X}_k(t) = X_0(t)^{-1} X_k(t)$. Given a self-financing portfolio, the discounted values are $\tilde{V}_\theta(t) = L_0(t)^{-1} V_\theta(t)$, and applying the Itô formula $d\tilde{V}_\theta(t) = \theta(t)' d\tilde{L}(t)$. Therefore, the self-financing portfolio property is not affected by the discount normalization. Furthermore, if $(\theta_0(t), \theta(t)')$ is admissible for $(L_0, L(t))$, then $(\theta_0(t), \theta(t)')$ is also admissible for the normalized market $(1, \tilde{L}(t))$, as $r(t)$ is bounded.

4.1 Valuation of boundary-linked derivatives

In this section we consider pricing of boundary-linked derivatives. In this context, standard Black-Scholes techniques cannot help to value derivatives of boundary-linked assets as these processes are not Markovian. However, an alternative approach based on the generalized Clark-Ocone formula can be considered. We illustrate this approach considering an European call option of boundary-linked assets.

Let X_T be the values of the d boundary-linked assets at the maturity date of the contract. By the Clark-Ocone formula,

$$X_T(\omega) = E[X] + \int_0^T E[D_t X_T(\omega) | \mathcal{F}_t] dW_t,$$

with $D_t X_T(\omega)$ is the Malliavin derivative of $X_T(\omega)$. The Clark-Ocone formula can be extended to study \mathcal{F}_T random variables $G(\omega)$ that are stochastic integrals respect to processes:

$$\widetilde{W}_t(\omega) = \int_0^T \alpha(s, \omega) ds + W_t(\omega),$$

where $\alpha(s, \omega)$ is an \mathcal{F}_t adapted stochastic process satisfying some appropriate regularity conditions. By the Girsanov's theorem, \widetilde{W}_t is a Wiener process under certain probability measure Q on \mathcal{F}_T , where $dQ(\omega) = Z_T(\omega) dP(\omega)$, with

$$Z_t(\omega) = \exp \left\{ - \int_0^t \alpha(s, \omega) ds - \int_0^t \alpha(s, \omega)^2 ds \right\}.$$

The generalization of the Clark-Ocone formula ensures that if $G(\omega)$ is a regular stochastic integral respect to \widetilde{W}_t , then

$$G(\omega) = E_Q[G] + \int_0^T \varphi_Q(t, \omega) d\widetilde{W}_t,$$

where $\varphi_Q(t, \omega) = E_Q \left[D_t G - G \int_0^T D_t \alpha(s, \omega) d\widetilde{W}_s | \mathcal{F}_t \right]$. The proof and other technical details can be found in Øksendal (1997).

This result can be applied to the valuation of derivatives in linear boundary-linked markets. As the value of a portfolio is given by $V_\theta(t) = \theta_0(t) X_0(t) + \theta(t)' X(t)$, we have,

$$\theta_0(t) = X_0(t)^{-1} (V_\theta(t) - \theta(t)' X(t)). \quad (12)$$

If the portfolio is self-financing,

$$dV_\theta(t) = \theta_0(t) dX_0(t) + \theta(t)' dX(t),$$

using (9), (11) and (12), we obtain that

$$dV_\theta(t) = (r(t) V_\theta(t) + (b(t) - r(t)) \theta(t)' X(t)) dt + \theta(t)' \sigma(t) dW(t).$$

Assuming that the solution of the boundary value problem is \mathcal{F}_t adapted, our aim is to find a portfolio $\theta(t)$ leading to the lower bounded \mathcal{F}_T measurable random variable $G(\omega)$, such that $G(\omega) = V_\theta(T)$ and the initial value is $V_\theta(0)$.

If $V_\theta(t)$ is \mathcal{F}_t adapted, taking $\alpha(t) = (b(t) - r(t)) \sigma(t)^{-1}$ and $\widetilde{W}_t = \int_0^T \alpha(s) ds + W_t$, we can express

$$dV_\theta(t) = r(t) V_\theta(t) dt + \theta(t)' \sigma(t) d\widetilde{W}(t).$$

Therefore, the discounted portfolio $\tilde{V}_\theta(t) = X_0(t)^{-1} V_\theta(t)$ satisfies

$$d\tilde{V}_\theta(t) = X_0(t)^{-1} \theta(t)' \sigma(t) d\tilde{W}(t).$$

By the generalized Clark-Ocone theorem, the discounted final value $G := \tilde{V}_\theta(T) = X_0(T)^{-1} V_\theta(T)$ verifies

$$G = E_Q[G] + \int_0^T E_Q \left[D_t G - G \int_0^T D_t \alpha(s) d\tilde{W}_s | \mathcal{F}_t \right] d\tilde{W}_t.$$

As a consequence, $\tilde{V}_\theta(0) = E_Q[G]$ and the required portfolio is,

$$\theta(s) = X_0(t) \sigma(t)^{-1} E_Q \left[D_t G - G \int_0^T D_t \alpha(s) d\tilde{W}_s | \mathcal{F}_t \right].$$

This expression can be applied to the analysis of derivative prices in boundary-linked markets in an analogous way to the Black-Scholes formula.

For example, consider an European call option which gives the owner the right to buy the stock with value X_T at exercise price p . Then, $G = (X_T - p)^+$ represents the payoff at time T . Clearly, $G = f_p(X_T)$ where $f_p(x) = (x - p)^+$. Note that $f_p(\cdot)$ is continuous but not differentiable at $x = p$ and $D_t G$ cannot be obtained applying the chain rule. However, $f_p \in C([0, T])$ can be approximated by a sequence $\{f_n\} \subset C^1([0, T])$ with $f_n(x) = f_p(x)$ for $|x - p| \geq 1/n$, and $0 \leq f_n \leq 1$. Taking $G_n = f_n(X_T)$ we have

$$D_t G = \lim_{n \rightarrow \infty} D_t G_n = I_{[p, \infty)}(X_T) \cdot D_t X_T = I_{[p, \infty)}(X_T) \cdot X_T \cdot \sigma(t).$$

Hence,

$$\theta(t) = X_0(t) \sigma(t)^{-1} E_Q \left[I_{[p, \infty)}(X_T) X_T \sigma(t) - f_p(X_T) \int_0^T D_t \alpha(s) d\tilde{W}_s | \mathcal{F}_t \right].$$

In particular, if $dX(t) = bX(t)dt + \sigma dW(t)$ and $r(t) = r > 0$, then $D_t \alpha = D_t(b - r)\sigma^{-1} = 0$ a.e., and

$$\theta(t) = X_0(t) E_Q [I_{[p, \infty)}(X_T) X_T | \mathcal{F}_t]. \quad (13)$$

When X_T follows a diffusion process, (13) leads to the classical Black-Scholes formula applying Markovian arguments. However, in case of boundary-linked assets markets, as these assets follow a boundary value stochastic differential equation, the expectation in (13) cannot be computed using Markovian arguments and numerical resolution methods are required.

In order to compute the portfolio $\{\theta_t\}$, associated to a given a realization of the underlying processes $\{(X_t, W_t)\}$, we propose the use of a Monte Carlo simulation-based estimation of (13) using independently generated realizations of the process $\{X_t\}$ conditioned to the information set \mathcal{F}_t , which is computed using the wavelet-collocation approach presented in Section 3. Three steps are involved. For each dyadic point $t_{i,n} = 2^{-n}i \in [0, T]$ with $i \in \mathbb{Z}$, we simulate M independent realizations of the Brownian motion, denoted by $W_{t_{i,n}}^j$ for all $j = 1, \dots, M$, such that $W_t^j = W_t$, for any dyadic point $t \in [0, t_{i,n}]$. The second step of the algorithm consists of solving, for each $j = 1, \dots, M$, the following system of boundary value stochastic differential equations,

$$\begin{aligned} dX_t^j &= bX_t^j dt + \sigma dW_t^j, \\ \beta(X^j) &= \rho, \end{aligned}$$

with the additional constraints $X_t^j = X_t$, for any dyadic point $t \in [0, t_{i,n}]$. In particular, we compute the solution of these problems $\{X_t^j\}_{j=1}^M$ by means of the wavelet-collocation algorithm presented in Section 3. In the third and final step, we compute Portfolio (13) at each $t_{i,n} \in [0, T]$ as

$$\theta^M(t_{i,n}) = \exp(rt_{i,n}) \frac{1}{M} \sum_{j=1}^M I_{[p,\infty)}(X_T^j) X_T^j. \quad (14)$$

For example, Figure 4 shows the numerical simulations of X_t conditioned to the available information at $t_{i,n} = 0.25$, when X_t follows the boundary-valued stochastic differential equation given in Example I and $M = 50$. Figure 5 shows the path of portfolio (13) computed as (14) with $r = 0.2$ and $p = 0.25$.

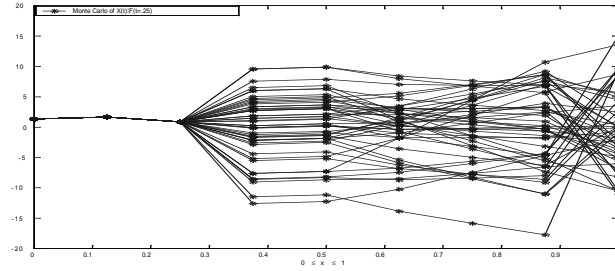


Figure 4. Monte Carlo simulations of $X_t \mid \mathcal{F}_{t=0.25}$.

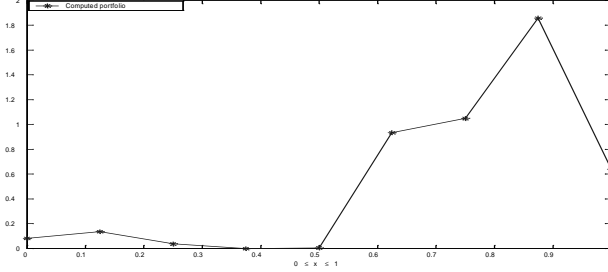


Figure 5. Path of the portfolio (13), with $r = 0.2$ and $p = 0.25$.

5 Appendix A: Proofs

A) Proof of Theorem 1

We will use the following Theorem,

Theorem 5 *Let B be a Banach space, $\{V_n\} \subset B$ a sequence of increasing linear subspaces, and Π_{V_n} a sequence of continuous projections converging pointwise to the identity operator on B . Let T define a (non linear) operator in B . If $(1 - T)u = 0$ has a solution u^0 , T is continuously Frechet differentiable at u^0 and $(1 - T'_{u^0})u = 0$ has only the trivial solution in B , then u^0 is unique in some sphere $B(u^0, \delta) = \{u \in B : \|u - u^0\| \leq \delta\}$ for some $\delta > 0$, and there exists an integer M such that for all $n > M$ the equation $\Pi_{V_n} \{(1 - T)u\} = 0$ has a unique solution $u_n \in V_n \cap B(u^0, \delta)$. Moreover, $\exists K > 0$ such that*

$$\|u_n - u^0\| \leq K \|\Pi_{V_n} u^0 - u^0\|.$$

Proof. See e.g. Vainikko (1967, Th. 5). ■

Using the properties of the Green function and the continuity of b , the functional T is continuous relative to the uniform norm on a neighborhood of $u^0 = G(x^0)$. Since for each realization of the white noise process $(1 - T)u = 0$ can be seen as an equation in $C_0([0, T], \mathbb{R}^d)$, we will consider the equation $\Pi_{V_n}(I - T)u_n = 0$ in V_n .

First, we check the continuously Frechet differentiability of T . For any $u \in B(u^0, \delta)$ define $h = u - u^0$. Notice that \mathcal{N} contains all line segments in \mathbb{R}^{R+1} such as $\{u^0 + \theta h : \theta \in [0, 1]\}$, since

$$x(t) - x^0(t) = \int_0^T D_t G(t, s) h(s) ds,$$

with $\|x - x^0\|_{L_\infty} < \varepsilon$ whenever δ is small enough, using that

$$\chi := \operatorname{ess\,sup}_{t \in [0, T]} \int_0^T |D_t G(t, s)| ds < \infty.$$

Recall that $u^0 = G(x^0)$. We will see that the Frechet derivative T at $u^0(t) = Dx^0(t)$ respect to the direction $h = (u - u^0)$ is given by

$$T'_{u^0}(h)(t) = \left(D_u b(t, u^0) + D_u \sigma(t, u^0) \dot{W}_t \right) \int_0^T D_t G(t, s) h(s) ds,$$

and the error term is given by

$$\begin{aligned} \epsilon_{u^0}(u)(t) &= \|h\|^2 \int_0^T (1 - \theta) b''(t, u^0(t) + \theta h(t)) dt \\ &\quad + \|h\|^2 \int_0^T (1 - \theta) \sigma''(t, u^0(t) + \theta h(t)) dW(t), \end{aligned}$$

being b'', σ'' the second directional derivatives of $b(t, \cdot), \sigma(t, \cdot)$ respectively, in the direction $h/\|h\|$, and $\|h\|^2 = \sum_{r=1}^R \|h_r\|^2$. Clearly $\|\epsilon_{u^0}(u)\|_{L_\infty} \leq c_1 \|u - u^0\|_{L_\infty}^2$, where c_1 is the maximum between χ and $\sup \{b''(t, x) + \sigma''(t, x) W_T\}$ over all directions on $\operatorname{adh}(\mathcal{N})$, which is finite with probability one as $\Pr(W_T = \infty) = 0$ for finite T .

Notice also that $T'_{u^0}(h)(t)$ can be expressed in the original domain as the operator

$$T'_{x^0}(x) = \left(D_x b(t, x^0(t)) + D_x \sigma(t, x^0(t)) \dot{W}_t \right) Dx.$$

Since $\det \left\{ \left(I - D_x b(t, x^0(t)) - D_x \sigma(t, x^0(t)) \dot{W}_t \right) \right\} \neq 0$, almost surely, for all $t \in [0, T]$, there exists a unique trivial solution for

$$\left(I - D_x b(t, x^0(t)) - D_x \sigma(t, x^0(t)) \dot{W}_t \right) Dx = 0,$$

with $\alpha(x) = c$. This implies the same result for $(I - T'_{u^0})u = 0$, hence assumptions of Theorem 5 are satisfied.

Thus, there exists an integer $M > 0$ such that, for all $n > M$ a solution $u_n \in V_n$ exists and is unique in the same sphere. Moreover, there exists a constant $c > 0$ such that $u_n = Dx_n, u^0 = Dx^0$ and

$$\|u_n - u^0\|_{L_\infty} \leq c \|\Pi_{V_n} u^0 - u^0\|_{L_\infty}.$$

By the Banach-Steinhaus theorem, for all $u \in V_n$,

$$\begin{aligned} \|\Pi_{V_n} u^0 - u^0\|_{L_\infty} &= \|\Pi_{V_n}(u^0 - u) - (u - u^0)\|_{L_\infty} = \|(1 - \Pi_{V_n})(u^0 - u)\|_{L_\infty} \\ &\leq c' \inf \left\{ \|u^0 - u\|_{L_\infty} : u \in V_n \right\} = O(2^{-n}), \end{aligned}$$

where the rate $O(2^{-n})$ follows from Assumption A.1.

The result follows noticing that $\|Dx_n - Dx^0\|_{L_\infty} = \|u_n - u^0\|_{L_\infty}$, and

$$\|x_n - x^0\|_{L_\infty} \leq \|G^{-1}\|_{L_\infty} \|u_n - u^0\|_{L_\infty}$$

using that $x_n - x^0 = G^{-1}(u_n - u^0)$.

B) Proof of Theorem 2

The problem of interpolation in V_n at points $t_{n,i} = 2^{-n}i$ can be reduced to solve the problem $g_0(i) = x(t_{n,i})$ in $g_0 \in V_0$ and then take $g_n(t) = 2^{n/2}g_0(2^nt)$. Therefore, assume that $g_0(t) = \sum_{k \in \mathbb{Z}} \theta_k \phi(t - k)$ solves this problem, i.e.

$$\sum_{k \in \mathbb{Z}} \theta_k \phi(i - k) = x(t_{n,i}).$$

Clearly, a unique solution exists since $\{\phi(t - k)\}_{k \in \mathbb{Z}}$ are linearly independent functions. To simplify the notation, we denote $x_i = x(t_{n,i})$, hence $\sum_{k \in \mathbb{Z}} \theta_k \phi(i - k) = x_i$. This is a convolution equation that we will solve in the spectral domain. Let define the discrete Fourier transform of ϕ by

$$\tilde{\Phi}(\omega) = \sum_{k \in \mathbb{Z}} \phi(k) e^{-ik\omega}.$$

The Poisson formula states that $\tilde{\Phi}(\omega) = \sum_{k \in \mathbb{Z}} \Phi(\omega + 2\pi k)$. If ϕ is regular of at least order 1, this series converges uniformly on compact sets. Furthermore, as $\tilde{\Phi}(\omega) > 0$ a.e. for $\omega \in [0, 2\pi]$, the inverse has a Fourier expansion $(1/\tilde{\Phi}(\omega)) = \sum_{k \in \mathbb{Z}} \beta_k e^{-ik\omega}$ where $b := \sum_{k \in \mathbb{Z}} |\beta_k| < \infty$, by the Wiener-Lévy theorem. Thus, we can explicitly evaluate the coefficients $\{\theta_k\}$ as,

$$\theta_k = \sum_{i \in \mathbb{Z}} \beta_{k-i} x_i.$$

Obviously,

$$\sum_{k \in \mathbb{Z}} |\theta_k|^2 = \sum_{k \in \mathbb{Z}} \left| \sum_{l \in \mathbb{Z}} \beta_{l-i} x_i \right|^2 \leq b^2 \sum_{k \in \mathbb{Z}} |x_i|^2 = b^2 \sum_{i \in \mathbb{Z}} |x(t_{n,i})|^2,$$

with $\sup_{n \geq 1} \sum_{i \in \mathbb{Z}} |x(t_{n,i})|^2 < \infty$ as x is continuous with compact support.

Next, we will prove that

$$\|\Gamma_{V_n}(x)\|_{L_2} \leq b \|x\|_n,$$

where $\|x\|_n = \left(2^{-n} \sum_{i \in \mathbb{Z}} |x(t_{n,i})|^2\right)^{1/2}$.

Notice that $\|g_n\|_{L_2} = 2^{-n} \|g_0\|_{L_2} = 2^{-n} \|\mathcal{F}(g_0)\|_{L_2}$, where $\mathcal{F}(g_0)(\omega)$ is the continuous Fourier transformed of g_0 . We will prove that $\|\mathcal{F}(g_0)\|_{L_2}^2 = \sum_{k \in \mathbb{Z}} |\theta_k|^2$ and the result follows. Let define $\tilde{c}(\omega) =$

$\sum_{k \in \mathbb{Z}} \theta_k e^{-ik\omega}$, then

$$\begin{aligned}
\|\mathcal{F}(g_0)\|_{L_2}^2 &= \int_{\mathbb{R}} \left| \mathcal{F} \left(\sum_{k \in \mathbb{Z}} \theta_k \phi_{0,k} \right) (\omega) \right|^2 d\omega = \int_{\mathbb{R}} \left| \Phi(\omega) \left(\sum_{k \in \mathbb{Z}} \theta_k e^{-ik\omega} \right) \right|^2 d\omega \\
&= \int_{\mathbb{R}} |\Phi(\omega) \tilde{c}(\omega)|^2 d\omega = \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} |\Phi(\omega) \tilde{c}(\omega)|^2 d\omega \\
&= \int_0^{2\pi} |\tilde{c}(\omega)|^2 \left| \sum_{k \in \mathbb{Z}} \Phi(\omega + 2\pi k) \right|^2 d\omega \\
&= \int_0^{2\pi} |\tilde{c}(\omega)|^2 d\omega = \sum_{k \in \mathbb{Z}} |\theta_k|^2,
\end{aligned}$$

as $\{\phi(t-k)\}_{k \in \mathbb{Z}}$ is orthonormal if and only if $\sum_{k \in \mathbb{Z}} |\Phi(\omega + 2\pi k)|^2 = 1$ a.e., for details see Daubechies (1992). Hence, we have that $\|\Gamma_{V_n}(x)\|_{L_2}^2 \leq b^2 \|x\|_n^2$.

Defining $x_n = \Pi_{V_n}(x)$, we have that $\Gamma_{V_n}(x_n) = x_n$ since $x_n \in V_n$ and has compact support. And as a consequence,

$$\begin{aligned}
\|\Gamma_{V_n}(x) - x\|_{L_2} &= \|\Gamma_{V_n}(x_n - x) + x_n - x\|_{L_2} \leq b^2 \|x_n - x\|_n + \|x_n - x\|_{L_2} \\
&= b^2 \|\Pi_{V_n}(x) - x\|_n + \|\Pi_{V_n}(x) - x\|_{L_2}.
\end{aligned}$$

Moreover, as for all $x \in W_2^r(\mathbb{R})$, with $r \geq 1$,

$$\|x\|_n^2 \leq C \left\{ \int_{-2^n\pi}^{2^n\pi} |\mathcal{F}(x)(\omega)|^2 d\omega + 2^{-nr} \|x\|_{W_2^r}^2 \right\}$$

see Thomée (1973, Lemma 4.4.), the result follows applying the same bound to $\|\Pi_{V_n}(x) - x\|_n^2$.

C) Proof of Proposition 4

Let X_n be the solution to (7), and \tilde{X}_n be the solution to (8), i.e. the proposed algorithm. We will prove that $E \left[\left\| X_n - \tilde{X}_n \right\|_{\infty}^2 \right] = O(h_n^2)$ and the result follows.

Consider first the autonomous case. By assumption $b, \sigma \in C^2(\mathcal{N})$. Define the operators,

$$L^0 = b \frac{\partial}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2}, \quad L^1 = \sigma \frac{\partial}{\partial x}.$$

By the Wagner and Platen expansion (see Kloden and Platen, 1999, for a review), X_n satisfies the equations

$$A_{i,n}(X_n) = R_{i,n}(X_n),$$

for all $i \in \mathbb{Z}$ such that $t_{i,n} = 2^{-n}i \in [0, T]$, where

$$\begin{aligned}
A_{i,n}(X_n) &= X_n(t_{i,n}) - X_n(t_{i-1,n}) - h_n b(X_n(t_{i-1,n})) - \sigma(X_n(t_{i-1,n}))(W_{t_{i,n}} - W_{t_{i-1,n}}) \\
&\quad - L^1 \sigma(X_n(t_{i-1,n})) \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s dW_z dW_s \\
R_{i,n}(X_n) &= \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s L^0 b(X_n(z)) dz ds + \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s L^1 b(X_n(z)) dW_z ds \\
&\quad + \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s L^0 \sigma(X_n(z)) dz dW_s \\
&\quad + \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s \int_{t_{i-1,n}}^z L^0 L^1 \sigma(X_n(u)) du dW_z dW_s \\
&\quad + \int_{t_{i-1,n}}^{t_{i,n}} \int_{t_{i-1,n}}^s \int_{t_{i-1,n}}^z L^1 L^1 \sigma(X_n(u)) dW_u dW_z dW_s
\end{aligned}$$

Let $A_n(X_n) = R_n(X_n)$ denote this system of nonlinear equations, where Let denote this system by $A_n(X_n) = R_n(X_n)$, where $E \left[\|R_n(X_n)\|_\infty^2 \right] = O(h_n^2)$.

On the other hand, \tilde{X}_n satisfies system (8); i.e. $A_n(\tilde{X}_n) = 0$. Then,

$$R_n(X_n) = A_n(X_n) = A_n(\tilde{X}_n) + DA_{\varphi_n}(X_n - \tilde{X}_n) = DA_{\xi_n}(X_n - \tilde{X}_n),$$

where DA_{φ_n} is the Frechet derivative at some intermediate point φ_n . Since $\|DA_{\varphi_n}(\cdot)\|_\infty^{-1} \geq \varepsilon > 0$ uniformly, it is satisfied that

$$E \left[\|X_n - \tilde{X}_n\|_\infty^2 \right] = O \left(E \left[\|R_n(X_n)\|_\infty^2 \right] \right),$$

and the result follows. For non autonomous systems the argument is analogous.

6 References

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