A Simple Finite Element Method for Boundary Value Problems with a Riemann-Liouville Derivative

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Abstract

We consider a boundary value problem involving a Riemann-Liouville fractional derivative of order $\alpha \in (3/2, 2)$ on the unit interval $(0, 1)$. The standard Galerkin finite element approximation converges slowly due to the presence of singularity term $x^{\alpha-1}$ in the solution representation. In this work, we develop a simple technique, by transforming it into a second-order two-point boundary value problem with nonlocal low order terms, whose solution can reconstruct directly the solution to the original problem. The stability of the variational formulation, and the optimal regularity pickup of the solution are analyzed. A novel Galerkin finite element method with piecewise linear or quadratic finite elements is developed, and $L^2(D)$ error estimates are provided. The approach is then applied to the corresponding fractional Sturm-Liouville problem, and error estimates of the eigenvalue approximations are given. Extensive numerical results fully confirm our theoretical study.

Keywords: finite element method; Riemann-Liouville derivative; fractional boundary value problem; Sturm-Liouville problem; singularity reconstruction.
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1. Introduction

In this work, we consider the following boundary value problem involving a Riemann-Liouville fractional derivative

$$
\begin{aligned}
-{}_0^R D_x^\alpha u + qu = f \quad \text{in } D \equiv (0,1), \\
u(0) = u(1) = 0,
\end{aligned}
\tag{1.1}
$$

where $f \in L^2(D)$, and ${}_0^R D_x^\alpha u$ denotes the Riemann-Liouville fractional derivative of order $\alpha \in (3/2, 2)$, defined in (2.1) below. The choice $\alpha \in (3/2, 2)$ is mainly technical, since for $\alpha \in (1, 3/2]$, the analysis below does not carry over, even though numerically the technique to be developed works well. For $\alpha = 2$, the fractional derivative ${}_0^R D_x^\alpha u$ recovers the usual second-order derivative u'', and thus the model (1.1) can be viewed as the fractional counterpart of the classical two-point boundary value problem.

Problem (1.1) arises in the mathematical modeling of superdiffusion process in heterogeneous media, in which the mean square variance grows faster than that in the Gaussian process. It has found applications in magnetized plasma [1, 2] and subsurface flow [3]. The numerical study of problem (1.1) is quite extensive. Among existing methods, the finite difference method based on the shifted Grünwald-Letnikov formula is predominant, since the earlier introduction [4]; and see also [5] for higher order schemes. However, in these interesting works, one standing assumption is that the solution is sufficiently smooth, which unfortunately is generally not justified [6]. To this date, the precise condition under which the solution to (1.1) is indeed smooth remains unclear. Recently, finite element methods (FEMs) [7, 8] were developed and analyzed.

One of the main challenges in accurately solving problem (1.1) is that the solution contains a singular term $x^{\alpha-1}$ (see [6] and Section 2 below), which in turn limits the global solution regularity and thus also the accuracy of numerical approximations. One way to resolve the issue is the singularity reconstruction technique recently developed by the first and fourth named authors [9] and inspired by [10], in which the solution is split into a singular part containing the term $x^{\alpha-1}$, and a regular part. A variational formulation of the regular part is derived, and the singularity strength is then reconstructed from the regular part. The numerical experiments in [9] indicate that the method converges well for problem (1.1), with provable $L^2(D)$ convergence rates, which improves that for the standard Galerkin FEM. However, the extension of the method to the related Sturm-Liouville problem seems not viable, due to the nonlinear nature of the eigenvalue problem.

In this work, we develop a novel approach for solving problem (1.1) based on transformation. It retains the salient features of the singularity reconstruction approach, i.e., resolving accurately the singularity, enhanced convergence rates and easy implementation. Meanwhile it can be extended straightforwardly to the related Sturm-Liouville problem with a Riemann-Liouville fractional derivative in the leading term, and the resulting linear system can be solved efficiently by a preconditioning technique. The approach is motivated by the following

observation: under the Riemann-Liouville integral transformation ${}_0I_x^{2-\alpha}u$, cf. (2.2), the leading singularity $x^{\alpha-1}$ is actually smoothed into a very smooth function x, which can be well approximated by the standard conforming finite elements or orthogonal polynomials. We shall derive a new formulation for the transformed variable, and analyze its stability and the finite element approximation. Further, the approach is extended to the related Sturm-Liouville problem, and the convergence rate is also established.

The rest of the paper is organized as follows. In Section 2 we recall preliminaries of fractional calculus, including properties of fractional integral and differential operators in Sobolev spaces. Then in Section 3, we derive the new approach, develop the proper variational formulation, and establish stability estimates. The Galerkin FEM with continuous piecewise linear and quadratic finite elements is discussed in Section 4. $L^2(D)$ error estimates are provided for the FEM approximations to (1.1). The approach is then extended to the Sturm-Liouville problem in Section 5. Finally, extensive numerical results are presented in Section 6 to verify the efficiency and accuracy of the new approach. Throughout, the notation c, with or without a subscript, denote a generic constant, which may differ at different occurrences, but it is always independent of the mesh size h.

2. Preliminaries

We first recall the definition of the Riemann-Liouville fractional derivative. For any $\beta > 0$ with $n - 1 < \beta < n$, $n \in \mathbb{N}$, the left-sided Riemann-Liouville fractional derivative ${}_0^R D_x^\beta u$ of order β of a function $u \in C^n[0,1]$ is defined by [11, pp. 70]:

$$
{}_0^R D_x^\beta u = \frac{d^n}{dx^n}\left({}_0I_x^{n-\beta}u\right). \tag{2.1}
$$

Here ${}_0I_x^\gamma$ for $\gamma > 0$ is the left-sided Riemann-Liouville fractional integral operator of order γ defined by

$$
({}_0I_x^\gamma f)(x) = \frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}f(t)dt, \tag{2.2}
$$

where $\Gamma(\cdot)$ is Euler's Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. The right-sided versions of the fractional-order integral operator ${}_xI_1^\gamma$ and derivative operator ${}_x^R D_1^\beta$ are defined analogously by

$$
({}_xI_1^\gamma f)(x) = \frac{1}{\Gamma(\gamma)}\int_x^1 (t-x)^{\gamma-1}f(t)\,dt \quad\text{and}\quad {}_x^R D_1^\beta u = (-1)^n\frac{d^n}{dx^n}\left({}_xI_1^{n-\beta}u\right).
$$

Now we introduce some function spaces. For any $\beta \geq 0$, we denote $H^\beta(D)$ to be the Sobolev space of order β on the unit interval D, and $\widetilde{H}^\beta(D)$ to be the set of functions in $H^\beta(D)$ whose extension by zero to $\mathbb{R}$ are in $H^\beta(\mathbb{R})$. Analogously, we define $\widetilde{H}_L^\beta(D)$ (respectively, $\widetilde{H}_R^\beta(D)$) to be the set of functions u whose extension by zero, denoted by $\tilde{u}$, is in $H^\beta(-\infty, 1)$ (respectively, $H^\beta(0, \infty)$). For

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$u \in \widetilde{H}_L^\beta(D)$, we set $\|u\|_{\widetilde{H}_L^\beta(D)} := \|\tilde{u}\|_{H^\beta(-\infty,1)}$, and analogously the norm in $\widetilde{H}_R^\beta(D)$.

The following theorem collects their important properties [11, pp. 73, Lemma 2.3] [6, Theorems 2.1 and 3.1]. In particular, Theorem 2.1(b) extends the domain of the operator ${}_0^R D_x^\beta$ from $C^n[0,1]$ to $\widetilde{H}_L^\beta(D)$.

Theorem 2.1. *The following statements hold.*

(a) *The integral operators ${}_0 I_x^\beta$ and ${}_x I_1^\beta$ satisfy the semigroup property.*

(b) *The operators ${}_0^R D_x^\beta$ and ${}_x^R D_1^\beta$ extend continuously to operators from $\widetilde{H}_L^\beta(D)$ and $\widetilde{H}_R^\beta(D)$, respectively, to $L^2(D)$.*

(c) *For any $s, \beta \geq 0$, the operator ${}_0 I_x^\beta$ is bounded from $\widetilde{H}_L^s(D)$ to $\widetilde{H}_L^{\beta+s}(D)$, and ${}_x I_1^\beta$ is bounded from $\widetilde{H}_R^s(D)$ to $\widetilde{H}_R^{\beta+s}(D)$.*

We shall also need an "algebraic" property of the space $\widetilde{H}^s(D)$, $0 < s < 1$ [6, Lemma 4.6].

Lemma 2.1. *Let $0 < s \leq 1$, $s \neq 1/2$. Then for any $u \in H^s(D) \cap L^\infty(D)$ and $v \in \widetilde{H}^s(D) \cap L^\infty(D)$, $uv \in \widetilde{H}^s(D)$.*

Now we describe the variational formulation. We first introduce the bilinear form

$$a(u,v) = -({}_0^R D_x^{\alpha/2} u, {}_x^R D_1^{\alpha/2} v) + (qu, v). \tag{2.3}$$

Then the variational formulation for problem (1.1) is given by: find $u \in \widetilde{H}^{\alpha/2}(D)$ such that

$$a(u,v) = (f,v) \quad \forall v \in \widetilde{H}^{\alpha/2}(D). \tag{2.4}$$

For trivial case $q \equiv 0$, the well-posedness follows from the boundedness and coercivity of $-({}_0^R D_x^{\alpha/2} \cdot, {}_x^R D_1^{\alpha/2} \cdot)$ in $\widetilde{H}^{\alpha/2}(D)$ (see [7, Lemma 3.1], [6, Lemma 4.2]). Simple computation shows that the variational solution u of (2.4) is given by

$$u(x) = -({}_0 I_x^\alpha f)(x) + ({}_0 I_x^\alpha f)(1) x^{\alpha-1}, \tag{2.5}$$

and it satisfies the strong formulation (1.1).

To study the bilinear form $a(\cdot, \cdot)$ in general case, i.e. $q \neq 0$, we make the following assumption.

Assumption 2.2. *Let the bilinear form $a(u,v)$ with $u, v \in \widetilde{H}^{\alpha/2}(D)$ satisfy*

(a) *The problem of finding $u \in \widetilde{H}^{\alpha/2}(D)$ such that $a(u,v) = 0$ for all $v \in \widetilde{H}^{\alpha/2}(D)$ has only the trivial solution $u \equiv 0$.*

(a*) *The problem of finding $v \in \widetilde{H}^{\alpha/2}(D)$ such that $a(u,v) = 0$ for all $u \in \widetilde{H}^{\alpha/2}(D)$ has only the trivial solution $v \equiv 0$.*

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Under Assumption 2.2, there exists a unique solution $u \in \widetilde{H}^{\alpha/2}(D)$ to (2.4) [6, Theorem 4.3]. In fact the variational solution is a strong solution. To see this, we consider the problem $-{}^R_0 D^\alpha_x u = f - qu$. A strong solution is given by (2.5) with a right hand side $\widetilde{f} = f - qu$. It satisfies the variational equation (2.4) and hence coincides with the unique variational solution. Further, the solution u satisfies the stability estimate $\|u\|_{\widetilde{H}^{\alpha-1+\beta}_L(D)} \leq c\|f\|_{L^2(D)}$, for any $\beta \in (2 - \alpha, 1/2)$. The representation (2.5) indicates that the global regularity of the solution u does not improve with the regularity of the source term f, due to the inherent presence of the term $x^{\alpha-1}$.

3. A new approach: Variational formulation and regularity

In this section, we develop a new approach for problem (1.1). We first motivate the approach, and then discuss the variational stability and regularity pickup. The adjoint problem is also briefly discussed.

3.1. Motivation of the new approach

First, we motivate the new approach. The basic idea is to absorb the leading singularity $x^{\alpha-1}$ into the problem formulation. To this end, we set

$$u = {}^R_0 D^{2-\alpha}_x w - ({}^R_0 D^{2-\alpha}_x w)(1) x^\mu, \tag{3.1}$$

where $\mu \geq \alpha$ is a parameter to be selected. The motivation behind the choice of the fractional derivative ${}^R_0 D^{2-\alpha}_x w$ is that the primitive of the singularity $x^{\alpha-1}$ under the "fractional" transformation is x (up to a multiplicative constant), which is smooth and can be accurately approximated by standard finite element functions. The second term in the expression is to keep the boundary condition $u(1) = 0$. From the condition $w(0) = 0$, we deduce that $u(0) = 0$ (for more details see the proof of Theorem 3.4). Upon substituting it back into (1.1), and noting that for $w \in \widetilde{H}^1(D)$

$$\left({}_0 I^{\alpha-1}_x w \right)' (x) = \left({}_0 I^{\alpha-1}_x w' \right) (x),$$

we arrive at

$$-{}^R_0 D^\alpha_x u + qu = -w'' + ({}^R_0 D^{2-\alpha}_x w)(1)(c_0 x^{\mu-\alpha} - q x^\mu) + q\,{}^R_0 D^{2-\alpha}_x w, \tag{3.2}$$

where the constant c_0 is defined as

$$c_0 = \Gamma(\mu + 1)/\Gamma(1 + \mu - \alpha). \tag{3.3}$$

Here the second line follows from the boundary condition $w(0) = 0$ and the identity

$$ {}^R_0 D^\alpha_x \, {}^R_0 D^{2-\alpha}_x w = ({}_0 I^{2-\alpha}_x {}^R_0 D^{2-\alpha}_x w)'' = w''.$$

Consequently, the transformed variable w solves the boundary value problem

$$-w'' + q\,{}^R_0 D^{2-\alpha}_x w + ({}^R_0 D^{2-\alpha}_x w)(1) \left(c_0 x^{\mu-\alpha} - q x^\mu \right) = f \quad \text{in } D, \tag{3.4}$$
$$w(0) = w(1) = 0.$$

Once problem (3.4) is solved, the solution u to problem (1.1) can be reconstructed from (3.1). Equation (3.4) is a boundary value problem for an integro-differential equation and has a number of distinct features:

(a) The leading term involves a canonical second-order derivative, and thus the solution w is free from singularity, if the source term f is smooth. This overcomes one of the main challenges inherent to the fractional formulation (1.1).

(b) In the resulting linear system from the Galerkin discretization of problem (3.4), the leading term is dominant and has a simple structure; it can naturally act as a preconditioner.

(c) The approach extends straightforwardly to the related Sturm-Liouville problem of finding the eigenpairs.

Remark 3.1. *Throughout, the condition $\mu \geq \alpha$ will be assumed below. Note that the choice $\mu = \alpha - 1$ is also of special interest, for which, with the identity ${}^{R}_{0}D^{\alpha}_{x}x^{\alpha-1} = 0$, the modified equation reads*

$$-w'' + q(x)\, {}^{R}_{0}D^{2-\alpha}_{x}w - ({}^{R}_{0}D^{2-\alpha}_{x}w)(1)\, q(x)\, x^{\alpha-1} = f(x) \quad in\ D,$$
$$w(0) = w(1) = 0.$$

Since $\alpha > 3/2$, the term $x^{\alpha-1}$ belongs to the space $H^1(D)$. Thus, the theoretical developments below, especially Theorem 3.3, remain valid for this choice.

3.2. Variational stability

Next we discuss the well-posedness of the formulation (3.4) for the case $\alpha \in (3/2, 2)$, by showing

(a) Problem (3.4) has a unique solution $w \in \widetilde{H}^1(D)$ and certain regularity pickup;

(b) $u = {}^{R}_{0}D^{2-\alpha}_{x}w - ({}^{R}_{0}D^{2-\alpha}_{x}w)(1)x^{\mu}$ is the solution of problem (1.1).

Further, we shall consider the following general problem: For $\alpha \in (3/2, 2)$, find w

$$-w'' + q\, {}^{R}_{0}D^{2-\alpha}_{x}w + p\, ({}^{R}_{0}D^{2-\alpha}_{x}w)(1) = f \quad in\ D, \tag{3.5}$$
$$w(0) = w(1) = 0,$$

where $f, p \in H^r(D)$ and q belongs to suitable Sobolev spaces to be specified below. The weak formulation of problem (3.5) is given by: find $w \in V \equiv \widetilde{H}^1(D)$ such that

$$A(w, \varphi) := a(w, \varphi) + b(w, \varphi) = (f, \varphi) \quad \forall \varphi \in V, \tag{3.6}$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined on $V \times V$ by

$$a(\psi, \varphi) = (\psi', \varphi') \quad \text{and} \quad b(\psi, \varphi) = ({}^{R}_{0}D^{2-\alpha}_{x}\psi, q\varphi) + ({}^{R}_{0}D^{2-\alpha}_{x}\psi)(1)(p, \varphi). \tag{3.7}$$

First we show that $A(\cdot,\cdot)$ is bounded on $V \times V$. For $b(\cdot,\cdot)$, by Theorem 2.1 we note that for $\psi \in V$

$$\|_0^R D_x^{2-\alpha}\psi\|_{L^2(D)} \le c\|\psi\|_{\widetilde{H}_L^{2-\alpha}(D)} \le c\|\psi'\|_{L^2(D)}.$$

By the identity $(\,_0 I_x^{\alpha-1}\psi)' = (\,_0 I_x^{\alpha-1}\psi')$ for $\psi \in V$ [6, Lemma 4.1] we have (with $\omega_{\alpha-1}(x) = (1-x)^{\alpha-2}/\Gamma(\alpha-1)$)

$$|(_0^R D_x^{2-\alpha}\psi)(1)| = |(\,_0 I_x^{\alpha-1}\psi')(1)| \le c\|\omega_{\alpha-1}\|_{L^2(D)}\|\psi'\|_{L^2(D)}.$$

Note that $\omega_{\alpha-1} \in L^2(D)$ for $\alpha \in (3/2,2)$. Hence

$$|b(\psi,\varphi)| \le \|q\|_{L^\infty(D)}\|_0^R D_x^{2-\alpha}\psi\|_{L^2(D)}\|\varphi\|_{L^2(D)} + |(_0^R D_x^{2-\alpha}\psi)(1)|\|p\|_{L^2(D)}\|\varphi\|_{L^2(D)}$$
$$\le c\|\psi\|_V\|\varphi\|_{L^2(D)}.$$
$$(3.8)$$

Now we turn to the well-posedness of the variational formulation (3.6). In case of $q \equiv p \equiv 0$, the bilinear form $A(\cdot,\cdot)$ is identical with $a(\cdot,\cdot)$ which recovers the standard Poisson equation and the well-posedness is well-known. Next we consider the general case when q and p are not identically zero. To this end, we make the following uniqueness assumption on the bilinear form $A(\cdot,\cdot)$.

Assumption 3.1. *Let the bilinear form $A(w,v)$ with $w,v \in V$ satisfy*

(a) *The problem of finding $w \in V$ such that $A(w,v) = 0$ for all $v \in V$ has only the trivial solution $w \equiv 0$.*

(a*) *The problem of finding $v \in V$ such that $A(w,v) = 0$ for all $w \in V$ has only the trivial solution $v \equiv 0$.*

Under Assumption 3.1, the variational formulation (3.6) is stable.

Theorem 3.2. *Let Assumption 3.1 hold, $q \in L^\infty(D)$ and $p \in L^2(D)$. Then for any $F \in V^*$, there exists a unique solution $w \in V$ to*

$$A(w,\varphi) = \langle F,\varphi \rangle \quad \forall \varphi \in V, \qquad (3.9)$$

where $\langle \cdot,\cdot \rangle$ denotes the duality between V and its dual space $V^* = H^{-1}(D)$.

Proof. The stability is proved by Petree-Tartar Lemma [12, pp. 469, Lemma A.38]. To this end, we define two operators $S \in \mathcal{L}(V;V^*)$ and $T \in \mathcal{L}(V;V^*)$ by

$$\langle Sw,\varphi \rangle = A(w,\varphi) \quad \text{and} \quad \langle Tw,\varphi \rangle = -b(w,\varphi),$$

respectively. Assumption 3.1(a) shows the injectivity of the operator S. Further,

$$(Tw)(x) = -\int_0^1 \frac{p(x)(1-y)^{\alpha-2}}{\Gamma(\alpha-1)}w'(y)\,dy - \int_0^1 \frac{q(x)(x-y)^{\alpha-2}\chi_{(0,x)}(y)}{\Gamma(\alpha-1)}w'(y)\,dy$$
$$=: (T_1 w)(x) + (T_2 w)(x).$$

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We note that both T_1 and T_2 are compact from V to $L^2(D)$, since for $\alpha \in (3/2, 2)$ both kernels are square integrable [13, pp. 277, example 2]. Thus the operator $T : V \to L^2(D)$ is compact. By the definition of $a(\cdot, \cdot)$, we obtain

$$\|w\|_V^2 = a(w, w) = A(w, w) - b(w, w) \leq c\left(\|Tu\|_{V^*} + \|Su\|_{V^*}\right)\|w\|_V,$$

Now Petree-Tartar Lemma immediately implies that there exists a constant $c_0 > 0$ satisfying the following inf-sup condition

$$c_0\|u\|_V \leq \sup_{v \in V} \frac{A(u, v)}{\|v\|_V}. \tag{3.10}$$

This and Assumption 3.1(a^*) yield the existence of a unique solution $u \in V$ to (3.9). □

Now we state an improved regularity result for the case $\langle F, v \rangle = (f, v)$, for some $f \in H^s(D)$, $0 \leq s \leq 1$.

Theorem 3.3. *Let Assumption 3.1 hold and $q \in L^\infty(D)$ and $f \in L^2(D)$. Then the solution w to problem (3.5) belongs to $\widetilde{H}^1(D) \cap H^2(D)$ and satisfies*

$$\|w\|_{H^2(D)} \leq c\|f\|_{L^2(D)}.$$

Further, if $q, f \in H^1(D)$, then it belongs to $H^3(D) \cap \widetilde{H}^1(D)$ and satisfies

$$\|w\|_{H^3(D)} \leq c\|f\|_{H^1(D)}.$$

Proof. The existence and uniqueness of a solution $w \in V$ follows directly from Theorem 3.2. Hence, it suffices to show the stability estimate. By Theorem 2.1, ${}_0^R D_x^{2-\alpha} w \in H^{\alpha-1}(D)$, and by Sobolev embedding theorem, $q {}_0^R D_x^{2-\alpha} w \in H^{\alpha-1}(D)$. Note that problem (3.5) can be rewritten as

$$-w'' = \widetilde{f},$$

where $\widetilde{f} = -q {}_0^R D_x^{2-\alpha} w - ({}_0^R D_x^{2-\alpha} w)(1)p + f$. The preceding discussion yields $\widetilde{f} \in L^2(D)$ and $\|\widetilde{f}\|_{L^2(D)} \leq c\|f\|_{L^2(D)}$. Hence, by standard elliptic regularity theory [14], we deduce $u \in H^2(D) \cap \widetilde{H}^1(D)$. Further, if $q, f \in H^1(D)$, with this improved regularity on w, repeating the preceding arguments gives $\widetilde{f} \in H^1(D)$ and $\|\widetilde{f}\|_{H^1(D)} \leq c\|f\|_{H^1(D)}$, and applying elliptic regularity theory again yields the desired estimate. □

The next result shows that Assumption 2.2 implies Assumption 3.1(a).

Lemma 3.1. *Let $p(x) = c_0 x^{\mu-\alpha} - qx^\mu$ where c_0 is defined in (3.3). Then Assumption 2.2 implies Assumption 3.1(a).*

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Proof. Let $f = 0$ in (2.4) and (3.6). Suppose that $w \in V$ satisfies (3.6). Then by construction $u = {}^R_0D^{2-\alpha}_x w - ({}^R_0D^{2-\alpha}_x w)(1)x^\mu \in \widetilde{H}^{\alpha-1}(D)$ and $(w', \varphi') = \langle -w'', \varphi \rangle$ for $\varphi \in C^\infty_0(D)$ we have

$$\langle -{}^R_0D^\alpha_x u + qu, \varphi \rangle = 0 \quad \forall \varphi \in C^\infty_0(D),$$

i.e., $-{}^R_0D^\alpha_x u + qu = 0$ in the sense of distribution and in view of Theorem 3.3, also in $L^2(D)$. Now Assumption 2.2 yields $u = 0$. Hence $w \in V$ satisfies

$$ {}^R_0D^{2-\alpha}_x w = ({}^R_0D^{2-\alpha}_x w)(1)x^\mu. \tag{3.11}$$

by setting ${}^R_0D^{2-\alpha}_x w = cx^\mu$, the solution $w \in V$ of (3.11) is of the form $w(x) = c({}_0I^{2-\alpha}_x x^\mu)(x)$. This together with the boundary condition $w(1) = 0$ yields $c = 0$ and hence $w = 0$. $\qquad\square$

Once the solution w to problem (3.5) is found, the solution to problem (1.1) can be found by the reconstruction formula (3.1).

Theorem 3.4. *Let $f \in L^2(D)$ and $q \in L^\infty(D)$, and w be the unique solution to (3.5). Then the representation u given in (3.1) is a solution of problem (1.1).*

Proof. For $f \in L^2(D)$, by Theorem 3.3, there exists a unique solution $w \in \widetilde{H}^1(D) \cap H^2(D)$ to (3.5). By Theorem 2.1(a), we deduce

$$w'' = ({}_0I^1_x w')'' = ({}_0I^{2-\alpha}_x ({}_0I^{\alpha-1}_x w'))'' = ({}_0I^{2-\alpha}_x ({}_0I^{\alpha-1}_x w)')'' = {}^R_0D^\alpha_x ({}^R_0D^{2-\alpha}_x w).$$

Upon substituting this into (3.5), we get

$$-{}^R_0D^\alpha_x ({}^R_0D^{2-\alpha}_x w) + q\,{}^R_0D^{2-\alpha}_x w + ({}^R_0D^{2-\alpha}_x w)(1)\left(c_0 x^{\mu-\alpha} - q(x)x^\mu\right) = f,$$

which together with the definition $u = {}^R_0D^{2-\alpha}_x w - ({}^R_0D^{2-\alpha}_x w)(1)x^\mu$ yields directly $-{}^R_0D^\alpha_x u + qu = f$ in $L^2(D)$. Clearly, by the definition of u, $u(1) = 0$, and further by Theorem 2.1 and the fact that $w \in \widetilde{H}^1(D)$, ${}^R_0D^{2-\alpha}_x w - ({}^R_0D^{2-\alpha}_x w)(1)x^\mu \in \widetilde{H}^{\alpha-1}_L(D)$, and thus $u(0) = 0$. Hence, u is the solution to problem (1.1). $\qquad\square$

3.3. Adjoint problem

To derive $L^2(D)$ error estimates for the Galerkin approximation below, we need the adjoint problem to (3.6). For any $F \in V^*$, the adjoint problem is to find $\psi \in V$ such that

$$A(\varphi, \psi) = \langle \varphi, F \rangle \quad \forall \varphi \in V. \tag{3.12}$$

In the case of $\langle \varphi, F \rangle = (\varphi, f)$ for some $f \in L^2(D)$, the strong form reads

$$-\psi'' + {}^R_xD^{2-\alpha}_1(q\psi) + \Gamma(\alpha - 2)^{-1}(1 - x)^{\alpha-3}(p, \psi) = f \quad \text{in } D, \tag{3.13}$$
$$\psi(0) = \psi(1) = 0.$$

We note that for $\alpha \in (3/2, 2)$, the term $(1 - x)^{\alpha-3}$ is not a function in $L^1(D)$, and it should be understood in the sense of distribution. In view of the identity $(1 - x)^{\alpha-3} = -((1 - x)^{\alpha-2})'/(\alpha - 2)$, and the fact that $(1 - x)^{\alpha-2}$ belongs to the space $\widetilde{H}^{\alpha-2+\beta}(D)$, with $\beta \in (2 - \alpha, 1/2)$. Hence, $(1 - x)^{\alpha-3}$ lies in the space $H^{\alpha-3+\beta}(D) \subset H^{-1}(D)$.

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Theorem 3.5. *Let Assumption 3.1 hold, $q \in H^1(D)$ and $f \in L^2(D)$. Then there exists a unique solution $\psi \in H^{\alpha-1/2}(D) \cap \widetilde{H}^1(D)$ to problem (3.12) and it satisfies for $\beta \in (2-\alpha, 1/2)$*

$$\|\psi\|_{H^{\alpha-1+\beta}(D)} \leq c\|f\|_{L^2(D)}.$$

Proof. The unique existence of a solution $\psi \in V$ follows from Theorem 3.2. To see the regularity, we rewrite the problem into

$$-\psi'' = -{}_x^R D_1^{2-\alpha}(q\psi) - \Gamma(\alpha-2)^{-1}(1-x)^{\alpha-3}(p,\psi) + f.$$

Under the given assumptions on the right hand side f and the potential term q, and by the preceding discussions, the right hand side belongs to $H^{\alpha-3+\beta}(D)$. Thus by the standard elliptic regularity theory [14], the desired estimate follows. □

Remark 3.2. *In Theorem 3.5, the regularity assumption on the source term f can be relaxed to $f \in H^{\alpha-3+\beta}(D)$.*

Last we recall Green's function to the adjoint problem, i.e., for all $x \in D$

$$-G''(x,y) + {}_y^R D_1^{2-\alpha}(qG(x,y)) + \Gamma(\alpha-2)^{-1}(1-x)^{\alpha-3}(p,\psi) = \delta_x(y) \quad \text{in } D,$$
$$G(x,0) = G(x,1) = 0.$$

By Sobolev embedding theorem, $\delta_x \in H^{-1+\beta}(D) \subset H^{-1}(D)$, $\beta \in (2-\alpha, 1/2)$, and thus the existence and uniqueness of $G(x,\cdot) \in \widetilde{H}^1(D)$ follows directly from the stability of the variational formulation. Moreover, by the argument in the proof of Theorem 3.5 and Remark 3.2, $G(x,\cdot) \in H^{\alpha-1+\beta}(D)$.

4. Galerkin finite element method

The variational formulation (3.6) enables us to develop a Galerkin FEM for problem (1.1): first we approximate the solution w to (3.5) by a Galerkin finite element approximation w_h, and then reconstruct the solution to (1.1) using (3.1), i.e.,

$$u_h = {}_0^R D_x^{2-\alpha} w_h - ({}_0^R D_x^{2-\alpha} w_h)(1)x^\mu. \tag{4.1}$$

To this end, we divide the domain D into quasi-uniform partitions with a maximum length h, and let V_h denote the resulting space of continuous piecewise polynomials of degree at most $k+1$, vanishing at both end points of D. Thus, the functions in $V_h \subset \widetilde{H}^1(D)$ are piecewise linear if $k = 0$, and piecewise quadratic if $k = 1$. Since we consider only a right hand side $f \in L^2(D)$ or $f \in H^1(D)$, we shall focus on the choice $k = 0, 1$ in our discussion. The space V_h has the following approximation properties.

Lemma 4.1. *If $v \in H^\gamma(D) \cap \widetilde{H}^1(D)$ with $1 \leq \gamma \leq 3$, then for $k = 0, 1$*

$$\inf_{v_h \in V_h} \|v - v_h\|_{\widetilde{H}^1(D)} \leq ch^{\min(\gamma-1, k+1)}\|v\|_{H^\gamma(D)}. \tag{4.2}$$

The Galerkin FEM is to find $w_h \in V_h$ such that

$$A(w_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{4.3}$$

The computation of the stiffness matrix and mass matrix is given in Appendix Appendix A. We next analyze the stability of the discrete formulation (4.3), and derive (suboptimal) error estimates for the approximations w_h and u_h. First we have the following stability result. The proof is identical with that in [6, Lemma 5.2], using a kick-back trick analogous to Schatz [15]. We sketch the proof for completeness.

Theorem 4.1. *Let Assumption 3.1 hold, $f \in L^2(D)$, and $q \in L^\infty(D)$. Then there is an h_0 such that for all $h \leq h_0$ the finite element problem (4.3) has a unique solution $w_h \in V_h$, and further*

$$\|w_h\|_{H^1(D)} \leq c\|f\|_{L^2(D)}. \tag{4.4}$$

Proof. We first define the Ritz projection $R_h : V \to V_h$ by $((R_h\varphi)', \psi') = (\varphi', \psi')$ for all $\psi \in V_h$. Then for $v_h \in V_h \subset V$ we have

$$c_0\|v_h'\|_{L^2(D)} \leq \sup_{\varphi \in V} \frac{A(v_h, \varphi)}{\|\varphi'\|_{L^2(D)}} \leq \sup_{\varphi \in V} \frac{A(v_h, \varphi - R_h\varphi)}{\|\varphi'\|_{L^2(D)}} + \sup_{\varphi \in V} \frac{A(v_h, R_h\varphi)}{\|\varphi'\|_{L^2(D)}} =: I + II.$$

Then by (3.8) and Theorem 3.3 we have

$$I = \sup_{\varphi \in V} \frac{b(v_h, \varphi - R_h\varphi)}{\|\varphi'\|_{L^2(D)}} \leq c \sup_{\varphi \in V} \frac{\|v_h'\|_{L^2(D)}\|\varphi - R_h\varphi\|_{L^2(D)}}{\|\varphi'\|_{L^2(D)}} \leq c_1 h\|v_h'\|_{L^2(D)}.$$

Further the second term II could be bounded as follows by using the inequality $\|(R_h\varphi)'\|_{L^2(D)} \leq \|\varphi'\|_{L^2(D)}$ and the fact that $R_h\varphi \in V_h$

$$II \leq \sup_{\varphi \in V} \frac{A(v_h, R_h\varphi)}{\|(R_h\varphi)'\|_{L^2(D)}} \leq \sup_{\varphi_h \in V_h} \frac{A(v_h, \varphi_h)}{\|\varphi_h'\|_{L^2(D)}}.$$

Now by choosing $h_0 = c_0/(2c_1)$ we derive the following *inf-sup* condition:

$$\|v_h\|_V \leq c \sup_{\varphi_h \in V_h} \frac{A(v_h, \varphi_h)}{\|\varphi_h\|_V}. \tag{4.5}$$

This shows that the corresponding stiffness matrix is nonsingular and the existence of a unique discrete solution $u_h \in V_h$ follows. The estimate (4.4) is a direct consequence of (4.5) and this completes the proof. $\square$

Now we turn to the error analysis, and focus on the case $f \in H^1(D)$.

Theorem 4.2. *Let Assumption 3.1 hold, and $f, q \in H^1(D)$. For the FEM of piecewise $(k+1)$'s degree polynomials (k=0,1), there is an h_0 such that for all $h \leq h_0$, the solution w_h to problem (4.3) satisfies with $\beta \in (2 - \alpha, 1/2)$*

$$\|w - w_h\|_{L^2(D)} + h^{\alpha-2+\beta}\|(w - w_h)'\|_{L^2(D)} \leq Ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.$$

Proof. The error estimate in the $\widetilde{H}^1(D)$-norm follows directly from Céa's lemma, (4.5) and the Galerkin orthogonality. Specifically, for all $h \le h_0$ and any $\chi \in V_h$ we have

$$\|w_h - \chi\|_V \le c \sup_{v_h \in V_h} \frac{A(w_h - \chi, v_h)}{\|v_h\|_V} \le c \sup_{v_h \in V_h} \frac{A(w - \chi, v_h)}{\|v_h\|_V} \le c\|w - \chi\|_V.$$

Then the desired $\widetilde{H}^1(D)$-estimate follows from Lemma the triangle inequality and 4.1 by

$$\|w - w_h\|_V \le c \inf_{\chi \in V_h} \|w - \chi\|_V \le ch^{k+1}\|f\|_{H^1(D)}.$$

Then we apply Nitsche's trick to establish the $L^2(D)$-error estimate. To this end, we consider the adjoint problem (3.12) with $f = w - w_h$, i.e.

$$\|w - w_h\|_{L^2(D)}^2 = A(w - w_h, \psi) = A(w - w_h, \psi - \psi_h),$$

for any $w_h \in V_h$. Then Lemma 4.1 and Theorem 3.5 yield for any $\beta \in [1 - \alpha/2, 1/2)$

$$\|w - w_h\|_{L^2(D)}^2 \le c\|w - w_h\|_V \inf_{\psi_h \in V_h} \|\psi - \psi_h\|_V$$
$$\le ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}\|w - w_h\|_{L^2(D)}.$$

This completes the proof of the theorem. $\square$

Below we analyze the convergence of the approximation u_h, reconstructed from w_h using (4.1). We divide the convergence analysis into several lemmas. First we estimate the leading term ${}^R_0D_x^{2-\alpha}w_h(x)$.

Lemma 4.2. *Let the assumptions in Theorem 4.2 hold, and w and w_h be solutions of (3.6) and (4.3), respectively. Then for $e = w - w_h$, there holds with $\beta \in (2 - \alpha, 1/2)$*

$$\|{}^R_0D_x^{2-\alpha}e\|_{L^2(D)} \le ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.$$

Proof. Recall that $\alpha \in (3/2, 2)$, $2 - \alpha \in (0, 1/2)$, and thus the spaces $\widetilde{H}^{2-\alpha}(D)$ and $H^{2-\alpha}(D)$ are equal, and further $\|{}^R_0D_x^{2-\alpha}\cdot\|_{L^2(D)}$ induces an equivalent norm on $H^{2-\alpha}(D)$ [16]. By a standard duality argument, we deduce

$$\|{}^R_0D_x^{2-\alpha}e\|_{L^2(D)} \le c\|e\|_{H^{2-\alpha}(D)} = c \sup_{\varphi \in H^{-2+\alpha}(D)} \frac{\langle e, \varphi \rangle}{\|\varphi\|_{H^{-2+\alpha}(D)}}$$
$$= c \sup_{\varphi \in H^{-2+\alpha}(D)} \frac{A(e, g_\varphi)}{\|\varphi\|_{H^{-2+\alpha}(D)}},$$

where g_φ is the solution to the adjoint problem $\langle v, \phi \rangle = A(v, g_\varphi)$, for all $v \in V$. By Theorem 3.5, $g_\varphi \in H^{\alpha-1+\beta}(D)$. Let $\Pi\varphi \in V_h$ be the standard Lagrange

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finite element interpolant of φ. Then by Galerkin orthogonality and the continuity of the bilinear form

$$
\begin{aligned}
A(e, g_\varphi) = A(e, g_\varphi - \Pi g_\varphi) &\leq c\|e'\|_{L^2(D)}\|(g_\varphi - \Pi g_\varphi)'\|_{L^2(D)} \\
&\leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}\|g_\varphi\|_{H^{\alpha-1+\beta}(D)} \\
&\leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}\|\varphi\|_{H^{-2+\alpha}(D)}.
\end{aligned}
$$

$\square$

Next we provide an $L^\infty(D)$ estimate on the term $e = w - w_h$.

Lemma 4.3. *Let the assumptions in Theorem 4.2 hold, and w and w_h be solutions of (3.6) and (4.3), respectively. Then for $e = w - w_h$ and $\beta \in (2-\alpha, 1/2)$, there holds*
$$
\|e\|_{L^\infty(D)} \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.
$$

Proof. Using the weak formulation of $G(x,y)$ and Galerkin orthogonality, we have for any $\varphi_h \in V_h$

$$
e(x) = A(e, G(x, \cdot)) = A(e, G(x, \cdot) - \varphi_h).
$$

Then by Theorem 4.2, we obtain for any $\beta \in (2 - \alpha, 1/2)$

$$
|e(x)| \leq c\|e\|_{H^1(D)} \inf_{\varphi_h \in V_h} \|G(x, \cdot) - \varphi_h\|_{H^1(D)} \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)},
$$

where the last inequality follows from $G(x, \cdot) \in H^{\alpha-1+\beta}(D) \subset H^1(D)$ and Lemma 4.1. $\square$

The next result gives an estimate on the crucial term $|({}^R_0 D_x^{2-\alpha} e)(1)|$.

Lemma 4.4. *Let the assumptions in Theorem 4.2 hold, and w and w_h be solutions of (3.6) and (4.3), respectively. Then for $e = w - w_h$, there holds with $\beta \in (2 - \alpha, 1/2)$*
$$
|({}^R_0 D_x^{2-\alpha} e)(1)| \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.
$$

Proof. By the Galerkin orthogonality, we have

$$
(e', \varphi_h') + ({}^R_0 D_x^{2-\alpha} e, q\varphi_h) + ({}^R_0 D_x^{2-\alpha} e)(1)(p, \varphi_h) = 0 \quad \forall \varphi_h \in V_h.
$$

Note that $p(x) = \frac{\Gamma(\mu+1)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha} - q(x)x^\mu$ is smooth for large μ. Without loss of generality, we may assume that $x = 1/2$ is a grid point and let $\varphi_h = x\chi_{[0,1/2)} + (1-x)\chi_{(1/2,1]} \in V_h$ with $|(p, \varphi_h)| := c_1 > 0$. Then we obtain

$$
c_1|({}^R_0 D_x^{2-\alpha} e)(1)| \leq |(e', \varphi_h')| + |({}^R_0 D_x^{2-\alpha} e, q\varphi_h)| =: I + II.
$$

It suffices to bound the terms on the right hand side. The second term II can be bounded using Lemma 4.2 as

$$
II \leq \|{}^R_0 D_x^{2-\alpha} e\|_{L^2(D)}\|\varphi_h\|_{L^2(D)}\|q\|_{L^\infty(D)} \leq c\|{}^R_0 D_x^{2-\alpha} e\|_{L^2(D)} \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.
$$

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and the first term I can be bounded by Lemma 4.3 by

$$I \leq |\int_0^{1/2} e'(x)dx - \int_{1/2}^1 e'(x)dx| = 2|e(1/2)| \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.$$

This completes the proof of the lemma. □

Now by the triangle inequality, we arrive at the following $L^2(D)$ estimate for the approximation u_h.

Theorem 4.3. *Let Assumption 3.1 hold, $f, q \in H^1(D)$. Then there is an h_0 such that for all $h \leq h_0$, the solution u_h satisfies that for any $\beta \in (2 - \alpha, 1/2)$*

$$\|u - u_h\|_{L^2(D)} \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}. \tag{4.6}$$

Remark 4.1. *By Remark 3.1, we may choose $\mu = \alpha - 1$, for which the error estimate follows similarly. The only difference is the bound on $|({}_0^R D_x^{2-\alpha}e)(1)|$ in case of $q = 0$. By the definition of $({}_0^R D_x^{2-\alpha}e)(1)$, we have*

$$|{}_0^R D_x^{2-\alpha}e(1)| = \frac{1}{\Gamma(\alpha-1)}\left|\int_0^1 (1-x)^{\alpha-2}e'(x)dx\right| = \frac{1}{\Gamma(\alpha)}\left|\int_0^1 ((1-x)^{\alpha-1})'e'(x)\,dx\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^1 ((1-x)^{\alpha-1} - (1-x))'e'(x)\,dx\right| + \frac{1}{\Gamma(\alpha)}\left|\int_0^1 e'(x)\,dx\right|.$$

The second term vanishes due to $e(0) = e(1) = 0$. Hence it suffices to establish an estimate on first term. Since the transformed problem reproduces Poisson's equation, by the Galerkin orthogonality $(e', \varphi_h') = 0$ and the fact that $\varphi = (1-x)^{\alpha-1} - (1-x) \in \widetilde{H}^1(D) \cap H^{\alpha-1+\beta}(D)$ with $\beta \in (2-\alpha, 1/2)$, we have by Lemma 4.1

$$|(\varphi', e'(x))| \leq c\|e'\|_{L^2(D)} \inf_{\varphi_h \in V_h} \|\varphi' - \varphi_h'\|_{L^2(D)} \leq ch^{\alpha+k-1+\beta}\|f\|_{H^1(D)}.$$

Thus the $L^2(D)$ estimate (4.6) holds also for the choice $\mu = \alpha - 1$.

Next, we derive an optimal $L^2(D)$ error estimate for all $\alpha \in (1,2)$ provided that $q = 0$, $\mu = \alpha - 1$ and f is smooth enough.

Theorem 4.4. *Assume $q = 0$ and $\mu = \alpha - 1$. Then for all $\alpha \in (1,2)$ there holds*

$$\|u - u_h\|_{L^2(D)} \leq ch^{\alpha+k}\|f\|_{W^{1,\infty}(D)}.$$

Proof. For $q = 0$ and $\mu = \alpha - 1$, the transformed problem is the standard one-dimensional Poisson's equation

$$-w'' = f \quad \text{in } D, \quad \text{with} \quad w(0) = w(1) = 0.$$

Then the solution w_h of the discrete problem (4.3) satisfies [17, 18]

$$\|w - w_h\|_{W^{s,\infty}(D)} + \|w - w_h\|_{W^{s,2}(D)} \leq ch^{k+2-s}\|f\|_{W^{1,\infty}(D)}, \quad s = 0, 1.$$

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Now let $e = w - w_h$ and we have by interpolation

$$\|_0^R D_x^{2-\alpha} e\|_{L^2(D)} \leq \|e\|_{H^{2-\alpha}(D)} \leq ch^{\alpha+k}\|f\|_{W^{1,\infty}(D)}. \tag{4.7}$$

Hence it suffices to bound $|(_0^R D_x^{2-\alpha} e)(1)|$. Since $e \in \widetilde{H}^1(D)$, we have for $\delta \in (0,1)$

$$|(_0^R D_x^{2-\alpha} e)(1)| = \frac{1}{\Gamma(\alpha-1)} \left| \int_0^1 (1-s)^{\alpha-2} e'(s) \right|$$
$$\leq c \left(\left| \int_0^{1-\delta} (1-s)^{\alpha-2} e'(s) \, ds \right| + \left| \int_{1-\delta}^1 (1-s)^{\alpha-2} e'(s) \, ds \right| \right).$$

Then the second term can be easily bounded by

$$\left| \int_{1-\delta}^1 (1-s)^{\alpha-2} e'(s) \, ds \right| \leq \int_{1-\delta}^1 (1-s)^{\alpha-2} \, ds \|e'\|_{L^\infty(D)} \leq c\delta^{\alpha-1} h^{k+1} \|f\|_{W^{1,\infty}(D)},$$

while the first term can be bounded using integration by parts

$$\left| \int_0^{1-\delta} (1-s)^{\alpha-2} e'(s) \, ds \right| \leq c \left(\left| (1-s)^{\alpha-2} e(s) \Big|_0^{1-\delta} \right| + \left| \int_0^{1-\delta} \frac{(1-s)^{\alpha-1}}{\alpha-1} e(s) \, ds \right| \right)$$
$$\leq c(\delta^{\alpha-2} + 1 - \delta^\alpha) h^{k+2} \|f\|_{W^{1,\infty}(D)}.$$

Now choosing $\delta = h$ yields the following estimate

$$|(_0^R D_x^{2-\alpha} e)(1)| \leq ch^{\alpha+k}.$$

This together with (4.7) gives an optimal $L^2(D)$-error estimate

$$\|u - u_h\|_{L^2(D)} \leq \|_0^R D_x^{2-\alpha} e\|_{L^2(D)} + c|(_0^R D_x^{2-\alpha} e)(1)| \leq ch^{\alpha+k} \|f\|_{W^{1,\infty}(D)}.$$

$\square$

5. Eigenvalue problem

Now we apply the new approach to the following fractional Sturm-Liouville problem (FSLP): find u and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} -_0^R D_x^\alpha u + qu &= \lambda u \quad \text{in } D, \\ u(0) &= u(1) = 0. \end{aligned} \tag{5.1}$$

The eigenvalue problem is important in studying the dynamics of superdiffusion processes. However, the accurate computation of the eigenvalues and eigenfunctions is challenging, due to the presence of a singularity in the eigenfunction. In [6], a finite element method with piecewise linear finite elements was developed for the problem. Numerically, a second-order convergence of the eigenvalue approximations is observed, but the theoretical convergence rate of eigenfunction approximations is of order $O(h^{\alpha-1})$ in the $L^2(D)$ norm which is very slow. In

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this part, we develop an efficient method for problem (5.1) by extending the new approach in Sections 3 and 4.

Proceeding like in section 3, we deduce that the weak formulation of the Sturm-Liouville problem reads: find $w \in V$ and $\lambda \in \mathbb{C}$ such that

$$A(w, \varphi) = \lambda({}^{R}_0D_x^{2-\alpha}w - ({}^{R}_0D_x^{2-\alpha}w)(1)x^{\mu}, \varphi) \quad \forall \varphi \in V. \qquad (5.2)$$

Then we define u by

$$u = {}^{R}_0D_x^{2-\alpha}w - ({}^{R}_0D_x^{2-\alpha}w)(1)x^{\mu}.$$

Then λ is the eigenvalue and u is the corresponding eigenfunction. Accordingly, the discrete problem is given by: find $w_h \in V_h$ and $\lambda_h \in \mathbb{C}$ such that

$$A(w_h, \varphi) = \lambda_h({}^{R}_0D_x^{\alpha-2}w_h - ({}^{R}_0D_x^{\alpha-2}w_h)(1)x^{\mu}, \varphi) \quad \forall \varphi \in V_h,$$
$$u_h = {}^{R}_0D_x^{\alpha-2}w_h - ({}^{R}_0D_x^{\alpha-2}w_h)(1)x^{\mu}. \qquad (5.3)$$

and $\{\lambda_h, w_h\}$ is an approximated eigenpair of the transformed FSLP (5.2).

We shall follow the notation and use some fundamental results from [19, 20]. To this end, we introduce the operator $T : L^2(D) \to \widetilde{H}^1(D)$ defined by

$$Tf \in \widetilde{H}^1(D), \quad A(Tf, \varphi) = (f, \varphi) \quad \forall \varphi \in V. \qquad (5.4)$$

Obviously, T is the solution operator of the source problem (3.5). By Theorem 3.3, the solution operator T satisfies the following smoothing property:

$$\|Tf\|_{H^2(D)} \le c\|f\|_{L^2(D)}.$$

Since $H^2(D)$ is compactly embedded into $H^1(D)$ [21], we deduce that the operator $T : L^2(D) \to \widetilde{H}^1(D)$ is compact. Next we define an operator $S : \widetilde{H}^1(D) \to L^2(D)$ by

$$Sw = {}^{R}_0D_x^{2-\alpha}w - ({}^{R}_0D_x^{2-\alpha}w)(1)x^{\mu}. \qquad (5.5)$$

Lemma 5.1. *The operator* $S : \widetilde{H}^1(D) \to L^2(D)$ *defined in* (5.5) *is compact.*

Proof. We observe that for $w \in \widetilde{H}^1(D)$

$$\|Sw\|_{L^2(D)} \le \|{}^{R}_0D_x^{2-\alpha}w\|_{L^2(D)} + |({}^{R}_0D_x^{2-\alpha}w)(1)|\|x^{\mu}\|_{L^2(D)}.$$

By Theorem 2.1, we have

$$\|{}^{R}_0D_x^{2-\alpha}w\|_{L^2(D)} \le c\|w\|_{H^{2-\alpha}(D)}.$$

Meanwhile, by Sobolev embedding theorem [21] and norm equivalence on the space $\widetilde{H}^s(D)$ [6], there holds for $\alpha - 1 > s > 1/2$, i.e., $1/2 < s + 2 - \alpha < 1$,

$$|({}^{R}_0D_x^{2-\alpha}w)(1)| \le c\|{}^{R}_0D_x^{2-\alpha}w\|_{H^s(D)} \le c\|{}^{R}_0D_x^s({}^{R}_0D_x^{2-\alpha}w)\|_{L^2(D)}$$
$$= c\|{}^{R}_0D_x^{s+2-\alpha}w\|_{L^2(D)} \le c\|w\|_{H^{s+2-\alpha}(D)}.$$

These two estimates implies that the operator is bounded from $\widetilde{H}^{s+2-\alpha}(D)$ to $L^2(D)$, which together the compactness of the embedding from $\widetilde{H}^1(D)$ into $\widetilde{H}^{s+2-\alpha}(D)$ yields the desired compactness. $\qquad \square$

Then the FSLP (5.2) can be rewritten as to find $w \in V$, such that $A(w, \varphi) = \lambda(Sw, \varphi) \; \forall \varphi \in V$ or equivalently $TSw = \lambda^{-1}w$. Now after applying the operator S to this equality and noting that $Sw = u \in L^2(D)$ we get the problem in oprator form: find $(\lambda, u) \in \mathbb{C} \times L^2(D)$ such that

$$\lambda^{-1}u = STu,$$

i.e., (λ^{-1}, u) is an eigenpair of the operator ST. By Lemma 5.1, the operator $S : \widetilde{H}^1(D) \to L^2(D)$ is bounded and compact, and thus $ST : L^2(D) \to L^2(D)$ is a compact operator. With the help of this correspondence, the properties of the eigenvalue problem (5.1) can be derived from the spectral theory for compact operators [13, 22]. Let $\sigma(ST) \subset \mathbb{C}$ be the set of all eigenvalues of ST (or its spectrum), which is known to be a countable set with no nonzero limit points. By Assumption 3.1 on the bilinear form $a(u; v)$, zero is not an eigenvalue of ST. Furthermore, for any $\mu \in \sigma(ST)$, the space $N(\mu I - ST)$, where N denotes the null space, of eigenvectors corresponding to μ is finite dimensional.

Now let $T_h : V_h \to V_h$ be a family of operators for $0 < h < 1$ defined by

$$T_h f \in V_h, \quad A(T_h f, \varphi) = (f, \varphi) \quad \forall \varphi \in V_h. \tag{5.6}$$

Then the discrete FSLP (5.3) can be written as: to find $w_h \in V_h$, such that $A(w_h, \varphi) = \lambda_h(Sw_h, \varphi) \; \forall \varphi \in V$ or equivalently $T_h Sw_h = \lambda_h^{-1}w_h$, with $u_h = Sw_h$. Hence the discrete problem in operator form reads: to find $(\lambda_h, u_h) \in \mathbb{C} \times L^2(D)$ such that

$$\lambda_h^{-1}u = ST_h u.$$

By Theorem 4.3, the operator ST_h converges to ST in $L^2(D)$. Further, the operator sequence $\{ST_h\}_{h>0}$ is collectively compact on $L^2(D)$, i.e., the set $\{ST_h f : \|f\|_{L^2(D)} \leq 1\}$ is compact in $L^2(D)$. To see this, we note that by the discrete inf-sup condition, $\|T_h f\|_{H^1(D)} \leq c$, cf. Theorem 4.1, and thus the set $\{T_h f : \|f\|_{L^2(D)} \leq 1\}$ is uniformly bounded in $\widetilde{H}^1(D)$, and the claim follows from the compactness of the operator $S : \widetilde{H}^1(D) \to L^2(D)$ from Lemma 5.1. Hence, we can apply the approximation theory [19] of compact operators. Specifically, let $\mu = \lambda^{-1} \in \sigma(ST)$ be an eigenvalue of ST with algebraic multiplicity m. Then m eigenvalues of ST_h, μ_h^j, $j = 1, 2, \ldots, m$, of ST_h will converge to μ, where the eigenvalues μ_h^j are counted according to the algebraic multiplicity of μ_h^j as eigenvalues of ST_h.

Now we state the main result for the spectral approximation. It follows directly from [19, Theorems 5 and 6] and Theorem 4.3.

Theorem 5.1. *Let Assumption 3.1 hold and $q \in H^1(D)$. For $\lambda^{-1} \in \sigma(ST)$, let δ be its ascent, i.e., the smallest integer m such that $N((\lambda^{-1} - ST)^m) = N((\lambda^{-1} - ST)^{m+1})$.*

(i) For any $\gamma < \alpha + k - 1/2$, there holds

$$|\lambda - \lambda_h^j| \leq Ch^{\gamma/\delta}.$$

(ii) Let λ_h^{-1} be an eigenvalue of ST_h such that $\lim_{h\to 0}\lambda_h = \lambda$ with $\lambda \in \sigma(ST)$. Suppose for each h, u_h is a unit vector satisfying $((\lambda_h^j)^{-1} - ST_h)^k u_h = 0$ for some positive integer $k \leq \delta$. Then, for any integer l with $k \leq l \leq \alpha$, there is a vector u such that $(\lambda^{-1} - ST)^l u = 0$ and for any $\gamma < \alpha + k - 1/2$,

$$\|u - u_h\|_{L^2(D)} \leq Ch^{\gamma/\delta}.$$

Remark 5.1. *It is known that in case of $q = 0$, all eigenvalues to (5.1) are simple [23, Section 4.4], i.e., $\delta = 1$ in Theorem 5.1. Numerically we observe that the eigenvalues to (5.1) are always simple. When using piecewise linear finite elements, the convergence rate of the new approach in Theorem 5.1 is better than that for the standard Galerkin method, which has a convergence rate $Ch^{\gamma/\delta}$, for any $\gamma < \alpha - 1$ [6, Theorem 6.1]. This shows the advantage of the new approach.*

6. Numerical results and discussions

In this section, we present numerical results to illustrate the efficiency and accuracy of the new approach and to verify our theoretical findings. We shall discuss the source problem and the Sturm-Liouville problem separately.

6.1. Source problem

For the source problem (1.1), we consider the following three different right hand sides:

(a) The source term $f(x) = x(1-x)$ belongs to $\widetilde{H}^{1+\beta}(D)$ for any $\beta \in [0, 1/2)$.

(b) The source term (b1) $f(x) = 1$ and (b2) $f(x) = (1-x)^{\frac{3}{5}}$ belong to the space $H^1(D) \cap \widetilde{H}^\beta(D)$ for any $\beta \in [0, 1/2)$.

(c) The source term $f(x) = \chi_{[0,1/2]}$ belongs to $\widetilde{H}^\beta(D)$ for any $\beta \in [0, 1/2)$.

The computations were performed on a uniform mesh with a mesh size $h = 1/2^m$, $m \in \mathbb{N}$. We note that if the potential q is zero, the exact solution u can be computed explicitly. For the case $q \neq 0$, the exact solutions are not available in closed form, and hence we compute the reference solution on a very refined mesh with a mesh size $h = 1/2^{12}$. For each example, we consider three different α values, i.e., 1.55, 1.75 and 1.95, and present the $L^2(D)$-norm of the error $e = u - u_h$.

6.1.1. Numerical results for example (a)

For this very smooth source, we consider the simple case $q = 0$. The exact solution $u(x)$ is given by $u(x) = \frac{1}{\Gamma(\alpha+2)}(x^{\alpha-1} - x^{\alpha+1}) - \frac{2}{\Gamma(\alpha+3)}(x^{\alpha-1} - x^{\alpha+2})$, and it belongs to $\widetilde{H}_L^{\alpha-1+\beta}(D)$ with $\beta \in (2 - \alpha, 1/2)$ due to the presence of the term $x^{\alpha-1}$, despite the smoothness of the right hand side f. Thus the standard Galerkin FEM converges slowly; see [6, Table 1]. Numerical results for the new

approach are presented in Table 1. In the table, $P1$ and $P2$ denote piecewise linear and piecewise quadratic FEMs, respectively. `rate` refers to the empirical convergence rate, and the numbers in the bracket denote theoretical rates. The numerical results show $O(h^\alpha)$ and $O(h^{\alpha+1})$ convergence for $P1$ and $P2$ FEMs, respectively. Hence, the $L^2(D)$-error estimate in Theorem 4.3 is suboptimal: the empirical ones are one half order higher than the theoretical one. The suboptimality is attributed to the low regularity of the adjoint problem (3.12), used in Nitsche's trick. Although not presented, we note that with the choice $\mu = \alpha - 1$, the optimal convergence rate in Theorem 4.4 can be fully confirmed.

Table 1: The $L^2(D)$-norm of the error for example (a) with $q = 0$, $\mu = 4$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/2^m$.

α	m	3	4	5	6	7	8	rate
1.55	$P1$	2.62e-3	9.28e-4	3.20e-4	1.09e-4	3.68e-5	1.22e-5	1.55 (1.05)
	$P2$	2.30e-5	3.96e-6	6.79e-7	1.16e-7	1.98e-8	3.39e-9	2.55 (2.05)
1.75	$P1$	7.89e-4	2.26e-4	6.47e-5	1.86e-5	5.34e-6	1.53e-6	1.80 (1.25)
	$P2$	1.11e-5	1.69e-6	2.54e-7	3.80e-8	5.66e-9	8.39e-10	2.74 (2.25)
1.95	$P1$	3.06e-4	7.74e-5	1.95e-5	4.93e-6	1.24e-6	3.11e-7	1.98 (1.45)
	$P2$	5.38e-6	7.03e-7	9.15e-8	1.18e-8	1.53e-9	1.98e-10	2.95 (2.45)

6.1.2. Numerical results for example (b)

In Table 2, we present numerical results for example (b1) with $q(x) = x$. Since both the source term f and the potential q belong to $H^1(D)$, by Theorem 3.3, w belongs to $H^3(D) \cap \widetilde{H}^1(D)$, and the $L^2(D)$-error achieves a rate $O(h^{\alpha+k-1/2})$ for $k = 0, 1$. The empirical $L^2(D)$ rate is one half order higher than the theoretical one. Next we compare the new approach with the singularity enhanced FEM developed in [9]. Since the regular part u_r (i.e., the part of the solution u apart from the leading singularity $x^{\alpha-1}$) only belongs to $H^{\alpha+\beta}(D)$ due to $f, q \in \widetilde{H}^\beta(D)$, even with the $P2$ FEM, the approach in [9] can only achieve a convergence rate slower than that in Theorem 4.3, and the new approach requires less regularity on the potential q and source term f. In Table 3, we show numerical results for $\alpha < 1.5$, which is not covered by our theory. Interestingly, the numerical results indicate that our scheme converges equally well with the order $O(h^{\alpha+k})$ in this case.

Further numerical results for different μ values are presented in Table 4. By Remarks 3.1 and 4.1, the choice $\mu = \alpha - 1$ achieves the rate $O(h^{\alpha+k-1+\beta})$. In theory, the choice of $\mu(\geq \alpha)$ does not affect the convergence of P1 method, and for the P2 method, the optimal convergence rate holds only for $\mu \geq \alpha + 1/2$. This is confirmed by Table 4: the choice $\mu = \alpha + 1/4$ fails to achieve the optimal order.

The numerical results for example (b2), i.e., $f(x) = (1-x)^{3/5}$, with $q(x) = x$, are shown in Table 5. In this case, the weak solution singularity appears at both

left and right end points. Like before we observe an optimal convergence order h^α for the $P1$ FEM. Interestingly, for the $P2$ FEM, the empirical orders are close to the theoretical ones when α is close to 1.5, whose precise mechanism awaits theoretical justification.

Table 2: The $L^2(D)$-norm of the error for example (b1) with $q = x$, $\mu = 4$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/2^m$.

α	m	3	4	5	6	7	8	rate
1.55	$P1$	1.47e-2	5.40e-3	1.91e-3	6.62e-4	2.26e-4	7.58e-5	1.52 (1.05)
	$P2$	2.21e-4	3.88e-5	6.71e-6	1.15e-6	1.98e-7	3.37e-8	2.54 (2.05)
1.75	$P1$	4.64e-3	1.41e-3	4.21e-4	1.25e-4	3.70e-5	1.08e-5	1.75 (1.25)
	$P2$	3.35e-5	5.05e-6	7.56e-7	1.13e-7	1.68e-8	2.52e-9	2.74 (2.25)
1.95	$P1$	1.64e-3	4.20e-4	1.08e-4	2.76e-5	7.07e-6	1.80e-6	1.93 (1.45)
	$P2$	2.92e-6	3.82e-7	4.96e-8	6.44e-9	8.36e-10	1.15e-10	2.95 (2.45)

Table 3: The $L^2(D)$-norm of the error for example (b1) with $q = x$, $\mu = 4$, $\alpha = 1.05, 1.25, 1.45$, $h = 1/2^m$.

α	m	3	4	5	6	7	8	rate
1.05	$P1$	5.13e-2	3.12e-2	1.73e-2	8.97e-3	4.41e-3	2.06e-3	1.02 (−−)
	$P2$	1.11e-2	2.92e-3	7.29e-4	1.78e-4	4.33e-5	1.03e-5	2.04 (−−)
1.25	$P1$	2.05e-2	1.01e-2	4.61e-3	2.01e-3	8.49e-4	3.47e-4	1.24 (−−)
	$P2$	2.55e-3	5.66e-4	1.22e-4	2.59e-5	5.46e-6	1.14e-6	2.25 (−−)
1.45	$P1$	7.38e-3	2.90e-3	1.10e-3	4.10e-4	1.50e-4	5.40e-5	1.43 (−−)
	$P2$	5.19e-4	9.85e-5	1.83e-5	3.38e-6	6.20e-7	1.13e-7	2.44 (−−)

Table 4: The $L^2(D)$-norm of the error for example (b1) with $q = x$, $\alpha = 1.75$, $h = 1/2^m$ and different μ.

μ	m	3	4	5	6	7	8	rate
3	$P1$	4.05e-3	1.20e-3	3.55e-4	1.05e-4	3.08e-5	8.96e-6	1.75 (1.25)
	$P2$	2.21e-4	3.88e-5	6.71e-6	1.15e-6	1.98e-7	3.37e-8	2.74 (2.25)
0.75	$P1$	3.07e-3	8.92e-4	2.60e-4	7.61e-5	2.22e-5	6.41e-6	1.75 (1.25)
	$P2$	3.35e-5	5.05e-6	7.56e-7	1.13e-7	1.68e-8	2.52e-9	2.74 (2.25)
2	$P1$	3.57e-3	1.05e-3	3.06e-4	8.95e-5	2.62e-5	7.58e-6	1.75 (1.25)
	$P2$	6.81e-6	1.12e-6	1.83e-7	2.98e-8	4.90e-9	8.27e-10	2.60 (−−)

Table 5: The $L^2(D)$-norm of the error for example (b2) with $q = x$, $\mu = 3$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/2^m$.

α	m	3	4	5	6	7	8	rate
1.55	$P1$	5.15e-3	1.72e-3	5.74e-4	1.91e-4	6.38e-5	2.12e-5	1.59 (1.05)
	$P2$	3.91e-5	1.03e-5	2.62e-6	6.42e-7	1.54e-7	3.59e-8	2.04 (2.05)
1.75	$P1$	1.98e-3	5.54e-4	1.55e-4	4.39e-5	1.24e-5	3.56e-6	1.82 (1.25)
	$P2$	2.02e-5	3.64e-6	6.74e-7	1.28e-7	2.46e-8	4.76e-9	2.38 (2.25)
1.95	$P1$	1.02e-3	2.59e-4	6.52e-5	1.65e-5	4.15e-6	1.04e-6	1.99 (1.45)
	$P2$	9.38e-6	1.27e-6	1.73e-7	2.34e-8	3.18e-9	4.33e-10	2.88 (2.45)

6.1.3. Numerical results for example (c)

Since the source term $f(x) = \chi_{[0,1/2]}$ is in $H^\beta(D)$, $\beta \in (2 - \alpha, 1/2)$, by Theorem 3.3, w belongs to $H^{2+\beta}(D)$. Hence by repeating the argument for Theorem 4.3, the P1 FEM achieves a convergence rate of $O(h^{\alpha-1+\beta})$, while that for the P2 FEM is $O(h^{\alpha-1/2+\beta})$, $\beta \in (2 - \alpha, 1/2)$. In Table 6, we show the results when the discontinuous point is supported at a grid point. The P1 FEM converges at a rate $O(h^\alpha)$, which is one half order higher than the theoretical one. However, the P2 FEM exhibits superconvergence, which is attributed to the fact that the solution is piecewise smooth and $\|(w - w_h)'\|_{L^2}$ is second order convergent. In Table 7, we show the error when the discontinuous point is not supported at a grid point. Then the empirical rate for P2 FEM is $O(h^{\alpha+1/4})$, i.e., one quarter order higher than the theoretical ones.

Table 6: The $L^2(D)$-norm of the error for example (c) with $q = x$, $\mu = 4$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/2^m$.

α	m	3	4	5	6	7	8	rate
1.55	$P1$	4.40e-3	1.54e-3	5.33e-4	1.83e-4	6.22e-5	2.09e-5	1.54 (1.05)
	$P2$	7.36e-5	1.28e-5	2.22e-6	3.80e-7	6.05e-8	1.11e-8	2.54 (2.05)
1.75	$P1$	1.84e-3	5.18e-4	1.46e-4	4.17e-5	1.20e-5	3.43e-6	1.81 (1.25)
	$P2$	1.20e-5	1.80e-6	2.68e-7	4.00e-8	5.96e-9	8.94e-10	2.74 (2.25)
1.95	$P1$	1.08e-3	2.72e-4	6.87e-5	1.73e-5	4.36e-6	1.09e-6	1.99 (1.45)
	$P2$	1.14e-6	1.49e-7	1.94e-8	2.51e-9	3.26e-10	4.51e-11	2.92 (2.45)

6.2. Fractional Sturm-Liouville problem

Now we illustrate the FSLP (5.1) with the following potentials:

(a) a zero potential $q_1 = 0$;

(b) a non-zero potential $q_2 = x$.

Table 7: The $L^2(D)$-norm of the error for example (c) with $q = x$, $\mu = 4$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/(2^m + 1)$.

α	m	3	4	5	6	7	8	rate
1.55	$P1$	1.43e-2	5.65e-3	2.08e-4	7.37e-4	2.55e-4	8.60e-5	1.54 (1.05)
	$P2$	1.56e-4	4.77e-5	1.42e-5	4.15e-6	1.21e-6	3.49e-7	1.83 (1.55)
1.75	$P1$	4.47e-3	1.48e-3	4.65e-4	1.42e-4	4.23e-5	1.24e-5	1.76 (1.25)
	$P2$	6.41e-5	1.81e-5	4.83e-6	1.24e-6	3.17e-7	8.00e-8	2.00 (1.75)
1.95	$P1$	1.72e-3	4.98e-4	1.36e-4	3.59e-5	9.32e-6	2.38e-6	1.96 (1.45)
	$P2$	2.98e-5	7.34e-6	1.71e-6	3.84e-7	8.51e-8	1.97e-8	2.20 (1.95)

Like before, we use a uniform mesh with a mesh size $h = 1/(2^m \times 10)$. We measure the accuracy of an approximate eigenvalue λ_h by the absolute error $|\lambda - \lambda_h|$ and the approximate eigenfunction u_h by the $L^2(D)$-error $\|u - u_h\|_{L^2(D)}$. It is well known that problem (5.1) with $q(x) = 0$ has a countable number of eigenvalues λ that are zeros of the Mittag-Leffler functions $E_{\alpha,\alpha}(-\lambda)$ [24] and the corresponding eigenfunction is given by $u(x) = x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^\alpha)$. However, accurately computing zeros of the Mittag-Leffler function remains a challenging task and it does not cover the interesting case of a general potential q. Thus we compute eigenvalues λ and eigenfunctions u on a very refined mesh with $h = 1/6000$ by P2 FEM. The resulting discrete eigenvalue problems are solved by built-in MATLAB function eigs.

The numerical results for the two potentials are presented in Tables 8-9 and 10-11, respectively, for $\alpha = 1.75$. Although not presented, we note that a similar convergence behavior is observed for other fractional orders. Since both q_1 and q_2 belong to $H^1(D)$, by Theorem 5.1, the theoretical rate is $O(h^{\alpha+k-1/2})$, $k = 0, 1$, for the approximate eigenvalues and eigenfunctions. The errors are identical for both potentials, i.e., the potential term influences the errors very little. For $\alpha = 1.75$, the first eight eigenvalues are all real. Surprisingly, the approximation exhibits a second-order convergence for P1 method, and the mechanism of superconvergence is to be analyzed. Further, P2 approximation converges almost at rate of $O(h^{\alpha+1})$. However, the eigenfunction approximation converges steadily at a standard rate $O(h^{\alpha+k})$.

6.3. Preconditioned algorithms

One advantage of the new approach is that the leading term can naturally act as a preconditioner, because it is dominant and has simple structure. We present the condition number of the systems in Table 12, in which P and W denotes with preconditioner and without preconditioner, respectively. The system is more stable when α close to 2. Interestingly, the preconditioned system is very stable for the choice $\mu = \alpha - 1$, which awaits theoretical justifications.

Table 8: The absolute errors of the first eight eigenvalues, which are all real, for $\alpha = 1.75$, q_1, $\mu = 3$, with mesh size $h = 1/(10 \times 2^m)$.

	$e\backslash m$	3	4	5	6	7	8	rate
	λ_1	1.73e-3	4.77e-4	1.33e-4	3.73e-5	1.05e-5	3.01e-6	1.83
	λ_2	1.15e-2	2.89e-3	7.30e-4	1.84e-4	4.68e-5	1.20e-5	1.98
	λ_3	5.34e-2	1.34e-2	3.39e-3	8.58e-4	2.18e-4	5.56e-5	1.98
P1	λ_4	1.51e-1	3.76e-2	9.38e-3	2.34e-4	5.87e-4	1.47e-4	2.00
	λ_5	3.57e-1	8.92e-2	2.24e-2	5.61e-3	1.41e-3	3.56e-4	2.00
	λ_6	6.89e-1	1.72e-1	4.28e-2	1.07e-2	2.66e-3	6.65e-4	2.01
	λ_7	1.26e0	3.16e-1	7.91e-2	1.99e-2	4.99e-3	1.25e-3	2.00
	λ_8	2.02e0	5.01e-1	1.25e-1	3.11e-2	7.75e-3	1.93e-3	2.01
	$e\backslash m$	1	2	3	4	5	6	rate
	λ_1	1.00e-4	1.50e-5	2.22e-6	3.17e-7	3.36e-8	8.37e-9	2.71
	λ_2	1.57e-3	2.46e-4	3.72e-5	5.54e-6	8.07e-7	1.02e-7	2.78
	λ_3	5.69e-3	9.93e-4	1.57e-4	2.36e-5	3.49e-6	4.86e-7	2.70
P2	λ_4	1.19e-2	2.55e-3	4.26e-4	6.60e-5	9.96e-6	1.49e-6	2.59
	λ_5	1.25e-2	4.77e-3	8.85e-4	1.41e-4	2.18e-5	3.39e-6	2.37
	λ_6	5.52e-3	7.34e-3	1.59e-3	2.67e-4	4.12e-5	6.17e-6	2.61
	λ_7	8.21e-2	7.92e-3	2.43e-3	4.37e-4	6.93e-5	1.03e-5	2.65
	λ_8	2.39e-1	6.07e-3	3.52e-3	6.83e-4	1.11e-4	1.75e-5	2.64

Table 9: The $L^2(D)$ errors of the first five eigenfunctions u_i, for $\alpha = 1.75$, q_1, $\mu = 3$, with mesh size $h = 1/(10 \times 2^m)$.

	$e\backslash m$	3	4	5	6	7	8	rate
	u_1	2.51e-4	7.48e-5	2.23e-5	6.66e-6	1.98e-6	5.91e-7	1.75
	u_2	7.19e-4	2.11e-4	6.23e-5	1.84e-5	5.45e-6	1.62e-7	1.76
P1	u_3	1.54e-3	4.49e-4	1.31e-4	3.86e-5	1.14e-5	3.36e-6	1.77
	u_4	2.68e-3	7.73e-4	2.25e-4	6.57e-5	1.93e-5	5.68e-6	1.78
	u_5	4.05e-3	1.16e-3	3.37e-4	9.81e-5	2.88e-5	8.46e-6	1.78
	$e\backslash m$	1	2	3	4	5	6	rate
	u_1	5.39e-5	8.12e-6	1.22e-6	1.83e-7	2.72e-8	4.05e-9	2.74
	u_2	4.01e-4	6.06e-5	9.11e-6	1.37e-6	2.04e-7	3.04e-8	2.74
P2	u_3	1.22e-3	1.86e-4	2.80e-5	4.21e-6	6.30e-7	9.40e-8	2.73
	u_4	2.68e-3	4.10e-4	6.21e-5	9.35e-6	1.40e-6	2.10e-7	2.73
	u_5	4.87e-3	7.52e-4	1.14e-4	1.73e-5	2.59e-6	3.89e-7	2.73

7. Concluding remarks

In this work, we have developed a new approach to the boundary value problem with a Riemann-Liouville fractional derivative of order $\alpha \in (3/2, 2)$ in the leading term. It is based on transforming the problem into a second-order boundary value problem (possibly with nonlocal lower-order terms), and eliminates several challenges with the classical formulation. The well-posedness of the formulation and the regularity pickup were analyzed, and a novel Galerkin finite element method with P1 and P2 finite elements have been provided. The

Table 10: The absolute errors of the first eight eigenvalues, which are all real, for $\alpha = 1.75$, q_2, $\mu = 3$, with mesh size $h = 1/(10 \times 2^m)$.

	$e \backslash m$	3	4	5	6	7	8	rate
	λ_1	1.69e-3	4.67e-4	1.30e-4	3.64e-5	1.02e-5	2.93e-6	1.83
	λ_2	1.11e-2	2.89e-3	7.29e-4	1.84e-4	4.68e-5	1.20e-5	1.99
	λ_3	5.34e-2	1.34e-2	3.39e-3	8.57e-4	2.17e-4	5.56e-5	1.99
P1	λ_4	1.51e-1	3.76e-2	9.38e-3	2.34e-4	5.87e-4	1.47e-4	2.00
	λ_5	3.56e-1	8.92e-2	2.24e-2	5.61e-3	1.41e-3	3.56e-4	2.00
	λ_6	6.89e-1	1.72e-1	4.28e-2	1.07e-2	2.66e-3	6.65e-4	2.01
	λ_7	1.26e0	3.16e-1	7.91e-2	2.00e-2	4.99e-3	1.25e-3	2.00
	λ_8	2.02e0	5.01e-1	1.25e-1	3.11e-2	7.75e-3	1.93e-3	2.01
	$e \backslash m$	1	2	3	4	5	6	rate
	λ_1	8.69e-4	1.30e-5	1.91e-6	2.64e-7	1.98e-8	1.65e-8	2.71
	λ_2	1.52e-3	2.38e-4	3.60e-5	5.36e-6	7.80e-7	9.80e-8	2.78
	λ_3	5.58e-3	9.76e-4	1.53e-4	2.32e-5	3.44e-6	4.74e-7	2.77
P2	λ_4	1.17e-2	2.53e-3	4.22e-4	6.53e-5	9.86e-6	1.47e-6	2.72
	λ_5	1.22e-2	4.73e-3	8.78e-4	1.41e-4	2.17e-5	3.37e-6	2.68
	λ_6	5.91e-2	7.28e-3	1.58e-3	2.64e-4	4.10e-5	6.14e-6	2.71
	λ_7	8.26e-2	7.84e-3	2.41e-3	4.35e-4	6.90e-5	1.02e-5	2.66
	λ_8	2.40e-1	5.97e-3	3.50e-3	6.80e-4	1.11e-4	1.75e-5	2.62

Table 11: The $L^2(D)$ errors of the first five eigenfunctions u_i, for $\alpha = 1.75$, q_2, $\mu = 3$, with mesh size $h = 1/(10 \times 2^m)$.

	$e \backslash m$	3	4	5	6	7	8	rate
	u_1	2.49e-4	7.44e-5	2.22e-5	6.63e-6	1.98e-6	5.90e-7	1.75
	u_2	7.27e-4	2.13e-4	6.29e-5	1.86e-5	5.50e-6	1.63e-7	1.76
P1	u_3	1.55e-3	4.52e-4	1.32e-4	3.88e-5	1.14e-5	3.38e-6	1.77
	u_4	2.70e-3	7.77e-4	2.26e-4	6.60e-5	1.94e-5	5.71e-6	1.77
	u_5	4.07e-3	1.17e-3	3.38e-4	9.84e-5	2.88e-5	8.49e-6	1.78
	$e \backslash m$	1	2	3	4	5	6	rate
	u_1	5.52e-5	8.34e-6	1.25e-6	1.88e-7	2.81e-8	4.21e-9	2.74
	u_2	4.06e-4	6.13e-5	9.22e-6	1.38e-6	2.07e-7	3.08e-8	2.74
P2	u_3	1.23e-3	1.87e-4	2.82e-5	4.24e-6	6.35e-7	9.48e-8	2.74
	u_4	2.69e-3	4.12e-4	6.25e-5	9.41e-6	1.41e-6	2.11e-7	2.73
	u_5	4.89e-3	7.56e-4	1.15e-4	1.73e-5	2.61e-6	3.90e-7	2.73

$L^2(D)$ error estimate of the approximation has been established. Further the approach was extended to the Sturm-Liouville problem, and convergence rates of the eigenvalue and eigenfunction approximations were provided. Extensive numerical experiments were provided to verify the convergence theory.

In our theoretical developments, the analysis is only for the case $\alpha > 3/2$. The interesting case $\alpha \in (1, 3/2]$ was not covered by the theory. However, our numerical experiments indicate that the approach converges equally well in this case. Further, the theoretical convergence rate is one half order lower than the empirical one, for both source problem and Sturm-Liouville problem. These

Table 12: condition number for P1, $q = x$, $\alpha = 1.55, 1.75, 1.95$, $h = 1/2^m$. (P - preconditioned, W - without preconditioner)

α	μ	m	3	5	7	9	11
	$\alpha - 1$	P	1.21e0	1.74e0	5.89e0	5.81e1	7.82e2
		W	2.32e1	3.80e2	6.08e3	9.73e4	1.56e6
1.55	3	P	2.10e0	1.12e1	1.34e2	1.85e3	2.58e4
		W	3.30e2	5.47e2	8.78e3	1.41e5	2.25e6
	4	P	2.26e0	1.35e1	1.67e2	2.32e3	3.23e4
		W	3.41e1	5.71e2	9.18e3	1.47e5	2.35e6
	$\alpha - 1$	P	1.10e0	1.19e0	1.53e0	3.08e0	1.31e1
		W	2.36e1	3.87e2	6.20e3	9.91e4	1.59e6
1.75	3	P	1.39e0	2.72e0	1.09e1	7.44e1	5.81e2
		W	2.85e1	4.69e2	7.51e3	1.20e5	1.92e6
	4	P	1.48e0	3.16e0	1.41e1	9.99e1	7.86e2
		W	2.93e1	4.82e2	7.73e3	1.24e5	1.98e6
	$\alpha - 1$	P	1.06e0	1.06e0	1.07e0	1.09e0	1.17e0
		W	2.40e1	3.93e2	6.30e3	1.01e5	1.61e6
1.95	3	P	1.05e0	1.13e0	1.32e0	1.81e0	3.38e0
		W	2.49e1	4.08e2	6.53e3	1.04e5	1.67e6
	4	P	1.06e0	1.16e0	1.40e0	2.05e0	4.02e0
		W	2.50e1	4.10e2	6.57e3	1.05e5	1.68e6

gaps are still to be closed. Last, it is of much interest to extend the approach to the time dependent case [25, 26] as well as the multi-dimensional analogue, for which a complete solution theory seems missing.

Acknowledgment

Appendix A. Computation of the stiffness matrix

In this appendix we discuss the implementation of the new approach, especially the computation of the stiffness matrix $A = [a_{ji}]$, with

$$a_{ji} = (\phi_i', \phi_j') + (_0^R D_x^{2-\alpha} \phi_i, q\phi_j) + (_0^R D_x^{2-\alpha} \phi_i)(1)(p, \phi_j),$$

with $\{\phi_i\}$ being the finite element basis functions. The computation of the leading term (ϕ_i', ϕ_j') is straightforward, and thus we focus on the last two terms.

Below we shall discuss the cases of piecewise linear and piecewise quadratic finite elements separately.

Appendix A.1. Piecewise linear finite elements

To simplify the notation, we denote $\gamma = \alpha - 1$. We first note the identity (with $h_i = x_i - x_{i-1}$)

$$
\begin{aligned}
{}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x) &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\phi'_i(t)dt \\
&= \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\left(\frac{\chi_{[x_{i-1},x_i]}}{h_i} - \frac{\chi_{[x_i,x_{i+1}]}}{h_{i+1}}\right)dt \\
&= \frac{1}{\Gamma(\gamma+1)}\left[h_i^{-1}((x-x_{i-1})^{\gamma}_+ - (x-x_i)^{\gamma}_+) - h_{i+1}^{-1}((x-x_i)^{\gamma}_+ - (x-x_{i+1})^{\gamma}_+)\right],
\end{aligned}
$$

where $(c)_+$ denotes the positive part, i.e., $(c)_+ = \max(c,0)$. In the case of a uniform mesh, it simplifies to

$$
{}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x) = \frac{1}{\Gamma(\gamma+1)h}\left((x-x_{i-1})^{\gamma}_+ + (x-x_{i+1})^{\gamma}_+ - 2(x-x_i)^{\gamma}_+\right).
$$

Hence, the term $b_{ji} = \int_0^1 {}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x)q(x)\phi_j(x)dx$ in the middle is of the form

$$
b_{ji} = \int_{x_{j-1}}^{x_j} q(x)\phi_j(x){}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x)dx + \int_{x_j}^{x_{j+1}} q(x)\phi_j(x){}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x)dx.
$$

The integrals on the right hand side can be evaluated accurately using an appropriate Gauss-Jacobi quadrature rule. The last term is a rank-one matrix, and it requires only computing two vectors. The quantity $({}^{R}_{0}D^{2-\alpha}_{x}\phi_i)(1)$ can be computed in closed form

$$
\begin{aligned}
({}^{R}_{0}D^{2-\alpha}_{x}\phi_i)(1) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1}\phi'_i(t)dt \\
&= \frac{1}{\Gamma(\gamma)}\left[h_i^{-1}\int_{x_{i-1}}^{x_i}(1-t)^{\gamma-1}dt - h_{i+1}^{-1}\int_{x_i}^{x_{i+1}}(1-t)^{\gamma-1}dt\right] \\
&= \frac{1}{\Gamma(\gamma+1)}\left[h_i^{-1}((1-x_{i-1})^{\gamma} - (1-x_i)^{\gamma}) - h_{i+1}^{-1}((1-x_i)^{\gamma} - (1-x_{i+1})^{\gamma})\right].
\end{aligned}
$$

In case of a uniform mesh, it simplifies to

$$
({}^{R}_{0}D^{2-\alpha}_{x}\phi_i)(1) = \frac{1}{\Gamma(\gamma+1)h}\left((1-x_{i-1})^{\gamma} + (1-x_{i+1})^{\gamma} - 2(1-x_i)^{\gamma}\right).
$$

For $h \ll x - x_i$, $(x-x_{i-1})^{\gamma}_+ + (x-x_{i+1})^{\gamma}_+ \approx 2(x-x_i)^{\gamma}_+$. Then the expression for ${}^{R}_{0}D^{2-\alpha}_{x}\phi_i(x)$ may suffer precision loss due to roundoff errors. We may improve the accuracy by writing

$$
(x-x_{i-1})^{\gamma}_+ - (x-x_i)^{\gamma}_+ =: A^{\gamma} - B^{\gamma} = B^{\gamma}\left[(A/B)^{\gamma} - 1\right] = B^{\gamma}\mathbf{expm1}(\gamma\log(A/B)),
$$

which allows stable computation in e.g., `MATLAB`. Last, given w_h, one needs to recover u_h, which involves only fractional-order differentiation of the basis $\{\phi_i\}$

$$u_h(x_j) = {}^R_0D_x^{2-\alpha}w_h(x_j) - ({}^R_0D_x^{2-\alpha}w_h)(1)x_j^\mu,$$

where the first term can be computed efficiently by (with $w_i = w_h(x_i)$)

$$u_h(x_j) = \frac{1}{\Gamma(\gamma+1)} \sum_{i=1}^{j-1} w_i \left[h_i^{-1}((x_j - x_{i-1})^\gamma - (x_j - x_i)^\gamma) \right.$$

$$\left. + h_{i+1}^{-1}((x_j - x_i)^\gamma - (x_j - x_{i+1})^\gamma) \right] + \frac{1}{\Gamma(\gamma+1)} w_j h_j^{-1}(x_j - x_{j-1})^\gamma.$$

Appendix A.2. Piecewise quadratic finite elements

Next we describe the case of piecewise quadratic finite elements, i.e.,

$$u = \sum_{i=1}^{N} u_i \phi_i(x) + \sum_{i=0}^{N-1} u_{i'} \phi_{i'}(x),$$

where for simplicity, we denote by $x_{i'} = (x_i + x_{i+1})/2$, the middle point of the interval $[x_j, x_{j+1}]$, and $\phi_{i'}$ denotes the basis function corresponding to the node $x_{i'}$. Then like before, we find

$$
\begin{aligned}
{}^R_0D_x^{2-\alpha}\phi_i(x) &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\phi_i'(t)dt \\
&= \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\left(\frac{\chi_{[x_{i-1},x_i]}}{h_i}\left(3 + 4\frac{t-x_i}{h_i}\right) + \frac{\chi_{[x_i,x_{i+1}]}}{h_{i+1}}\left(-3 + 4\frac{t-x_i}{h_{i+1}}\right) \right)dt \\
&= \frac{1}{\Gamma(\gamma+1)} \left[3h_i^{-1}((x-x_{i-1})_+^\gamma - (x-x_i)_+^\gamma) - 3h_{i+1}^{-1}((x-x_i)_+^\gamma - (x-x_{i+1})_+^\gamma) \right] \\
&\quad + \frac{1}{\Gamma(\gamma)} \left[4h_i^{-2}(\gamma^{-1}(x-x_i)((x-x_{i-1})^\gamma - (x-x_i)^\gamma) \right. \\
&\quad \left. - (\gamma+1)^{-1}((x-x_{i-1})^{\gamma+1} - (x-x_i)^{\gamma+1}) \right] \\
&\quad + \frac{1}{\Gamma(\gamma)} \left[4h_{i+1}^{-2}(\gamma^{-1}(x-x_i)((x-x_i)^\gamma - (x-x_{i+1})^\gamma) \right. \\
&\quad \left. - (\gamma+1)^{-1}((x-x_i)^{\gamma+1} - (x-x_{i+1})^{\gamma+1})) \right].
\end{aligned}
$$

For a uniform mesh, the expression simplifies to

$$
\begin{aligned}
{}^R_0D_x^{2-\alpha}\phi_i(x) &= \frac{3}{\Gamma(\gamma+1)}\left(3h^{-1} + 4h^{-2}(x-x_i)\right)\left((x-x_{i-1})_+^\gamma + (x-x_{i+1})_+^\gamma - 2(x-x_i)_+^\gamma\right) \\
&\quad - \frac{4}{\Gamma(\gamma)(\gamma+1)h^2}((x-x_{i-1})^{\gamma+1} + (x-x_{i+1})^{\gamma+1} - 2(x-x_i)^{\gamma+1}).
\end{aligned}
$$

Likewise, with $\phi_{i'} = 1 - 4\frac{(x-x_{i'})^2}{h_{i+1}^2}$, we have

$$\begin{aligned}
{}_0^R D_x^{2-\alpha} \phi_{i'}(x) &= \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \phi'_{i'}(t)dt \\
&= \frac{1}{\Gamma(\gamma)} \int_0^x -8h_{i+1}^{-2}(t-x_{i'})\chi_{[x_i,x_{i+1}]}(x-t)^{\gamma-1}dt \\
&= \frac{-8}{\Gamma(\gamma)h_{i+1}^2}\left[(\gamma+1)^{-1}((x-x_{i+1})_+^{\gamma+1} - (x-x_i)_+^{\gamma+1})\right. \\
&\qquad\qquad\left. -\gamma^{-1}(x-x_{i'})((x-x_{i+1})_+^{\gamma} - (x-x_i)_+^{\gamma})\right].
\end{aligned}$$

The computation of the remaining terms is similar to the case of piecewise linear finite elements, and thus omitted.

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