Non-cooperative game theoretic approaches to bilateral exchange networks

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Abstract

Bilateral exchange networks are structures in which a finite set of players have a restricted framework of bargaining opportunities with each other. The key restrictions are that each player may participate in only one 'exchange' and each of these may only involve a pair of players. There is a large sociology literature which investigates these networks as a simplified model of social exchange. This literature contains many predictions and experimental results, but not a non-cooperative game theoretic analysis. The aim of the thesis is to provide this.

The analysis builds on the economic theory literature on non-cooperative bargaining, principally the alternating offers and Nash demand games. Two novel perfect information models based on the alternating offers game are considered and it is demonstrated that they suffer from several difficulties. In particular, analysis of an example network shows that for these two models multiple subgame perfect equilibria exist with considerable qualitative differences. It is argued that an alternating offers approach to the problem is therefore unlikely to be successful for general networks.

Models based on Nash demand games also have multiple solutions, but their simpler structure allows investigation of equilibrium selection by evolutionary methods. An agent based evolutionary model is proposed. The results of computer simulations based on this model under a variety of learning rules are presented. For small networks the agents often converge to unique long-term outcomes which offer support both for theoretical predictions of 2 and 3 player alternating offers models and experimental results of the sociology literature. For larger networks the results become less precise and it is shown they sometimes leave the core. It is argued that a modified evolutionary model has scope for avoiding these difficulties and providing a constructive approach to the problem for large networks.

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Dedication,

To my parents for their love and support

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Chapter 1

Introduction and Overview

Social exchange is a framework for studying a wide range of ongoing mutually profitable social interactions which take place between pairs of participants. These interactions typically take place within networks of many possible exchanges. Network position can be a crucial determinant of the outcomes that a participant receives from social exchange. As a simple example of social exchange, suppose that one person in a community has a monopoly on a valuable skill (e.g. literacy or medical training) whereas all the others have identical skills and resources and are incapable of acting collectively. The monopolist will receive many offers to exchange their skill (e.g. requests to read letters or provide medical aid) and over time exchange outcomes will develop which they find favourable (e.g. monetary rewards, pleasant company, actions denoting social status).

The relationship between network structure and the outcome of social exchange has recently been the focus of considerable research by sociologists. One research direction has concentrated on a simplified model of social exchange networks in which there are many discrete rounds and in each round every participant may take part in at most one exchange with another participant. There is a network of opportunities for exchange which is fixed over all rounds. The bilateral exchange networks of the title mathematically describe such settings. Sociologists have proposed many theories predicting exchange in these settings and carried out extensive laboratory experiments with human subjects.

Bilateral exchange networks can also be used to model special cases of economic exchange. There exists a substantial economic theory literature on non-cooperative game theoretic models of bargaining between two participants. This thesis develops extensions of these models which apply to general bilateral exchange networks. This is of use to the theories of both economic and social exchange. In particular, it allows an investigation of whether the sociological theories mentioned above can be supported by rigorous non-cooperative game theoretic models. Also, the extensive experimental results of the sociology literature provide a convenient test for the predictions of game theoretic models. The overall aims of this research are to find models which:

a) Adequately support experimental results and satisfy other reasonable properties[1],

b) Produce predictions for large networks representing typical social networks.

c) Explain the relationship between network parameters and outcome.

There are two leading approaches to non-cooperative models of bargaining between a pair of participants in the economic theory literature. The first is based on the *alternating offers game* of Rubinstein [56]. It models the bargaining process as a sequence of proposals made alternately by the two participants. Each proposal must be accepted or rejected before a counter-proposal is made. The first accepted proposal is binding. The bargaining process entails costly delays which, although typically small, turn out to be the mechanism providing the game with a unique outcome under a solution concept motivated by assumptions of players' rationality, namely subgame perfect equilibrium. However, the complicated strategies spaces of this game preclude an evolutionary approach. The second approach is based on the

[1] Some 'reasonable properties' are developed throughout the thesis and collated in section 9.1.1.

Nash demand game proposed by Nash [53]. This game abstracts away many details of the bargaining process and simply requires each player to simultaneously demand a utility value. If it is feasible for both players to receive their demand then they do so. Otherwise both receive nothing. Under solution concepts motivated by players' rationality, this model supports a very wide range of outcomes. However, its simple strategy structure is well suited to an evolutionary approach.

It is easy to state many plausible sounding extensions of the alternating offers game to the setting of bilateral exchange networks. However, seemingly innocent variations in the rules of these extensions can hide significant implicit assumptions about the bargaining opportunities available to players. In this thesis, the features of such extensions are investigated and, based on this, two novel models are proposed which apply to general bilateral exchange networks and allow appropriate bargaining opportunities to players.

Once the possibility of more than one exchange is introduced, the analysis of models based on the alternating offers game using subgame perfect equilibrium typically requires considerable effort. Even intuitively obvious results can require complicated proofs[2]. However, this thesis succeeds in proving several results describing the SPE behaviour of the two novel models mentioned above for several small networks. The models are shown to support a wide range of solution outcomes for certain networks involving significant qualitative payoff differences. It is argued that the underlying causes of this multiplicity are likely to also apply to most large networks. It is concluded that, except for the smallest networks, only weak predictions are likely to be produced by the alternating offers approach, such as loose upper and lower bounds on the payoffs of certain positions. These do not match the more precise experimental results of the sociology literature, failing aim a) above.

Evolutionary models based on the approach of the Nash demand game are shown to be a much more successful approach. In this thesis, an evolutionary model is de-

[2]For example, many such results are contained in section 5.4.

veloped and implemented as a computer simulation. The simulation results give strong predictions for the outcomes of bargaining in several bilateral exchange networks. These predictions match the patterns found by sociology experiments for some networks, such as line networks, as required by aim a) above. Furthermore they complement the experimental results by showing that in some settings they remain valid over much longer time scales. Also, very strong support is found for the von Neumman-Morgenstern triple solution to 3 player ring networks. This complements the theoretical support this solution receives from an alternating offers approach of Binmore [3] (where this solution is first proposed). A theoretical result, theorem 7.3, on the evolutionary model is also proved, predicting the payoffs of certain networks positions under various conditions representing a limiting case (low mutation) of the evolutionary model. The predictions of this theorem are supported qualitatively by the simulation results but not quantitatively. Thus theorem 7.3 reveals one mechanism by which network parameters can drive the results of bargaining, matching aim c) above, but results are also affected by other evolutionary pressures.

For a particular network the simulation results offer support to non-core solutions. It is argued that the support for these solutions is due to the bargaining opportunities available to players being overly restricted in the game underlying the evolutionary model. This hinders the potential usefulness of this particular evolutionary model for large networks. However, in general the evolutionary approach based on the Nash demand game shows considerable potential for development to study large networks; aim b) above.

Both approaches mentioned above provide support for some qualitative features of theoretical predictions of the sociology literature, and many of the experimental results. Also the theoretical results described above provide support for some of the intuitions described in the sociology papers for factors that drive the outcome of bargaining. However no support is found for the ad-hoc assumptions at the heart of the theories of the sociology literature. This research suggests that theories and experiments of social exchange should seek to investigate how network level outcomes are generated from individual level behaviour rather than such assumptions.

Chapter 2 is a literature review of social exchange, focusing on network effects. In particular, it contains a summary of experimental results. Chapter 3 consists of some preliminary mathematical material, such as a definition of a bilateral exchange network and an outline of the game theoretic concepts which are used later. Chapter 4 contains a literature review of the alternating offers game and various extensions of this game which model bargaining situations with more players. Some novel material is included as well, exploring various features of these models and their effects on subgame perfect equilibrium behaviour. Chapter 5 defines two novel extensions of the alternating offers game and analyses them for various example networks with a small number of players. Chapters 6 – 8 are concerned with the evolutionary approach based on the Nash demand game. Chapter 6 defines an extension of the Nash demand game to large bilateral exchange networks and gives an overview of an evolutionary model. A review of related literature is also contained. Chapter 7 discusses the details of this model in more depth and also contains some limited theoretical analysis of it. Chapter 8 contains the results of simulations using this model.

Chapter 9 is the conclusion. One section of this chapter summarizes the theoretical results and simulation data obtained for various bilateral exchange networks throughout the thesis. These results are compared with each other and with the theories and experimental data of the sociology literature. There is also some discussion on what this reveals about the forces driving the outcome of bargaining in particular networks. The other main section of the conclusion is a discussion of the suitability of the models proposed throughout the thesis. Also, some future research directions are proposed, including possible ways to adapt the evolutionary model of chapter 6 to allow the investigation of larger networks.

Finally, note that the more technical material of many chapters, mainly lengthy proofs, is relegated to appendices. These appear at the end of the corresponding chapters.

Chapter 2

Sociological Background

This chapter reviews the sociology literature on social exchange. Section 2.1 is on early work on social exchange and also serves to describe what is usually meant by the term 'social exchange'. Section 2.2 focuses on the simple case of *dyadic exchange*; exchange between a pair. Section 2.3 discusses extending the investigation of social exchange from dyads to networks and also briefly mentions one area of application. Many social exchange researchers have concentrated on studying *negatively connected* networks. These are discussed in section 2.4 and form the basis for the bilateral exchange networks investigated in this thesis. Section 2.5 describes several predictive theories from the sociology literature for negatively connected networks. The remaining sections discuss laboratory experiments in this setting. Section 2.6 describes typical features of the experimental designs used. Section 2.7 summarizes some experimental results obtained from the literature. Section 2.8 briefly discusses some issues raised by these results. In chapter 9 this experimental data is compared to the results of this thesis.

2.1 Early Work

The roots of the literature on social exchange include the work of Homans [35] (chapters 3 and 4), Blau [12], and Emerson [25]. The area of study is any form of pairwise exchange in a social setting. Indeed, Emerson sees social exchange as a framework for the investigation of any observations about reciprocal social behaviour based on interactions between pairs. This allows a very wide definition of what is exchanged. However, requiring a *social* setting puts some constraints on the domain of study. Some conditions which are typically mentioned are that exchanges are face-to-face and bilateral, opportunities for exchange are repeated rather than one-offs, and there is no mechanism for participants to enter into formal binding contracts. Thus, only special cases of economic exchange can be considered as social exchange.

An example of social exchange considered by all three authors mentioned above is a workplace where workers sometimes ask each others' advice[1]. Homans describes this as social exchange where advice is exchanged for "approval", e.g. flattery. Emerson offers an interpretation where exchanges take place with the expectation of future reciprocal exchange. If the exchange relationship is to continue, the advisee must supply something the advisor values, e.g. help around the office or pleasant company. Blau interprets help as being exchanged for "status", a social signal which plays several roles. It signals that the advisor is a good source of help on this subject, which presumably is valuable for the receiver. It also signals that the advisee is under certain social obligations. If these are reneged upon then he is liable to some form of punishment by his social group, e.g. he will not be supplied with advice by anyone else. Thus possible sources of reciprocity are due to social pressure or to maintain a

[1]This is based on a field study by Blau [11] of an office of federal agents auditing firms to enforce certain laws. Discussing details of a case is officially forbidden. However, the cases are quite complex and the agents often feel they require assistance. They are reluctant to go to their supervisor, believing it may adversely influence their annual rating. Instead they often ask the advice of a more experienced agent, which is unofficially permitted.

valuable flow of exchange.

Other examples of social exchange given by these authors include the exchange of favours between neighbours e.g. the loan of items, or the exchange of invitations to participate in some social activity such as dinner parties or tennis matches. A more large scale example is the "Kula ring" studied by Malinowski [41]. This was a complicated system of ceremonial exchange between Melanesian islanders which entailed considerable social obligations and offered opportunities for strategic behaviour. The custom of exchange indirectly linked a large ring of islands many of which had little direct contact.

Blau discusses some limits to what can be modelled by social exchange. He contrasts the local influence that can be achieved through these means with "impersonal power on a large scale". This is split into economic power and political authority. Differences between economic and social exchange have already been mentioned. Political authority requires institutions to transmit commands. These may act partly through networks of social exchange, but also partly through economic action or through actions that affect large numbers of people without social contact, e.g. mass media.

For the purposes of the discussion in this chapter it is convenient to assume that a player's[2] outcome from exchange can be easily quantified by a numerical measure, which I shall refer to as their *payoff*. The authors mentioned above construct various theories of how this can be achieved which differ from standard economic theories of utility. The measures used by Homans and Emerson[3] are related to the frequency with which a valued action is performed by a potential exchange partner, and so have some basis in concrete experimental data. The exact details of what is meant by a

[2]The sociology literature generally uses the term 'actor' for a participant in exchange. I use 'player' for consistency with the game theoretic terminology used in the other chapters of the thesis.

[3]This refers to the measure Emerson used in [25]. In [21] a frequency based measure is no longer used.

payoff can be glossed over in this chapter, since most of the discussion below requires only a qualitative notion of payoff. The exception is the laboratory experiments of sections 2.6 and 2.7. In these payoffs have a concrete meaning in terms of the payments the experimental subjects receive.

2.2 Dyadic Exchange and Power

Early research pays particular attention to dyadic (i.e. two player) exchange. Emerson's approach to dyadic exchange influences much of the subsequent literature. His theory developed over time and the version I describe here is taken from [22][4]. An exchange relation between players A and B is considered in which these players have resources x and y respectively which they can exchange. The following definitions are made[5]:

"The dependence (D_{AB}) of A on B in a dyadic exchange relation ... is a joint function (1) varying directly with the value of y to A, and (2) varying inversely with the availability of y to A from alternate sources."

"... the power of A over B (P_{AB}) is the potential of A to obtain favorable outcomes at B's expense."

The latter definition is supported (in [25]) by a quote from Weber [69] (page 152):

"Power is the probability that one actor with a social relationship will be in a position to carry out his own will despite resistance."

[4]The earlier version of [25] is in terms of what is referred to in section 2.3 as a reciprocal setting; the players occasionally have opportunities to make rewarding actions to each other. The later version of [22] which is presented in the text is in terms of what is referred to in section 2.3 as a negotiated setting; in each time period the players must agree the terms of a bilateral exchange. One reason for the switch to a negotiated setting is that this matches the experimental setup used in [22].

[5]Emerson does not appear to explicitly define a value for D_{BA}, although in [25] he writes "this can readily be accomplished with considerable precision" of the definition of a closely related quantity.

Emerson equates P_{AB} and D_{BA}. Furthermore, in [21], in the context of two players who must split a unit of payoff, Emerson and Cook argue that if $P_{AB} = P_{BA}$ then the payoff is split equally[6], and if there is an inequality then this indicates which player takes the majority of the payoff.

Markovsky et al [45] make the following similar but more straightforward definition of power:

"Power is ... [a] potential for obtaining relatively favourable resource levels."

In this and Emerson's definitions of power it is important to note the use of the word 'potential'. The literature commonly makes a distinction between power and *exercised power* (or power use). For example, Markovsky et al [44] claim that a player with a power advantage (under a measure of power defined in section 2.5.1 below) could obtain the maximum available gains from exchange if they chose to fully exert their power. Some reasons which are given in the literature (e.g. see Cook and Emerson [21]) for why not all power is exercised include equity norms or other psychological biases, players who are not fully rational, and social pressure.

The above discussion of the definition of power is given because many sociological predictions on social exchange in networks are phrased in terms of power differences between network positions. On the other hand, the laboratory experiments of the literature reveal only exercised power.

Homans [35] makes several qualitative propositions on the frequency of actions in an ongoing dyadic relationship where the players may take actions which reward each other. These propositions are not detailed here as they do not generate a specific prediction for a dyadic exchange. However, he does conclude from them that in a situation where one central player has an opportunity to exchange with two outlying players, the outliers do worse than they would if the other outlier did

[6]This is a vague conclusion as it is not robust to a reinterpretation of the meaning of 'payoff'. In [21] details are not provided of how a payoff scale representing preferences is constructed. However in an experiment later in [21], experimental subjects earn cash payments proportional to their total payoffs over many exchanges.

not exist whereas the central player does better.

2.3 Exchange Networks

In practice, social exchange is not restricted to two player settings. An interesting question to sociologists is the effect of the structure of a network of exchange opportunities on the payoffs that the players receive. Emerson [25] provides some ideas on how to move from studying exchange in pairs to studying networks. He proposes a classification of *connections* between exchanges. A pair of exchange opportunities with a player in common can have a *positive* or *negative* connection. Two exchange opportunities have a positive connection if exchange in one "facilitates exchange" in the other[7]. They are said to be negatively connected if exchange in one will "diminish or prohibit exchange" in the other[8]. Emerson acknowledges that this classification does not include all possible connections.

In many subsequent papers, the terms positive and negative are usually used – and often defined – to mean the following stronger forms of these definitions. Positively connected exchanges may only form together and negatively connected exchanges may never form together. These strong definitions are useful in a *negotiated* exchange setting. In this there are a series of rounds. In each the players must come to a set of bilateral agreements amongst themselves giving terms of exchange, constrained by the connections between exchange opportunities. An alternative is a *reciprocal* setting in which players make each other unilateral gifts in the hope of future reciprocity[9]. Most of the subsequent work has concentrated on the negotiated setting. An exception is the work of Molm et al (e.g. [48, 50]) which investigates network effects in a reciprocal framework. The definitions of negotiated and reciprocal settings are taken from Molm et al [50].

[7]These quotes are taken from the summary of Emerson's classification in [73].

[8]According to Emerson's scheme these are the *bilateral* versions of connections. In *unilateral* connections, the effects mentioned are one-way.

[9]It is hard to see how the strong definitions could be applied in this setting.

One reason for an interest in power in exchange networks is its influence on network formation, as discussed by Emerson in [25][10]. One example he considers is a network in which all exchange opportunities are between a central player and a large set of outlying players, and all exchanges are negatively connected. This is a unilateral monopoly in which the central player can extract a large payoff from the outliers. Emerson argues that this situation provides incentives for an outlier to diversify what they can exchange with the central player in order that their exchange opportunity becomes less negatively connected to the others. Alternatively, there is also an incentive for the outliers to bargain collectively. Emerson draws parallels between these processes and possible paths in the development of a society.

2.4 Negatively Connected Networks

Most sociological research has concentrated on studying *negatively connected exchange networks*[11]. These use the negotiated setting of the previous section and the restriction that the strong form of negative connection exists between any pair of exchanges involving a common player. Thus in a round of play a player may be involved in at most one exchange. A rationale for concentrating on this case is given by Emerson and Cook in [21]:

"...a) the relation of power and dependence to position in negatively connected networks is relatively straightforward, and b) negative connections are easily operationalized in the laboratory."

Another standard assumption is that each exchange opportunity takes the form of splitting a number of payoff points. This number is usually constant across all exchange opportunities[12]. Also note that networks are assumed to stay constant

[10] Network formation has not been a major interest of subsequent research. Two exceptions are Cook and Emerson [21] and Walker et al [68] which contain some material on network formation.

[11] Papers which consider other types of exchange network include Markovsky et al [45], Skvoretz and Willer [66] and Yamaguchi [75].

[12] Some papers, such as Cook and Emerson [21] and Molm et al [50], investigate negatively

over time. That is, the exchange opportunities do not vary.

Several competing predictive theories have emerged, a few of which are briefly summarized in section 2.5 below. These approaches typically rely on some parameter values or functions which are chosen on an ad-hoc basis[13]. A common characteristic of these approaches is that they are not generated directly from assumptions about individuals. Indeed the outcome is often claimed to be fairly robust to the specification of individual behaviour. For example Lovaglia et al [39] state of a certain class of networks:

"...structural determinants are so powerful . . . that actor cognitions can introduce only minor variations at best."

However, assumptions about the behaviour of individual players are sometimes made. For example, it is often assumed that players who are excluded from exchanging will lower their subsequent demands, and those who are included in an exchange will raise theirs. These sometimes occur as assumptions used in simulations (e.g. in Cook et al [22]) and sometimes as general assumptions (e.g. in Markovsky et al [45][14]).

Many laboratory experiments have also been carried out as empirical tests of the competing theories of the literature. Section 2.6 describes the design of these experiments. They concentrate on investigating the outcomes produced in different network settings, and test hypotheses about aggregate outcomes rather than individual behaviour. The literature also mentions many simulation results. These are not discussed here because usually only a few derived statistics are published rather than detailed results and the models implemented by the simulations are usually

connected networks in which the number of payoff points available varies in different exchange opportunities.

[13]For example the functional form of D_{BA} mentioned earlier, or the weights in the GPI function described in section 2.5.1.

[14]Here they take the form of 'scope conditions' which delimit the domain of applicability of the theory. It follows that these scope conditions are assumed to capture an aspect of players' behaviour in some interesting situations.

not documented in much depth[15].

A central concern for a non-cooperative game theoretic approach to modelling bargaining is the structure of the bargaining process. For example, is there a time limit at which bargaining must cease? How much communication between players is allowed? Can a player bargain with two others simultaneously? These details can greatly affect the outcomes supported by such models as is made clear in the following chapters[16]. The sociology literature is generally agnostic on most of these points. It may be that the solutions proposed by the literature are thought to be robust to these details; this is often the case with information, as discussed shortly. However, in the majority of laboratory experiments, subjects interact through computer terminals and thus it is necessary to have a detailed specification of a bargaining process. Such specifications are described in section 2.6. These may give some insight into the assumptions made by researchers. However, it is quite possible that many of these assumptions are made for experimental expediency rather than on a theoretical basis, so too much should not be read into them.

One element of the bargaining process which is discussed explicitly in the literature is information. However, there is not a consensus on the effect of information. For example in [45] Markovsky et al state:

"Having information on negotiations other than one's own is expected to accelerate the use of power, but not affect relative power."

On the other hand in [22] Cook et al state:

"An important feature of our laboratory research is that actors. . . have no knowledge of the network beyond their own opportunity set. . . . This feature allows us to examine 'purely' structural determinants of behaviour."

In other words, they do not rule out the possibility that information could affect the outcome. The role of information has been explored experimentally (e.g. in Lovaglia [39]). However, as discussed in section 2.8 below, the results are inconclusive.

[15] An exception is Markovsky [42].

[16] For example see the discussion of section 4.4.5.

The sociology literature generally makes little use of game theory to model social exchange networks. One exception is the work of Bienenstock and Bonacich (e.g. [1, 2]) who apply cooperative game theoretic solution concepts to the problem. However, they do not probe the assumptions underlying these different solution concepts to form a view on which, if any, is most appropriate to the situation. Also, Willer and Skvoretz [72] consider a simplified non-cooperative model of bargaining in which some players have limited strategy choices while others act "parametrically" i.e. according to fixed rules. For example they consider a negatively connected exchange network in which a central player can split 24 payoff units with one of two outlying players in each round. In their model, in each round the outlying players must choose simultaneously from two possible strategies: demand d units or $d-1$ units. These correspond to sticking to a convention or undercutting. The central player simply (randomly) chooses a player with the lowest demand, gives them the number of points which they demanded and keeps the rest. In the following round d is set to whatever value the preceding lowest demand was[17]. This illustrates a mechanism by which the outlying players undercut each other in this network, driving down their payoffs over time.

2.5 Predictive Theories

This section briefly sketches some theories from the sociology literature which predict the outcome of negatively connected exchange networks in which all exchanges involve splitting the same number of payoff points. The main motivation for this is to allow some similarities between parts of these theories and the results of this thesis to be highlighted in the discussion of the concluding chapter. Thus the theories discussed are those mentioned in later discussions.

[17]This is essentially a version of an extension of the Nash demand game to 3 players with very simplified strategy sets played repeatedly. Chapter 6 discusses extensions of the Nash demand game to bargaining situations with more than 2 players.

Many other theories exist in the literature. Some continue to be developed by their authors, whereas others have been abandoned following poor performance in experiments[18]. One active branch of research concerns theories based on the power-dependence arguments outlined in section 2.2. Some examples are Molm et al [49] and Yamagishi et al [73].

2.5.1 NET: GPI

In [45] (1988), Markovsky, Willer et al introduce a predictive theory which they name 'network exchange theory'. Later, other researchers often used this term to describe the whole field of social exchange in networks, so in [43] (1997) Markovsky coins the acronym NET to refer to the particular research program based on [45]. I follow this convention here. Over the course of several papers, NET has undergone several revisions and grown quite complicated; various techniques must be applied in different cases. This section discusses one particular technique from the original paper in the context of negatively connected networks. Section 2.5.2 discusses some other aspects of NET.

The *graph-theoretic power index* (GPI) is defined in [45] as follows:

$$p_x = \sum_{i \geq 1} (-1)^{i-1} m_{ix}$$

where m_{ix} is the number of elements in any set of paths[19] starting from x of length i such that no vertex other than x occurs in two paths[20].

[18] An example is the network vulnerability measures proposed in Cook et al [22] which performed badly in an experiment of Markovsky et al [45].

[19] This entails viewing the network as a graph whose vertices are the set of players and whose edges are the set of unordered player pairs which have exchange opportunities. A path is defined in section 3.2. Cycles are not viewed as paths for this definition (this can be deduced from the GPI values given for the kite and stem networks in Markovsky et al [44]).

[20] m_{ix} is not well defined. For example consider a network with players $\{1, 2, 3, 4, 5\}$ and exchange opportunities between the pairs $\{12, 13, 23, 24, 35\}$. For $x = 1$ and $i = 3$, two sets of paths as described in the definition are $\{123\}$, $\{124, 135\}$. If m_{ix} was defined as the *maximum* number of elements in any such set then it would be well defined.

The rationale for this expression is that odd length paths are beneficial to the power of a player x but even length paths are detrimental. For example, consider a player x in a negatively connected exchange network. Suppose a new player y is added whose only exchange opportunity is with x. This can only strengthen the bargaining position of x. However suppose another new player z is now added whose only exchange opportunity is with y. This improves the bargaining position of y and thus weakens x. This argument can be extended inductively. The weighting that the formula gives to these effects is ad-hoc. A subset of paths are given weights of identical magnitude, 1, and the others are given no weight (i.e. those that would produce at least one overlapping path if they were included) .

A possible motivation for the 'non-overlapping' condition in the GPI definition, based on an example given in [45] is as follows. Consider a player x in a negatively connected exchange network. Suppose a player y is added whose only exchange opportunity is with x. Then player x receives an added contribution of 1 to p_x. If y were connected to another bargaining network, then the contribution to p_x of the branch including y must be between 0 and 1. That is, player x does better than if y did not exist, but worse than if x could monopolise y. The non-overlapping condition ensures that the contribution of the y branch falls within these limits. However, this argument is only persuasive in a tree setting.

The GPI values are assumed to predict which player receives a greater payoff conditional on an exchange occuring. Only a qualitative prediction about outcomes is made in [45]: given that players x and y exchange, x receives a greater share of the payoff than y if and only if $p_x > p_y$. In [44], Markovsky et al state that in these cases:

"exchange outcomes approach maximum differentiation across positions, constrained only by the size of the resource pools"

The question of which exchanges form is addressed in other NET papers such as Lovaglia et al [38].

2.5.2 NET: Other Predictions

Various papers have pointed out networks for which versions of NET perform badly. For example Yamagishi and Cook [74] contains simulation results for two networks (including the stem network, a diagram of which is given in section 2.7) which GPI methods match poorly. These have prompted various modifications to NET. For example Markovsky et al [44] introduces an 'iterative extension' of GPI to deal with these networks.

An extension to NET which is particularly relevant to the results of this thesis is the concept of *weak and strong power*. This is introduced in Markovsky et al [44]. A summary of this theory is as follows. *Strong power* results in near-total payoff differentiation and is more characteristic of small sparsely connected networks. It is stated in [44] that the source of large differentiation is that:

"strong power structures exhibit a 'ratcheting' process whereby actors in structurally disadvantaged positions serially outbid one another..."

In Markovsky et al [44] strong power effects are predicted by an iterative version of GPI. Rules are given which classify players as having high and low strong power network positions based on their GPI values. Players in high and low strong power positions are predicted to receive payoffs of 1 and 0 respectively. When these GPI rules predict no strong power differences, weak power effects may result in mild payoff differentiation. Weak power is said to be more characteristic of large densely connected networks which are more typical of social relations. It is claimed in [44] that:

"Weak power differentials have the same microfoundation as strong power differentials: Actors seeking to avoid exclusion from exchanges accept deals... unfavorable to themselves."

Weak power is predicted by a method that gives a "probability of exclusion". Lovaglia et al [39] extend this weak power model to give quantitative predictions of payoffs.

Lovaglia et al [38] give the following heuristics on strong power as a simpler alternative to iterative GPI calculations. Note that a "relation" means an exchange opportunity, a "break" means an exchange opportunity which never forms in practice, and an "equal power structure" is one in which neither strong nor weak power effects exists and all exchanges which form involve an equal split of the available payoff. The comment in brackets is mine.

"i) Adding a relation between a low strong-power position and a high strong-power position does not change the type of power [i.e. high or low] of any position in the network.

ii) Adding a relation between two high strong-power positions does not change the type of power of any position in the network.

iii) Adding a relation between two low strong-power positions creates a weak or equal power structure.

iv) Adding a relation between weak or equal power positions cannot create a strong power structure.

v) Breaks occur between high strong-power positions or between high strong-power positions and equal or weak power positions, but not between equal or weak power positions."

In Lovaglia et al [40], the authors admit that the version of NET current at the time of writing (2001) typically produces poor predictions for large networks. They argue that a reason for this is because interior high strong power positions have significant chances of being excluded, and therefore payoffs do not reach maximum differentiation[21]. Also they suggest that as more players are included in networks,

[21]The experimental results in section 2.6 below illustrate that in odd length line networks, in which strong power is predicted by GPI, payoff differences become lower for longer lengths. The only players in such networks who are guaranteed to exchange are those with an exchange opportunity with a player at the end of the line.

there is a greater chance of at least one deviating from standard behaviour and thus disrupting the expected outcome.

2.5.3 Degree Dependence

A player's degree is defined to be the number of exchange opportunities they have. Lovaglia et al [39] state the following prediction, based on Marsden [46], which I refer to as *degree dependence*:

"The higher an actor's degree, the higher the actor's expected profit."

In Lovaglia et al [39], the following argument is presented as one possible explanation for this effect. Players are not fully rational and base their decisions on information which seems particularly salient to the situation at hand. This includes their degrees and the degrees of their neighbours. This argument seems plausible in the short-run for inexperienced players. It is less obvious whether it applies in the long term as players are able to learn about their bargaining opportunities in the network. Therefore whether experimental results support this effect is an interesting question.

2.6 Experimental Design

The experimental designs of laboratory studies of social exchange networks in the sociology literature vary considerably. Also, full details of the bargaining procedures used are not always given. However some features are almost always present. The experimental subjects participate in a number of rounds. In each round a subject is associated with each position in the network under investigation. The network does not change between rounds, although subjects are occasionally moved to other positions (e.g. Skvoretz and Willer [66]). In each round any subject may participate in at most one exchange. Every exchange opportunity is represented by a number of payoff points which can be split between the two subjects. At the end of the experiment subjects receive cash payments depending on the points they have won.

The number of rounds used in the experiments detailed in section 2.7 varies from 16 (for some networks in Skvoretz and Willer [66]) to 60 (Lucas et al [40]). Typically there are 24 points in each exchange opportunity. However Lucas et al [40] and one experiment in Lovaglia et al [39] use 30 points in each exchange opportunity, and Cook and Emerson [21] and Cook et al [22] contain some exchange opportunities with 24 points and others with only 8 points. The payment per payoff point is usually constant but sometimes increases in later rounds (e.g. Lucas et al [40]). Multiple sessions are played replicating the experiment with different subjects.

The remainder of this section details some of the variations in experimental design which occur. In a few experiments (e.g. Markovsky et al [45]) subjects bargain face-to-face, and choose their own bargaining process, under a few restrictions. However, typically subjects communicate through computer terminals. In this case, the experimenters must design a bargaining process, effectively choosing a non-cooperative game to model bargaining. The details of this process are not always fully described in the experimental papers. The features which are given are quite diverse. Some experiments (e.g. experiment 2 in Lovaglia et al [39]) require subjects to make simultaneous[22] demands, and exchanges form when demands are jointly feasible[23]. This is a similar approach to the Nash demand game (described in chapter 6). Some such experiments (e.g. Lucas et al [40]) also allow a 'second chance' bargaining round for subjects who do not exchange immediately. Other experiments (e.g. Bienenstock and Bonacich [1]) require subjects to repeatedly make offers to each other, and exchanges form when an offer is accepted. This is a similar approach to the alternating offers game (described in chapter 4). In some experi-

[22]Decisions are 'simultaneous' in this context if they are not revealed or acted on until everyone has made one. That is, the computer program waits until a decision has been received from everyone before allowing subjects to make further input.

[23]Sometimes many configurations of exchanges may be possible under this restriction. In experiment 2 of Lovaglia [39] the computer uses an exogenous rule to decide which exchanges form in such cases.

ments these offers must be made simultaneously and in others they may be made at any time. A fairly complicated system is sometimes used to decide acceptance, requiring several signals being sent between the subjects. This is especially necessary in the case where acceptance decisions are made simultaneously. Subjects are sometimes (e.g. Lovaglia et al [39]) restricted to only changing their offers by 1 point from that of the previous round.

The level of information given to subjects by experimenters differs. Indeed, the aims of some experiments (e.g. Lovaglia et al [39]) include investigating the effects of information. Others (e.g. Lucas et al [40]) restrict information in an attempt to avoid the use of fairness norms; subject preferences which depend not just on payoffs earned, but also on whether the payoffs to other subjects are perceived to be 'fair'. Some pieces of information that are withheld from subjects include: the actions of other subjects, the global structure of the network, and the payoffs of other subjects – sometimes even the payoffs of subjects' exchange partners were disguised.

The treatment of subjects outside the experiment also varies. Many papers (e.g. Cook and Emerson [21]) describe allowing the subjects to meet beforehand for instruction about the experiment. It can be argued that this may encourage the use of fairness norms and reputational effects; subjects acting as if they would meet the others again and playing to establish a good reputation. The rationale given by Cook and Emerson [21] for this is to reassure subjects that they are playing against humans not computer programs[24].

2.7 Experimental Results

This section summarizes experimental results from the sociology literature. As mentioned above, there is considerable variation in the experimental designs used to generate these results. The results are included only as a guide to the qualitative

[24]Indeed in Willer and Skvoretz [72], this method was used because subjects sometimes *were* playing against computer programs!

features they reveal in these networks, so only particularly striking variations in design are noted. However it is noted whether subjects are given 'complete' or 'limited' information[25]. This is done to illustrate that it is not obvious whether this choice has a powerful effect on the outcomes.

The level of detail to which experimental results are given in the corresponding papers varies. The results in this section have been taken from papers which at least give average *payoff splits* in most exchange opportunities. A payoff split is the average number of payoff points received by each player in an exchange opportunity conditional on that exchange forming. Some other papers only publish variables derived from this data. Even some of the results below have been slightly modified from the raw data (e.g. Markovsky et al [44]). Where it has been published, the frequency of each exchange is also included. Blanks in the tables below represent information which is not provided. Note that results for symmetric positions in networks are often aggregated. For example, for the 4 player line network discussed below most papers do not give the average payoff splits in each exchange but give the average of any split between an inner player – i.e. player 2 or 3 in figure 2.3 – and an outer player – i.e. player 1 or 4 – and the average of any split between inner players[26]. Another variation is in the rounds of play that average results are given for. Sometimes they are given over all rounds and sometimes (e.g. Lucas et al [40]) only over a final portion of rounds. The experiments below use cakes of 24 payoff points in all exchanges except where mentioned otherwise.

Most papers also include statistics on the distribution of the data around the mean values. This is used to check that the results are significant compared to various null hypotheses based on network position being unimportant. This is not included as these results are only used as a rough qualitative guide to behaviour in this thesis. Note that the issue of whether play has converged to a stable pattern which will survive for future rounds is not directly investigated by these experiments.

[25] Papers are not always precise in what they mean by these terms.

[26] This makes it difficult to assess the extent to which average payoffs are symmetric.

Most of the data in the tables below lies in average split columns. The heading of such a column gives the players involved in the split e.g. '1-2'. The data in this column is of the form 'x_1-x_2' where $x_1 + x_2 = 1$. These are the average proportions of the available payoff points that the players receive conditional on the exchange forming. Sometimes the heading contains two exchanges e.g. '1-2 or 3-4'. In this case the data of the column is also of the form 'x_1-x_2' where $x_1 + x_2 = 1$. This time x_1 is the average proportion of the available payoff points that player 1 or 3 receives in any exchange with player 2 or 4 respectively.

Star Networks

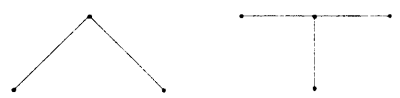

Figure 2.1: 3 and 4 player star networks

I refer to networks of at least 3 players with the property that all exchanges involve one common player as *star networks.* I refer to the common player as the central player and the others as the outliers. Both [40] and [64] contain experiments on a 3 player star network. The other papers listed in table 2.1 contain results for a 4 player star network. In [40], 30 point cakes and a limited information setting are used. A complete information setting is used in [66] and [64], but I am unsure about the remaining experiments. In [72] the central position was played by a computer program which always accepted the best offer made to it.

Stem Network

In [39] two experiments are performed. That labelled b) in table 2.2 uses 30 point cakes and has a limited information setting. That labelled a) uses 24 point cakes and a complete information setting, as do the remaining experiments listed in table 2.2.

Paper	Average centre-outlier split
[65]	0.793 – 0.207
[66]	0.901 – 0.089
[72]	0.988 – 0.012
[64]	0.665 – 0.335
[40]	0.832 – 0.168

Table 2.1: Star network results

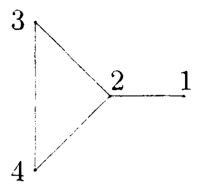

Figure 2.2: Stem network

Paper	Average split			Number of
	2–1	2–3 or 2–4	3–4	2–3 or 2–4 exchanges
[66]	0.637 – 0.363	0.687 – 0.313		8 (of 64 possible)
[44]	0.601 – 0.399	0.639 – 0.361		
[39] a)	0.663 – 0.337	0.583 – 0.417		2
[39] b)	0.671 – 0.332	0.582 – 0.418	0.606 – 0.397	"infrequent"

Table 2.2: Stem network results

4 Player Line

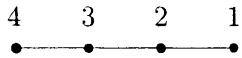

Figure 2.3: 4 player line network

The experiment of [40] uses a limited information setting and 30 point cakes. A complete information setting is used in [66] and [39]. It is unclear what the informational assumptions of the remaining experiments listed in table 2.3 are. In [72] the central positions, 2 and 3, were played by a computer programs which always accepted the best offers made to them and sometimes made offers of 12 payoff points to each other.

Paper	Average split		Frequency of 2 − 3 exchange
	2 − 1 or 3 − 4	2 − 3	
[65]	0.522 − 0.478		
[1]	0.597 − 0.403	0.517 − 0.483	0.16% [27]
[66]	0.585 − 0.415		0.18
[39]	0.600 − 0.400		
[72]	0.542 − 0.458		
[40]	0.647 − 0.337	0.501 − 0.499	0.11

Table 2.3: 4 player line network results

5 Player Line

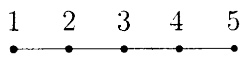

Figure 2.4: 5 player line network

[27] This seems surprisingly low. Possibly the % symbol in [1] is a typographical error.

The experiment of [40] uses a limited information setting and 30 point cakes. A limited information setting is also used in [22], whereas [64] uses complete information. The network of [22] is not strictly a 5 player line; it also contains an exchange opportunity for players 1 and 5 to split 8 payoff points. This exchange was rarely used.

	Average split	
Paper	2–3 or 4–3	2–1 or 4–5
[22]	0.556 – 0.444	0.600 – 0.400
[64]	0.608 – 0.392	0.640 – 0.360
[40]	0.831 – 0.169	0.879 – 0.121

Table 2.4: 5 player line network results

In [64] the frequencies of each exchange are also recorded. The other experiments listed in table 2.4 do not give any data about the frequency of exchanges.

Exchange	12	23	34	45
Frequency	0.70	0.29	0.38	0.62

7 Player Line

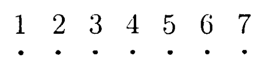

Figure 2.5: 7 player line network

Two experiments were carried out on this network. In [64], 40 rounds of play are used whereas [40] uses 60. The outcomes in [40] are from the last 20 rounds of play, whereas [64] does not mention whether or not its results are similarly taken from the later part of the experiment. As before, [40] uses 30 point cakes and limited information and [64] has complete information.

Paper	Average split		
	2-1 or 6-7	2-3 or 6-5	4-3 or 4-5
[64]	0.581 –0.419	0.582 –0.418	0.523 – 0.473
[40]	0.792 –0.208	0.745 –0.255	0.708 –0.292

Table 2.5: 7 player line network results

In [64], the frequencies of each exchange are also given:

Exchange	12	23	34	45	56	67
Frequency	0.73	0.25	0.47	0.50	0.25	0.74

In [40], average payoffs for each position are given. The following table presents these as a proportion of the maximum available payoff.

Player	1	2	3	4	5	6	7
Payoff propotion	0.260	0.750	0.279	0.708	0.296	0.807	0.187

Other Networks

In [64] experiments are carried out on several other networks. Partial results of one network, the "Strong4" network, are given here as they lend some support to the degree dependence hypothesis. This experiment uses complete information.

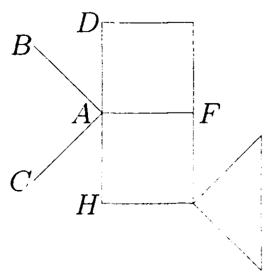

Figure 2.6: Strong4 network

Exchange opportunity	Average split	Frequency
BA or BC	0.874 -0.126	BA: 0.49 BC: 0.36
BD	0.815 -0.185	0.05
BF	0.812 -0.188	0.03
BH	0.914 -0.086	0.06

Table 2.6: Strong4 results

2.8 Discussion

The most clear-cut experimental results are those of Lovaglia et al [40]. That is, these have the greatest relative differences between average payoffs for different players. Producing such results was a deliberate aim of this paper. Two features of the design which were intended to aid this are that a larger than typical number of experimental rounds is used with results only taken from the final third of rounds, and that limited information is used. The motivation for limiting information is to avoid fairness norms.

The experimental data given above provides some support for the degree dependence hypothesis. Positions A B and C in the Strong4 network studied in Skvoretz [64] form a 3 player star, but player A also has other neighbours. Player A does better than the central player in any star network experiment, including those carried out in the same paper under the same experimental design[28].

The experiments do not give any clear indications about the effect of limiting information. Lovaglia et al [39] contains two experiments on the stem network under different informational settings and finds no significant variations. However, there is a lot of variation between some results above and the available evidence does not allow a strong view to be taken on whether information has a significant effect.

[28]The only variation in experiment desgin mentioned in [64] is that 12 sessions of 30 rounds are used for the 3 player line and 8 sessions of 32 rounds are used for the Strong4 network.

In the networks for which NET predicts strong power, such as odd length lines, high and low strong power players do not receive payoffs of 1 and 0 as predicted. Payoff differences between high and low power players (respectively even and odd numbered players in odd length line networks) seem smaller for larger networks, for example for the 7 player line in comparison to the 3 player line.

Finally, the experiments do not give unconditional support to symmetric outcomes forming. The experiment of [39] on the stem network, which is labelled b) in the table above, the average payoff split between players 3 and 4 is significantly unequal.

Chapter 3

Mathematical Preliminaries and Definitions

This chapter contains definitions which are repeatedly used later and summarizes relevant mathematical background material. Section 3.1 defines bilateral exchange networks. These are mathematical descriptions of the negatively connected social exchange networks discussed in the previous chapter, and form the main focus of study in this thesis. Section 3.2 is comprised of relevant graph theoretic definitions. Section 3.3 contains background material on game theory. Appendix 3.4 develops the game theory material more formally. This level of formality is required for various results, but for the purpose of clarity is only used in appendices and footnotes. The appendix also presents a theorem of Harris on the existence of subgame perfect equilibria. A corollary is proved showing existence for a class of games commonly used in this thesis.

3.1 Bilateral Exchange Networks

Section 3.1.1 contains the definition of a bilateral exchange network, and other related definitions. Section 3.1.2 discusses how these definitions represent the negatively connected networks described in section 2.4. This principally involves a discussion of the use of utility functions. Finally, section 3.1.3 defines notation for some example networks which are often used.

3.1.1 Definitions and Notation

Definition 3.1. $\mathbb{R}^+$ is the non-negative real interval $[0, \infty)$.

Definition 3.2. A (2 player) *utility cake* is a compact convex non-empty subset of $\mathbb{R}^{+2}$ which allows free disposal i.e. if (a, b) is contained then so is every (c, d) such that $c \leq a$, $d \leq b$.

Let $\mathcal{K}^*$ be the set of all utility cakes.

Definition 3.3. A bilateral exchange network is a triple $N = (P, E, K)$, where P is a finite set of *players*, E is a set of *exchange opportunities*, which are unordered pairs of distinct players, and $K : P \times P \to \mathcal{K}^* \cup \{\emptyset\}$ is a *utility cake function* satisfying

1. $K(a, b) = \emptyset$ if and only if $(a, b) \notin E$.

2. $K(b, a) = \{(\sigma_b, \sigma_a) \mid (\sigma_a, \sigma_b) \in K(a, b)\}$

The set P is the set of distinct bargainers. Since an aim of this thesis is to represent the process of bargaining as a game, P is referred to as the set of *players*. For simplicity, P takes the form $\{1, 2, 3, \ldots, n\}$ unless specified otherwise.

An exchange opportunity represents a pair of players who have the possibility of forming an exchange[1]. An exchange opportunity (a, b) is often referred to by the shorthand ab and $K(ab)$ by $\mathcal{K}^{ab}$, where this will not cause confusion. The utility cake function K maps two players with an exchange opportunity to the set of feasible von Neumann-Morgenstern utility pairs for that coalition[2]. In the expression $(\sigma_a, \sigma_b) \in \mathcal{K}^{ab}$, σ_x refers to the utility of player x. Thus condition 2 above means that $\mathcal{K}^{ab}$ and $\mathcal{K}^{ba}$ effectively refer to the same set of utility pairs.

[1]In the terminology of economic theory, E represents the set of feasible coalitions.

[2]Section 3.1.2 discusses the use of von Neumann-Morgenstern utility functions.

Note that E is not strictly necessary in the definition above. It could be defined in terms of K and P as[3] $\{(a, b) \mid a, b \in P \text{ and } K(a, b) \neq \emptyset\}$. However, it is convenient to include E. For example, this allows a network to be defined by stating P, E and a single utility cake which applies to all exchange opportunities.

A cake $\mathcal{K}^{ab} = \{(0, 0)\}$ can be interpreted as showing that the only possible interactions between a and b are non-profitable. It seem intuitively obvious that the outcome of a bargaining situation should be robust to whether or not any such opportunities exist[4]. The possibility of such cakes is only included in the definition because they *do* affect the outcomes of some later bargaining models[5], indicating that they are not robust in this sense[6].

Definition 3.4. For $ab \in E$, the *boundary function* $f^{a,b} : [0, M^{ab}] \to \mathbb{R}^+$ is given by

$$f^{a,b}(x) = \max\{y \mid (x, y) \in \mathcal{K}^{ab}\}$$

where $M^{ab} = \max\{M \mid (M, 0) \in \mathcal{K}^{ab}\}$.

Recall that utility cakes are compact. Thus the sets used in this definition have maximum elements as required.

As a shorthand for the composition of boundary functions, let

$$f^{a,b,c,\ldots,y,z}(\sigma_a) = f^{a,b} o f^{b,c} o \ldots o f^{y,z}(\sigma_a)$$

Note that the domain of such a function may be empty[7].

[3] Also, it would be necessary to replace condition 1 on K with $K(a, a) = \emptyset$.

[4] This assumes that transmission of information does not occur through these interactions. This possibility is outside the scope of this thesis and the bargaining models studied do not include any mechanisms for such information transfer.

[5] See section 4.4.4 for example.

[6] The same results can be usually be found using cakes containing only utilities of less than some sufficiently small ϵ. Sometime it is also necessary to take the limit $\epsilon \to 0$. Cakes of the form $\{(0, 0)\}$ simply permit straightforward examples.

[7] For example suppose $f^{1,2}(x) = f^{3,1}(x) = 1 - x$ and $f^{2,3}(x) = 3 - x$. The respective domains of these functions are $[0, 1]$, $[0, 1]$ and $[0, 3]$. The function $f^{1,2,3}(x) = 2 + x$ has domain $[0, 1]$ and range $[2, 3]$, which does not intersect the domain of $f^{3,1}$. Thus the domain of $f^{1,2,3,1}$ is empty.

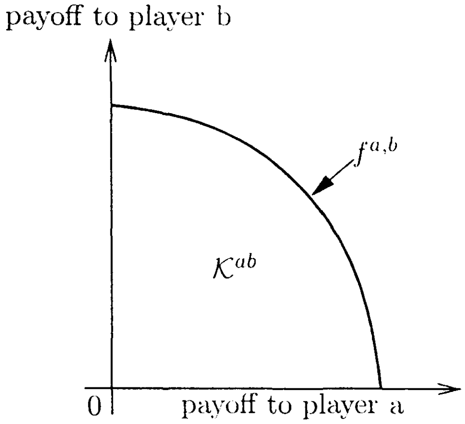

Figure 3.1: A utility cake and its boundary function

Stating a boundary function defines a corresponding utility cake. In fact, giving an extension of a boundary function – e.g. some polynomial $f^{a,b} : \mathbb{R} \to \mathbb{R}$ – is usually the easiest way to define a utility cake[8].

Definition 3.5. The *m-unit cake* is generated by the boundary function $m - x$. The (one) unit cake is written as $\mathcal{K}_{\text{unit}}$. The unit cake function is $K_{\text{unit}}(ab) \equiv \mathcal{K}_{\text{unit}}$.

Such cakes are referred to as m-unit cakes since they correspond to situations where 2 players have an opportunity to split m units of utility.

Definition 3.6. The *outer boundary* of a utility cake $\mathcal{K}^{ab}$ is the set

$$\{(x, y) \in \mathcal{K} \mid x' > x \text{ and } y' > y \Rightarrow (x', y') \notin \mathcal{K}^{ab}\}$$

Note that the outer boundaries of $\mathcal{K}^{ab}$ and $\mathcal{K}^{ba}$ represent the same set of utility pairs for players a and b.

Definition 3.7. The cake $\mathcal{K}^{ab}$ is said to be *insatiable* if $f^{a,b}$ and $f^{b,a}$ are strictly monotonic.

[8]In this case the domain of the actual boundary function should be taken to be $[0, r]$ where $r = \min\{x \geq 0 | f^{a,b}(x) = 0\}$.

An equivalent condition is that $f^{a,b,a}$ is the identity. A consequence of the definition is that the outer boundary of $\mathcal{K}^{ab}$ contains no straight line segments.

An interpretation of this condition is as follows. For any pair $\sigma = (\sigma_a, \sigma_b) \in \mathcal{K}^{ab}$ such that $\sigma_a > 0$, there exists another pair $(\lambda_a, \lambda_b) \in \mathcal{K}^{ab}$ such that $\lambda_a < \sigma_a$ and $\lambda_b > \sigma_b$. Thus from any bargaining outcome σ as described, player a has an available *concession*; a utility pair which reduces his own share of the cake and increases that of player b.

Definition 3.8. An *outcome* for a bilateral exchange network $N = (P, E, K)$ is a pair $o = (F, q)$ where $F \subseteq E$ is the set of *realised exchanges* and $q : P \to \mathbb{R}^+$ is the *share function.* An outcome is *feasible* if it satisfies the following conditions:

1. A player may only exchange once. i.e.

$$ab \in F \text{ and } ac \in F \Rightarrow b = c$$

2. The shares of any pair of exchanging players must be in their utility cake. i.e.

$$ab \in F \Rightarrow (q_a, q_b) \in \mathcal{K}^{ab}$$

3. Non-exchanging players receives zero shares. i.e.

$$ac \notin F \ \forall c \in P \Rightarrow q_a = 0$$

A value $x \in \mathbb{R}^+$ is said to be *feasible from i to j* for some players i and j if $ij \in E$ and $x \leq f^{i,j}(0)$.

3.1.2 Utility Theory

To make any investigation into players' behaviour in a bargaining situation it is necessary to know something about their preferences over the possible outcomes. This section briefly discusses how assumptions about preferences and other considerations lead to the situations described in section 2.4 as negatively-connected networks being represented by bilateral exchange networks.

It is assumed that each player has a transitive and complete preference relation over all possible outcomes of bargaining[9]. A player's preferences can then be represented by a *utility function* assigning a real number to each outcome. One outcome is strictly preferred to another if and only it has a higher associated utility.

A two player bargaining problem can then be represented by a set $\mathcal{K}$ of utility pairs for all possible outcomes. However, note that many different representations are possible, as utility functions under the definition above only represent *ordinal* preferences. Under the assumption of a non-discrete set $\mathcal{K}$, it can be shown that a reasonable resolution of the 2 player bargaining problem cannot be based on ordinal utility functions alone (see section 4.3.2 of Shubik [63] for example). A non-discrete utility cake seems desirable in a social exchange setting because the intensity of player's actions may vary continuously, and in an economic setting because contracts may allow outcomes constructed as lotteries over other outcomes.

To make any progress on analysis of general a bargaining problem it is necessary to introduce more preference structure. One resolution is to use cardinal *von Neumann-Morgenstern utility functions.* These also require players' preferences on lotteries of outcomes to be specified. That is, they encapsulate attitudes toward risk. Various extra axioms on preferences over lotteries must be satisfied (see Myerson [52] for example). The result is a utility function in which a player's utility of a lottery is equal to the expectation of their realised utility value in that lottery. Another important property is that von Neumann-Morgenstern utility functions are unique up to positive affine transformations[10]. In this thesis, players' von Neumann-Morgenstern utility scales are normalised so that a payoff of zero corresponds to the payoff of not taking part in an exchange[11].

[9]These assumptions and some of the other axioms of von Neumann-Morgenstern utility mentioned below can be criticised on experimental grounds. For example, see section 1.7 of Myerson [52] for a summary of some experimental results.

[10]A transformation of the form $x \mapsto \alpha x + \beta$ where $\alpha > 0$.

[11]It is assumed that players are indifferent between all outcomes in which they do not exchange.

Consider the conditions on a utility cake given in definition 3.2. Utility cakes are assumed to be in $\mathbb{R}^{+2}$ by only considering outcomes such that no player strictly prefers not to exchange. Utility cakes are assumed to be non-empty because otherwise there is no opportunity for an exchange which both players view as at least as good as not exchanging. In an economic context, free disposal corresponds to allowing players to sign contracts agreeing to 'throw away' some of the proceeds of exchange e.g. by burning money. In the context of social exchange, as mentioned above, the intensity of player's actions may vary continuously. This goes some way towards generating free disposal[12]. Utility cakes are assumed to be convex by allowing any outcome which is a lottery over other outcomes and applying the first property of von Neumann-Morgenstern utility mentioned above. Utility cakes can be defined as the minimal sets satisfying the above properties and containing a finite set of points corresponding to the 'basic outcomes' of bargaining. Under this definition the cakes also satisfy compactness.

Consider a negatively connected network in the sense of section 2.4. Assume that each player's utility depends only on his own exchange. That is, a player is indifferent between different global outcomes in which he makes the same exchange on the same terms. This is reasonable if players have little knowledge of how their exchange may affect the pattern of exchange elsewhere in the network[13]. The set of outcomes of the network can then be represented in terms of the utility cakes representing the possible outcomes of each exchange opportunity. This generates the definitions of a bilateral exchange network and the outcome of such a network

[12]Free disposal is not just included in the list of conditions for convenience. It is necessary for some later results such as that of footnote 20 of chapter 4. Also, note that insatiability (see definition 3.7), convexity and the inclusion of $(0,0)$ in a utility cake imply free disposal in any case.

[13]In small networks this may be unlikely. For example consider a four player line network with exchange opportunities 12, 23, 34 (L_4 in the notation of section 3.1.3). If players 2 and 3 know the structure of the network then they know that in the case that they form an exchange with each other, players 1 and 4 receive utilities of zero. Should players 2 and 3 have some preference for 'fairness', their utility in the exchange 23 is reduced.

used in section 3.1.1.

3.1.3 Example Networks

This subsection defines networks which are used later. Recall from section 3.1.1 that $P = \{1, 2, \ldots, n\}$.

Definition 3.9. An n-player *line network* satisfies $E = \{12, 23, \ldots, (n-1, n)\}$. The *$n$-player unit line network* L_n also satisfies $K = K_{\text{unit}}$.

Definition 3.10. An n-player *ring network* satisfies $E = \{12, 23, \ldots, (n-1, n), n1\}$. The *$n$-player unit ring network* R_n also satisfies $K = K_{\text{unit}}$.

Definition 3.11. For $n \geq 3$, an n-player *star network* satisfies $E = \{1k | 2 \leq k \leq n\}$. Player 1 is called the *central player*, and the others are called *outliers*.

Note that a 3 player line network is a star network.

Definition 3.12. A network is *bipartite* if P can be partitioned into *sides* P_1 and P_2 such that all exchanges $ab \in E$ contain one player from each side.

Note that $2n$-player ring networks and all line and star networks are bipartite.

3.2 Graph Theoretic Definitions

Definition 3.13. A *graph* is a pair (V, E) where V is a set of *vertices* and E is a set of *edges*, pairs of distinct elements of V. A graph is said to be *directed* if the pairs in E are ordered and *undirected* if not.

In all the graphs considered in this thesis V is a finite set. An edge (a, b) is often written as ab where this will not cause confusion. A *subgraph* of (V, E) is any graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. The subgraph of (V, E) *induced* by $W \subseteq V$ is (W, F) where $F = \{ab \in E \mid \{a, b\} \subseteq W\}$.

Given a graph (V, E), $b \in V$ is said to be a *neighbour* of $a \in V$ if $ab \in E$ or $ba \in E$. The number of neighbours of $v \in V$ is called its *degree*.

A *walk* in graph (V, E) is a sequence $(v_i)_{0 \leq i \leq n}$ such that $v_i v_{i+1} \in E$. A *path* in graph (V, E) from a to b is a walk $(v_i)_{0 \leq i \leq n}$ such that $v_0 = a$, $v_n = b$ and all vertices in the sequence are distinct with the permissible exception that v_0 may equal v_n. In the latter case if also $n > 0$ then the path is called a *cycle*.

An undirected graph is said to be *connected* if there is a path between every distinct pair of vertices. A *connected component* of an undirected graph is a maximal connected subgraph. That is, it is a connected subgraph S such that there exists no distinct connected subgraph S' such that S is a subgraph of S'. Any graph may be partitioned into connected components.

An undirected graph is called a (undirected) *tree* if it is connected and contains no cycles. A directed graph $T = (V, E)$ is called a (directed) tree if contains no cycles and there exists a *root* $r \in V$ such that there is a path in T from r to every other node in V.

3.3 Game Theoretic Definitions and Results

Section 3.3.1 informally defines a game. The formal details are contained in section 3.4.1 in the appendix to this chapter. Section 3.3.2 contains some other game theoretic material. Again, many of the formal details are relegated to the appendix, mainly section 3.4.2. Section 3.3.3 describes how games can represent the outcome of bargaining in bilateral exchange networks. Section 3.3.4 discusses subgame perfect equilibrium (SPE), the main game theoretic solution concept used in chapters 4 and 5. It also contains the statement of an existence result for SPEs which applies to most of the games considered in these chapters. The proof is contained in section 3.4.3 in the appendix, and is a corollary of a SPE existence theorem of Harris [33].

3.3.1 A Summary of the Definition of a Game

There is a finite set P of *players*. The set of *periods* is given by the non-zero natural numbers $\mathbb{N}^+$. In each period, every player chooses some *action* independently of the other players. A vector of actions for all players is called an *action profile.*

An *infinite history* of the game is a sequence of action profiles for each period and also a 'zero period', which is included for notational convenience. The definition of a game is based on the set of possible infinite histories of the game, H^∞.

A *finite history* is a subsequence of the first n elements of any infinite history for any n. Finite histories are often referred to below simply as histories, since they are discussed more often than infinite histories.

Given a finite history of length n, consider the set of action profiles which can be appended to this sequence to produce a valid finite history of length $n+1$. Since players choose their actions independently, this set factorises into sets of actions for each player called *action sets.*

A (pure) *strategy* for a player maps from a finite history to an action in the corresponding action set. A vector of strategies for each player is called a *strategy profile.* A strategy profile specifies a unique infinite history of the game which results if the players choose actions according to these strategies[14].

A *payoff function* π maps from a infinite history to a vector containing a *payoff* in $\mathbb{R}$ for each player. The payoff for player i is written π_i. In this thesis, these values typically represent utilities available to players in utility cakes and thus lie in $\mathbb{R}^+$.

A *game* is a pair (H^∞, π) as described above. A game of *perfect information* is one such that given any finite history, at most one action set is not a singleton. That is, only one player may take a non-trivial action in each period.

3.3.2 Further Game Theoretic Terms

A *mixed strategy* maps from a finite history to a probability distribution over the corresponding action set. These are rarely used in this thesis.

The *decision function* D maps from a finite history to the player whose action set is not a singleton. In the case where no such player exists $D(h)$ can be defined

[14] A technical condition is introduced in section 3.4.1 to ensure that this is the case.

as $\emptyset$.

In a game of perfect information, a history can effectively be represented by the sequence of actions made in each period by the player with a non-singleton action set (no entries need be made for periods where no such player exists). The action profile at each period can be inferred from this, as described in section 3.4.2. This is usually the most convenient method of representing histories.

It is often useful to allow *terminal* finite histories. Such a history has an associated payoff and empty action sets; the game is over once a terminal history is reached. Section 3.4.2 shows that the above definition of a game allows a method of expressing terminal finite histories. In the remainder of the thesis (except the appendix of this chapter) games are often defined assuming terminal histories are possible without reference to the details of this method. In such a definition payoffs must be given for both terminal and non-terminal infinite histories. In making these definitions, the phrase 'infinite history' is often used as a shorthand for 'non-terminal infinitie history', where this does not cause confusion.

A 2×2 game is a 2 player game such that both players have two actions in their action set in the first period and all one period histories are terminal. The payoffs of such a game can be represented by functions $p_i(x_1, x_2)$ giving the payoff to player i if players 1 and 2 make actions x_1 and x_2 respectively in the first period. A 2×2 game is *symmetric* if both players have the same action set in the first period and $p_1(x_1, x_2) = p_2(x_2, x_1)$ for all actions (x_1, x_2). A definition of more general symmetric games can be made similarly.

Given a finite or infinite history x, a *subhistory* is a subsequence of the first n elements of x for any n less than the length of x. A subhistory is a finite history.

Given a game $\mathcal{G}$ and a non-terminal finite history x of length t, the *subgame* $\mathcal{H}$ of $\mathcal{G}$ generated by x is (informally) constructed as follows. Take all histories of $\mathcal{G}$ with x as a subhistory, and delete the first t terms. Use the resulting set as the set of histories of $\mathcal{H}$. Payoffs, strategies and other terms must be defined accordingly for $\mathcal{H}$. The subgame $\mathcal{H}$ represents a game beginning after the history x has taken place. A more formal definition is given in section 3.4.2.

A game with *random moves* is as follows. Following certain finite histories an action is taken at random according to a fixed probability distribution rather than by a player. Introducing random moves complicates the proof of corollary 3.1. However the only games with random moves analysed in this thesis[15] have a simple form so that this result is not required. Some more complicated games with random moves are mentioned in passing.

Given a pure strategy profile f for a game $\mathcal{G}$ without random moves, a (pure) *best reply* (or best response) to this profile for player i is any pure strategy for player i which maximises the payoff to player i when all other players play according to f. If f contains mixed strategies or the game includes random moves then a (pure) best reply to f for player i is any pure strategy which maximises the expected payoff to player i when all other players play according to f. A best reply for i can also be given if a strategy is only specified for every player other than i.

A *Nash equilibrium* of a game is a strategy profile f satisfying the following property. No player can increase their payoff by deviating from f while all other players play according to f. A *strict* Nash equilibrium is a strategy profile in which any player receives a lower payoff by deviating in this way. A *subgame perfect equilibrium* (SPE) of a game is a strategy profile satisfying the following stronger property. *In any subgame* no player can increase their payoff by deviating from f while all other players play according to f.

Definition 3.14. Suppose $\mathcal{G}(\epsilon)$ is a family of games indexed by parameter ϵ. A *limiting SPE payoff* of this family under the limit $\epsilon \to \epsilon^*$ is a vector $p = (p_i)_{i \in P}$ with $p_i \in \mathbb{R}$ satisfying the following property. For any sequence $(\epsilon_j)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} \epsilon_j = \epsilon^*$, there exists a sequence $(p^j)_{j \in \mathbb{N}}$ such that there is a SPE of $\mathcal{G}(\epsilon_j)$ with payoff vector p^j and $\lim_{j \to \infty} p^j = p$.

[15] That is, the extensions of the Nash demand game defined in chapter 6.

3.3.3 Bargaining Games and Models

A *bargaining game* on a network $N = (P, E, K)$ is a game with players P such that each terminal and infinite history of the game is associated with a feasible outcome of N as well as a payoff. For any terminal or infinite history h of the game, $q_i(h)$ represents the share of player i in the corresponding feasible outcome. The payoff of each player i must satisfy $\pi_i(h) \leq q_i(h)$. This bound ensures that the feasibility constraints of definition 3.8 apply to payoffs as well as shares. Associating an outcome with terminal and infinite histories also allows discussion of which exchanges form in such histories.

A share represents the utility a player places on a particular exchange agreement. A payoff represent the utility also taking into account the cost of participating in the bargaining process. For example an agreement reached immediately would typically be preferred to the same agreement reached after a lengthy period of bargaining. A terminal or infinite history has both shares and payoffs defined for later convenience.

A *bargaining model* is a function which maps a bilateral exchange network and certain extra structure to a bargaining game. Such a model represents specific rules for bargaining which can apply to many networks[16]. The extra structure is split into two parts. *Endogenous structure* is an integral part of the bargaining situation which is not described by the bilateral exchange network[17] (e.g. discount factors representing the time preferences of players). *Exogenous structure* is not an integral part of the bargaining situation but is necessary to provide a well defined game (e.g. specification of a first mover).

The division between exogenous and endogenous structure is subjective. Indeed, so is the division between exogenous structure and the model itself; a complete specification of the bargaining model could be included in the exogenous structure!

[16]This is similar to Muthoo's notion of a 'procedure' in [51].

[17]Typically endogenous structure contains only functions with domain P or E i.e. properties of individual players or exchange opportunities.

The choice of where to draw the line is a modelling choice to aid in interpretation. Section 4.4.1 on the market bargaining game provide an example of the usefulness of endogenous and exogenous structure.

In this thesis bargaining games are represented by script letters – e.g. $\mathcal{G}$ – whereas bargaining models are represented by plain text e.g. M. Note that a model and its arguments also represents a bargaining game e.g. $M(N, E, X) = \mathcal{M}$.

3.3.4 Subgame Perfect Equilibrium

SPE is the usual solution concept used for games of perfect information. The usual motivation for its use is as follows (see Binmore [5] or Fudenberg and Tirole [30] for more details). Nash equilibrium is a necessary requirement for a strategy profile to represent a stable solution of any game. Given any other strategy profile there exists a player who would prefer to unilaterally deviate from it. However some games possess multiple Nash equilibria, and some seem more plausible as solutions of the game than others. The results in an *equilibrium selection problem*. One reason for some Nash equilibria being less plausible is that they allow players to make 'incredible threats'. An example of this in a 2 player bargaining situation is that one bargainer may make an initial demand and threaten to never exchange should it be refused. Should this bargainer be put in the position where he must carry out this threat then it is clearly not in his own interest to do so. However, in an appropriate game modelling 2 player bargaining (e.g. the alternating offers game of section 4.2) a Nash equilibrium can be constructed in which one player uses a strategy corresponding to this incredible threat and the other accepts the initial demand. The motivation for SPE is to avoid incredible threats by requiring strategies to form Nash equilibria of every subgame. In the example just given this rules out the threat of never exchanging.

Note that for a game in which a terminal history is always reached within a finite number of a periods, an equivalent definition of SPE to that given in section 3.3.2

is as follows. In any subgame no player can increase their payoff by unilaterally deviating to another *action* in any single period. In this thesis, this distinction is of little conceptual importance and can usually be neglected except in proofs[18].

The existence of SPEs is straightforward to prove for games of perfect information in which all infinite histories have a terminal subhistory and there are a finite number of terminal histories[19] using Zermelo's algorithm (see theorem 3.2 of Fudenberg and Tirole [30] for example). This condition does not hold for many of the bargaining games in chapters 4 and 5 of this thesis. In this case, the existence of SPEs is a more complicated issue. Indeed it is easy to define games in which no SPE exists[20]. For most games in chapters 4 and 5, a SPE existence theorem of Harris [33] resolves this issue. Bargaining literature generally does not require such an existence theorem. Instead, a typical resolution is as follows. Existence is assumed, and some properties of a SPE are deduced – e.g. a unique SPE outcome is found. These properties are then used to construct a simple strategy profile – e.g. a stationary strategy profile – which can easily be verified to be a SPE. This method does not always suffice in this thesis because in some cases it is not straightforward to explicitly construct example SPEs (see section 5.4.7 for example).

For several bargaining games used later[21], the method just described does suffice. Section 3.4.3 of the appendix to this chapter defines a class $\mathcal{E}$ which captures the remaining bargaining games of perfect information which are used below. The following corollary of Harris' theorem is then proved:

Corollary 3.1. *There exists a SPE for all games in the class* $\mathcal{E}$.

[18]In the proofs of section 5.4, lemma 5.8 is used to take care of the technicalities relating to this distinction.

[19]That is, games of perfect information with a finite set of infinite histories.

[20]For example, consider the one-player game in which a player must choose an element $x \in [0, 1)$ and receives payoff x.

[21]That is, the alternating offers game, the telephoning model and Herrero's model.

3.3.5 Evolutionary Game Theory

Non-cooperative game theory typically concentrates on finding the equilibria of games, especially Nash equilibria. This raises the positive question of whether these equilibria describe how players actually play the game and if so, how players concentrate on an equilibrium. One traditional answer is that players are rational and concentrate on an equilibrium through a process of introspection based on common knowledge of the details of the game[22]. This is problematic for several reasons. Firstly, common knowledge of the details of the game seems a heavy requirement[23]. Secondly, finding an equilibrium through introspection may be a very difficult task[24]. Thirdly, in a game for which multiple equilibria exist, a mechanism must be provided for all players to coordinate their play upon a particular equilibrium. Another difficulty is the equilibrium selection problem mentioned in the previous section. *Equilibrium refinements* strengthen the conditions of Nash equilibrium in an attempt to select the more plausible equilibria. Subgame perfect equilibrium is an example. However a large number of refinements exist in the literature (e.g. see chapter 5 of Myerson [52]) and it is often difficult to decide which one is appropriate, especially when intuitive insight into the situation being modelling is hard to come by.

Evolutionary game theory offers a different approach to the questions mentioned above. This posits a process where players repeatedly play a game and use trial and error methods to decide what strategies to play. Strategies which earn players higher payoffs flourish and eventually a stable pattern of play may emerge. This

[22]See chapter 1 of Fudenberg and Tirole [30] for a development of this argument.

[23]In particular, in the case that mixed strategies are considered and von Neumann-Morgenstern utilities are used, these details must include the preferences of all players over all lotteries of terminal or infinite histories of the game.

[24]For example, two player zero-sum games are guaranteed a unique SPE outcome (see section 3.8 of Myerson [52] for example) but for complicated many games in this class (e.g. chess) this outcome is not known and it is certainly not the case that players always coordinate on it.

setting can be formalized mathematically as a dynamical system.

Simple trial and error methods or *learning rules* have the advantage that they do not necessarily require a large amount of information about the game. In a social setting they are also attractive because they capture the intuition that people have limited cognitive resources to apply to a large number of decisions and thus often use simple heuristics. A problem is which learning rules to use. This suggests the application of psychological results and the search for results which are robust for many learning rules.

Evolutionary game theory typically postulates a large population of players for player position in the underlying game. One approach is to study the expected behaviour of these populations under particular dynamics. This generates deterministic population equations, such as the replicator dynamics. See Weibull [70] for a survey of results following this approach. Another approach is to study models retaining stochastic features[25].

Evolutionary game theory provides some support to the equilibrium concepts mentioned at the start of this section. For example see sections 3.3 and 5.2 of Weibull [70] on the connection between Nash equilibria and the stationary states of the replicator dynamics, or section 4.2 of Samuelson [57] for more general dynamics. Also, some stochastic models provide methods of selecting between multiple equilibria in certain settings. For example see Binmore et al [8] and Kandori et al [37]. The approach of the latter forms the basis for the evolutionary model of chapter 6 and is discussed in more depth in section 6.1. Also, for some situations evolutionary game theory offers an explanation for the departure of behaviour from that predicted by equilibrium concepts. For example Gale et al [31] and Seymour [60] investigate evolutionary models of the ultimatum game. This game has a unique SPE but it is not supported by evidence from laboratory experiments, whereas the papers mentioned

[25]Many papers investigate the connection between these two approaches, for example Binmore et al [8] and Seymour [61].

contain predictions from evolutionary models which are a close match.

3.4 Appendix: Formal Game Theoretic Details

Section 3.4.1 contains the definition of a game used by Harris in [33]. Section 3.4.2 contains brief note on the application of this definition and makes some further definitions in this setting. Section 3.4.3 contains Harris's theorem on SPE existence and proves corollary 3.1 showing existence for a class of games used in this thesis.

3.4.1 Full Definition of a Game

This section is based on the setting used by Harris in [33]. Some terms are renamed for later convenience. In particular Harris's use of 'history' is replaced with 'infinite history' to allow 'history' to refer to finite subhistories, since these are most often under discussion outside this section.

The definition is based on the set of infinite histories of the game, H^∞. For convenience in stating theorem 3.2, this is embedded in a larger product space S, which is defined as follows.

There is a finite set P of *players*, indexed in this appendix by i. The set of periods is given by the non-zero natural numbers $\mathbb{N}^+$, indexed here by t. In each period t, each player i chooses, independently of the other players, some action which can be represented by an element of S_{ti}. The outcome of each period can therefore be represented by an element of

$$S_t = \prod_{i \in P} S_{ti}$$

Play begins in period 1. An *infinite history* of the game can be represented as an element of

$$S = \{0\} \times \prod_{t \in \mathbb{N}^+} S_t$$

where $\{0\}$ is included for later notational convenience.

Any $x \in S$ can be written as $x = (x_t)_{t \in \mathbb{N}}$ where $x_0 = 0$, and $x_t \in S_t$ for $t \in \mathbb{N}^+$. Any $x_t \in S_t$ can be written as $x_t = (x_{ti})_{i \in P}$ where $x_{ti} \in S_{ti}$. Finally, given $x \in S$, define $\lambda_t x = (x_s)_{s \in \mathbb{N}, 0 \leq s \leq t}$.

The set H^∞ of infinite histories of the game is a non-empty subset of S. Let

$$\lambda_t H = \{\lambda_t x | x \in H^\infty\}$$

be the set of all *finite histories* of length t. Finite histories are usually referred to simply as histories.

The set of outcomes possible in period t depends upon the initial history up to period $t-1$:

$$A_t(\lambda_{t-1}x) = \{y_t | y \in H^\infty, \lambda_{t-1}y = \lambda_{t-1}x\}$$

Players choose their actions independently so $A_t(\lambda_{t-1}x)$ factorises as

$$A_t(\lambda_{t-1}x) = \prod_{i \in P} A_{ti}(\lambda_{t-1}x)$$

$A_{ti}(\lambda_{t-1}x)$ is the *action set* of player i following history $\lambda_{t-1}x$. Note that A_t and A_{ti} are correspondences.

In any period t a *period-strategy* for player i in that period is a function

$$f_{ti} : \lambda_{t-1}H \longrightarrow S_{ti}$$

which satisfies

$$f_{ti}(\lambda_{t-1}x) \in A_{ti}(\lambda_{t-1}x)$$

for all $x \in H^\infty$. That is, period-strategies must always specify actions in the appropriate action set. Given a period-strategy for each player in each period, a *strategy* for player i is

$$f_i = (f_{ti})_{t \in \mathbb{N}^+}$$

and a *strategy profile* is

$$f = (f_i)_{i \in P}$$

Let $F(H^\infty)$ be the set of all strategy profiles. The notation reflects that this set is defined by the choice of H^∞. Denote by $F_i(H^\infty)$ the set of strategies of player i. If $f \in F(H^\infty)$ and $g_j \in F_j(H^\infty)$, let

$$f\backslash g_j = (h_i)_{i\in P}$$

$$\text{where } h_i = \begin{cases} g_j & \text{for i = j} \\ f_i & \text{otherwise} \end{cases}$$

Given a strategy profile f, $x \in H^\infty$ and $t \in \mathbb{N}$, define $\alpha(f, x, t)$ as the infinite history resulting from the strategy profile f being used following history $\lambda_t x$. In other words, $\alpha(f, x, t)$ has the recursive definition:

$$\alpha_{si}(f, x, t) = x_{si} \qquad \text{for } s \leq t$$

$$\alpha_{si}(f, x, t) = f_{si}[(\alpha_\tau)_{0\leq\tau\leq s}] \qquad \text{for } s \geq t$$

To guarantee that $\alpha(f, x, t) \in H^\infty$, the following condition is introduced. For any $x \in S$ such that $\lambda_t x \in \lambda_t H$ for all $t \in \mathbb{N}$, it is the case that $x \in H^\infty$. That is, if every finite initial subsequence of x is a history, then x is an infinite history.

A *payoff function* $\pi = (\pi_i)_{i\in P}$ is made up of functions of the form $\pi_i : H^\infty \to \mathbb{R}$. Each function π_i describes an individual's payoff in each outcome of the game.

Definition 3.15. A *game* is a pair (H^∞, π) as described above.

Definition 3.16. A game of *perfect information* is one that satisfies the following condition. Given any $h \in H^\infty$ and any $t > 0$, there is at most one $i \in P$ such that the action set $A_{ti}(\lambda_{t-1}x)$ is not a singleton.

Definition 3.17. Given a game (H^∞, π), a strategy profile f is called a *subgame perfect equilibrium* (SPE) if it satisfies the following condition for all $x \in H^\infty$, $t \in \mathbb{N}$, $i \in P$ and all strategies $g_i \in F_i(H^\infty)$:

$$\pi_i(\alpha(f, x, t)) \geq \pi_i(\alpha(f\backslash g_i, x, t)) \tag{3.1}$$

3.4.2 Further Definitions and Notes

Note that the only form of imperfect information that definition 3.15 can describe is simultaneous actions. This is sufficient for the games used in this thesis. Also note that to construct a game in the form of definition 3.15, the choice of S_{ti} is not important. For example this set could be constructed as the union of all actions that might be taken in any history of the game. The choice of S_{ti} only becomes important in applying the conditions of theorem 3.2 in section 3.4.3.

Definition 3.15 requires all finite histories to have actions sets and only assigns payoffs to infinite histories. The possibility of a finite terminal history is not directly allowed. However, a finite history $\lambda_t x$ can effectively represent a terminal history if the only $y \in H^\infty$ such that $\lambda_t y = \lambda_t x$ is $y = x$. For $s > t$, the action set $A_{si}(\lambda_{s-1}x)$ is a singleton, so the infinite history x and its associated payoffs are guaranteed to be realised.

The *decision function* from finite histories to P can be defined as follows. For $x \in H^\infty$ and $t \in \mathbb{N}^+$, $D(\lambda_{t-1}x)$ is defined to be the unique $i \in P$ such that $A_{ti}(\lambda_{t-1}x)$ is not a singleton, where such an i exists. In the case where no such player exists, $D(\lambda_{t-1}x)$ can be defined as $\emptyset$.

Given $x \in H^\infty$ for a game of perfect information, let $\hat{x}_t = x_{tD(\lambda_{t-1}x)}$ (or if $D(\lambda_{t-1}x) = \emptyset$, let $\hat{x}_t = \emptyset$). The sequence $\hat{x} = (\hat{x}_t)_{t\in\mathbb{N}^+}$ then fully describes the history x, as follows. Suppose that the first $t-1$ action profiles are known. Then $D(\lambda_{t-1}x)$ can be found. The action of this player in period t is $\hat{x}_t$. Every other player[26] has a singleton action set, so they have only one possible action at period t. Thus the action profile at period t can be constructed. A sequence of the form of $\hat{x}$ is usually the most convenient method of representing histories, especially in this thesis where the rules governing the order of play are always reasonably simple.

A Nash equilibrium of a game (H^∞, π) can be defined as a strategy profile f which satisfies equation (3.1) for any $x \in H^\infty$, $i \in P$, all strategies $g_i \in F_i(H^\infty)$ and

[26]In the case $D(\lambda_{t-1}x) = \emptyset$, this means every player.

$t = 0$.

A *subgame* of (H^∞, π) can be defined as follows. For $x \in H^\infty$ and $t \in \mathbb{N}$, the subgame generated by the history $\lambda_t x$ is $(\tilde{H}^\infty, \tilde{\pi})$ where $\tilde{H}^\infty = \{y \in H^\infty | \lambda_t y = \lambda_t x\}$ and $\tilde{\pi}$ is the restriction of π to domain $\tilde{H}^\infty$. Strategies can also be mapped to restricted forms for subgames; it is only necessary to ensure that the actions for periods up to t correspond to those of x. The notational details are omitted. Using these definitions, it is possible to state the equivalent definition of a SPE as a strategy profile which is a Nash equilibrium of every subgame.

3.4.3 Proof of Corollary 3.1

First the class of games $\mathcal{E}$ mentioned in corollary 3.1 is defined. The first condition on this class is:

$\mathcal{E}.1$ $S_{ti} \subseteq A \cup B$ and $B = I \times C$, where A and C are finite sets and I is a real interval $[0, M]$ with $M > 0$.

An interpretation of this condition is given below. To define the remaining conditions on $\mathcal{E}$ some more notation is required. Assume that condition $\mathcal{E}.1$ holds on a game (H^∞, π) of perfect information. Given $x \in H^\infty$, let

$$Q(x) = \{t \in \mathbb{N}^+ | \hat{x}_t \in B\}$$

be the set of periods t at which an action in B is taken by[27] $D(\lambda_{t-1} x)$. For $t \in Q$, $\hat{x}_t$ can be written as $(\mu_t, c_t) \in I \times C$. Let

$$J = \{(\eta_t)_{t \in \mathbb{N}^+} | \eta_t \in I\}$$

For $\eta \in J$ and $x \in H^\infty$, define $b(x, \eta)$ by replacing μ_t with η_t in x for all $t \in Q(x)$. Let

$$K(x) = \{\eta \in J | b(x, \eta) \in H^\infty\}$$

be the set of all sequences in J which generate a valid history in this way.

[27] It does not matter that this set excludes periods where all action sets are singletons.

Definition 3.18. The class $\mathcal{E}$ is the set of those games of perfect information which satisfy the condition $\mathcal{E}.1$ and:

$\mathcal{E}.2$ For any $x \in H^\infty$ there exists $p_t(x) \in \mathbb{R}$ for each $t \in Q(x)$ such that

$$K(x) = \{\eta \in J | \forall t \in Q(x)\ \eta_t \in [0, p_t]\}$$

$\mathcal{E}.3$ For any $x \in H^\infty$, under the subspace topology induced on $K(x)$ by the weak topology, the function $p_i : K(h) \to \mathbb{R}^+$ defined by $p_i(\eta) = \pi_i(b(x, \eta))$ is continuous.

This class is meant to represent those bargaining games of chapters 4 and 5 which are games of perfect information and for which SPE existence cannot be easily be proved by construction. It is now shown that these games do indeed lie in $\mathcal{E}$. This is done informally but the details are straightforward to check for each individual game.

Condition $\mathcal{E}.1$ is straightforward. The actions in A represent acceptance or refusal decisions, the values in I represent numerical demand levels[28], and the elements in C represent decisions about to whom demands are made. C can be taken to be a singleton if no decisions of this sort are required.

[28]The set I is restricted to a closed and bounded interval to satisfy the first condition of theorem 3.2 below. If I were allowed to be any compact set in the definition of $\mathcal{E}$, then the class would include the alternating offers game, the telephoning game and Herrero's model. Corollary 3.1 could be proved by a similar method to that used below. This is not done in order to keep the notation required simple and because SPE existence can be proved by construction for the models mentioned.

It seems plausible that SPE existence is conserved for the bargaining games under investigation if I were an unbounded interval. However, there seems no great benefit in extending the allowed form of I thus since this would only amount to extending the range of non-feasible demands which players can make.

Next, condition $\mathcal{E}.3$ is demonstrated. For the games in question

$$\pi_i(b(x,\eta)) = \begin{cases} kf^{j,i}(\eta_t) & \text{in case i} \\ k\eta_t & \text{in case ii} \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant which is independent of η. Case i is 'in x player i accepts a demand made by player j in period t' and case ii is 'in x a demand made by player i in period t is later accepted'.

By the definition of a utility cake, $f^{j,i}$ is continuous (in the Euclidean topology) over its range for all values of i and j. In case i, for $\eta \in K(x)$, η_t must be in the range of $f^{j,i}$. Otherwise $b(x,\eta)$ would not be a valid history, contradicting $\eta \in K(x)$. This shows that condition $\mathcal{E}.3$ holds.

It remains to demonstrate condition $\mathcal{E}.2$. The bargaining games in question fall into one of two classes. For each class values of $p_t(x)$ are given to satisfy the condition.

The first class[29] comprises games where a proposer (a player making a demand) may make any numerical demand in I (as defined in condition $\mathcal{E}.1$). For this class $p_t(x) = f^{i,j}(0)$ in the case that in x player i accepts a demand made by player j in period t, and $p_t(x) = M$ otherwise. That is, numerical demands that are accepted can be changed to any value which is feasible to the acceptor. Numerical demands which are not accepted can be changed to any value in I.

[29]Typically this class contains games with exogenous orders of play. Games in this class are: the market bargaining game, unilateral demand exogenous order models and the exogenous ordering model.

The second class[30] comprises games where a proposer i must specify[31] a set of responders $R \subseteq P$ and may make any numerical demands up to $\min_{j \in R} f^{j,i}(0)$. That is, numerical demands must be feasible to all players in R. These games satisfy condition $\mathcal{E}.2$ with

$$p_t(x) = \min_{j \in R} f^{j,i}(0)$$

The following result is theorem 1 of Harris [33]. I have slightly altered the statement of the theorem but made no alterations to the content.

Theorem 3.2. *Suppose that (H^∞, P) is a game of perfect information. The following is a sufficient condition for the existence of a subgame perfect equilibrium in (H^∞, P). Topologies on the sets S_{ti} exist such that the resulting product topology on S satisfies:*

1. *For all $t \in \mathbb{N}^+$ and $i \in P$, S_{ti} is compact.*
2. *For all $t \in \mathbb{N}^+$ and $i \in P$, S_{ti} is Hausdorff.*
3. *H^∞ is a closed subset of S.*
4. *For all $t \in \mathbb{N}^+$, the correspondence A_t is lower hemicontinuous.*
5. *For all $i \in P$, π_i is continuous.*

Notes:

1. The continuity conditions 4 and 5 use appropriate topologies induced by those on S and S_{ti}. For example, condition 5 uses the subspace topology induced on H^∞ by that on S.

[30] Typically this class contains games with endogenous orders of play. Games in this class are: the telephoning game, the perfect information models of Calvó-Armengol, and the endogenous ordering model.

[31] Formally, it could be required that there is a function from C (as defined in condition $\mathcal{E}.1$) to the set of subsets of P.

2. Condition 4 means that given any $x \in H^\infty$ and any open set $U \subseteq S_t$ containing x_t, there exists an open set $V \subseteq \lambda_{t-1}H$ containing $\lambda_{t-1}x$ such that for all $\lambda_{t-1}v \in V$, $A_t(\lambda_{t-1}v) \cap U \neq \emptyset$.

The proof of the following corollary can now be given.

Corollary 3.1. *There exists a SPE for all games in the class $\mathcal{E}$.*

Proof. Let T_I be the subspace topology on I induced by Euclidean space. Let T_A and T_C be the discrete topologies on A and C. Let T_B be the topology on B given by the product topology of T_I and T_C. Let T' be the topology on S_{ti} given by arbitrary unions of elements of $T_A \cup T_B$. Let T be the resulting product topology on S.

It is immediate from this definition that T' satisfies properties 1 and 2 of theorem 3.2. Property 5 is a consequence of conditions $\mathcal{E}.1$ and $\mathcal{E}.3$.

To prove property 3, consider a sequence $(x_a)_{a\in\mathbb{N}}$ with $x_a \in H^\infty$ for all a, which is convergent in T. Let x be its limit. By definition of T there exists some $\bar{a}$ such that for $a > \bar{a}$, $x_a = b(x_{\bar{a}}, \eta^a)$ for some η^a. Furthermore, for $t \in Q(h)$, $(\eta_t^a)_{a\in\mathbb{N}}$ is convergent under T_I. By condition $\mathcal{E}.2$, $\eta_t^a \in [0, p_t]$. It follows that $\eta_t = \lim_{a\to\infty} \eta_t^a \in [0, p_t]$ and thus $x = b(x_N, \eta) \in H^\infty$ where $\eta = (\eta_t)_{t\in Q(h)}$.

To prove property 4, suppose that $x \in H^\infty$ and $t \in \mathbb{N}^+$. By definition of T' and condition $\mathcal{E}.2$, there exists an open set $V \subseteq \lambda_{t-1}H$ such that $\lambda_{t-1}v \in V$ implies $\lambda_{t-1}v = \lambda_{t-1}b(x, \eta)$ for some η. Fix some such V. Select any $v \in H^\infty$ such that $\lambda_{t-1}v \in V$. Define $w = (w_s)_{s\in\mathbb{N}}$ by $w_s = v_s$ for $s < t$ and $w_s = x_s$ for $s \geq t$. Note that $w = b(x, \eta)$ for some η such that $\eta_t \in [0, p_t(x)]$, so by condition $\mathcal{E}.2$, $w \in H^\infty$. Since $\lambda_{t-1}w = \lambda_{t-1}v$, $x_t \in A_t(\lambda_{t-1}v)$ as required. □

Chapter 4

The Alternating Offers Game and Multi-Player Extensions in the Literature

Rubinstein's *alternating offers game* [56] is a highly successful model of 2 player bargaining over a utility cake. In the notation of the previous chapter, this situation is a 2 player bilateral exchange network with the single exchange opportunity 12; the simplest non-trivial bilateral exchange network. The structure of the game is follows. A player, the *proposer*, proposes a feasible utility pair from the cake. The other player, the *responder*, then chooses whether to accept or refuse it. If she refuses it becomes her turn to make a proposal. If she accepts, the game ends. The game continues in this way until a proposal is accepted. The initial proposer is player 1.

The accepted utility pair represents players' shares. In the alternating offers game the bargaining process is costly. There is a discount factor $\delta_i \in (0, 1)$ for each player i modelling their time preferences, and bargaining incurs a delay, τ. The payoff of player i is found by multiplying their share by δ_i^τ. In an infinite history in which no proposal is accepted payoffs are zero. Rubinstein [56] proves that this game has a unique SPE. A version of the game with costless bargaining given by taking $\delta_i \equiv 1$ can easily be shown to support a wide range SPE outcomes.

It is greatly desirable that games with a high level of bargaining detail, such as the alternating offers game, possess a unique SPE, or at least that their SPE outcomes lie close together, since equilibrium selection is problematic. Imposing a subjective choice of equilibrium refinement relies on having a strong intuitive grasp of the situation being modelled. In all but the simplest multi-player bargaining situations this is not the case. The alternative is to use evolutionary methods. However, current evolutionary methods cannot easily be applied to such games since they have strategy spaces of large dimension[1]. A similar problem applies to computer simulations of evolutionary models[2]. It is hard for a computer to store or access quickly highly complicated strategies. For example in the alternating offers game for each player there are infinitely many subgames in which proposals must be made, resulting in strategy spaces of infinite dimension.

The simple structure and unique SPE of the alternating offers game makes it an attractive candidate for generalisation to other bargaining situations. This chapter investigates the alternating offers game and various generalisations which have been proposed in the literature. The purpose is to prove results and develop concepts that are used in chapter 5 to construct extensions applicable to general bilateral exchange networks.

An outline of this chapter is as follows. Section 4.1 describes various concepts from the economic theory approach to bargaining which are required in this and later chapters. Section 4.2 is on the alternating offers game. Sections 4.3 and 4.4 describe two generalisations of this model to 3 player ring networks which are defined

[1]See Seymour [59] for an approach to constructing and analysing dynamics for 2 player games of infinite dimension. Also, see Seymour [60] for an application of these dynamics to the ultimatum game which does not support the unique SPE of that game under all conditions. This cautions against viewing a unique SPE outcome in the alternating offers game or its extensions as necessarily also being an exact prediction in an evolutionary setting.

[2]To make any progress on this approach it seems necessary to make simplifying assumptions about the form of strategies.

by Binmore in [3]: the telephoning game and the market bargaining game. These games both have unique SPEs, which suggests that it may be possible to extend them to model bargaining in larger networks. Sections 4.2–4.4 each begin with a literature review. Further discussion and analysis of the alternating offers and market bargaining games relevant to later work is also provided. This includes a discussion of Binmore's argument for preferring the approach of the market bargaining game to that of the telephoning game, and a case for altering the delay scheme of the market bargaining game. Section 4.5 briefly summarizes other relevant bargaining models from the literature.

4.1 Definitions from Bargaining Theory

The outcomes of a general bargaining situation with set of players P can be described by giving a *multiplayer utility cake* $C(Q)$ for each $Q \subseteq P$. The elements of a multiplayer utility cake are of the form $(x_i)_{i \in Q}$ with $x_i \in \mathbb{R}^+$ for all $i \in Q$. They represent the utility vectors which can be realised by the players in Q if all members of Q agree[3].

Suppose $C(Q)$ is a multiplayer utility cake. A *Pareto improvement* on $x = (x_i)_{i \in Q}$ is a vector $y = (y_i)_{i \in Q}$ such that $y_i \geq x_i$ for all $i \in Q$ and $y \neq x$. A utility vector $x \in C(Q)$ is *Pareto optimal* if it has no Pareto improvement in $C(Q)$. The *Pareto boundary* of a utility cake is its Pareto optimal subset. For example, the Pareto boundary of a 2 player utility cake is its outer boundary, as defined in definition 3.6, minus any vertical or horizontal line segments except the end points which do not lie on an axis.

[3] Various properties can be placed on the function C to capture reasonable features of bargaining situations. Some such properties are convexity and compactness of utility cakes, and superadditivity (informally; if players in Q can realise a utility vector if all members of Q agree, they can also realise this vector under an agreement by all members of $Q' \supset Q$). These properties are not required for the limited discussion of multiplayer utility cakes in this thesis.

Definition 4.1. Given a set of players P and a multiplayer utility cake $C(Q)$ for all $Q \subseteq P$, the *core* of this bargaining situation is the set of outcomes $(x_i)_{i \in P}$ such that $(x_i)_{i \in Q}$ is Pareto optimal in $C(Q)$ for all $Q \subseteq P$.

A bilateral exchange network (P, E, K) can be described[4] in terms of multiplayer utility cakes by defining $C(Q)$ from $\mathcal{K}^{ab}$ in the case that $Q = \{a, b\}$ and $ab \in E$, and $C(Q) = \emptyset$ otherwise. It is then straightforward to see that the core of a bilateral exchange network N is the set of all share vectors $x = (x_i)_{i \in P}$ corresponding to feasible outcomes of N such that for all $ab \in E$, (x_a, x_b) has no Pareto improvement in $\mathcal{K}^{ab}$.

In a non-core bargaining outcome, some subset Q of players have incentives (or at least no disincentives) to switch multilaterally to different behaviour. Nonetheless, non-core bargaining solutions are not automatically implausible. Players in Q may have disincentives to switch to different behaviours unilaterally. This depends on the details of the bargaining process. For example, switching to a Pareto improvement may be an involved task if Q is large. It could entail a risk of miscoordination: if some players in Q do not participate in the switch and instead make agreements with players outside Q then the remainder might be left with poor available utilities. In this thesis, bargaining solutions of small networks are expected to usually lie in the core. If not, a plausible explanation is required.

Definition 4.2. An (asymmetric) *Nash bargaining solution* is a function $g(\mathcal{K}, \xi)$ from a (2 player) utility cake $\mathcal{K}$ and a *status-quo* element $\xi \in \mathcal{K}$ satisfying axioms 1 – 4 below. If axiom 5 is also satisfied then the function is called a *symmetric Nash bargaining solution.*

1. *Individual rationality*

   For $i = 1$ or 2, $g_i(\mathcal{K}, \xi) \geq \xi_i$.

[4] A more natural description would be to define non-empty values of $C(Q)$ for all Q satisfying superadditivity (see footnote 3). However the description given in the text suffices to allow the core of N to be found.

2. *Pareto optimality*

   $g(\mathcal{K}, \xi)$ is Pareto optimal in $\mathcal{K}$.

3. *Independence of irrelevant alternatives*

   If $\mathcal{L} \subset \mathcal{K}$ and $g(\mathcal{K}, \xi) \in \mathcal{L}$ then $g(\mathcal{L}, \xi) = g(\mathcal{K}, \xi)$.

4. *Scale independence*

   For any positive affine transformation[5] $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$:

$$g(\alpha\mathcal{K}, \alpha\xi) = \alpha g(\mathcal{K}, \xi)$$

5. *Symmetry*

   For the transformation[6] β which maps (x_1, x_2) to (x_2, x_1):

$$g(\beta\mathcal{K}, \beta\xi) = \beta g(\mathcal{K}, \xi)$$

It can be shown that $g(\mathcal{K}, \xi)$ is a Nash bargaining solution if and only if

$$g(\mathcal{K}, \xi) = \arg\max_{(x_1, x_2)} (x_1 - \xi_1)^\gamma (x_2 - \xi_2)^{1-\gamma}$$

where the maximisation is taken over the subset of $\mathcal{K}$ containing elements on which ξ is not a Pareto improvement, and $\gamma \in (0, 1)$. There is a unique Nash bargaining solution associated with each γ. The values γ and $1 - \gamma$ are called the *bargaining powers* of players 1 and 2 respectively. Equal bargaining powers, $\gamma = \frac{1}{2}$, gives the unique symmetric Nash bargaining solution. Proofs of these facts are included in Roth [55].

The status-quo point ξ represents the outcome if bargaining breaks down. Recall from section 3.1.2 that the von Neumann-Morgenstern utility functions used in utility cakes are chosen so that $\xi = (0, 0)$. Henceforth any reference to a Nash bargaining solution assumes this value of ξ.

[5] And its corresponding extension to subsets of $\mathbb{R}^2$.

[6] And its corresponding extension to subsets of $\mathbb{R}^2$.

Many of the axioms in definition 4.2 can be criticised on both experimental and conceptual grounds (for example see Roth [55]), and other solutions exist which use different axioms. An example is the bargaining solution of Kalai and Smorodinski [36].

4.2 Alternating Offers Game

4.2.1 Literature Review

Given a utility cake $\mathcal{K}^{12}$ and a discount factor vector $\Delta = (\delta_1, \delta_2) \in (0,1)^2$, the alternating offers game $A(\mathcal{K}^{12}, \Delta)$ is as follows.

1. Player 1 is the first proposer.
2. The proposer, p, makes a proposal $\sigma \in \mathcal{K}^{12}$.
3. The other player, r, is the responder and must either accept or refuse. Accepting terminates the game.
4. Following a refusal, the responder becomes the next proposer and the game returns to step 2.

Delay: The delay, $\tau(h)$, of any finite history h is equal to the number of refusals that have occurred.

Payoffs: If proposal σ is accepted in history h then payoffs are $(\delta_1^{\tau(h)}\sigma_1, \delta_2^{\tau(h)}\sigma_2)$. In an infinite history (i.e. one in which no proposal is accepted) payoffs are zero.

Given a proposal by player p of $\sigma = (\sigma_1, \sigma_2)$, σ_p is referred to as the *demand* of p and σ_r as the *offer* of p to the other player r (the next responder). If the proposal σ is accepted then in the terminology of section 3.3.3, σ_1 and σ_2 are the shares of players 1 and 2.

Lemma 4.1. *$A(\mathcal{K}_{unit}, \Delta)$ has the unique SPE payoff:*

$$(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}) \tag{4.1}$$

The following proof is essentially that of Shaked and Sutton [62]. The statement below highlights methods which are used repeatedly later. The theorem of [62] also applies to non-unit cakes and similarly finds a unique SPE payoff. Only the unit cake case is given here to avoid introducing too much notation, but few extra details are required[7].

Proof. Fix Δ and let $\mathcal{A} = A(\mathcal{K}_{\text{unit}}, \Delta)$. Assume that $\mathcal{A}$ has a SPE. By the definition of SPE it follows that all subgames of $\mathcal{A}$ have SPEs. Define a *pre-proposal* subgame of $\mathcal{A}$ to be one at the start of step 2 of the game. Let B_i be the set of pre-proposal subgames of $\mathcal{A}$ with proposer i.

For a subgame $\mathcal{B}$, let $\tau(\mathcal{B})$ be the associated delay. Let $P_i(\mathcal{B})$ be the set of all values $\delta_i^{-\tau(\mathcal{B})}\pi_i$ such that π_i is a SPE payoff to player i in $\mathcal{B}$. Let $\Pi_i = \bigcup_{\mathcal{B}\in B_i} P_i(\mathcal{B})$. Let $\bar{\pi}_i = \sup \Pi_i$ and $\underline{\pi}_i = \inf \Pi_i$.

Let $(i,j) = (1,2)$ or $(2,1)$. The following relations are now proved:

$$\bar{\pi}_i \leq 1 - \delta_j \underline{\pi}_j \tag{4.2}$$

$$\underline{\pi}_i \geq 1 - \delta_j \bar{\pi}_j \tag{4.3}$$

Consider any subgame $\mathcal{B} \in B_i$. Suppose player j refuses the initial proposal made in $\mathcal{B}$. Then the set of SPE payoffs to player j in the resulting game is a subset of $\{\delta_j^{\tau(\mathcal{B})+1} x \mid x \in \Pi_j\}$. So in $\mathcal{B}$ the SPE payoff of player j is at least $\delta_j^{\tau(\mathcal{B})+1}\underline{\pi}_j$. If player i received a SPE share of more than $1 - \delta_j \underline{\pi}_j$ in $\mathcal{B}$ such a payoff would not be possible. This proves equation (4.2).

Suppose player i offers $\lambda > \delta_j \bar{\pi}_j$ in $\mathcal{B}$. If player j refuses, her payoff in any SPE is at most $\delta_j^{\tau(\mathcal{B})+1}\bar{\pi}_j$. Thus in any SPE of $\mathcal{B}$ she accepts the proposal involving λ. This proves equation (4.3).

[7]The only significant extra detail is to show that the analogues of equations (4.4) and (4.5) yield a unique solution.

Combining these inequalities gives:

$$\bar{\pi}_i \leq 1 - \delta_j(1 - \delta_i \bar{\pi}_i) \tag{4.4}$$

$$\underline{\pi}_i \geq 1 - \delta_j(1 - \delta_i \underline{\pi}_i) \tag{4.5}$$

Solving these yields $\bar{\pi}_i \leq \bar{n}_i \leq \underline{\pi}_i$ where $\bar{n}_i = \frac{1-\delta_j}{1-\delta_i\delta_j}$. By definition $\underline{\pi}_i \leq \bar{\pi}_i$, so $\underline{\pi}_i = \bar{\pi}_i = \bar{n}_i$.

Consider a SPE of $\mathcal{A}$ in which the initial proposal of player 1 is refused. Let (p_1, p_2) be the payoffs to players 1 and 2. By the above it must be the case that $p_1 = \bar{n}_1$ and $p_2 = \delta_2 \bar{n}_2$ which gives $p_1 + p_2 = 1$. But, since a proposal is refused in the history generated by this SPE, it must be the case that

$$(p_1, p_2) = (\delta_1^\tau \sigma_1, \delta_2^\tau \sigma_2) \tag{4.6}$$

for some $(\sigma_1, \sigma_2) \in \mathcal{K}_{\text{unit}}$ and $\tau \geq 1$. Thus $p_1 + p_2 \leq \delta$ which is a contradiction. Therefore in any SPE, players 1 and 2 exchange immediately, and the SPE payoff must be $(\bar{n}_1, 1 - \bar{n}_1)$, as required.

It remains to prove that $\mathcal{A}$ has a SPE as assumed above. It is easy to confirm that the following strategy profile is a SPE[8]. Let $(i, j) = (1, 2)$ or $(2, 1)$ as before. Player i always makes the proposal (p_1, p_2) where $p_i = \bar{n}_i$ and $p_j = 1 - \bar{n}_i$, and accepts offers if and only if they are $1 - \bar{n}_j = \delta_i \bar{n}_i$ or better. □

Binmore [4] argues that the most important case of the alternating offers game is where the costs of bargaining are small. The justification is that following a refusal, players have an incentive to make new offers as soon as possible. Situations in which this is not possible seem rare. This case can be investigated by setting[9] $\delta_i = \eta_i^\epsilon$ and

[8]Indeed, it can be shown that this is the unique SPE.

[9]See Osborne and Rubinstein [54] for an axiomatic approach to preferences over time/share pairs which yields a utility function using this form of discounting. A general method of solution to the alternating offers game which also applies to other specifications of time/share preferences is given in Binmore [4]. This shows that under some other time/share preferences the characterisation of the limiting outcomes made here does not hold.

taking the limit $\epsilon \to 0$. In this relation, δ_i is the discount factor for a refusal, η_i represents the discount factor for a unit of delay, and ϵ represents the length of delay following a refusal.

In this case it can easily be shown that the outcome of equation (4.1) converges in the limit $\epsilon \to 0$. The same is true of the analogous result for an arbitrary utility cake $\mathcal{K}$[12]. This limiting outcome has two important features[10]. Firstly, it is equal to the corresponding limiting outcome to a variation on the alternating offers game in which player 2 acts first. This shows that the exogenous choice of first mover has no influence in this limiting case. Secondly, it is equal to the asymmetric Nash bargaining solution when player 1 has bargaining power $\frac{\ln \delta_2}{\ln \delta_1 + \ln \delta_2}$. This shows that time preferences over incurred delays can select a unique outcome[11] even in the case where delays are arbitrarily small. This is in contrast to the case of costless delays – i.e. $\delta_1 = \delta_2 = 1$ – in which any utility pair on the Pareto boundary can be supported as a SPE outcome[12].

A variation on the alternating offers game is to allow *outside options*, as follows. Each player has an outside option share of $m_i \in \mathbb{R}^+$. At step 3 of the game, the responder has additional choice of opting out of bargaining. If this option is exercised by player i in history h then the payoff of player i is $\delta_i^{\tau(h)} m_i$ and the other player receives payoff zero. This setting is especially interesting with respect to bilateral exchange networks. It can be viewed as a simplified model of a case where either player has a chance to participate in an alternative exchange[13], but only one player may take this opportunity.

[10]For a proof see Binmore [4].

[11]Indeed, these time preferences can select any asymmetric Nash bargaining solution and any outcome on the Pareto boundary.

[12]Let $\sigma = (\sigma_1, \sigma_2)$ be on the Pareto boundary. The following strategies form a SPE yielding the payoff σ. Both players always propose σ. Player i accepts a proposal $\lambda = (\lambda_1, \lambda_2)$ if and only $\lambda_i \geq \sigma_i$.

[13]A player who has no such alternative can simply be endowed with $m_i = 0$.

It can be shown (see Binmore [3] and Muthoo [51]) that, with discount factors again taking the form $\delta_i = \eta_i^\epsilon$, the limiting SPE payoffs[14] of this variation as $\epsilon \to 0$ are the same as those of the alternating offers game where the cake $\mathcal{K}^{12}$ is replaced by[15]

$$\{(x_1, x_2) \in \mathcal{K}^{12} \mid x_1 \geq m_1, x_2 \geq m_2\}$$

These payoffs can be represented algebraically as $(\phi, 1 - \phi)$ where:

$$\phi = m_1 \vee [f^{2,1}(m_2) \wedge n_1]$$

and n_1 is the limiting SPE payoff to player 1 in the alternating offers game without outside options. Note that $\vee$ and $\wedge$ are infix maximum and minimum operators respectively.

4.2.2 Discussion and a Variation

In the terms of section 3.3.3, $A(\mathcal{K}^{12}, \Delta)$ is a bargaining model with endogenous structure Δ. There is also exogenous structure; the choice of the first proposer. This is embedded in the choice of which player is labelled as 1, but could easily be made explicit.

The core of the proof of lemma 4.1 is robust to many variations of the game rules (for example see Binmore [4]). The following variation, which introduces *personal delays*, is especially relevant to later work. In particular, the models of chapter 5 reduce to this game for 2 player networks. The motivation for using personal delays is the subject of section 4.4.4.

The *alternating offers game with personal delays*, $A^{\text{personal}}(\mathcal{K}^{12}, \Delta)$, is the same as $A(\mathcal{K}^{12}, \Delta)$ except for the specification of delay and payoffs. In any finite history h, the personal delay of player i, $\tau_i(h)$, is the number of times player i has refused

[14] In the sense of definition 3.14.

[15] If m_1 or m_2 is non-zero then this new set is not strictly a utility cake as it does not satisfy free disposal. However the same proof applies to this case. Alternatively the minimal utility cake containing this set could be used to give the same result.

in the course of the history. If proposal σ is accepted in finite history h then payoffs are $(\delta_1^{\tau_1(h)}\sigma_1, \delta_2^{\tau_2(h)}\sigma_2)$. In an infinite history payoffs are zero.

Corollary 4.2. *$A^{personal}(\mathcal{K}^{12}, \Delta)$ has the same unique SPE as $A(\mathcal{K}^{12}, \Delta)$.*

Proof. The proof of lemma 4.1 applies here with the following modifications. Let $\mathcal{A} = A^{\text{personal}}(\mathcal{K}_{\text{unit}}, \Delta)$. For a subgame $\mathcal{B}$, let $\tau_i(\mathcal{B})$ be the associated personal delay of player i. For $x \in \{i, j\}$, replace each occurrence of $\tau(\mathcal{B})$ in the exponent of δ_x with $\tau_x(\mathcal{B})$. Equation (4.6) should be replaced with

$$(p_1, p_2) = (\delta_1^{\tau_1}\sigma_1, \delta_2^{\tau_2}\sigma_2)$$

where $\tau_1 + \tau_2 \geq 1$. It is still the case that $p_1 + p_2 < 1$, as required. □

4.3 The Telephoning Game

The telephoning game is defined by Binmore in [3]. In the terminology of section 3.3.3, this is a bargaining model for 3 player ring networks. It requires a discount factor vector $\Delta = (\delta_1, \delta_2, \delta_3) \in (0, 1)^3$.

The telephoning game for network N, $T(N, \Delta)$, is defined as follows:

1. Player 1 is the first proposer.
2. Denote the proposer by p. The proposer selects a different player to be responder r and makes a proposal $\sigma \in \mathcal{K}^{pr}$.
3. The responder must either accept or refuse. Accepting terminates the game.
4. Following a refusal the responder becomes the next proposer and the game returns to step 2.

Delay: The delay, $\tau(h)$, of any finite history h is equal to the number of refusals that have occurred.

Payoffs: If player r accepts the proposal σ of player p in history h, then players p and r receive payoffs $\delta_p^{\tau(h)}\sigma_p$ and $\delta_r^{\tau(h)}\sigma_r$ respectively. The remaining player receives payoff zero. In an infinite history all payoffs are zero.

Recall from definition 3.10 that R_3 is a 3 player ring network with unit cakes. Binmore [3] proves that for $\delta_1 < \delta_2 < \delta_3$, the game $\mathcal{T} = T(R_3, \Delta)$ has a SPE in which the payoffs of players 1 and 2 are the same as in the alternating offers game on $\mathcal{K}_{\text{unit}}$ with the same discount factors for players 1 and 2. Furthermore it is shown that in a game with the same definition as $\mathcal{T}$ except that player 3 is the first proposer, there is a SPE in which player 3 earns only as much as player 2 does in the SPE of $\mathcal{T}$ just described. It is argued that this is an unreasonable property of a bargaining model since player 3 is the most patient player.

The telephoning model is not considered as a good bargaining model in this thesis due to the existence of the SPEs just mentioned. Further discussion of this case is postponed until section 4.4.5 as it is more fruitful to discuss it in parallel with the market bargaining game. Part of this discussion explicates the arguments against the use of the telephoning model in more depth.

4.4 The Market Bargaining Game

4.4.1 Literature Review

Binmore [3] defines a bargaining model for 3 player ring networks which he names the market bargaining game. It is referred to here as *the market bargaining game with public delays* to distinguish it from a modified version which will be introduced in section 4.4.4. Sometimes I refer simply to 'the market bargaining game' in contexts where the differences between these versions are irrelevant. The model requires a discount factor vector $\Delta = (\delta_1, \delta_2, \delta_3) \in (0, 1)^3$.

The market bargaining game with public delays for network N, $M^{\text{public}}(N, \Delta)$, is as follows.

1. Player 1 is the first proposer.

2. Denote the proposer by p. The responder, r, is the player satisfying[16] $r \equiv p + 1 (\text{mod } 3)$.

3. The proposer makes a demand[17] $\sigma_p \in [0, m(N)]$.

4. The responder may accept the most recent demand of any other player if one exists and it is feasible, or refuse all demands. Accepting a demand terminates the game.

5. Following a refusal, the responder becomes the new proposer and the game returns to step 2.

Recall from section 3.1.1 that a demand σ_p by player p is said to be feasible to player r if $\sigma_p \leq f^{r,p}(0)$. Define $m(N)$ to be the maximum demand which is feasible from some player to another in N.

In [3], Binmore does not explicitly define how delays occur and affect payoffs in this game. It seems reasonable to assume that he intended the use of public delays as in the original alternating offers game. That is, as follows.

Delay: The delay, $\tau(h)$, of any finite history h is equal to the number of refusals which have occurred.

Payoffs: If a demand σ_p made by player p is accepted by player r in history h then players p and r receive payoffs $\delta_p^{\tau(h)}\sigma_p$ and $\delta_r^{\tau(h)} f^{p,r}(\sigma_p)$ respectively. The third player receives payoff zero. In an infinite history all payoffs are zero.

The solution to this model involves the following set. Recall definition 3.6 of an outer boundary.

[16] In other words, for $p = 1$ or 2, $r = p + 1$. For $p = 3$, $r = 1$ since this is the unique element of $\{1, 2, 3\}$ equivalent to $p + 1$ modulo 3. This construction is used several times in this thesis.

[17] $m(N)$ is defined following the definition of the game. The restriction on demands to a closed interval is a technical condition required for corollary 3.1 to hold. A greater value of $m(N)$ would not affect the following analysis.

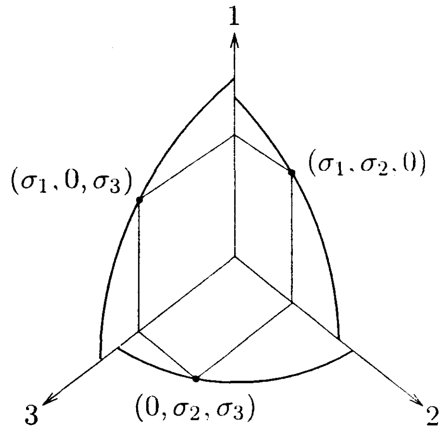

Figure 4.1: A 3 player network with a von Neumann-Morgenstern triple

Definition 4.3. A *von Neumann-Morgenstern triple* is a set of points $\{(\sigma_1, \sigma_2, 0), (\sigma_1, 0, \sigma_3), (0, \sigma_2, \sigma_3)\}$ such that (σ_i, σ_j) lies on the outer boundary of $\mathcal{K}^{ij}$. The values σ_i are referred to as the *components* of the triple.

It is proved in [3] that for cakes which are insatiable in the sense of definition 3.7, if a von Neumann-Morgenstern triple exists then it is unique[18].

Binmore argues that the market bargaining game with public discounting has a unique SPE in the limit $\epsilon \to 0$ in the case where discount factors are of the form $\delta_i = \eta_i^\epsilon$. The following theorem summarizes one case of his results. Recall that definition 3.14 defines a limiting SPE payoff.

Theorem 4.3. *Let N be a 3 player ring network with insatiable cakes in the sense of definition 3.7 and an empty core. For $i \in P$ fix $\eta_i \in (0, 1)$ and let $\Delta = (\delta_i)_{i \in P}$ where $\delta_i = \eta_i^\epsilon$ for $\epsilon > 0$. There is a unique limiting SPE payoff of $(\sigma_1, \sigma_2, 0)$ to $M^{public}(N, \Delta)$ as $\epsilon \to 0$, where $(\sigma_1, \sigma_2, \sigma_3)$ are the components of the unique von Neumann-Morgenstern triple of N.*

A proof is given in the next section. It is essentially that of Binmore [3] but explicates the limiting process in slightly more detail.

[18] This is straightforward since such a von Neumann-Morgenstern triple must satisfy $\sigma_1 = f^{1,2,3,1}(\sigma_1)$ and the right hand side is a strictly decreasing function of σ_1.

In the case that the core of a 3 player ring network is non-empty, then there exists a player who receives zero in all core outcomes[19]. Suppose without loss of generality that this is player 3. This case is illustrated by figure 4.2 (page 104). Binmore argued that in this case the unique limiting SPE of the market bargaining game with public discounting is equivalent to that of the alternating offers game between players 1 and 2 on $\mathcal{K}^{12}$ with outside options $f^{3,1}(0)$ and $f^{3,2}(0)$ for players 1 and 2 respectively. In section 4.6 it is proved that this conclusion is *false*; it requires personal discounting.

It is important to note that the unique SPE of the market bargaining game when a von Neumann-Morgenstern triple exists does not correspond to a unique solution of the bargaining situation. This is because the unique SPE which is produced is dependent on the identity of the first player i.e. on which player is labelled as 1. Binmore argues:

"...one may ask which coalition would be expected to form. The question is clearly unanswerable... without further structure being applied".

In the market bargaining game, this extra structure is supplied by the numbering of the players.

In a major difference to the telephoning game, the market bargaining game players allows player to commit to *multilateral demands*. That is, a player may make a demand which either neighbour may choose to accept. Binmore argues that allowing such demands is more natural from the following premise:

"...one cannot expect players to submit to constraints that limit their payoffs unless there is some mechanism that forces the constraint on them."

He goes on to say that the instability he found in the telephoning game suggests that:

"...if it were the custom to deal exclusively by telephone (or bilaterally through

[19]Suppose otherwise. Then there must exist distinct core outcomes σ and σ' such that $(\sigma_i, \sigma_j) \in \mathcal{K}^{ij}$ and $(\sigma'_j, \sigma'_k) \in \mathcal{K}^{jk}$ for some distinct $i, j, k \in P$. Whichever has the higher utility for j is a Pareto improvement on the other for a pair of players.

private conversations), then there would be players who would wish to disturb the custom by advertising or shouting or whatever was necessary to gain attention for their offers."

The use of multilateral demands and the preceding argument are discussed further in section 4.4.5.

4.4.2 Further Analysis

Lemma 4.4. *In a 3 player ring network with insatiable cakes in the sense of definition 3.7, the core is empty if and only if a von Neumann-Morgenstern triple with non-zero components exists.*

Proof. Suppose the core is non-empty. Let $c = (c_1, c_2, c_3)$ be an element of the core. Choose $k \in P$ such that $c_k = 0$. Let i and j be the other players. For $p \in \{i, j\}$ and any $x \in (0, f^{p,k}(0)]$, it must be the case that $f^{k,p}(x) < c_p$. Otherwise $(f^{k,p}(x), x)$ is a Pareto improvement on (c_p, c_k) in $\mathcal{K}^{pk}$. Therefore $(f^{k,i}(x), f^{k,j}(x))$ does not lie on the outer boundary of $\mathcal{K}^{ij}$ for any $x > 0$, and no von Neumann-Morgenstern triple can exist in which player k has a non-zero component.

Suppose the core is empty. Select $i = 1$ or 2 to minimise $f^{i,3}(0)$. Let j be the other element of $\{1, 2\}$. Let $\alpha = f^{i,3}(0)$. Let $g(x) = (f^{3,1}(x), f^{3,2}(x))$ for $x \in [0, \alpha]$. It must be the case that $g(0) \notin \mathcal{K}^{12}$. Otherwise $(a, b, 0)$ is in the core where (a, b) lies on the Pareto boundary of $\mathcal{K}^{12}$ and either equals $(f^{3,1}(0), f^{3,2}(0))$ or a Pareto improvement on it. It must be the case that $g(\alpha) \in \text{int } \mathcal{K}^{12}$ otherwise the vector (c_1, c_2, α) is in the core, where $c_i = 0$ and $c_j = f^{3,j}(\alpha)$. Since g is continuous and strictly monotonic, it follows that some $g(\sigma_3) = (\sigma_1, \sigma_2)$ lies on the outer boundary of $\mathcal{K}^{12}$ and $\sigma_1, \sigma_2, \sigma_3$ are all positive. These are the components of the required von Neumann-Morgenstern triple. □

As mentioned above, the following proof of theorem 4.3 contains slightly more detail about the limiting process than that in Binmore [3].

Theorem 4.3. *Let N be a 3 player ring network with insatiable cakes in the sense of definition 3.7 and an empty core. For $i \in P$ fix $\eta_i \in (0,1)$ and let $\Delta = (\delta_i)_{i \in P}$ where $\delta_i = \eta_i^\epsilon$ for $\epsilon > 0$. There is a unique limiting SPE payoff of $(\sigma_1, \sigma_2, 0)$ to $M^{\text{public}}(N, \Delta)$ as $\epsilon \to 0$, where $(\sigma_1, \sigma_2, \sigma_3)$ are the components of the unique von Neumann-Morgenstern triple of N.*

Proof. Let $\mathcal{M} = M^{\text{public}}(N, \Delta)$ for some Δ. Note that by corollary 3.1 a SPE of $\mathcal{M}$ exists.

Suppose player 1 makes an initial demand of $\lambda_1 < \sigma_1$ in $\mathcal{M}$. Let $\mathcal{A}$ be the resulting subgame. First it is proved that in any SPE this is accepted by either player 2 or 3. Suppose otherwise. Fix a SPE e of $\mathcal{A}$ in which 2 and 3 both refuse λ_1. Let β and γ be the payoffs of players 2 and 3 in e. Then $\beta \geq f^{1,2}(\lambda_1)$ and $\gamma \geq f^{1,3}(\lambda_1)$. Hence $\beta > \sigma_2$ and $\gamma > \sigma_3$ which is a contradiction since (σ_2, σ_3) is on the outer boundary of $\mathcal{K}^{23}$. Thus in any SPE of $\mathcal{M}$ the payoff of player 1 is at least σ_1.

Suppose player 2 makes an initial demand of $\lambda_1 \geq \sigma_1$ in $\mathcal{M}$. Let $\mathcal{B}$ be the resulting subgame. Suppose player 2 refuses and makes a demand of $\lambda_2 < \sigma_2$ in $\mathcal{B}$. Let $\mathcal{B}'$ be the resulting subgame. Observe that $f^{1,3}(\lambda_1) \leq \sigma_3 < f^{2,3}(\lambda_2)$. Hence in any SPE of $\mathcal{B}'$, player 3 does not accept the demand λ_1. There cannot be a SPE of $\mathcal{B}'$ in which players 3 and 1 both refuse the demand λ_2 by a similar argument to the previous paragraph. Hence in any SPE of $\mathcal{B}$ the payoff of player 2 is at least $\delta_2\sigma_2$.

Thus in any SPE of $\mathcal{M}$ the payoff of player 1 is in the interval $[\sigma_1, f^{2,1}(\delta_2\sigma_2)]$. In the limit $\delta_2 \to 1$ both bounds tend to σ_1 so this is the unique limiting SPE payoff to player 1 in $\mathcal{M}$.

Note that in $\mathcal{A}$ player 2 may earn $f^{1,2}(\lambda_1) > \sigma_2$ by accepting the initial demand. Thus the SPE payoff of player 2 in $\mathcal{M}$ is at least $\delta_2\sigma_2$. Therefore σ_2 is the unique limiting SPE payoff to player 2 in $\mathcal{M}$. A higher limiting payoff would result in an non-feasible SPE payoff for all sufficiently small ϵ. □

In the case that the insatiability condition does not hold, the proof still holds under the condition that a unique von Neumann-Morgenstern triple exists[20]. The only additions necessary are as follows. In the second paragraph, the case that $f^{1,2}(\lambda_1) = \sigma_2$ and $f^{1,3}(\lambda_1) = \sigma_3$ is not possible because otherwise $(\lambda_1, \sigma_2, \sigma_3)$ would then be the components of a second von Neumann-Morgenstern triple. In the third paragraph the case that $f^{1,3}(\lambda_1) = f^{2,3}(\lambda_2) = \sigma_3$ is not possible. Otherwise it must be the case that σ_3 is the maximum feasible payoff to player 3, and $(\lambda_1, f^{1,2}(\lambda_1), \sigma_3)$ are the components of a second von Neumann-Morgenstern triple.

4.4.3 General Notes

The market bargaining game demonstrates that multiple solutions of a bargaining network can be described by a game with a unique SPE. The choice of player numbering selects between these outcomes. In the terms of section 3.3.3, the model has exogenous structure which is embedded in the player labelling. This exogenous structure could be made explicit by instead requiring a bijection q from P to $\{1, 2, 3\}$ representing the order in which players act.

The market bargaining game can provide a prediction for any network of up to 3 players. If the network does not contain 3 cakes, the missing cakes can be replaced by[21] $\{(0, 0)\}$. Adding non-profitable exchange relations to a bargaining situation should not change the solution[22].

The market bargaining and telephoning games could be defined for any network

[20] Multiple von Neumann-Morgenstern triples can exist when the insatiability condition fails. It can be shown that should this occur, it must be the case that a pair of components from each triple must lie on a vertical or horizontal part of the outer boundary of the corresponding utility cake. It can be proved that theorem 4.3 still holds under the additional condition that $(\sigma_1, \sigma_2, \sigma_3)$ are the components of the von Neumann-Morgenstern triple maximising the component of player 1. However, case seems of little interest as the outcome is not robust to many small variations in the outer boundary of the cake.

[21] An alternative is to replace them with ϵ-unit cakes and take the limit $\epsilon \to 0$.

[22] See footnote 4 of chapter 3.

of more than 3 players. In most such networks multiple exchanges may form. However, the market bargaining and telephoning games do not provide mechanisms for bargaining to continue after a single exchange. The models of chapter 5 introduce such mechanisms.

A feature of the model is that the proposer is allowed to make demands which are non-feasible to either neighbour[23]. It is certainly necessary to allow demands which are non-feasible to *one* player (e.g. suppose $\mathcal{K}^{12}$ is a unit cake and $\mathcal{K}^{23} = \mathcal{K}^{13} = \{(0,0)\}$). An alternative approach would be to only allow demands which are feasible to at least one neighbour. If this approach were used, corollary 3.1 can no longer be used to prove the existence of SPEs. If existence is assumed, the proof of theorem 4.3 still applies so the SPE outcome is unchanged. The SPE analysis of the market bargaining game[24] shows that players choose not to make demands which are non-feasible to both neighbours (except for the case of a player who does not exchange in any SPE, whose demand does not affect play). For this reason and the difficulty of proving SPE existence, the alternative rule seems unnecessarily complicated, especially if it must later be generalized to the case of more than 3 players.

Another feature that arguably seems unrealistic is that if a player is a responder and all her offers are infeasible then she must still make the action of refusing them and incur a delay. This feature is a technical condition required to apply corollary 3.1 on SPE existence[25], although intuitively it still seems likely that SPEs exist in

[23]The specification of $m(N)$ means that there exists a player who must make demands which are feasible to at least one neighbour. This fact does not have any significance; $m(N)$ can be increased without affecting any results.

[24]That is, the proof of theorem 4.3 above and that of lemma 4.6 below.

[25] Suppose a responder did not incur a delay should she have no feasible demands in step 4 of the game. Consider the following (infinite) history h. In period t_2 the responder r has a single feasible demand, made by player p in period t_1. The demand is $\lambda = f^{r,p}(0)$. Player r refuses this demand and eventually receives share s_r after delay τ. Any open set (under the topology of S described in corollary 3.1) containing h contains a history h' which is the same as h except that the demand in

models without it[26]. There are some plausible arguments in support of this feature. For example, the delay could be viewed as modelling the time required to prepare a proposal.

4.4.4 Public and Personal Delays

This section argues the case for using personal delays rather than public delays. The argument can be summed up as follows. A perfect information model with public delay may require players to wait and accumulate delay while distant players act. The use of personal delays captures the intuition that each player is only affected by delays caused by local actions.

The market bargaining game with personal delays, $M^{\text{personal}}(N, \Delta)$ has the same definition as $M^{\text{public}}(N, \Delta)$ except for the specification of delays and payoffs. In any finite history h, the personal delay of player i, $\tau_i(h)$, is the number of times player i has refused in the course of the history. If a demand σ_p made by player p is accepted by player r in history h then players p and r receive payoffs $\delta_p^{\tau_p(h)}\sigma_p$ and $\delta_r^{\tau_r(h)} f^{p,r}(\sigma_p)$ respectively. The third player receives payoff zero. In an infinite history all payoffs are zero.

Using personal rather than public delays has little effect if the conditions of theorem 4.3 hold. This is not surprising as the solution described there is independent of the relative time preferences of players, and delay only features briefly in the proof.

Corollary 4.5. *Under the conditions of theorem 4.3, the limiting SPE outcome of $M^{personal}(N, \Delta)$ is the same as that of $M^{public}(N, \Delta)$ given by theorem 4.3.*

Proof. As for theorem 4.3. □

period t_1 is $\lambda + \epsilon$ for $\epsilon > 0$ sufficiently small. In h' player r does not incur a delay in period t_2. Thus $\pi_r(h) = \delta_r^{\tau} s_r$ but $\pi_r(h')\delta_r^{\tau-1} s_r$. This violates continuity on π_r; condition 5 of theorem 3.2.

[26]As discussed in the next section, such SPEs would have some advantageous properties; there would be no need to introduce personal delays for the market bargaining game. However, as argued in section 4.4.4, it is still desirable to introduce them for bargaining models on larger bilateral exchange networks.

At first sight, personal delays do not seem appropriate in a realistic bargaining model. After all, why should it be possible for two players to exchange with each other and experience non-equal delays? The following lemma illustrates the motivation for the use of personal delays. It describes the SPE behaviour of both variations of the market bargaining game for a case where theorem 4.3 does not apply.

Lemma 4.6. *Suppose N is a 3 player ring network such that $\mathcal{K}^{12} = \mathcal{K}^{unit}$ and the core of N contains an element $c = (c_1, c_2, 0)$. For each $i \in P$ fix $\eta_i \in (0,1)$. For $\epsilon > 0$, let $\Delta = (\delta_1, \delta_2, \delta_3)$ where $\delta_i = \eta_i^\epsilon$. Let $\mathcal{M}$ be either $M^{public}(N, \Delta)$ or $M^{personal}(N, \Delta)$. The unique limiting SPE outcome of $\mathcal{M}$ in the limit $\epsilon \to 0$ is $(\phi, 1-\phi, 0)$ where*

$$\phi = f^{3,1}(0) \vee [[1 - f^{3,2}(0)] \wedge n_1]$$
$$n_1 = \lim_{\epsilon \to 0} \frac{1 - \delta_2^w}{1 - \delta_1 \delta_2^w}$$

and w is 1 in the case of personal delay and 2 in the case of public delay.

The infix operators $\vee$ and $\wedge$ represent the maximum and minimum operations respectively. An equivalent definition of ϕ is 'the element of the closed interval bounded by $f^{3,1}(0)$ and $1 - f^{3,2}(0)$ closest to n_1'. It is easy to shown that the conditions on δ_i guarantee that the limit given for n_1 converges. Figure 4.2 shows a network where the conditions of this lemma are met.

Proof. See section 4.6. □

This lemma states that the solution of M^{personal} coincides with that of the alternating offers game on $\mathcal{K}^{12}$ with the same discount factors, and outside options $f^{3,i}(0)$ for players $i = 1$ and 2. For M^{public} the solution coincides with that of the same alternating offers game with outside options, except that the discount factor of player 2 is δ_2^2.

It can be proved that the characterisation of the result just made also holds in the more general case where $\mathcal{K}^{12} \not\equiv \mathcal{K}^{\text{unit}}$. In the case where the core is non-empty and contains an element c such that $c_3 > 0$ and $c_i = 0$, a similar result can be proved[27]. The details of the proof of the general case are along the same lines as the proof given in section 4.6. Only the special case is given to minimise the length of the proof.

A sketch of the proof of lemma 4.6 is as follows. In a subgame at the start of step 4 with responder $i = 1$ or 2, the responder is guaranteed a SPE share of $f^{3,i}(0)$. For example suppose $i = 2$. Player 2 may accept the demand of player 1 if it yields a share of at least $f^{3,2}(0)$. Otherwise, the demand of player 1 cannot be feasible to player 3. In this case player 2 may refuse the demand of player 1 and make any demand feasible to player 3 and it will be accepted in SPE. If it were refused by players 1 and 3 in a SPE they must both receive higher payoffs, but no such outcome is feasible. Players 1 and 2 therefore effectively have outside options equal to the lowest payoff they could receive in the core. Now the arguments of the alternating offers game with outside options can be used. However, in the case of public demands, if player 2 refuses and the most recent demands of players 1 and 2 are not feasible to player 3, then the delay is incremented by 2 since player 2 has refused and player 3 must also refuse. Thus the arguments of the alternating offers game must be used but with player 2 effectively having the discount factor δ_2^2.

Lemma 4.6 illustrates that the solution of the market bargaining game with public discounting is not consistent with that of the alternating offers game. As discussed in section 3.1.1, this can be seen from the case where $\mathcal{K}^{13}$ and $\mathcal{K}^{23}$ are both the trivial cake containing only the origin. The reason for the inconsistency is that player 2 is forced to incur an extra delay by waiting for player 3 to act.

It can be argued that the inconsistent solution produced by the public discounting version of the market bargaining game is simply an artifact of requiring player

[27]There is one case which displays a novel SPE. Let j be the player other than i and 3. If the payoff of player j in the alternating offers game with player 3 on cake $\mathcal{K}^{j3}$ with the appropriate discount factors is less than any payoff he could receive in the core, then the exchange ij forms in SPE, the limiting SPE payoff of player j is $f^{i,j}(0)$ and the others receive zero.

3 to refuse and incur a delay when she has no feasible demands. If this were not the case then all resulting limiting SPEs would match those of the personal delay model. Indeed, perhaps this rule is what was intended by Binmore in [3]. This resolution has two problems. Firstly, as noted above (in footnote 25 of this chapter), corollary 3.1 can no longer be used to prove the existence of SPEs. Secondly, it is not obvious that this resolution works successfully for larger networks, although it certainly does not seem impossible. If it is the case that exchanges form immediately in SPE – and this usually seems to be the case for perfect information models – then there may not be players available to provide disruptive delays after the first round. The inconsistent result of lemma 4.6 hinges on the fact that player 3 provides such a delay in every round. However, it may not be straightforward to prove this immediate exchange result[28]. I prefer to use personal delays because it rules out this possible source of inconsistent solution from the outset.

Note that games using personal delays cannot be referred to as *temporal monopoly* games. This term is useful for games in which players have time preferences, and there is a time value associated with each history of the game satisfying appropriate conditions such as monotonicity. In a temporal monopoly game, only one player may act at a particular time value. However, in a game using personal delays a single history may represent different time values to different players.

Another application of lemma 4.6 is to the network given by $\mathcal{K}^{12} = \mathcal{K}^{23} = \mathcal{K}_{\text{unit}}$ and $\mathcal{K}^{31} = \{(0,0)\}$. The limiting SPE outcome is[29] $(0, 1, 0)$. As discussed in section 4.4.3, this can be viewed as a prediction for the network L_3 defined in section 3.1.1.

[28]The proof of this result *is* typically quite straightforward for games with unilateral demands, such as the telephoning game and various models discussed in section 4.5. However, it is argued in section 4.4.5 that it is desirable to allow multilateral demands to be made in models.

[29]This outcome is in a von Neumman-Morgenstern triple. It can also be demonstrated that this is the outcome by a method based on theorem 4.3.

4.4.5 Multilateral Demands

It is problematic to justify multilateral demands of the sort used in the market bargaining game as part of a realistic bargaining process. One problem is that making proposals to several neighbours with the guarantee that each neighbour will have a chance to accept before the proposer takes any further actions requires a high level of commitment. Another problem occurs if the outcomes in a bargaining situation are more complicated objects than a single numerical value. It may then be a difficult task to create proposals to several neighbours which the proposer is indifferent between, and certainly not one that can be performed immediately in a bargaining situation[30].

In light of these difficulties, I interpret the argument given at the end of section 4.4.1 which Binmore made for allowing the use of multilateral demands as follows. Multilateral demands are not intended as a literal description of the bargaining process. Instead they operate as a device allowing ongoing bilateral bargaining between players to be interrupted. The proof of theorem 4.3 illustrates that there exists a situation in which an individual player has an incentive to do so, as discussed presently. In a more realistic bargaining model, other devices could be employed for this purpose. One candidate is to use a random order of play. Another is to use unilateral demands and a well chosen order of play, as discussed in section 4.5 below. However, multilateral demands have the advantage of producing a tractable and concise model which is consistent with the solution of the alternating offers game.

In the proof of theorem 4.3, it is shown that in the case where the core is empty (and the cakes are insatiable), if the first player in a market bargaining game makes any demand less than their SPE payoff then it is accepted by the following player in SPE. This central argument of the proof is quite robust to variations of the rules. It only requires that both neighbours of the proposer have a chance

[30] This task may be easier in a setting where the bargaining situation is repeated. Bargainers then have an opportunity to become familiar with the available proposals of this sort.

to accept his demand. Indeed, it holds even if no further multilateral offers are permitted. This can be interpreted as meaning that if only a single player is willing to bargain non-bilaterally then they can secure at least as much as in the von Neumann-Morgenstern triple outcome described in theorem 4.3. Since in any other outcome at least one player who exchanges can do better in a von Neumann-Morgenstern triple outcome, this player has an incentive to break a bilateral bargaining convention. Thus maintaining such a convention requires exogenous pressure and it can be seen as the less usual case. This is the main argument in this thesis against the use of telephoning game.

The market bargaining game assumes that proposers must demand the same utility from each possible responder. One may reasonably wonder whether this assumption of *public demands* is necessary since, as noted above, the difficulty to players of producing such proposals may be significant. That is, the case for public demands as part of a literal description of a bargaining process is weak. However, as argued above, the intention of introducing multilateral demands is not to make such a literal description, but to capture the realistic possibility of bilateral bargaining being interrupted. The use of public multilateral demands also allows players the opportunity to commit to unrealistic threats, as illustrated by the following example. Suppose all cakes are unit. Player 1 can initially demand utility $x < 1$ from player 2 and utility 0 from player 3. Player 2 must accept this demand in SPE because otherwise player 3 is guaranteed to receive a SPE payoff of 1, giving player 2 a SPE payoff of zero. In effect player 1 has given player 2 an ultimatum that he will capitulate to player 3 unless player 2 accepts his terms.

4.5 Other Models

Herrero's Model

Herrero's bargaining model of [34] is for a bargaining situation with a set of players P and the single multiplayer utility cake[31]

$$\mathcal{K} = \{(x_i)_{i \in P} \mid x_i \in \mathbb{R}^+, \sum_{i \in P} x_i = 1\}$$

A single discount factor $\delta \in (0, 1)$ which applies to all players is also required. The model is the following straightforward extension of the alternating offers game.

1. Player 1 is the first proposer p.
2. The proposer, p, makes a proposal $\sigma \in \mathcal{K}$.
3. Each player other than the proposer sequentially decides whether to accept or refuse the proposal. If all accept then the game is terminated.
4. Following a refusal, the next proposer is the player satisfying $p' \equiv p + 1 \pmod{|P|}$ and the game returns to step 2.

Delay: The delay, $\tau(h)$, of any finite history h is equal to the number of refusals which have occurred.

Payoffs: If the proposal σ is accepted in history h, then the payoff of player i is $\delta^{\tau(h)}\sigma_i$. In an infinite history all payoffs are zero.

Herrero proves the following result (proposition 4.1 in [34]).

Theorem 4.7. *If $\delta \geq \frac{1}{|P|-1}$ then any $\sigma \in \mathcal{K}$ is a SPE outcome of the model.*

The details of the proof are omitted as they are not required in later arguments. The crucial step is to construct SPEs for each $i \in P$ in which player i receives share 1 and all other players receive zero. If a player deviates from this SPE, he is punished

[31] In fact this definition of a game holds for any utility cake $\mathcal{K} \subseteq (\mathbb{R}^+)^{|P|}$. Herrero's proof of theorem 4.7 also holds for this case with only a few cosmetic modifications.

by the play of a SPE in which the deviator receives zero. If the deviation involved making a proposal, then in the punishment SPE a player for whom the deviator proposed the lowest share receives a share of 1. To tempt the other players into not playing this punishment SPE, the deviator must offer them each a payoff of more than δ. But this is not possible given the condition placed on δ.

The reason that this argument does not apply in bargaining games on bilateral exchange networks is that not all the other players have a veto. To avoid a punishment SPE of the sort described, the deviator need offer only one other player more than δ, which is always possible[32].

Herrero's theorem shows that generalisations of the alternating offers game to general bargaining problems suffer from indeterminate solutions. The result that the market bargaining game possesses a unique SPE suggests that generalisations to the restricted setting of bilateral exchange networks may avoid this problem. This is a motivation for the attempt to develop such models in chapter 5.

Unilateral Demand Exogenous Order Models

In [3], Binmore briefly considers a generalisation of the alternating offers game with the same rules as the market bargaining game (with public delay) except that the responder may accept only the demand of the most recent proposer. In this model, only unilateral demands may be made, but, unlike the telephoning game, the order of play is exogenously fixed. The nth proposer is the player p_n satisfying $p_n \equiv$

[32] A variation on Herrero's model is to require the proposer to only make a demand, and let an exchange form once the sum of the most recent demands is no more than 1. Then such a punishment SPE could not always exist. For example in the 3 player case, consider a strategy profile in which the initial demands of players 1, 2, and 3 are 0, 0, and 1. Player 2 could instead demand $\lambda < 1 - \delta$ and in SPE an exchange would form after player 3's demand. This argument holds under various delay schemes e.g. delay for player i equals a) number of demands of player i minus one (a personal scheme) or b) number of demands of player 1 minus one (a public scheme). In Osborne and Rubinstein [54] (section 3.13) it is mentioned that for a version of this model with more restrictions on play no complete analysis is available.

$n \pmod 3$). Binmore shows that in the case where the core is empty the solution of this model coincides with that of the market bargaining game. However, in the case where the core is non-empty, the model has a serious deficiency. Consider the case where $\mathcal{K}^{12}$ is a unit cake and $\mathcal{K}^{23}$ and $\mathcal{K}^{31}$ are ϵ-unit cakes. Player 1 has a large advantage in this case since he can make demands to player 2, but player 2 cannot make counterdemands. This produces[33] a SPE payoff strongly biased towards player 1. The bias is due to the choice of ordering rather than any aspect of the bargaining situation, so this is not a good candidate for a bargaining model.

For the network just discussed, an alternative player ordering which allows players 1 and 2 to make demands to each other and treats them symmetrically seems more appropriate. However, for this model to be useful, a player ordering is required which is appropriate for any network. Thus it must implement the von Neumann-Morgenstern triple solution in the case of a network whose core is empty. A model involving such an intricate ordering seems a less appealing candidate for a bargaining model than the market bargaining game on grounds of concision, especially as it seems likely that the necessary player ordering would become even more complicated for larger networks.

A Model of Corominas-Bosch

In [23], Corominas-Bosch introduces a model of bargaining for a setting in which players are partitioned into a set of buyers and a set of sellers and all exchange opportunities involve one player from each set. It is assumed that each seller owns an indivisible good and each buyer possesses money. If a seller and a buyer trade at price p and time t, they receive utilities of $\delta^t p$ and $\delta^t(1-p)$ respectively. In the terminology of section 3.1.1, this setting is a bipartite bilateral exchange network with unit cakes.

[33]This can easily be proved along the lines of the proof of lemma 4.1. Define $\bar{\pi}_i$ and $\underline{\pi}_i$ as in that proof. Then $\underline{\pi}_1 \geq 1 - \delta_2\bar{\pi}_2$ and $\bar{\pi}_2 \leq f^{3,2}(0) \vee [1 - \underline{\pi}_1]$. Combining these gives $\bar{\pi}_2 \leq f^{3,2}(0) \vee \delta_2\bar{\pi}_2$, so it must be the case that $\bar{\pi}_2 \leq f^{3,2}(0) = \epsilon$.

An outline of the model is as follows. Either the buyers or sellers are chosen to be the initial proposers. The other players are the responders. The proposers simultaneously make demands. Then each responder must choose to either accept one proposed demand value or refuse them all. This decision is made simultaneously by all responders. A deterministic matching rule is given that selects which exchanges take place[34]. This matching rule guarantees that the maximum possible number of exchanges form. In cases where no two responders accept the same demand, this rule is straightforward, but in other cases it can be quite complex. Agents who exchange are removed from the network. The process is then repeated with the players who responded most recently now taking the proposing role. The time of an exchange corresponds to how many times this process was repeated before the exchange took place.

Corominas-Bosch shows that for many networks there exists a unique SPE outcome under this model. However, there are also networks for which multiple SPE outcomes exist. She gives the example of L_5 (as defined in section 3.1.3). This is a bilateral network with players 2 and 4 on one side and players 1, 3 and 5 on the other. In the case where players 2 and 4 propose first, for certain parameters used by the matching rule there are multiple SPE payoffs[35]. The set of SPE payoffs to players 2 and 4 is $[\frac{1}{1+\delta}, 1]$. Recall that $\frac{1}{1+\delta}$ is the unique SPE payoff to the first proposer in an alternating offers game on a unit cake with common discount factor δ.

This model allows simultaneous actions. An equivalent representation is in terms of imperfect information. Proposers do not necessarily make proposals simultane-

[34]The alternative where responders may choose to accept a particular neighbour is also considered in Corominas-Bosch [23] (in section 3.8). The details of the argument mentioned below showing that L_5 produces multiple SPEs continue to hold.

[35]In the case where players 1, 3 and 5 propose first there is a unique SPE outcome in which they receive payoff zero and player 2 and 4 receive payoff 1. Since all players thus have some interest in not proposing first, this suggests incorporating the decision of when to enter the market as part of the model.

ously, but do make them without knowledge of the other proposals (e.g. proposals are placed in sealed envelopes). This seems a more realistic description of a bargaining situation than either simultaneous actions or perfect information. Indeed, this avoids some unrealistic features of perfect information bargaining models. For example there is no necessity to use personal delays since every player gets to act in every time period. Also it provides a method to avoid the problems of instantly adaptive exchange (discussed below in section 5.3.1). On the other hand, given the presence of simultaneous actions, it seems natural to allow players to use mixed strategies. This suggests SPE analysis may not find all the solutions.

A difficulty with applying the Corominas-Bosch model to the setting of bilateral exchange networks is how to deal with non-bipartite networks. There are many such networks (for example, complete networks) with no obvious structure suggesting a rule to determine which players are the proposers in a given round. Since this rule is likely to have a significant influence on the outcome of the model, an arbitrary choice does not seem satisfactory. It seems more natural to extend the model in alternative bargaining settings which do not restrict players to a single exchange. This would allow some players to be both proposers and responders.

Chatterjee and Dutta [19] investigate similar models to those of Corominas Bosch in the case of a 4 player bipartite network in which each player is connected to both players on the other side of the network. The principal difference is that is no longer assumed that all utility cakes are unit cakes[36]. It is shown that for each of their models there is a case where either the model has no SPE or any SPE involves a delay in reaching agreement. This is in contrast to the behaviour of the model of Corominas-Bosch on the unit cake versions of this networks and casts doubt upon the robustness of the unit cake solutions.

[36]There are other differences. For example, players choose to accept demands of particular players rather than simply demand values.

Models of Calvó-Armengol

Calvó-Armengol [16, 17, 18] proposes a series of bargaining models for bilateral exchange networks with unit cakes under the constraint that all negotiations are bilateral. In [18], a perfect information model is presented in which a neighbour of the proposer is randomly selected to be the responder, and the proposer must then make a unilateral offer to the responder. The responder may accept the offer or refuse and become the next proposer. This model has the restriction that the game ends when a single exchange forms and non-exchanging players receive payoff zero[37]. A unique stationary SPE is found for any network. In [16], a similar model of bargaining in a 3 player line network is proposed, with the difference that the responders are chosen according to a pre-specified order. On the other hand [17] contains a 2-stage model. In the first stage each player selects a single neighbour as their bargaining partner. In the second stage a randomly chosen initial proposer makes a demand. Their bargaining partner is the responder and may accept the demand or refuse and become the next proposer. Again this game terminates once a single exchange forms and non-exchanging players receive zero. The usual approach to discounting is used in all these models. None of these models reproduce the limiting prediction of the market bargaining game for the L_3 given in section 4.4.4 above. Instead they produce an outcomes in which the central player receives an outcome identical to that in the alternating offers game over one of the two cakes. This underlines the discussion in section 4.4.5 that the market bargaining game supports qualitatively different outcomes to the case of purely bilateral negotiations by providing a mechanism for them to be interrupted.

[37]This allows the models to illustrate features arising from the bargaining situation without having to deal with complications of instantly adaptive exchange as discussed below in section 5.3.1 (although a one-exchange rule could be viewed as an extreme case of instantly adaptive exchange).

4.6 Appendix: Proof of Lemma 4.6

Lemma 4.6. *Suppose N is a 3 player ring network such that $\mathcal{K}^{12} = \mathcal{K}^{unit}$ and the core of N contains an element $c = (c_1, c_2, 0)$. For each $i \in P$ fix $\eta_i \in (0, 1)$. For $\epsilon > 0$, let $\Delta = (\delta_1, \delta_2, \delta_3)$ where $\delta_i = \eta_i^\epsilon$. Let $\mathcal{M}$ be either $M^{public}(N, \Delta)$ or $M^{personal}(N, \Delta)$. The unique limiting SPE outcome of $\mathcal{M}$ in the limit $\epsilon \to 0$ is $(\phi, 1 - \phi, 0)$ where*

$$\phi = f^{3,1}(0) \vee [[1 - f^{3,2}(0)] \wedge n_1]$$
$$n_1 = \lim_{\epsilon \to 0} \frac{1 - \delta_2^w}{1 - \delta_1 \delta_2^w}$$

and w is 1 in the case of personal delay and 2 in the case of public delay.

Recall that $\vee$ and $\wedge$ are infix maximum and minimum operators. Figure 4.2 shows a network where the conditions of this lemma are met.

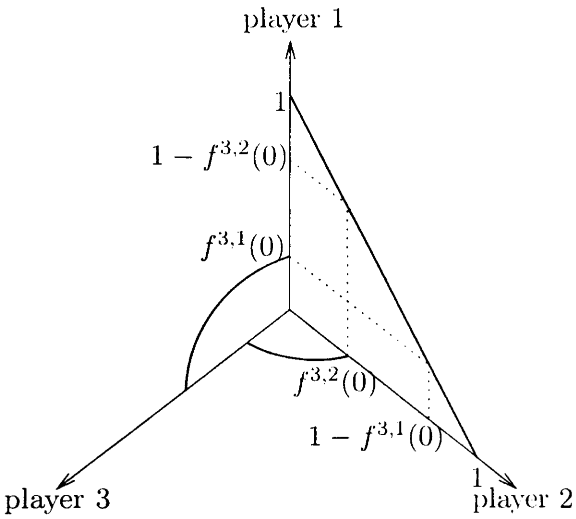

Figure 4.2: A 3 player network with a non-empty core

Proof. Let $w_1 = 1$ and $w_2 = w$. Fix Δ. Define *pre-proposal* and *post-proposal* subgames of $\mathcal{M}$ to be those at the start of, respectively, steps 3 and 4 of the game. Let B_i be the set of pre-proposal subgames of $\mathcal{M}$ with proposer i.

Let $\mathcal{B}$ be a subgame of $\mathcal{M}$. In the case of personal delay, let $\tau_i(\mathcal{B})$ be the associated personal delay of player i. In the case of public delay, let $\tau_i(\mathcal{B})$ be the associated delay. Let $P_i(\mathcal{B})$ be the set of all values $\delta_i^{-\tau_i(\mathcal{B})}\pi_i$ such that π_i is a SPE payoff to player i in $\mathcal{B}$. Note that by corollary 3.1, $\mathcal{M}$ has a SPE and therefore so does $\mathcal{B}$. Hence P_i is non-empty.

Let $\Pi_1 = \bigcup_{\mathcal{B}\in B_1} P_1(\mathcal{B})$. Let $\Pi_2 = \bigcup_{\mathcal{B}} P_2(\mathcal{B})$ where the union is taken over the subset of B_2 such that the most recent demand of player 1 is more than $f^{3,1}(0)$. Let $\bar{\pi}_i = \sup \Pi_i$ and $\underline{\pi}_i = \inf \Pi_i$.

It must be the case that for $x \in \{1, 2\}$, $c_x \geq f^{3,x}(0)$. Otherwise $(0, c_x)$ would be Pareto dominated by $(0, f^{3,x}(0)) \in \mathcal{K}^{3x}$ and c would not be in the core. This gives $1 - f^{3,2}(0) \geq 1 - c_2 = c_1 \geq f^{3,1}(0)$ and so:

$$1 - f^{3,2}(0) \geq f^{3,1}(0) \tag{4.7}$$

Let $(i, j) = (1, 2)$ or $(2, 1)$. Let $\mathcal{A}$ be a pre-proposal subgame of $\mathcal{M}$ with proposer i. In the case $i = 2$ suppose also that the most recent demand of player 1 in $\mathcal{A}$ is more than $f^{3,1}(0)$.

Suppose player i demands $\lambda_i < f^{3,i}(0)$ in $\mathcal{A}$. Let $\mathcal{B}$ be the resulting subgame. Suppose that in a SPE of $\mathcal{B}$ a post-proposal subgame $\mathcal{B}'$ is reached with responder j in which the demand λ_i is available. By (4.7), $\lambda_i < 1 - f^{3,j}(0)$. Thus the share of player j in any SPE of $\mathcal{B}'$ is more than $f^{3,j}(0)$ so the exchange ij must form. Suppose that in a SPE of $\mathcal{B}$ a post-proposal subgame $\mathcal{B}''$ is reached with responder 3 in which the demand λ_i is available. Then the share of player 3 in any SPE of $\mathcal{B}''$ is non-zero. Thus in any SPE of $\mathcal{B}$ the first responder, r, must accept a demand otherwise a contradiction is produced. If $r = 3$ then the only feasible demand is that of player i. If $r = j$ then accepting the demand of player i results in a better payoff than accepting any demand of player 3 since $1 - \lambda_i > f^{3,j}(0)$. Thus the demand λ_i is accepted in SPE and so:

$$\underline{\pi}_i \geq f^{3,i}(0) \tag{4.8}$$

Suppose player i demands $\lambda_i < [1 - f^{3,j}(0)] \wedge [1 - \delta_j^{w_j} \bar{\pi}_j]$ in $\mathcal{A}$. Let $\mathcal{C}$ be the resulting subgame. In the case where $\lambda_i < f^{3,i}(0)$ it has already been shown that this demand is accepted in any SPE of $\mathcal{C}$. So suppose $\lambda_i \geq f^{3,i}(0)$. If player 3 is the responder in $\mathcal{C}$ then j must be 1 and player 3 cannot accept the demand of player j since it is infeasible. Thus in any SPE of $\mathcal{C}$, either player 3 accepts the demand of player i, or a post-proposal subgame of $\mathcal{C}$ with responder j, $\mathcal{C}'$, is reached in which the demand λ_i is available (In the case $i = 1$, then $\mathcal{C}' = \mathcal{C}$). Since $1 - \lambda_i > f^{3,j}(0)$, it must be the case that the exchange $3j$ does not form in any SPE of $\mathcal{C}'$. If player j refuses in $\mathcal{C}'$ then the delay she incurs when an exchange forms is at least $\tau_j(\mathcal{C}') + 1$. In the case that $j = 2$, if player 2 refuses in a SPE of $\mathcal{C}'$ then player 3 must also refuse before an exchange forms in that SPE. If public discounting is used then the delay that player 2 incurs when an exchange forms is at least $\tau_2(\mathcal{C}') + 2$. Thus if player j refuses in a SPE of $\mathcal{C}'$ then her SPE payoff is at most $\delta_j^{t+w_j} \bar{\pi}_j$, which is less than that of accepting λ_i in $\mathcal{C}'$. This shows that λ_i is accepted in any SPE of $\mathcal{A}$. However, if $i = 2$ and public discounting is used, the delay may be incremented by 1 before it is accepted. Thus

$$\underline{\pi}_i \geq \delta_i^{w_i - 1} \left\{ [1 - f^{3,j}(0)] \wedge [1 - \delta_j^{w_j} \bar{\pi}_j] \right\} \tag{4.9}$$

Suppose there is a SPE e of $\mathcal{A}$ in which player i receives a share of

$$\mu_i > f^{3,i}(0) \vee [1 - \delta_j^{w_j} \underline{\pi}_j]$$

Suppose the play e involves a post-proposal subgame of $\mathcal{A}$ with responder 3 being reached. Then it must be the case that in e player 3 refuses in this subgame for the following reason. If the exchange $3i$ forms then the share of player i is no more than $f^{3,i}(0)$. If the exchange $3j$ forms then the share of player i is zero. Therefore it must be the case that the play of e involves a post-proposal subgame $\mathcal{D}$ with responder j being reached. In the play of e, player j receives a payoff of less than $\alpha_j = \delta_j^{\tau_j(\mathcal{D})}(1 - \mu_i)$. Let λ_i be the most recent demand of player i in $\mathcal{D}$. If $\lambda_i \leq f^{3,i}(0)$ then player j could earn a higher payoff than α_j in $\mathcal{D}$ by accepted the demand of player i. If $\lambda_i > f^{3,i}(0)$ then player j could earn a payoff of at least $\delta_j^{\tau_i(\mathcal{D})+w_j}\underline{\pi}_j$ in SPE of $\mathcal{D}$. This contradicts the existence of e. Thus

$$\overline{\pi}_i \leq f^{3,i}(0) \vee [1 - \delta_j^{w_j}\underline{\pi}_j] \tag{4.10}$$

Let

$$m_i = \frac{1 - \delta_j^{w_j}}{1 - \delta_i^{w_i}\delta_j^{w_j}}$$

Note that m_i has the property that $x - [1 - \delta_j^{w_j}(1 - \delta_i^{w_i}x)]$ has the same sign as $x - m_i$. Thus substituting (4.10) into (4.9) and combining the result with (4.8) gives:

$$\underline{\pi}_i \geq \delta_i^{w_i - 1}\left\{f^{3,i}(0) \vee [[1 - f^{3,j}(0)] \wedge m_i]\right\} \tag{4.11}$$

Substituting this into (4.10) gives:

$$\overline{\pi}_i \leq f^{3,i}(0) \vee A \vee B \tag{4.12}$$

where

$$A = [1 - \delta^* f^{3,j}(0)] \wedge [1 - \delta^* + \delta^* f^{3,i}(0)]$$

$$B = [1 - \delta^* f^{3,j}(0)] \wedge [1 - \delta^* m_i]$$

$$\delta^* = \delta_j^{2w_j - 1}$$

Observe that taking the limit $\epsilon \to 0$ in equations (4.11) and (4.12) yields $\lim_{\epsilon\to 0}\underline{\pi}_1 \geq \phi$ and $\lim_{\epsilon\to 0}\overline{\pi}_1 \leq \phi$. Since $\underline{\pi}_1 \leq \overline{\pi}_1$, it must be the case that $\lim_{\epsilon\to 0}\underline{\pi}_1 = \lim_{\epsilon\to 0}\overline{\pi}_1 = \phi$. Thus ϕ is the unique limiting SPE payoff to player 1 as required.

Let $\mathcal{E}$ be the subgame resulting from an initial demand of λ_1 by player 1 in $\mathcal{M}$. In the case that $\lambda_1 \leq f^{3,1}(0)$ the SPE payoff of player 2 in $\mathcal{E}$ is at least $1 - \lambda_1 \geq 1 - \phi$. In the case that $\lambda_1 > f^{3,1}(0)$ the SPE payoff of player 2 in $\mathcal{E}$ is at least $\delta_2\underline{\pi}_2$. Using equation (4.11) and taking the limit $\epsilon \to 0$, this also gives a lower bound of $1 - \phi$. Thus the limiting SPE payoff of player 2 in $\mathcal{M}$ is at least $1 - \phi$. It cannot be higher or for some $\epsilon > 0$ there must be a SPE with payoffs that are not feasible in N. □

Chapter 5

Novel Extensions of the Alternating Offers Game

This chapter contains two bargaining models which extend the market bargaining game of the previous chapter to model bargaining in bilateral exchange networks with more than 3 players. Section 5.1 presents the *exogenous ordering model*. This is a straightforward extension of the market bargaining game. It requires an exogenously specified ordering on the players which represents the order in which they play. This model does not produce as precise a prediction as the market bargaining game; the SPE outcome is shown to be highly dependent on the exogenous ordering. This is illustrated for the network L_5 as defined in section 3.1.3.

The existence of multiple solutions motivates the *endogenous ordering model* of section 5.2. In this model players' actions determine the order of play, although the first player to act must still be exogenously chosen. It is shown that this model also supports a wide range of SPE outcomes for L_5. Also, proving this result requires exhaustive consideration of many cases. This suggests that there may be many larger networks for which solving this model is not practical. Finally, the rules that are required to allow an endogenous ordering while retaining the character of the market bargaining game seem quite unrealistic.

Section 5.3 discusses the multiple solutions found for both models in greater depth. It also introduces and discusses the concept of *instantly adaptive exchange*, an often undesirable feature of many perfect information bargaining models including those of this chapter. Sections 5.4 is an appendix containing most of the proofs for this chapter.

Due to the problems detailed in this chapter, an approach to modelling bargaining in general bilateral exchange networks based on the market bargaining game does not seem feasible. This conclusion is discussed in more detail in section 9.1.2 of the concluding chapter. However, the models of this chapter do produce some predictions, especially for small networks. Interpretation of these results is postponed until section 9.2 of the concluding chapter, where they are compared with the predictions of other chapters.

5.1 The Exogenous Ordering Model

5.1.1 Definition

This is a direct generalisation of the market bargaining game in that it also uses an exogenous order of play. As in the market bargaining game, the player ordering is embedded in the labelling of the players: recall the assumption that $P = \{1, 2, \ldots, n\}$. As well as a network $N = (P, E, \mathcal{K})$, the model requires a vector of discount factors $\Delta = (\delta_i)_{i \in P}$ where $\delta_i \in (0, 1)$. Define $m(N)$ to be the maximum demand which is feasible[1] from some player to another in N.

The exogenous ordering model produces the following game, $X(N, \Delta)$:

1. Initially all players are active. Players 1 and 2 are respectively the first proposer and responder.

[1]Recall from section 3.1.1 that a demand σ_p by player p is said to be feasible to player r if $\sigma_p \leq f^{r,p}(0)$.

2. The proposer p makes a demand[2] $\sigma_p \in [0, m(N)]$.

3. The responder r may either accept the most recent demand of any active neighbour if it is feasible, or refuse all demands.

4. a) If r accepts then players p and r exchange and become inactive. Any players with no active neighbours also become inactive. If no active players remain then the game terminates. Otherwise the new responder r' is the minimal active player $i > r$, or if no such player exists, simply the minimal active player i. The game returns to step 3.

   b) If r refuses, the new proposer p' is r and the new responder r' is as defined in 4 a). The game returns to step 2.

Delay: In any finite history h, the personal delay of player i, $\tau_i(h)$, is the number of times player i has refused in the course of the history.

Payoffs: Let h be a terminal or infinite history. If a demand σ_p made by player p was accepted by player r in a subhistory h' of h then the payoffs of players p and r in h are $\delta_p^{\tau_p(h')}\sigma_p$ and $\delta_r^{\tau_r(h')}f^{p,r}(\sigma_p)$ respectively. All players who are not allocated a payoff in this way receive zero.

Note that once players exchange their personal delays do not increase. Thus payoffs are well defined. The definition of a payoff is slightly more complicated than in the previous chapter because there is now the possibility of an infinite history in which some players exchange but others continue to bargain indefinitely.

The definition above of an active player aims to describe those who have not yet exchanged but still have a possibility of doing so. Thus a player can become inactive either by taking part in an exchange or by all their neighbours doing so and thereby losing the possibility of taking part in an exchange.

A subgame in which the next action to be taken must be in step 2 is called a

[2]The restriction on demands to a closed interval is a technical condition required for corollary 3.1 to hold. Defining $m(N)$ to take a greater value would not affect the following analysis.

pre-proposal subgame. One in which the next action must be in step 3 is called a *post-proposal subgame.* Thus the subgames of a game $\mathcal{X}$ generated by this bargaining model are partitioned into pre- and post-proposal subgames. Note that there exists another game which is clearly equivalent[3] to $\mathcal{X}$ but does not permit such a partition to be made. This game requires a responder to either accept a demand or make a new demand. The latter case implies that the responder has rejected all feasible demands. Choosing a representation of the model which allows a partition into pre- and post- proposal subgames simplifies SPE analysis. For example, observe that the proof of lemma 4.1 on the SPE behaviour of the alternating offers game is based on the SPE payoffs in similarly defined pre-proposal subgames.

Observe that in a post-proposal subgame where the responder has no feasible demand it is necessary for her to refuse and incur a delay cost. A similar feature is found in the market bargaining game. The reasons for this are discussed in section 4.4.4.

For 2 and 3 player networks, the exogenous ordering model gives bargaining games which are the personal delay versions of the alternating offers and market bargaining games respectively. Hence it is consistent with earlier results on these networks. Also note that for games generated by this model, if a situation is reached in which a connected component of the subgraph induced by active players contains only 2 or 3 players then in this component play continues[4] as in the alternating offers or market bargaining game. This observation is often crucial to the SPE behaviour of this model.

[3]In the sense that there is a payoff preserving bijection between the sets of infinite histories which also preserves the identities of players who must make actions. See the definition of equivalence up to discounting in section 5.4.1 for a more precise definition.

[4]That is, the players in this component continue using the rules of the alternating offers or market bargaining game. However, they may already have made some demands which can still be accepted, and have already incurred some delays. Also, play may also be occurring outside this component.

5.1.2 Analysis for L_5

This section investigates the SPE behaviour of the exogenous ordering model for the network L_5. In particular, the effect of different orders of play is explored. Since the order of play is embedded in the player numbering, this requires using other networks which are equivalent to L_5 except for this numbering. For this section, let $P = \{1, 2, 3, 4, 5\}$. Given a sequence $p = (p_i)_{1 \leq i \leq 5}$ such that $\{p_i\} = P$, let $E(p) = \{p_1p_2, p_2p_3, p_3p_4, p_4p_5\}$. Then $L_5(p) = (P, E(p), K_{\text{unit}})$ is a 5 player line network with unit cakes. The sequence p can be written $p_1p_2p_3p_4p_5$ as there is no risk of confusion. For example, 54321 represents the sequence such that $p_i = 6 - i$.

By corollary 3.1, $X(N, \Delta)$ has a SPE, and hence so does every subgame. The following two lemmas are on the SPE outcomes of $X(L_5(p), \Delta)$ for two values[5] of p. The proofs of these results rely on a number of supporting lemmas and so are relegating to section 5.4.3 in the appendix of this chapter. However the main arguments are straightforward and are sketched below.

Lemma 5.1. *For each $i \in P$ fix $\eta_i \in (0, 1)$. For $\epsilon > 0$. let $\Delta = (\delta_i)_{i \in P}$ where $\delta_i = \eta_i^{\epsilon}$. Then $X(L_5(31524), \Delta)$ has a unique limiting SPE payoff[6] in the limit $\epsilon \to 0$ in which players 1 and 2 receive payoff 1 and the others receive zero.*

The key part of the proof is as follows. Suppose players 1 and 2 initially demand less than 1. If play reaches player 5 then player 5 is guaranteed a non-zero SPE payoff and it must be the case that either player 3 or 4 receives a SPE payoff of zero. This player would have preferred to accept the initial demand of their neighbour, so this is not SPE play. Thus in SPE both players 3 and 4 accept.

Note that this argument hinges on the fact that players 3 and 4 know which exchange will form if they both refuse. If player 5 randomised between accepting players 1 and 2 this would not be true. Then in the case where players 3 and 4 both

[5] For some other orderings I could not solve the corresponding bargaining game. An example is 54123.

[6] Recall that definition 3.14 defines a limiting SPE payoff.

refuse they would both have non-zero expected payoffs. This would disrupt the SPE argument given and support a solution in which the payoffs to the players are less extreme than 0 and 1.

Lemma 5.2. *Let* $\Delta_\delta = (\delta_i)_{i \in P}$ *such that* $\delta_i = \delta$ *for all* i*. Then* $X(L_5(41325), \Delta_\delta)$ *has two SPE outcomes,* $(1-\bar{n}, \bar{n}, 1-\bar{n}, \bar{n}, 0)$ *and* $(0, \bar{n}, 1-\bar{n}, \bar{n}, 1-\bar{n})$ *where* $\bar{n} = \frac{1}{1+\delta}$.

Recall from lemma 4.1 that $\bar{n}$ is the SPE payoff to the first mover in an alternating offers game on a unit cake in which both discount factors are δ.

A sketch of the proof is as follows. Suppose players 1 and 2 initially demand less than 1. If play reaches players 4 and 5, then they are both guaranteed non-zero SPE payoffs, and player 3 receives zero. Hence in SPE player 3 accepts the lowest demand from players 1 and 2. Suppose player 1 demands $\lambda \leq \bar{n}$. Then either player 3 accepts this, or players 2 and 3 exchange in SPE. In the latter case player 1 is effectively left in an alternating offers game with player 4 who thus accepts the demand of λ in SPE. Suppose player 1 demands $\lambda > \bar{n}$ initially. If player 2 demands more than λ then players 1 and 3 exchange, leaving player 2 effectively in an alternating offers game with player 5 so player 2 receives a SPE payoff of $\delta\bar{n}$. A better action for player 2 is to demand slightly less than λ since this is accepted by player 3 in SPE. This leaves player 1 in an alternating offers game with player 4 in which player 1 receives a SPE payoff of $\delta\bar{n}$. Thus the initial action of player 1 in SPE is to demand $\bar{n}$. Using the arguments just given it is quite easy to show that player 2 then also demands $\bar{n}$ in SPE. Which of the two SPE outcomes described in the lemma occurs depends on which neighbour player 3 chooses to accept.

5.2 The Endogenous Ordering Model

The endogenous ordering model is defined in section 5.2.2. Section 5.2.1 is a preliminary section discussing the motation for the rules of this model. Section 5.2.3 discusses some features of these rules. Amongst other things, it is proved that they produce a well defined game. Finally, section 5.2.4 describes the SPE behaviour of games generated by this model for various networks. The proofs are contained in the appendix to this chapter.

5.2.1 Motivation

At first sight the rules of the endogenous ordering model described in section 5.2.2 below seem an unnatural choice. This preliminary section explains how these rules arise from the motivation of producing a perfect information model allowing an endogenous ordering of play while retaining the character of the market bargaining game.

By the latter statement I mean that players must be able to make multilateral demands as described in section 4.4.5. In a perfect information setting such demands must entail a degree of forward commitment; the proposer commits to making no further action until all players to whom the demand was made have had a chance to consider it.

Consider the question of how the next player to act in a perfect information bargaining model is decided endogenously. In a model based on the alternating offers game, the natural mechanism by which a player can influence the future order of play is by making somebody a proposal. Therefore in the endogenous ordering model, the proposer chooses one player to whom his multilateral demand is made as the candidate for next responder.

However such a model allows a situation where a proposer is surrounded by neighbours who have made forward commitments and thus cannot immediately consider the proposer's next demand. In the endogenous ordering model, the proposer chooses a *pseudo-responder*. If the pseudo-responder has made a multilateral demand to some players who have not yet considered it, the right to act next is passed on to one of these. Which of these players receives this right has a crucial effect on the order of play. Thus this choice is endogenised. Proposers must choose an ordering over the set of players to whom their multilateral demand is made. If they later become pseudo-responder, the right to act is passed on to the first player in this set who has not yet had a chance to respond.

In the endogenous ordering game, players are allowed to make multilateral demands to any set of neighbours. There seems no reason to force players to make demands to all neighbours. Players with many neighbours would then be forced into much longer-term forward commitment than those with few neighbours which may well be a significant disadvantage.

The model resulting from the argument in this section does not appear a natural model of bargaining. It seems overly complicated and has artificial seeming features which do not obviously correspond to anything from the original bargaining situation. For example following a proposal it is quite possible for there to be a sequence of pseudo-responders ending in a next responder far from the proposer and SPEs can depend on the opportunity to set up such sequences. This model is investigated anyway to find out whether the two motivating features given at the beginning of this section produce an interesting SPE outcome despite these drawbacks.

5.2.2 Definition

The endogenous ordering model is for a network $N = (P, E, \mathcal{K})$ such that (P, E) is a connected graph[7]. The model also requires a vector of discount factors $\Delta = (\delta_i)_{i \in P}$ where $\delta_i \in (0, 1)$, and a first proposer $p_1 \in P$.

The model produces the following game, $F(N, \Delta, p_1)$:

1. Initially all players are active. p_1 is first proposer.

2. The proposer p makes a demand $\sigma_p \in \mathbb{R}^+$ and chooses an ordered non-empty sequence V_p of distinct active neighbours. σ_p must be feasible to all players in

[7]There is no reason to investigate a non-connected bilateral exchange network rather than study its connected components individually. The condition is imposed simply because it is required for the game to be well defined.

V_p. The pseudo-responder is the first element of V_p.

3. The pseudo-responder ψ becomes the responder if V_ψ contains no players. Otherwise the first element of V_ψ is chosen as the next pseudo-responder and this step is repeated.

4. Let R be the set of active neighbours q of the responder r such that V_q contains r. r may accept the most recent demand of any player in R or refuse all demands.

5. a) If r accepts, players p and r exchange and become inactive. Any players with no active neighbours also become inactive. For all $x \in P$, p and r are removed from the sequence V_x if they are contained in it. If no active players remain the game terminates. Otherwise, the new pseudo-responder is chosen from the set of active players who are neighbours of either x or y, where x and y are the most recently exchanged pair such that either has an active neighbour. An unspecified rule is used to make this choice deterministic. The game returns to step 3.

   b) If r refuses then, for all $x \in P$, r is removed from the sequence V_x if it is contained in it. The new proposer is r. The game returns to step 2.

Delay: In any finite history h, the personal delay of player i, $\tau_i(h)$, is the number of times player i has refused in the course of the history.

Payoffs: Let h be a terminal or infinite history. If a demand σ_p made by player p was accepted by player r in a subhistory h' of h then the payoffs of players p and r in h are $\delta_p^{\tau_p(h')}\sigma_p$ and $\delta_r^{\tau_r(h')}f^{p,r}(\sigma_p)$ respectively. All players who are not allocated a payoff in this way receive zero.

In step 2 an action must be taken of the form giving values of σ_p and V_p. Such a pair is referred to as a *proposition*. A proposition is written in the form $[\sigma_p, (v_1, v_2, \ldots)]$ where $(v_1, v_2, \ldots) = V_p$. A proposition of the form $[\sigma_p, (v_1)]$ is often referred to below as a *unilateral demand of* σ_p *to* v_1. A proposition of the form $[\sigma_p, V_p]$ where V_p includes all active neighbours of p is often referred to a *multilateral demand of* σ_p.

Note that in this description V_i refers to the most recent value of this variable. Formally, it should be thought of as a function whose domain is the set of finite histories of the game. In the analysis of this model the notation $V_x(h)$ is sometimes used to make it clear which value is referred to.

A subgame in which the next action to be taken is in step 2 is called a pre-proposal subgame. One in which the next action is in step 4 is called a post-proposal subgame. As for the exogenous ordering model, the subgames of a game generated by this bargaining model are partitioned into pre- and post-proposal subgames.

5.2.3 Discussion

First it is shown that the model produces well defined games. The choice of a new pseudo-responder in step 5a) is well defined since (P, E) is connected. The following argument shows that step 3 terminates and thus the selection of a responder is well defined. Let h be a finite history which ends with a proposition. Let $\Psi = (\psi_0, \psi_1, \psi_2, \ldots)$ be the sequence of pseudo-responders produced by step 3 following h. By step 3, ψ_{i+1} occurs in $V_{\psi_i}(h)$. If ψ_{i+1} proposed more recently than ψ_i in h then ψ_{i+1} was removed from V_{ψ_i} in step 5. Thus it must be the case that ψ_i proposed more recently than ψ_{i+1} in h, and the sequence Ψ must be finite as required.

The following example for a 4 player line network illustrates a potential problem with the rules of the model. Suppose player 1 and 2 use strategies in which they refuse all demands and always make propositions such that $V_1 = (2)$ and $V_2 = (1)$. This results in an infinite history in which no exchanges form and all players therefore receive payoff zero. Players 3 and 4 are denied an opportunity to exchange even though they take no actions and incur no delays. A possible resolution of this problem is that if such strategies are used then players 1 and 2 are deemed to have exchanged, both receiving payoff zero, and the game continues as in step 5a). A rule which covers all cases of this problem would be complicated. However, this is not required as the problem is unlikely to emerge in SPE: it would require neighbouring players to refuse all demands from each other and prefer to receive payoff zero[8]. It certainly does not affect SPE behaviour in the results of this section.

By corollary 3.1, $F(N, \Delta, p_1)$ has a SPE and hence so do all its subgames. The restriction on the demand σ_p to be feasible to players included in V_p is a technical condition required by this corollary. It seems intuitively unlikely that players would wish to make non-feasible demands in SPE. This is especially so for the unit cake networks studied in this section since a demand is either feasible to all neighbours or to none. In general however it cannot be ruled out since propositions involving non-feasible demands could allow the future order of play to be influenced in such a way as to alter SPE behaviour.

The method of choosing a new pseudo-responder following an exchange which is outlined in step 5a) is natural for tree networks such as those investigated in this section. In this case, following an exchange the active players are split up into separate connected components. Play in each component should continue, and since there are no exchange opportunities between components, which component plays first is irrelevant. Consider any component. Amongst all players in this component with an exchanged neighbour, let a be that with an exchanged neighbour whose exchange was most recent. The choice is unique since (P, E) is a tree. Had an exchange not taken place then a would have been the next pseudo-responder in this component. So it is consistent to let a be the first pseudo-responder in its component following the exchange. The choice of pseudo-responder is deterministic because, as discussed in section 3.3.2, the introduction of random moves complicates the proof

[8]Problematic SPEs are thus only possible in a network with a cake of the form $\{(0, 0)\}$. Even in a network where such a SPE exists there would be another SPE in which the players in question receive zero by exchanging. The problematic SPE could then simply be ignored rather than adding extra rules to eliminate it.

of corollary 3.1 on SPE existence. However, it is intuitively obvious that for tree networks a random rule will give the same SPE structure.

Although the choice of a pseudo-responder is well-defined for non-tree networks, it is rather ad-hoc. In addition the particular deterministic (or random) rule used may well affect the SPE structure. Further work would be required to determine which, if any, rule is suitable for these networks.

5.2.4 Analysis for Particular Networks

This section describes the SPE behaviour of the endogenous ordering model for various small networks. The proof of the results in this section are placed in sections 5.4.4 – 5.4.7 of the appendix to this chapter as they are quite lengthy. However sketches of the key parts of these proofs are given. Note that discussion and interpretation of this behaviour is postponed to section 5.3.2, which discusses the results of both models in this chapter, and the conclusion, chapter 9.

In the case of a 2 player bilateral exchange network, F reduces to an alternating offers game with personal delays. In the case of a 3 player line network, lemma 5.4 shows that the limiting SPE payoffs are $(0, 1, 0)$. In the case of a 3 player ring network, F does not reduce to the market bargaining game with personal delays. Section 4.4 contains results giving the limiting SPE outcomes of the market bargaining game in two situations. These results are theorem 4.3 and lemma 4.6. Under the conditions of these results, F has limiting SPE outcomes which represent essentially the same solutions[9]. The proofs for these cases are by similar methods to theorem 4.3 and lemma 4.6. However under the conditions of lemma 4.6 another class of SPEs exists for F, providing other possible limiting SPE payoffs.

[9] There are two differences. First, in F the first actor may be any player, rather than 1. Secondly, under the conditions of theorem 4.3 – a von Neumann-Morgenstern triple exists – the result for F does not predict whom the first actor chooses to exchange with. However, the exchange reached is the same as in theorem 4.3 if the first actor were renumbered as player 1 and the other exchanging player were renumbered as player 2.

An example of a SPE in this class is as follows. Recall that the conditions of lemma 4.6 are that $\mathcal{K}^{12} = \mathcal{K}_{\text{unit}}$ and there is an element of the core in which player 3 receives payoff zero. See diagram 4.2 (page 104) for an illustration of this case. Suppose player 3 is the first to act and makes an initial proposition of $(\sigma_3, [1,2])$ for any value of σ_3. Player 1 then makes the proposition $(f^{3,1}(0), [3])$. This results in a post-proposal subgame with responder 2 in which player 3 has an offer of payoff zero on the table from player 1. In the SPE under discussion, player 2 now makes a particular demand of σ_2 to player 1. Player 3 then refuses the offer of zero, makes some other proposition and player 1 accepts the demand of player 2. If player 2 makes an initial demand of more than σ_2 then player 3 accepts the offer of zero from player 1 and player 2 receives zero. The value of σ_2 can be chosen sufficiently small so that player 1 receives a better payoff from his initial action than by acting as in the SPE described in lemma 4.6.

These SPEs involve player 3 deciding to accept and refuse offers of zero as it benefits player 1. They therefore do not seem very robust[10].

For the network L_4 and uniform discount factors of δ, multiple limiting SPE outcomes again exist under F. However, there are only two: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2}, 0)$.

In the case of L_5, the endogenous ordering model supports a wide range of multiple SPE outcomes. However, these are much more plausible than those described above for L_3. In section 5.4.7 of the appendix it is shown that at least two SPEs exist for the game $\mathcal{A}_3 = F(L_5, \Delta_\delta, 3)$. Recall that Δ_δ refers to discount factor vector in which all discount factors equal δ. In the high payoff SPE, player 3 receives a payoff of $\bar{n}$. In the low payoff SPE, player 3 receives a payoff in an interval close to 0, both bounds of which tend to 0 as $\delta \to 1$.

The key difference in behaviour in these two SPEs is as follows. Suppose player 3 makes an initial proposition of $(\sigma_3, [2, 4])$ where $\sigma_3 < \bar{n}$. Suppose that player 2

[10]For example they seem unlikely to be stable evolutionarily. Also, suppose players' payoffs from exchanging were subject to small random positive perturations. Then player 3 would not be indifferent between an offer of zero and not exchanging.

eventually makes a counter demand of $\sigma_2 \geq \bar{n}$ to player 3, either unilaterally or multilaterally. The resulting subgame is a post-proposal subgame with responder player 4. Suppose player 4 refuses. It turns out that the resulting subgame effectively has the same SPE behaviour as a 3 player line network with players 3,4,5 in which player 4 is the central player and proposer and player 3 has an outside option of $1-\sigma_2$. That is, in any SPE of this subgame player 4 makes a multilateral demand of σ_2 and it is accepted. However, player 4 can choose to exchange with either player 3 or 5 and is indifferent between these choices. If the exchange 34 forms then player 2 is left in an alternating offers game and would have preferred to accept the initial demand of σ_3. This supports the high payoff SPE of $\mathcal{A}_3$. If the exchange 45 forms instead then player 3 is left in bargaining game on a 3 player line network with players 1,2,3 and would wish to have made a very low initial offer. This supports the low payoff SPE of $\mathcal{A}_3$[11].

It is also proved that in any bargaining game generated by the endogenous ordering model on L_5 with discount factors Δ_δ, players 1,3,5 receive limiting SPE payoffs of no more than $\frac{1}{2}$ and players 2,4 receive limiting SPE payoffs of at least $\frac{1}{2}$. Indeed, it can be shown that the limiting SPE payoffs of player i in $F(L_5, \Delta_\delta, i)$ in the limit $\delta \to 1$ are $[0, \frac{1}{2}]$ for $i = 1,3,5$ and $[\frac{1}{2}, 1]$ for $i = 2,4$. The proof of this is omitted as it is very lengthy and does not add much to the discussion of this model.

5.3 Discussion

Section 5.3.1 introduces and discusses the concept of *instantly adaptive exchange*, a feature of many perfect information models of bargaining, including those of this chapter, which limits their potential usefulness in modelling bargaining in large networks. Section 5.3.2 attempts to interpret the multiple solutions that have been found for L_5 under the bargaining models of this chapter and discusses possible

[11]Other possible initial propositions of player 3 are also considered in the full proof of the existence of this SPE.

resolutions to this problem.

Five player line networks with unit cakes are often under discussion in this section. Throughout the players in these networks are numbered according to the definition of L_5 from section 3.1.3, not the alternative definitions of the form $L_5(p)$ used in section 5.1.2.

5.3.1 Instantly Adaptive Exchange

Once bilateral exchange networks with more than 3 players are considered, the possibility is introduced that more than one exchange occurs. This section describes a difficulty of modelling this within the framework of a perfect information game. The problem is that such games do not seem able to capture time lag in the transmission of information about the formation of exchanges. I use the term *instantly adaptive exchange* to describe a situation in which exchanges all form in different periods of the game and the identity or the terms of the exchanges that form in SPE are highly sensitive to the structure of the active network in the period at which the exchange forms. In such a situation players must be able to instantly adapt their behaviours to take account of the reduced network of active players that remains. It is also necessary that any opportunities for forward commitment do not override this adaptation. This definition of instantly adaptive exchange is somewhat vague[12]. However, the most important feature is that it precludes a situation in which two exchanges in a bargaining network form based on the same knowledge of active network structure. Such an outcome seems a likely feature of bargaining situations in large networks where exchanges may form near-simultaneously in distant parts of the network. A consequence of instantly adaptive exchange is that the order of play

[12] For example, any game can be expressed in an equivalent form (i.e. strategic form) in which players all simultaneously choose a strategy in the first period and this decides the outcome. In this case all exchanges form in the first period. To avoid this problem the phrase 'under some representation of the game' could be added to the definition above. The resulting definition would be hard to apply in practice.

becomes crucial in deciding the SPE outcome. This section discusses these points.

Consider an extension of the telephoning game of section 4.3 in which any bargaining network can be used and there is a rule choosing a new proposer following an exchange. The structure of any subgame of this model in which a demand must be made depends entirely on the active network remaining and which player is the proposer[13]. This model can clearly only support instantly adaptive exchange. For models based on the market bargaining game there is some scope for avoiding instantly adaptive exchange. The structure of subgames in these models also often depends on the most recent demands of some players. In other words, multilateral demands allow an element of forward commitment. However, such a demand must still be accepted by a responder who is fully informed of the current active network structure. Whether this mechanism can avoid instantly adaptive exchange must be resolved by SPE analysis.

Instantly adaptive exchange is crucial in generating many of the solutions of this chapter. For example, consider the SPE described in lemma 5.2 for the exogenous ordering model on network L_5. As discussed in section 5.1.2, on her first turn player[14] 3 accepts the demand of either player 2 or 4. The player that is not accepted faces a 2 player bargaining situation in which their SPE payoff is at most $\delta \bar{n}$. It is this payoff which drives the initial interaction between players 2 3 and 4. Also there are many examples in the arguments of sections 5.4.6 and 5.4.7 where player 1 3 or 5 accepts a demand because otherwise they would be left as the outlying player of 3 remaining active players and thus receive a SPE payoff close to zero.

In a bargaining network of many players, it is intuitively likely that some ex-

[13]More formally, any two subgames in which the active network and proposing player are the same could be said to be equivalent up to discounting, as defined in section 5.4.1. Lemma 5.3 of that section then proves that these subgames have the same SPE structure. To use the definition of equivalence up to discounting would require analogues of some terms defined for the exogenous and endogenous models to be defined for the telephoning model under discussion, but this is not complicated.

[14]Recall that the players are numbered as in L_5 here, not $L_5(p)$ as in lemma 5.2.

changes may form 'as-if simultaneously'. That is, players can form an exchange without realising that an exchange in a distant part of the network formed very recently. This may be because it is physically impossible due to the distances involved, or simply because the players cannot pay attention to all aspects of the situation at once. In any case, this behaviour is not possible under instantly adaptive exchange.

It is not obvious that instantly adaptive exchange has a significant effect on the outcome of an exchange in a large bargaining network. For example consider a bipartite unit cake network (as defined in section 3.1.3) in which each player has an exchange opportunity with all players on the other side and both sides contain a large number of players. Intuitively, the effect of any removing any one exchange is very small. However for a more sparsely connected network intuitively the formation of a single exchange can have a significant effect at least on a few players. For example a player's position can be significantly strengthened by the removal of a neighbour's only alternative partner.

An important consequence of instantly adaptive exchange develops from its relation with perfect information. A common feature of perfect information bargaining models is that players who exchange in SPE do so without incurring delay. That is, their first action is either to accept a demand or to make a demand that is later accepted. In this case the structure of the subnetwork of active players at any period in the game is highly influenced by the order of play. This can have a large influence on the solution through instantly adaptive exchange. This is especially true for the last few players to exchange. For example in a unit cake network there is a wide difference between being left in an active subnetwork which is a 3 player line and one with 2 players.

One method of avoiding instantly adaptive exchange is to use a model allowing simultaneous actions so that more than one exchange can form in a period of the game[15]. The model of Corominas-Bosch in section 4.5 does this while retaining

[15]This should be done so that it naturally represents simultaneous bargaining rather than com-

some of the flavour of the alternating offers game. Alternatively, chapter 6 offers an extension of the Nash demand game to general bilateral exchange networks. This model abstracts away much of the detail of bargaining and produces games in which players all take a single simple action simultaneously. Another method to avoid instantly adaptive exchange is to use models with more complex specifications of imperfect information. However these demand more complicated solution concepts such as sequential equilibrium (see e.g. Myserson [52]), which make them unlikely to be amenable to analysis.

Instantly adaptive exchange seems more realistic in situations containing features such as a small number of players, fast transfer of information about the formation of exchanges to all bargainers, a slow pace of bargaining, or a team of people in each bargaining position (on the grounds that teams will be able to keep track of more information than individuals). A possible example is firms competing for a small number of contracts. A situation lacking these features is a busy marketplace. Given their small size, there is a case that instantly adaptive exchange is more relevant for the networks discussed in this chapter. Thus the comments of this section mainly raise concerns about modelling large networks using games of perfect information.

Finally, note that the design of computer based laboratory experiments may often influence whether play can match instantly adaptive exchange. For example if the computer program allows subjects to act at any time and gives them full information about exchanges then this may make it unlikely for subjects to exchange without being aware of all information about prior exchanges. On the other hand if subjects must take simultaneous actions or are given limited information about other exchanges then this possibility may often occur. Therefore experimental results should not be used to infer results about the conditions in which instantly adaptive exchange takes place unless this source of possible bias has been taken into account.

pressing non-simultaneous actions into a single period by use of complicated strategies, as in the example of footnote 12 of this chapter.

5.3.2 Multiple Solutions

This section begins with an interpretation of the results in this chapter demonstrating the existence of multiple solutions for L_5. The wider question of whether the problem of multiple solutions for this network can be resolved using these models or other methods is then addressed.

The SPE for the exogenous ordering model described in lemma 5.2 can be interpreted as follows. Players 2 and 4 cooperate so that players 1 and 5 are both faced with the threat that their only neighbour will exchange with player 3 if they do not meet his terms. In the SPE described in lemma 5.1 if, say, player 2 attempted to make a similar threat by demanding more than $\bar{n}$, then player 4 would undercut this demand and exchange with player 3, leaving player 2 a payoff of slightly less than $\bar{n}$ from the resulting alternating offers game with player 1. In this case players 2 and 4 compete rather than cooperate and this drives down their payoffs.

For a solution of the exogenous ordering model to describe a stable outcome in an ongoing bargaining process, it must be the case that the corresponding ordering remains constant. In an actual bargaining process this seems unrealistic. For example the exogenous factors determining order might change very easily. Also the players have strong incentives to alter their position in the ordering. For example given the ordering of lemma 5.1, player 3 would wish to act before players 1 and 5 to produce the ordering of lemma 5.2. This raises the possibility that a model with an endogenous order of play might select among the multiple solutions mentioned.

However, the endogenous ordering model on L_5 can support solutions with similar interpretations to those of the exogenous ordering game. Section 5.2.4 states that in a game in which player 3 is the first proposer, she can attain a limiting SPE payoff of $\frac{1}{2}$ or 0. A crucial difference between behaviour in these SPEs that generates this difference in payoff for player 3 occurs in a subgame in which player 4 is proposer for the first time. Player 4 can exchange immediately with either player 3 or 5 and is indifferent between these choices. If player 4 undercuts player 2 to exchange with player 3 then player 2 is left in an alternating offers game with player 1 and receives a poor SPE payoff. If player 4 instead exchanges with player 5 then a player 2 is left at the centre of a three player subnetwork of active players and receives a high SPE payoff. The former case can be interpreted as player 4 choosing to compete with player 2 to exchange with player 3, whereas the latter case can be interpreted as player 4 instead cooperating with player 2. Thus this model does not resolve the tension between cooperation and competition[16].

This interpretation of the multiple SPE outcomes in L_5 suggests that for both models the existence of SPE outcomes with large qualitative differences in payoffs is likely to be robust to many variations in the cakes[17]. Recall that the model of Corominas Bosch described in section 4.5 can also support limiting SPE payoffs in the range $[\frac{1}{2}, 1]$ for players 2 and 4 in L_5. This is interesting because it suggests that the existence of multiple solutions may not be driven by instantly adaptive exchange.

The results of section 5.2.4 show that the endogenous ordering model supports a near-unique SPE outcome for the network L_4. However it seems unlikely that the models of this chapter can support such SPE behaviour for sufficiently large networks. This is a consequence of instantly adaptive exchange. As argued in section 5.3.1, under instantly adaptive exchange the order of play has a large effect on the order of exchange. If an order of exchange is possible in SPE leaving a connected component sufficiently similar to L_5, then the arguments earlier in this section suggest that this has a wide range of SPE outcomes. This possibility could easily cause a wide range of multiple SPE payoffs in many other positions in the network.

The existence of a certain kind of diversity in bargaining outcomes for large

[16]Indeed the full characterisation of the SPEs of this model mentioned at the end of section 5.2.4 shows that these extremes can be used to generate a wide range of intermediate SPE outcomes.

[17]The proofs for the exogenous ordering game offer scope for adaptation to settings with other cakes. That of the endogenous ordering game is too complicated to easily allow this.

networks is natural. Since there are exogenous or random factors determining which players get to act first and exchanges are formed by local decisions, there appears to be no mechanism to enforce the appearance of the same global pattern of which exchanges form. Also, since players are only interested in their payoff upon exchange they can thus be indifferent between some alternatives which have a significant effect on the future order of play. As described above, this provides a mechanism[18] for generating multiple SPE outcomes in the endogenous ordering model on L_5. These arguments suggest that multiple solutions are not necessarily unrealistic, especially for large networks. However, the results for L_5 in this chapter allow only a very weak characterisation of possible solutions[19]. The allowed solutions are not necessarily all of equal relevance. For example, the experimental results on L_5 in section 2.7 are sharper. One possible resolution is to use an evolutionary approach: construct a dynamic model of behaviour in a bargaining situation repeated over time. Chapter 6 outlines such an approach.

It could be argued that for particular networks certain exogenous orders of play based on the structure of the network are more natural than other orderings. An example is for play in a line network to begin at one end and move along the line towards the other end. However, it is not clear that the ease of stating this order corresponds to a natural order of play. Also this network structure vanishes, or at least becomes less clear, if other negligibly small exchange opportunities are added, so this is not a robust resolution.

Another possible resolution is to endogenise the ordering by introducing a 'pre-bargaining' game which decides it. However, the wide difference in solutions supported by different orderings for L_5 means that the pre-bargaining game must itself effectively solve a substantial bargaining problem. Furthermore it seems more natural to allow players to make ordering decisions as part of the bargaining process

[18]Note that instantly adaptive exchange also plays a major role in this mechanism.

[19]That is, corollary 5.6, which states that players 1 3 and 5 receive payoffs of no more than $\frac{1}{2}$ whereas player 2 and 4 receive payoffs of no less than $\frac{1}{2}$.

rather than committing themselves beforehand.

5.4 Appendix: Proofs

Section 5.4.1 introduces some definitions and notation for use in this appendix. These involve concepts which are easy to deal with in an ad-hoc manner in the proofs in chapter 4. Section 5.4.2 contains various supporting lemmas for both the exogenous and endogenous ordering models which are used throughout the remaining material. Section 5.4.3 proves the lemmas of section 5.1.2 on the exogenous ordering model. Sections 5.4.4 – 5.4.7 prove the results of section 5.2.4 on the endogenous ordering model.

5.4.1 Definitions

Relative Payoffs

In each of the bargaining models of this chapter, all pre-proposal subgames have a similar structure. Players' roles may change (e.g. the players who are proposer and responder change), as may various other properties of the game (e.g. the set of active players or the set of demands which have been made and not yet refused). One particular such property is the values of the delays which players have already incurred. The following definition allows for comparisons between subgames without having to take these delays into account.

Consider a bargaining game $\mathcal{G}$ generated by either of the bargaining models of this chapter from the network $N = (P, E, \mathcal{K})$ and discount factors $(\delta_i)_{i\in P}$. Let h be a terminal or infinite history of $\mathcal{G}$ and h' be a subhistory of h. The *relative payoff* to player i in h with respect to h' is[20]

$$\pi_i(h|h') = \delta_i^{-\tau_i(h')}\pi_i(h)$$

[20]Recall that the expression on the right hand side of this formula has already been used in the proofs of lemmas 4.1 and 4.6.

Let $\mathcal{J}$ be the subgame generated by h'. Then the relative payoff to player i from h with respect to J is

$$\pi_i(h|\mathcal{J}) = \pi_i(h|h')$$

Note that $\pi_i(h|h')$ is occasionally referred to simply as 'the relative payoff to player i in h' when there is no confusion about the value of h'. Also, observe that for terminal h:

$$\pi_i(h|h') = \delta_i^{\tau_i(h)-\tau_i(h')} q_i(h)$$

That is, $\pi_i(h|h')$ is the share player i receives from the history h discounted by only the delay that has been incurred after the subhistory h'.

Suppose that $\mathcal{H}$ is a subgame of $\mathcal{G}$ and $\mathcal{J}$ is subgame of $\mathcal{H}$. Define $\bar{\pi}_i(\mathcal{J}|\mathcal{H})$ and $\underline{\pi}_i(\mathcal{J}|\mathcal{H})$ as the supremum and infinum of the set of relative payoffs with respect to $\mathcal{H}$ to player i in any SPE of the subgame $\mathcal{J}$. Define $\bar{\pi}_i(\mathcal{H})$ and $\underline{\pi}_x(\mathcal{H})$ as shorthand for $\bar{\pi}_i(\mathcal{H}|\mathcal{H})$ and $\underline{\pi}_x(\mathcal{H}|\mathcal{H})$. Recall that for the bargaining models considered in this chapter, corollary 3.1 can be used to prove that the existence of a SPE of $\mathcal{G}$ which implies the existence of a SPE of $\mathcal{J}$. Thus these definitions are well-defined.

Equivalence up to Discounting

Suppose bargaining games $\mathcal{G}^1$ and $\mathcal{G}^2$ are generated by bargaining models of this chapter from networks with player sets P^1 and P^2 using discount factors $\Delta^1 = (\delta_i^1)_{i\in P^1}$ and $\Delta^2 = (\delta_i^2)_{i\in P^2}$. Two subgames $\mathcal{A}^1$ and $\mathcal{A}^2$ of $\mathcal{G}^1$ and $\mathcal{G}^2$ respectively are *equivalent up to discounting* if the following conditions are met:

1. There is a bijection c from the set of active players of $\mathcal{A}^1$ to the set of active players of $\mathcal{A}^2$.
2. If player i is active in $\mathcal{A}^1$ then $\delta_i^1 = \delta_{c(i)}^2$.
3. There is a bijection b from the set of infinite histories[21] of $\mathcal{A}^1$ to that of $\mathcal{A}^2$.

[21]This definition is made from the formal definition of a game of section 3.4.1, not the informal version including terminal histories.

4. For $n = 1$ or 2, define γ^n as the minimal value such that there exists an infinite history x of $\mathcal{A}^n$ such that some player has a non-singleton action set in $\mathcal{A}^n$ following history[22] $\lambda_{\gamma^n} x$ (or let $\gamma^n = 0$ if no such value exists). Then the following condition must hold for any infinite history x of $\mathcal{A}^1$ and $t \in \mathbb{N}$. If player i has a non-singleton action set following history $\lambda_{t+\gamma^1} x$ in $\mathcal{A}^1$ then player $c(i)$ has a non-singleton action set following history $\lambda_{t+\gamma^2} b(x)$ in $\mathcal{A}^2$.

5. For any infinite history x of $\mathcal{A}$ and active player i of $\mathcal{A}$ the payoff functions of $\mathcal{A}^1$ and $\mathcal{A}^2$, π^1 and π^2 satisfy

$$\pi_i^1(x|\mathcal{A}^1) = \pi_{c(i)}^2(b(x)|\mathcal{A}^2) \tag{5.1}$$

This relation is intended to capture situations where two subgames have the same structure apart from the delays active players have already incurred. Relatively few conditions are needed for this definition because under the definition of a game introduced in section 3.4.1, most properties of a game are defined from its set of infinite histories. Condition 4 essentially means that[23] players' action sets are conserved under the bijection b. This ensures that players do not 'swap' their action sets while retaining their payoffs. The first lemma of the next section shows that two subgames which are equivalent up to discounting have the same SPE structure[24].

[22] Recall the definition from section 3.4.1 that $\lambda_t x$ represents the initial subsequence of x up and including period t.

[23] The length of this condition is due to the possibility that the set of infinite histories of one game is obtained by appending a fixed finite sequence to the start of each infinite history of the other game. This could easily occur if the games $\mathcal{A}^1$ and $\mathcal{A}^2$ are constructed by the method described in section 3.4.2 for representing subgames.

[24] This fact was essentially used in the proof of lemma 4.1 on the alternating offers game. For that game any two pre-proposal subgames (as defined in the proof) are equivalent up to discounting. The similarity of their SPE structures allows the recursive nature of the proof.

5.4.2 Preliminary Lemmas

Lemma 5.3. *Suppose subgames $\mathcal{A}^1$ and $\mathcal{A}^2$ are equivalent up to discounting, e^1 is a SPE of $\mathcal{A}^1$ and b is the bijection described in condition 3 of the definition of equivalence up to discounting. Then a SPE of $\mathcal{A}^2$, e^2, can be constructed as follows. Let γ^1 and γ^2 be as defined in condition 4 of equivalence up to discounting. Suppose e^1 specifies that following history $\lambda_{t+\gamma^1}x$ an action is made producing history $\lambda_{t+\gamma^1+1}y$, then e^2 specifies that following history $\lambda_{t+\gamma^2}b(x)$ the action made produces history $\lambda_{t+\gamma^2+1}b(y)$.*

Proof. Given any strategy profile f^1 of $\mathcal{A}^1$, let $\theta(f^1)$ be the strategy profile of $\mathcal{A}^2$ constructed from f^1 by the method described in the statement of this lemma. Recall the definition of α from section 3.4.1. Then given a infinite history x of $\mathcal{A}^1$ and $t \in \mathbb{N}$:

$$b(\alpha(f^1, x, t)) = \alpha(\theta(f^1), b(x), t + \gamma^2 - \gamma^1)$$

Let c be the bijection described in condition 2 of the definition of equivalence up to discounting. Expanding equation (5.1) gives:

$$\pi_i^1(x) = \delta_i^{t-s} \pi_{c(i)}^2(b(x))$$

where t is the initial delay[25] for player i in subgame $\mathcal{A}^1$ and s is the initial delay for player $c(i)$ in subgame $\mathcal{A}^2$.

Substituting these two relations into equation (3.1) and noting that θ is a bijection between the sets of strategy profiles for $\mathcal{A}^1$ and $\mathcal{A}^2$ shows that if the SPE conditions hold for e^1 in $\mathcal{A}^1$ they also hold for e^2 in $\mathcal{A}^2$. □

The remaining lemmas establish some results which are common for both the exogenous and endogenous ordering models. Recall that these models are represented by X and F respectively. Note that these results also hold for many similar bargaining models. For example it is easy to extend most[26] of them to the market bargaining game.

[25] Strictly speaking, the delay for player i in the finite history generating this subgame.

[26] The exception is corollary 5.6 which is particular to the network L_5.

Lemma 5.4. *Let $\mathcal{G}$ be a bargaining game on a network $N = (P, E, K_{unit})$ generated by the bargaining model $Q = X$ or F. Suppose $\mathcal{G}_b$ is a pre-proposal subgame of $\mathcal{G}$ with proposer b and exactly 3 active players $\{a, b, c\}$ such that $\{ab, bc\} \subseteq E$, and $ac \notin E$. Then $\underline{\pi}_b(\mathcal{G}_b) = \bar{\pi}_b(\mathcal{G}_b) = 1$.*

Proof. Suppose player b makes a multilateral demand of $\sigma_b < 1$ in $\mathcal{G}_b$. Note that if $Q = F$ then there is a choice between multilateral and unilateral demands and if $Q = X$ then only multilateral demands are allowed. Let $\mathcal{H}$ be the resulting subgame. $\mathcal{H}$ is a post-proposal subgame with responder a or c. Should the responder accept, then they receive a share of $1 - \sigma_b > 0$. Should they refuse the other of players a and c becomes the responder. If this new responder accepts then they receive a share of $1 - \sigma_b > 0$. Suppose that there exists a SPE of $\mathcal{H}$ in which players a and c both refuse. Then players a and c both receive a non-zero SPE payoff. This is a non-feasible outcome, so it must be the case that in any SPE of $\mathcal{H}$ σ_b is accepted by player a or c.

By corollary 3.1, $\mathcal{G}_b$ has a SPE. Suppose that $\underline{\pi}_b(\mathcal{G}_b) = p < 1$. Select $p < p' < 1$. Then the previous paragraph shows that following a multilateral demand of p', player b receives a relative payoff with respect to $\mathcal{G}_b$ of more than p in any SPE, which is a contradiction. □

Notes:

1. This lemma is independent of the choice of discount factors.

2. This lemma can be extended to the case of non-unit cakes. Observe that if player b makes a multilateral demand which earns players a and c non-zero payoffs should they accept it, then the same argument can be used to show that it is accepted in SPE. Hence player b earns at least as much as the maximum demand which is feasible to both players a and c.

3. This lemma can also be extended to all cases where a, b and c are active players, $\{ab, bc\} \subseteq E$ and there are no other exchange opportunities for players a and

c with active players. Observe that if player b makes a multilateral demand to a and c of $\sigma_b < 1$ then the argument that player a and c cannot both refuse remains valid. Thus in this case player b earns at least as much as if a, b and c were the only active players.

4. It is straightforward to use this lemma to deduce the unique SPE payoff in any subgame with 3 active players in a line formation. This task is done for the bargaining model F in corollary 5.9 below.

The following lemma places an upper bound on the SPE payoff of an isolated player, in the sense that they have a single neighbour, for the case where the isolated player has a unit exchange cake. The proof is based on the method of proof used for the alternating offers game in lemma 4.1.

Lemma 5.5. *Consider a network $N = (P, E, K)$ and players $p, q \in P$ such that $pq \in E$, $\mathcal{K}^{pq} = \mathcal{K}_{unit}$ and $px \notin E$ for any other $x \in P$. Let $\mathcal{G}$ be a bargaining game on a network N generated by the bargaining model $Q = X$ or F and discount factors such that $\delta_p = \delta q = \delta \in (0,1)$. Let $\mathcal{G}_i$ be a pre-proposal subgame of $\mathcal{G}$ with proposer i. Then $\underline{\pi}_q(\mathcal{G}_q) \geq \bar{n}$ and $\bar{\pi}_q(\mathcal{G}_q) \leq \bar{n}$ where $\bar{n} = \frac{1}{1+\delta}$.*

Note that equation (5.3) in the following proof is stronger than is necessary. This is to facilitate the proof of corollary 5.6.

Proof. Let P_x be the set of all pre-proposal subgames of $\mathcal{G}$ with proposer x. Define

$$\bar{\chi}_x = \sup_{\mathcal{H} \in P_x} \bar{\pi}_x(\mathcal{H})$$

$$\underline{\chi}_x = \inf_{\mathcal{H} \in P_x} \underline{\pi}_x(\mathcal{H})$$

The following relations hold[27]:

$$\underline{\chi}_q \geq 1 - \delta \bar{\chi}_p \tag{5.2}$$

$$\bar{\chi}_x \leq \max_{y | xy \in E} g^{y,x}(\delta \underline{\chi}_y) \quad \text{for any } x \in P \tag{5.3}$$

[27]Recall $\vee$ is an infix maximum operator.

where $g^{y,x}$ is an extension of $f^{y,x}$ onto $\mathbb{R}$ taking the value zero when $f^{y,x}$ is not defined.

Equation (5.2):

Fix $\mathcal{H} \in P_q$. In $\mathcal{H}$, player q is active so player p must also be. Let A be the quantity on the right hand side of the inequality. In the case $Q = X$, suppose player q demands $\sigma_q < A$. In the case $Q = F$, suppose player q demands $\sigma_q < A$ unilaterally to player p. In either case, let $\mathcal{H}'$ be the resulting subgame. Consider a subgame $\mathcal{H}''$ of $\mathcal{H}'$ in which player p is the responder and the demand of σ_q from player q is available. If player p accepts then she earns a relative payoff with respect to $\mathcal{H}'$ of more than $\delta\bar{\chi}_p$. By definition, this is more that her supremum SPE relative payoff from refusing. So in any SPE of $\mathcal{H}'$, the demand σ_q is accepted by some player.

Equation (5.3):

Let B be the quantity on the right hand side of this equation. Fix $\mathcal{H} \in P_x$. Suppose that there exists a SPE e of $\mathcal{H}$ such that player x receives a relative payoff with respect to $\mathcal{H}$ of more than B. As this value is non-zero, player x must exchange with some neighbour y under e. Then under e, player y receives a relative payoff with respect to $\mathcal{H}$ of less than $\delta\underline{\chi}_y$. Since y exchanges under e, its play involves some post-proposal subgame with responder y. Let $\mathcal{H}'$ be the first such subgame. Suppose player y refuses in this subgame. Then her SPE payoff with respect to $\mathcal{H}$ is at least $\delta\underline{\chi}_y$. This is higher than the SPE relative payoff under e which is a contradiction.

Combining equations (5.2) and (5.3) gives:

$$\underline{\chi}_q \geq 1 - \delta(1 - \delta\underline{\chi}_q)$$

Solving this gives $\underline{\chi}_q \geq \bar{n}$. Substituting this into equation (5.3) yields $\bar{\chi}_p \leq \bar{n}$. □

Corollary 5.6. *For $N = L_5$, $\mathcal{G}_i$ as described in the previous lemma satisfies: $\underline{\pi}_i(\mathcal{G}_i) \geq \bar{n}$ for $i = 2$ or 4, and $\bar{\pi}_i(\mathcal{G}_i) \leq \bar{n}$ for $i = 1, 3$ or 5.*

Proof. For $i \neq 3$, the results follow by direct application of the lemma. The case $i = 3$ follows by substituting the results for $i = 2$ and $i = 4$ into equation (5.3). □

The final lemma of this section describes situations in which a player is guaranteed to accept a demand in SPE play.

Lemma 5.7. *Let $\mathcal{G}$ be a bargaining game on a network $N = (P, E, K_{unit})$ generated by the bargaining model $Q = X$ or F. Let $\mathcal{H}$ be a post-proposal subgame of $\mathcal{G}$ with responder r. If the most recent demand of any neighbour of r is no more than (less than) $1 - \delta_r$ in $\mathcal{H}$ then in some (any) SPE of $\mathcal{H}$ the action of r is to accept the demand of a neighbour whose most recent demand is lowest.*

Proof. By corollary 3.1, a SPE of $\mathcal{H}$ exists. Let $\mathcal{H}'$ be the subgame which results if player r refuses in $\mathcal{H}$. The maximum share player r can achieve in $\mathcal{H}'$ is 1. Since player r incurs a delay of 1 by refusing in $\mathcal{H}$, the maximum relative payoff with respect to $\mathcal{H}$ that r can achieve in $\mathcal{H}'$ is δ_r. Hence if the most recent demand of a neighbour of r is $\lambda \leq 1 - \delta_r$ in $\mathcal{H}$, then there is a SPE of $\mathcal{H}$ in which player r accepts a lowest most recent demand. If $\lambda < 1 - \delta_r$ then this action is taken in any SPE of $\mathcal{H}$. □

5.4.3 The Exogenous Ordering Model on L_5

Lemma 5.1. *For each $i \in P$ fix $\eta_i \in (0, 1)$. For $\epsilon > 0$, let $\Delta = (\delta_i)_{i \in P}$ where $\delta_i = \eta_i^\epsilon$. Then $X(L_5(31524), \Delta)$ has a unique limiting SPE payoff in the limit $\epsilon \to 0$ in which players 1 and 2 receive payoff 1 and the others receive zero.*

Proof. Let $\mathcal{X}_1$ be a pre-proposal subgame of $X(L_5(31524), \Delta)$ with proposer 1 in which all players are active and the most recent demands of players 4 and 5, if any, are $\sigma_4 > 1 - \delta_2$ and $\sigma_5 > 1 - \delta_2$. Let $\mathcal{X}_3$ be a post-proposal subgame of $\mathcal{X}_1$ with responder 3 such that all players are active. Let σ_1 and σ_2 be the most recent demands of players 1 and 2 respectively in this subgame. Let $\mathcal{X}_3'$ be the subgame resulting from refusal in $\mathcal{X}_3$.

Suppose player 3 makes a demand of zero in $\mathcal{X}_3'$. Consider a SPE e of the resulting subgame. In the case that e specifies that player 5 on her next turn as responder accepts the most recent demand of player 1 then player 3 receives payoff zero. In the case that e does not specify this then by lemma 5.7 in this SPE player 1 accepts a demand on his next turn as proposer. Whichever player he accepts, player 3 receives payoff zero.

Suppose player 3 makes a demand $\sigma_3 > 0$ in $\mathcal{X}_3'$. Let $\mathcal{X}_4$ be the resulting subgame. The following argument shows that players 4 and 5 receive non-zero SPE payoffs in $\mathcal{X}_4$ and thus player 3 must receive a SPE payoff of zero by feasability constraints. Suppose in $\mathcal{X}_4$ player 4 refuses and demands σ_4. Let the resulting subgame be $\mathcal{X}_5$. Suppose in $\mathcal{X}_5$ player 5 refuses and demands less than $\min[1-\delta_2, \sigma_3]$. Then by lemma 5.7, player 1 accepts this demand. So $\underline{\pi}_5(\mathcal{X}_5) \geq \delta_5 \min[1-\delta_2, \sigma_3]$. Now suppose $\sigma_4 < \delta_5 \min[1-\delta_1, 1-\delta_2, \sigma_3]$. Then $\underline{\pi}_2(\mathcal{X}_5) \geq 1-\sigma_4 > 1-\bar{\pi}_5(\mathcal{X}_5)$, so the exchange 15 must form in SPE of $\mathcal{X}_5$. By lemma 5.7, when SPE play of $\mathcal{X}_5$ reaches player 2 with such a value of σ_4, player 2 must accept a demand in SPE. Since the exchange 15 forms in any SPE, it must be the case that player 2 accepts the demand of player 4. Hence player 4 is guaranteed a non-zero SPE payoff in $\mathcal{X}_4$. Note that if player 4 accepts in $\mathcal{X}_4$, then a demand of less than $\min[1-\delta_2, \sigma_3]$ still guarantees player 5 a non-zero SPE payoff (this can be seen using lemma 5.4).

The above shows that if $\sigma_1 < 1$ in $\mathcal{X}_3$ then player 3 accepts in any SPE. Suppose player 1 makes a demand $\sigma_1 < 1$ in $\mathcal{X}_1$. Let $\mathcal{X}_2$ be the resulting subgame. In the case that players 4 and 5 have already made demands this is a post-proposal subgame with responder 2. If player 2 refuses and make a demand of $\sigma_2 < 1$ then in any SPE player 3 accepts the demand σ_1 and player 2 is left at the centre of a 3 player line of active players. Hence by lemma 5.4 the demand σ_2 of player 2 is accepted in SPE. This yields player 2 a payoff of δ_2 relative to $\mathcal{X}_2$, so his SPE action in $\mathcal{X}_2$ cannot be to accept one of the demands σ_4 or σ_5. Hence in any SPE of $\mathcal{X}_2$ a game of the form $\mathcal{X}_3$ is reached and therefore the demand σ_1 of player 1 is accepted. This proves that the SPE demand and relative payoff of player 1 in $\mathcal{X}_1$ must be 1.

Let $\mathcal{Y}_2$ be the post-proposal subgame resulting from player 1 making a demand of 1 in $\mathcal{X}_1$. Let $\mathcal{Y}_3$ be the post-proposal subgame of $\mathcal{X}_1$ resulting from players 1 and 2 making the demands 1 and σ_2. Consider the case

$$\sigma_2 < A = \min[1 - \delta_5(1 - \delta_1), 1 - \delta_5(1 - \delta_2), 1 - \delta_4(1 - \delta_2)]$$

It is shown below that in any SPE of $\mathcal{Y}_3$ player 4 accepts the demand of σ_2. So in any SPE of $\mathcal{X}_1$ player 1 earns relative payoff 1 and the relative payoff of player 2 is bounded below by A which tends to 1 in the limit $\epsilon \to 0$, as required.

Consider a SPE of $\mathcal{Y}_3$ in which player 3 accepts. Player 4 must accept the demand σ_2 in this SPE otherwise player 5 is guaranteed a non-zero payoff in SPE giving player 4 a payoff of zero. Consider a SPE of $\mathcal{Y}_3$ in which players 3 and 4 refuse but player 5 accepts (The possibility that player 3 refuses in a SPE of $\mathcal{Y}_3$ cannot be ruled out since player 2 may have made a non-SPE demand.). Since $\sigma_2 < 1$, player 5 must accept the demand of player 2 rather than player 1. Player 4 therefore receives a SPE payoff of zero. However this is a contradiction as she refused the demand σ_2 earlier in the play of this SPE.

Finally, consider a SPE e of $\mathcal{Y}_3$ in which players 3 4 and 5 refuse. Then under e player 4 must receive a payoff of at least $1 - \sigma_2 > \delta_4(1 - \delta_2)$ relative to $\mathcal{Y}_3$ and player 5 must receive a payoff of at least $1 - \sigma_2 > \delta_5(1 - \delta_1) \vee \delta_5(1 - \delta_2)$ relative to $\mathcal{Y}_3$. Therefore e cannot specify that in $\mathcal{Y}_3$ player 4 or 5 makes a demand of less than or equal to $1 - \delta_4$ or $1 - \delta_5$ respectively since if these demands were accepted they would contradict the payoff bounds just stated and in SPE players 1 and 2 would only refuse such demands if they could receive equal or higher payoffs, also breaking the bounds mentioned. Also, the play of e must reach a post-proposal subgame $\mathcal{Y}_1$ with responder 1 in which all players are active. Let $\mathcal{Y}_1'$ be the subgame resulting from a refusal in $\mathcal{Y}_1$. It has just been shown that $\mathcal{Y}_1'$ is equivalent up to discounting to $\mathcal{X}_1$. Thus by the argument above and lemma 5.3, in any SPE of $\mathcal{Y}_1'$ player 1 makes the demand 1 and it is accepted. Thus under e the relative payoff of player 1 in $\mathcal{Y}_3$ is at least δ_1. Since player 5 refuses at least once under e, her share must be more than $1-\delta_1$. Thus it cannot be the case that the exchange 15 forms and it is infeasible for players 4 and 5 to both receive non-zero payoffs. This is a contradiction as both refused the demand σ_2 earlier in the play of this SPE. □

Lemma 5.2. *Let $\Delta_\delta = (\delta_i)_{i\in P}$ such that $\delta_i = \delta$ for all i. Then $X(L_5(41325), \Delta_\delta)$ has two SPE outcomes, $(1-\bar{n}, \bar{n}, 1-\bar{n}, \bar{n}, 0)$ and $(0, \bar{n}, 1-\bar{n}, \bar{n}, 1-\bar{n})$ where $\bar{n} = \frac{1}{1+\delta}$.*

Proof. Let $\mathcal{W}_1 = X(L_5(41325), \Delta_\delta)$. Consider a post-proposal subgame $\mathcal{U}_3$ of $\mathcal{W}_1$ in which player 3 is the responder for the first time and the most recent demands of players 1 and 2 are less than 1. Suppose there is a SPE of $\mathcal{U}_3$ in which player 3 refuses. In this SPE, players 3 4 and 5 all have a chance to accept a non-zero offer, and hence must all have non-zero payoffs. This is infeasible. Thus in any SPE of $\mathcal{U}$ player 3 accepts the highest initial offer.

By lemma 5.5, any initial demand by player 1 of $\sigma_1 < \bar{n}$ in $\mathcal{W}_1$ is accepted. Suppose player 1 makes a demand of $\sigma_1 \geq \bar{n}$ in $\mathcal{W}_1$. Let the resulting subgame be $\mathcal{W}_2$. Suppose player 2 makes a demand $\sigma_2 < 1$ in $\mathcal{W}_2$. Then in SPE player 3 accepts whichever of demands σ_1 and σ_2 is lower and in the case $\sigma_1 = \sigma_2$ there are SPEs in which either is accepted by player 3. Suppose $\sigma_2 > \sigma_1$. Then player 3 accepts σ_1 in SPE. Let the resulting subgame be $\mathcal{W}_5$. This a post-proposal subgame with responder player 5. If player 5 refuses in $\mathcal{W}_5$ then the resulting subgame is essentially an alternating offers between players 2 and 5 so the SPE payoff of player 5 is $\delta\bar{n} > 1-\sigma_2$. Thus player 2 receives a SPE payoff of only $\delta(1-\bar{n}) < \sigma_1$ in $\mathcal{W}_5$.

Hence the SPE payoff of player 2 in $\mathcal{W}_2$ is σ_1. In the case $\sigma_1 > \bar{n}$, it must also be the case that in any SPE of $\mathcal{W}_2$ player 2 demands σ_1 and player 3 accepts this demand (from player 2 rather than player 1). Consider the resulting post-proposal subgame with responder 4, $\mathcal{W}_4$. If player 4 refuses then the resulting subgame is essentially an alternating offers game between players 4 and 1 so the SPE payoff of player 4 is $\delta\bar{n} > 1-\sigma_1$. Thus in any SPE of $\mathcal{W}_4$ player 4 refuses. By lemma 5.3 and corollary 4.2 again, the payoff to player 1 in this SPE must be $1 - \bar{n}$.

Hence in any SPE of $\mathcal{W}_1$ players 1 and 2 demand $\bar{n}$ and these demands are accepted. There are two SPEs depending on whom player 3 exchanges with. □

5.4.4 The Endogenous Ordering Model on 3 Player Networks

Conditions of Theorem 4.3

Recall that the conditions of theorem 4.3 are as follows. Let N be a 3 player ring network with insatiable cakes in the sense of definition 3.7 and an empty core. For $i \in P$ fix $\eta_i \in (0, 1)$ and let $\Delta = (\delta_i)_{i \in P}$ where $\delta_i = \eta_i^\epsilon$ for $\epsilon > 0$. Recall from corollary 4.5 that the limiting SPE payoff to $\mathcal{M} = M^{\text{personal}}(N, \Delta)$ as $\epsilon \to 0$ is $(\sigma_1, \sigma_2, 0)$ where $(\sigma_1, \sigma_2, \sigma_3)$ are the components of the unique von Neumann-Morgenstern triple of N.

This section shows that under the conditions of theorem 4.3, $\mathcal{F} = F(N, \Delta, 1)$ either has a limiting SPE outcome of $(\sigma_1, \sigma_2, 0)$ or a limiting SPE outcome of $(\sigma_1, 0, \sigma_3)$, or both. The proof below proceeds by applying various cases of the proof of theorem 4.3.

Suppose player 1 makes an initial multilateral demand of $\lambda_1 < \sigma_1$ in $\mathcal{F}$. Then the responder accepts this demand in any SPE of the resulting subgame. This can be proved by the argument in the proof of theorem 4.3 for subgame $\mathcal{A}$ of that proof. Suppose player 1 makes an initial multilateral demand of $\lambda_1 \geq \sigma_1$ in $\mathcal{F}$. In any SPE of the resulting subgame the responder r receives a payoff of at least $\delta_r \sigma_r$ by the argument of theorem 4.3 for subgame $\mathcal{B}$ of that proof. Suppose player 1 makes an initial unilateral demand in $\mathcal{F}$. Let the resulting subgame be $\mathcal{F}'$. Suppose the responder r in $\mathcal{F}'$ refuses and demands $\lambda_r < \sigma_r$ multilaterally. Then in any SPE of the resulting subgame this is accepted by the argument in the proof of theorem 4.3 for subgame $\mathcal{A}$ of that proof. Thus in any SPE of $\mathcal{F}'$ player r receives a payoff of at least $\delta_r \sigma_r$.

Therefore in any SPE of $\mathcal{F}$ the payoff of player 1 is in the interval $[\sigma_1, f^{2,1}(\delta_2 \sigma_2) \vee$

$f^{3,1}(\delta_3\sigma_3)]$. In the limit $\epsilon \to 0$, both bounds tend to σ_1 so this is the unique limiting SPE payoff to player 1 in $\mathcal{F}$. Also, in any SPE of $\mathcal{F}$ the payoff of the first responder r is at least $\delta_r\sigma_r$. Therefore in any limiting SPE payoff of $\mathcal{F}$, σ_i is the unique limiting SPE payoff to the player $i \neq 1$ who receives a non-zero payoff. A higher limiting payoff would result in an infeasible SPE payoff for all sufficiently small ϵ.

Conditions of Lemma 4.6

This section describes the limiting SPE outcomes of $\mathcal{G} = F(N, \Delta, 1)$ under the conditions of lemma 4.6.

Recall that the conditions of lemma 4.6 are as follows. Suppose N is a 3 player ring network such that $\mathcal{K}^{12} = \mathcal{K}^{\text{unit}}$ and the core of N contains an element $c = (c_1, c_2, 0)$. For each $i \in P$ fix $\eta_i \in (0, 1)$. For $\epsilon > 0$, let $\Delta = (\delta_1, \delta_2, \delta_3)$ where $\delta_i = \eta_i^\epsilon$. Recall from lemma 4.6 that the unique limiting SPE outcome of $\mathcal{M} = M^{\text{personal}}(N, \Delta)$ in the limit $\epsilon \to 0$ is $(\phi, 1-\phi, 0)$ where[28]

$$\phi = f^{3,1}(0) \vee [[1 - f^{3,2}(0)] \wedge n_1]$$
$$n_1 = \lim_{\epsilon \to 0} \frac{1 - \delta_2}{1 - \delta_1\delta_2}$$

For a subgame $\mathcal{H}$ of $\mathcal{G}$ define a *restricted subgame perfect equilibrium* (RSPE) as a SPE e such that player $x \in \{1, 2\}$ never makes the proposition $[f^{3,x}(0), (3)]$ if e is played in $\mathcal{H}$. The following argument shows that an RSPE of $\mathcal{G}$ exists and the limiting RSPE payoff of player 1 as $\delta \to 1$ is ϕ. It also shows that if $\phi > f^{3,1}(0)$ then player 2 receives a limiting RSPE payoff as $\delta \to 1$ of $1 - \phi$. In the case $\phi = f^{3,1}(0)$ a RSPE may exist in which player 1 exchanges with player 3. It is not shown whether this occurs or not.

Recall equation (4.7) from the proof of lemma 4.6

$$1 - f^{3,2}(0) \geq f^{3,1}(0) \tag{5.4}$$

[28] Recall that $\wedge$ and $\vee$ are respectively infix maximum and minimum operators.

Let (i,j) be $(1,2)$ or $(2,1)$. Let B_i be the set of pre-proposal subgames of $\mathcal{G}$ with proposer i such that $V_j = \emptyset$.

For $\mathcal{B} \in B_i$, let $P_i(\mathcal{B})$ be the set of relative payoffs with respect to $\mathcal{B}$ to player i in any RSPE of $\mathcal{B}$. Let $\Pi_i = \bigcup_{\mathcal{B}\in B_i} P_i(\mathcal{B})$. Let $\bar{\chi}_i = \sup \Pi_i$ and $\underline{\chi}_i = \inf \Pi_i$. It is shown at the end of this section that a RSPE of $\mathcal{G}$ exists so that these quantities are well defined.

It is shown below that the following relations hold:

$$\bar{\chi}_i \leq f^{3,i}(0) \vee [1 - \delta_j \underline{\chi}_j] \tag{5.5}$$

$$\underline{\chi}_i \geq [1 - f^{3,j}(0)] \wedge \{f^{3,i}(0) \vee [1 - \delta_j \bar{\chi}_j]\} \tag{5.6}$$

Combining these relations are taking the limit $\delta \to 1$ yields

$$\bar{\chi}_i = \underline{\chi}_i = f^{3,i}(0) \vee [[1 - f^{3,j}(0)] \wedge n_i]$$

where $n_i = \frac{1-\delta_j}{1-\delta_i\delta_j}$. This is sufficient to support the characterisation of RSPE outcomes made above.

Equation (5.5):

Let A be the quantity on the right hand side of the equation. Fix $\mathcal{D} \in B_i$. Suppose there is an RSPE e of $\mathcal{D}$ in which player i receives a relative payoff with respect to $\mathcal{D}$ of more than A. Then it must be the case that the exchange 12 forms. Hence in the play of $\mathcal{D}$ under e a post-proposal subgame must be reached with responder j. Let $\mathcal{D}'$ be the first such subgame.

In $\mathcal{D}'$, if V_i does not contain 3 then player j can refuse and earn a relative payoff with respect to $\mathcal{D}$ of at least $\delta_j \underline{\chi}_j$ in any SPE of the resulting subgame. In $\mathcal{D}'$, if V_i contains 3 and j then it must be the case that the most recent demand of player i is no more than $f^{3,i}(0)$. Then player j can earn a payoff of at least $f^{3,j}(0)$ by accepting this demand (by equation (5.4)). In either of these cases player i receives a relative payoff with respect to $\mathcal{D}$ of less than A.

The remaining case is that in $\mathcal{D}'$ the value of V_i is (3). Since j is the responder in $\mathcal{D}'$, it must also be the case that $V_3 = (j)$. It must also be the case that the most recent demand of player i is less than $f^{i,3}(0)$, otherwise e would not be a RSPE[29]. If player j accepts the demand of player 3 in $\mathcal{D}'$ then the exchange $j3$ forms. If player j refuses the demand of player 3 in $\mathcal{D}'$ then player 3 is guaranteed a non-zero payoff so the exchange 12 cannot form. Thus no RSPE of the form described for e can exist.

Equation (5.6):

Let B be the quantity on the right hand side of the equation. Fix $\mathcal{E} \in B_i$.

Consider the case $B \leq f^{3,i}(0)$. Suppose in $\mathcal{E}$ player i makes the proposition $[\sigma_i, (3, j)]$ where $\sigma_i < B$. Let $\mathcal{E}'$ be the resulting subgame. By equation (5.4), $1 - \sigma_i > f^{3,j}(0)$. Thus in any SPE of $\mathcal{E}'$ if a post-proposal subgame with responder j is reached then the exchange 12 forms. However if a post-proposal subgame with responder 3 is reached then player 3 exchanges. Hence it must be the case that one of players 3 or j accepts the demand σ_i.

Consider the case $B \leq 1 - \delta_j \bar{\chi}_j$. Suppose in $\mathcal{E}$ player i makes the proposition $[\sigma_i, (j)]$ where $\sigma_i \leq B$. Then $1 - \sigma_i > f^{3,j}(0)$ so player j does not accept the demand of player 3 in any SPE of the resulting subgame, $\mathcal{E}'$. Also $1 - \sigma_i \geq \delta_j \bar{\chi}_j$ so given that an RSPE of $\mathcal{E}'$ exists, there exists an RSPE in which player j accepts.

RSPE existence:

Finally, the existence of a RSPE for $\mathcal{G}$ is demonstrated. Consider any SPE e of $\mathcal{G}$. Consider any subgame $\mathcal{G}_3$ of $\mathcal{G}$ such that player 3 must take an action for the first time. Let $e(\mathcal{G}_3)$ be the SPE of $\mathcal{G}_3$ induced by e. If player 3 receives a non-zero payoff in this SPE then let $e'(\mathcal{G}_3) = e(\mathcal{G}_3)$. Otherwise it must be the case that player 3 can receive only zero by accepting a demand in $\mathcal{G}_3$. Let $e'(\mathcal{G}_3)$ be the SPE $e(\mathcal{G}_3)$ modified so that player 3 accepts a demand of zero in $\mathcal{G}_3$. This is also a SPE.

Consider the following strategy profile f of $\mathcal{G}$. In any pre-proposal subgame of $\mathcal{G}$ with proposer i in which player 3 has never made an action, player i makes the

[29] The failure of this part of the proof if e is a SPE but not a RSPE generates the class of SPEs sketched in section 5.2.4 which are interpreted there as not robust.

proposition $[n_i, (j)]$. In any post-proposal subgame of $\mathcal{G}$ with responder i in which player 3 has never made an action, player i accepts the best demand which yields a payoff of at least δn_i or, if no such demand exist, refuses all demands. In any other subgame, play is according to the corresponding value of $e'(\mathcal{G}_3)$. If f is a SPE then it is also a RSPE.

If f is not a SPE, it must be the case that some player $k = 1$ or 2 prefers to make a demand of $f^{k,3}(0)$ to player 3. Let k' be the player 1 or 2 other than k. The following strategy profile g of $\mathcal{G}$ then forms a RSPE and it can be checked that it also forms a SPE. Let $\lambda_k = f^{3,k}(0)$. Let $\lambda_{k'} = 1 - \delta_k \lambda_k$. In any pre-proposal subgame of $\mathcal{G}$ with proposer k in which player 3 has never made an action, player k makes the proposition $[\lambda_k, (3, k')]$. In any pre-proposal subgame of $\mathcal{G}$ with proposer k' in which player 3 has never made an action, player k' makes the proposition $[\lambda_{k'}, (k)]$. In any post-proposal subgame of $\mathcal{G}$ with responder $i = 1$ or 2 in which player 3 has never made an action, player i accepts the best demand which yields a payoff of at least $\delta\lambda_i$ or, if no such demand exist, refuses all demands. In any other subgame, play is according to the corresponding value of $e'(\mathcal{G}_3)$.

5.4.5 The Endogenous Ordering Model on L_4

This section is on the SPE behaviour of games of the form $\mathcal{A}_i = F(L_4, \Delta_\delta, i)$.

It proves that for $i = 2$ or 3, the limiting SPE payoffs of $\mathcal{A}_i$ as $\delta \to 1$ are $(0, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For $i = 1$ or 4 the latter is the unique limiting SPE payoff. The proof is as follows.

Let S_i be the set of all subgames of $\mathcal{A}_i$ and $S = \bigcup_i S_i$. Let B_j be the set of pre-proposal subgames in S such that all players are active, the proposer is j, and for $x \in \{2, 3\}$ either $V_x = \emptyset$ or the most recent demand of player x is at least $\bar{n}$. For $\mathcal{B} \in B_j$, let $P_j(\mathcal{B})$ be the set of relative payoffs with respect to $\mathcal{B}$ to player j in any SPE of $\mathcal{B}$. Let $\Pi_j = \bigcup_{\mathcal{B} \in B_j} P_j(\mathcal{B})$. Let $\bar{\chi}_j = \sup \Pi_j$ and $\underline{\chi}_j = \inf \Pi_j$. By corrolary 3.1 a SPE of $\mathcal{A}_i$ exists for each i, so these quantities are well defined. By lemma 5.5,

$\bar{\chi}_1 \leq \bar{n}$, $\underline{\chi}_2 \geq \bar{n}$, $\underline{\chi}_3 \geq \bar{n}$ and $\bar{\chi}_4 \leq \bar{n}$ (recall that $\bar{n} = \frac{1}{1+\delta}$).

Consider $C_2 \in B_2$. Suppose $\bar{\chi}_2 > \bar{n}$. Consider a SPE e of C_2 in which player 2 receives a relative payoff with respect to C_2 of $\bar{\chi}_2 - \epsilon > \bar{n}$. Since $\underline{\chi}_3 \geq \bar{n}$, it must be the case that the exchange 12 forms in the play of e. Thus in the play of e a post-proposal subgame C_1 must be reached with responder 1. Let σ_2 be the most recent demand of player 2 in this subgame. In the case that $\sigma_2 < \bar{\chi}_2 - \epsilon$, player 1 receives a share of more than $1 - (\bar{\chi}_2 - \epsilon)$ in e which contradicts the relative payoff with respect to C_2 of player 2 in e.

So it must be the case that $\sigma_2 \geq \bar{\chi}_2 - \epsilon$. Suppose player 1 refuses in C_1 and demands $\sigma_1 < 1 - \delta\bar{\chi}_2$. Let C_1' be the resulting subgame. In the play of e in C_1' player 3 does not accept the demand σ_2 since $\underline{\chi}_3 \geq \bar{n}$. Thus the play of e reaches a post-proposal subgame of C_1' with responder 2, $\mathcal{D}_2$, in which the demand σ_1 is available. The following shows that player 2 accepts the demand σ_1 in $\mathcal{D}_2$ under e.

case i)

Suppose exchange 34 has taken place in $\mathcal{D}_2$. Then $\mathcal{D}_2$ is essentially an alternating offers game. Under e player 2 accepts in $\mathcal{D}_2$ since $\sigma_1 < \bar{n}$.

case ii)

Suppose the responder in C_1' is player 3 or 4 and the most recent proposition of player 3 in $\mathcal{D}_2$ is $[\sigma_3, (2)]$. Let $\mathcal{D}_3$ be the subgame of C_1' in which player 3 made this proposition. By refusing in $\mathcal{D}_2$ player 2 enters a game $\mathcal{D}_2' \in B_2$. The relative SPE payoff of player 2 in $\mathcal{D}_2'$ with respect to $\mathcal{D}_2$ is at most $\delta\bar{\chi}_2$. Thus under e player 2 accepts the lowest demand in $\mathcal{D}_2$. If this is σ_3 then player 3 receives a relative payoff with respect to $\mathcal{D}_3$ of less than $\bar{n}$. Note that $\mathcal{D}_3 \in B_3$ so this contradicts $\underline{\chi}_3 \geq \bar{n}$.

case iii)

Suppose the most recent proposition of player 3 in $\mathcal{D}_2$ is multilateral. Let $\mathcal{D}_3$ be the subgame in which it was made. Note that $\mathcal{D}_3 \in B_3$. The most recent demand of player 3 in $\mathcal{D}_2$ must be more than $\bar{n}$. Otherwise under e at least one of players 2 or 4 would accept this by lemma 5.5, contradicting $\underline{\chi}_3 \geq \bar{n}$. Thus by the argument in case ii), under e player 2 accepts the demand σ_1 in $\mathcal{D}_2$.

case iv)

Suppose the responder in $\mathcal{C}_1'$ is player 2 and either the most recent proposition of player 3 in $\mathcal{D}_2$ is $[\sigma_3, (2)]$ or player 3 has made no propositions in $\mathcal{D}_2$. It is proved below that $\sigma_3 \geq 1 - \delta\bar{\chi}_2$. Thus by the argument in case ii) under e player 2 accepts the demand σ_1 in $\mathcal{C}_1'$.

The claim in case iv) can be proved as follows. Let $\mathcal{D}_3$ be the subgame of $\mathcal{C}_2$ in which player 3 made her most recent proposition relative to $\mathcal{C}_1'$. Suppose $\sigma_3 < 1 - \delta\bar{\chi}_2$. Then by the argument in case ii), under e in $\mathcal{C}_1'$ player 2 accepts the lowest demand. Hence in $\mathcal{C}_1$ under e player 1 must either accept or make a demand which is accepted, as the alternative is to receive payoff zero. This results in a subgame which is essentially an alternating offers game between players 3 and 4. So under e player 3 receives relative payoff $\delta\bar{n}$ with respect to $\mathcal{D}_3$. But $\mathcal{D}_3 \in B_3$ so this contradicts $\underline{\chi}_3 \geq \bar{n}$.

It has been shown that player 1 earns a realtive payoff with respect to $\mathcal{C}_2$ of at least $\delta(1 - \delta\bar{\chi}_2)$ in any SPE of $\mathcal{C}_2$. It must therefore be the case that $\bar{\chi}_2 - \epsilon < 1 - \delta(1 - \delta\bar{\chi}_2)$. If $\bar{\chi}_2 > \bar{n}$ this is a contradiction for sufficiently small ϵ.

This proves that $\bar{\chi}_2 = \underline{\chi}_2 = \bar{n}$. Similarly, $\bar{\chi}_3 = \underline{\chi}_3 = \bar{n}$. It follows that in any game $\mathcal{D} \in B_1$ if player 1 demands $\sigma_1 < \bar{n}$ it is accepted in SPE. Hence $\bar{\chi}_1 = \underline{\chi}_1$. Similarly $\bar{\chi}_4 = \underline{\chi}_4$. The limiting payoff vectors described above are the only ones compatible with these values. To prove that both can be supported it is sufficient to note that in any game $\mathcal{D} \in B_i$ for $i \in P$ there is a SPE in which any unilateral demand of $\bar{n}$ is accepted.

5.4.6 Supporting Lemmas

This section contains various lemmas which are necessary to prove the results on the endogenous ordering model for the network L_5 contained in section 5.4.7. The game $\mathcal{A}_3 = F(L_5, \Delta_\delta, 3)$ is of particular interest in that section. Thus the lemmas of this section often refer to this game in their statements. However, many of them can be generalised to other settings.

Note that appeals to the symmetry of L_5 are often used. That is, the numbering of players in any result can be reversed so that $i \mapsto 6 - i$. The proofs then hold if the same transformation of player numbering is used.

The first lemma is on constructing a SPE of a game $\mathcal{A}$ from SPEs of its immediate subgames. It states that if the supremum payoff to the player who must act in $\mathcal{A}$ in these subgame SPEs is attainable in one subgame SPE then $\mathcal{A}$ has a SPE in which the subgame SPEs are played[30]. The long statement of the lemma is necessary to define 'the player who must act in $\mathcal{A}$'; the game could begin with several periods in which all action sets are singletons. This could easily occur for a game $\mathcal{A}$ which is constructed by the method described in section 3.4.2 for representing subgames.

Lemma 5.8. *Given a game of perfect information, $\mathcal{A}$, let Z be the set of its subgames which are generated by finite histories such that only one action is made from a non-singleton action set. Let i be the player who has this non-singleton action set. Let E be a function mapping each $\mathcal{Z} \in Z$ to a SPE of $\mathcal{Z}$. Let $\pi_i(E(\mathcal{Z}))$ be the payoff to player i when $E(\mathcal{Z})$ is played in $\mathcal{Z}$.*

There exists a SPE of $\mathcal{A}$ in which $E(\mathcal{Z})$ is played in $\mathcal{Z}$ for all $\mathcal{Z} \in Z$ if and only if the following condition holds for some $\mathcal{Y} \in Z$:

$$\pi_i(E(\mathcal{Y})) = \sup_{\mathcal{Z} \in Z} \pi_i(E(\mathcal{Z})) \tag{5.7}$$

Proof. If (5.7) holds then a SPE satisfying the required conditions is as follows. In $\mathcal{A}$ player i takes the action which induces $\mathcal{Y}$. In any $\mathcal{Z} \in Z$ the SPE $E(\mathcal{Z})$ is played. If (5.7) does not hold then suppose f is a SPE of $\mathcal{A}$ in which for any $\mathcal{Z} \in Z$ the SPE

[30] An example of when the condition fails is in the ultimatum game. In this game player 1 must make a demand in $x \in [0, 1]$, then player 2 may either accept or refuse. On acceptance the payoffs to 1 and 2 are x and $1 - x$ respectively. On refusal they are both zero. Let $\mathcal{Z}(x)$ be the subgame in which the demand x has been made. Then ACCEPT is a SPE of $\mathcal{Z}(x)$ for $x < 1$ and REFUSE is a SPE of $\mathcal{Z}(1)$. But there is no SPE of the overall game consistent with these.

$E(\mathcal{Z})$ is played. Then player i can choose an action in $\mathcal{A}$ inducing a subgame in Z in which he receives a higher payoff under f than his SPE payoff. □

The following lemma is a corollary to lemma 5.4 summarising some particularly useful cases.

Corollary 5.9. *Consider a subgame $\mathcal{B}$ of $\mathcal{A}_3$ such that the set of active players is $\{a, a+1, a+2\}$. Let $(i, j) = (a, a+2)$ or $(a+2, a)$. In the case that $\mathcal{B}$ is a pre-proposal subgame with proposer i or a post-proposal subgame with responder $a+1$, then in any SPE of $\mathcal{B}$, player i receives a relative payoff with respect to $\mathcal{B}$ of no more than $1-\delta$. In the case that $\mathcal{B}$ is a post-proposal subgame with proposer j then there is a SPE of $\mathcal{B}$ in which player i receives a payoff of zero.*

Proof. Suppose $\mathcal{B}$ is a post-proposal subgame with responder $a+1$. Should player $a+1$ refuse in $\mathcal{B}$ then in any SPE she attains a relative payoff with respect to $\mathcal{B}$ of δ by lemma 5.4. Thus in any SPE of $\mathcal{B}$, player i receives a relative payoff with respect to $\mathcal{B}$ of no more than $1-\delta$.

Suppose $\mathcal{B}$ is a pre-proposal subgame with proposer i. In any SPE of $\mathcal{B}$, either the exchange $(a+1, j)$ forms and player i receives payoff zero, or a post-proposal subgame with responder $a+1$ is reached and the previous case gives the desired result.

Suppose $\mathcal{B}$ is a post-proposal subgame with responder j. Let the most recent demand of player $a+1$ be λ. In the case that $\lambda < 1$, player j is guaranteed a non-zero SPE payoff in $\mathcal{B}$ and so player i must receive payoff zero in any SPE. In the case that $\lambda = 1$ suppose there exists a SPE e of $\mathcal{B}$ in which player i receives payoff σ_1. Then player j receives payoff zero. Alter the strategy profile e so that player j accepts in $\mathcal{B}$. This is also a SPE and has the required property. □

Lemma 5.10. *Consider a pre-proposal subgame $\mathcal{C}$ of $\mathcal{A}_3$ with proposer 2. Suppose there exists a SPE e of $\mathcal{C}$ with the following property. Should player 2 make any action in $\mathcal{C}$ other than a unilateral demand to player 1 then he receives a relative* *payoff with respect to C of no more than $M \geq \bar{n}$ under e. In this case there also exists a SPE f of C in which player 2 receives a payoff with respect to C of no more than M.*

Proof. Suppose $\underline{\pi}_2(C) = l > M$. Consider a SPE a of C in which player 2 receives a relative payoff with respect to C of $m = l + \epsilon$ for some $\epsilon \geq 0$. Construct a SPE b of C using lemma 5.8 and the following SPEs of each subgame resulting from an action of player 2 in C. If the initial action of player 2 in C is not a unilateral demand to player 1 then play proceeds as in e. Let $m' = 1 - \delta(1 - \delta m)$. If in C player 2 makes a unilateral demand to player 1 of m' or less then player 1 accepts. If in C player 2 makes a unilateral demand to player 1 of more than m' then player 1 refuses and makes a unilateral demand of $1 - \delta m$ to player 2 who accepts. To show that this is SPE behaviour it is sufficient to use lemma 5.8 and the following fact. Let C_2' be the subgame that results from a unilateral demand of player 2 in C followed by a refusal and demand by player 1 and a refusal by player 2. This subgame is equivalent up to discounting to C. So by application of lemma 5.3 there is a SPE of C_2' in which player 2 receives a relative payoff with respect to C_2' of m.

Note that in the SPE b player 2 can receive a payoff of no more than $\max[m', M]$. For ϵ sufficiently small, $m' < l$ so by contradiction, it must be the case that $\underline{\pi}_2(C) \leq M$. For $M > \bar{n}$ it is possible to construct a SPE of the form required for f by the method above, taking $m = M + \epsilon$ for ϵ sufficiently small.

For $M = \bar{n}$ a SPE of the form required for f can by applying lemma 5.8 and the following SPE behaviour for each subgame of C following an action of player 2. If the initial action of player 2 in C is not a unilateral demand to player 1 then play proceeds as in e. If in C player 2 makes a unilateral demand to player 1 of $\bar{n}$ or less then player 1 accepts. If in C player 2 makes a unilateral demand to player 1 of $\sigma_2 > \bar{n}$ then player 1 refuses and makes a demand of $1 - \delta\lambda$ where $\lambda = \bar{n} + \epsilon$ for some $\epsilon(\lambda) > 0$ such that $\delta(1 - \delta\lambda) > 1 - \sigma_2$. Player 2 accepts in the resulting subgame. In the case that player 2 instead refuses a SPE is played in the resulting subgame

C_2^* which gives player 2 a relative payoff of no more than $\bar{n} + \frac{1}{2}\epsilon$. Such a SPE exists since C_2^* is equivalent up to discounting to $\mathcal{C}$ and $\underline{\pi_2}(\mathcal{C}) \leq \bar{n}$. Application of lemma 5.8 shows that this is SPE behaviour. □

Lemma 5.11. *Consider a game $\mathcal{D}$ which is a pre-proposal subgame of $\mathcal{A}_3$ in which player 1 is the proposer and $V_2 = \emptyset$. There is a SPE of $\mathcal{D}$ in which player 1 makes a demand to player 2 and it is accepted.*

Proof. In the case that, in $\mathcal{D}$, V_3 contains 2, let σ_3 be the most recent demand of player 2. Otherwise let $\sigma_3 = 1$.

Suppose player 1 makes a demand σ_1 in $\mathcal{D}$. Let $\mathcal{D}_2(\sigma_1)$ be the resulting subgame. Suppose player 2 then refuses. Let $\mathcal{D}_2'(\sigma_1)$ be the resulting subgame. Note that for any σ_1, $\mathcal{D}_2'(\sigma_1)$ is equivalent up to discounting to $\mathcal{D}_2'(0)$.

Applying lemma 5.3, there is a strategy profile e which is a SPE of $\mathcal{D}_2'(\sigma_1)$ for all σ_1. Let λ_2 be the payoff of player 2 in $\mathcal{D}_2'(0)$ under e. Define a strategy profile $f(\sigma_1)$ of $\mathcal{D}_2(\sigma_1)$ as follows. Should the subgame $\mathcal{D}_2'(\sigma_1)$ be reached, play is according to e. In $\mathcal{D}_2$, player 2 accepts a lowest demand if it is no more than $1 - \delta\lambda_2$ and otherwise refuses. In the case that $\sigma_1 = \sigma_3 \leq \delta\lambda_2$, player 2 accepts the demand of player 1.

The demand profile $f(\sigma_1)$ is a SPE of $\mathcal{D}_2(\sigma_1)$. Applying lemma 5.8, there is a SPE, g, of $\mathcal{D}$ in which player 1 makes the demand $\min[\sigma_3, (1 - \delta\lambda_2)]$ and following a demand σ_1 of player 1, the SPE $f(\sigma_1)$ is played. The SPE g satisfies the requirements of the lemma. □

Corollary 5.12. *Consider a game $\mathcal{E}$ which is a post-proposal subgame of $\mathcal{A}_3$ in which player 1 is the responder and $V_2 = (1)$. There is a SPE of $\mathcal{E}$ in which player 1 either accepts the demand of player 2 or refuses and makes a demand which player 2 accepts.*

Proof. Let $\mathcal{E}'$ be the subgame of $\mathcal{E}$ in which player 1 refuses. By lemma 5.11, there is a SPE e of $\mathcal{E}'$ in which player 1 makes a demand which player 2 accepts. By lemma

5.8, there is a SPE f of $\mathcal{E}$ in which e is played should $\mathcal{E}'$ be reached. The SPE must satisfy the conditions of the corollary. □

Lemma 5.13. *Consider a post-proposal subgame $\mathcal{F}_3$ of $\mathcal{A}_3$ with proposer 3 such that the current value of V_2 is $(3,1)$ and the most recent demand of player 2 is $\sigma_2 \leq 1-\delta+\delta^2$. There exists an SPE of $\mathcal{F}_3$ in which player 3 accepts the demand of player 2 or 4.*

Proof. Let $\mathcal{F}_3'$ be the subgame resulting from a refusal by player 3 in $\mathcal{F}_3$. It is shown that following any action by player 3 in $\mathcal{F}_3$ there is a SPE of the resulting subgame in which player 3 receives a relative payoff with respect to $\mathcal{F}_3'$ of no more than $1-\delta$. By lemma 5.8 there is therefore a SPE of $\mathcal{F}_3$ in which player 3 accepts the lowest available demand.

case i

Suppose in $\mathcal{F}_3'$ player 3 makes a proposition $[\sigma_3,(2)]$ or $[\sigma_3,(2,4)]$. Let $\mathcal{F}_1$ be the resulting post-proposal subgame with responder 1. By corollary 5.12 there is a SPE of $\mathcal{F}_1$ in which player 1 accepts the demand of player 2 or refuses and makes a demand to player 2 which is accepted. By corollary 5.9 in any SPE of the resulting subgame player 3 receives a relative payoff with respect to $\mathcal{F}_3'$ of no more than $1-\delta$.

case ii

Suppose in $\mathcal{F}_3'$ player 3 makes a proposition $[\sigma_3,(4,2)]$. In the case that $\sigma_3 < 1-\delta$ then by lemma 5.7 it must be the case that in any SPE either this demand is accepted or player 3 does not exchange.

Suppose $\sigma_3 \geq 1-\delta$. Following the proposition of player 3 mentioned, there are two possibilities in SPE play. The first is that the exchange 45 forms and there results a post-proposal subgame with responder 1 and active players 1 2 3. By corollary 5.9 player 3 receives SPE payoff zero in this case. The second possibility is that SPE results in a post-proposal subgame $\mathcal{G}_4$ with responder 4.

Suppose in $\mathcal{G}_4$ player 4 refuses and makes the proposition $[1,(3,5)]$. Let $\mathcal{G}_1$ be the resulting post-proposal subgame with responder 1. By corollary 5.12 there is a

SPE of $\mathcal{G}_1$ in which player 1 accepts the demand of player 2 or refuses and makes a demand to player 2 which is accepted. By corollary 5.9 there exists a SPE of the resulting subgame in which player 3 receives payoff zero and, by lemma 5.4, player 4 receives a relative payoff with respect to $\mathcal{G}_4$ of δ. A SPE of $\mathcal{G}_4$ in which player 3 receives payoff zero can thus be constructed using lemma 5.8.

case iii

Suppose in $\mathcal{F}'_3$ player 3 makes a proposition $[\sigma_3, (4)]$. In the case that $\sigma_3 < 1 - \delta$ then by lemma 5.7 it must be the case that in any SPE either this demand is accepted or the exchange 45 forms and player 3 is the proposer in a 3 player active subnetwork. In the latter case, by lemma 5.9 player 3 receives a SPE payoff relative to $\mathcal{F}'_3$ of no more than $1 - \delta$.

Suppose $\sigma_3 \geq 1 - \delta$. Following the proposition of player 3 mentioned, there are two possibilities in SPE play. The first is that the exchange 45 forms and there results a post-proposal subgame with responder 1 and active players 1 2 3. By corollary 5.9 player 3 receives SPE payoff zero in this case. The second possibility is that SPE results in a post-proposal subgame $\mathcal{H}_4$ with responder 4.

Suppose in $\mathcal{H}_4$ player 4 refuses and makes the proposition $[1, (3, 5)]$. Let $\mathcal{H}_3$ be the resulting subgame. Should player 3 refuse in $\mathcal{H}_3$ and make a proposal then a post-proposal subgame $\mathcal{H}$ with responder 1 or 5 is reached. By corollary 5.12 and symmetry there is a SPE of $\mathcal{H}$ in which the responder either accepts or refuses and makes a demand to their neighbour which is accepted. By corollary 5.9 there is a SPE of the resulting subgame in which player 3 receives payoff zero. A SPE e of $\mathcal{H}_3$ in which player 3 receives payoff zero can thus be constructed using lemma 5.8. If player 3 does not already do so, alter e so that she accepts the demand of player 4 in $\mathcal{H}_3$. This is also a SPE. In this SPE player 4 receives a relative payoff with respect to $\mathcal{H}_4$ of $1 - \delta$. Hence a SPE of $\mathcal{H}_4$ in which player 3 receives payoff zero can be constructed by lemma 5.8. □

Lemma 5.14. *Consider a pre-proposal subgame $\mathcal{J}_2$ of $\mathcal{A}_3$ with proposer 2 such that*

$V_1 = V_3 = V_5 = \emptyset$ *and* $V_4 = \emptyset$ *or* (3). *Then there is a SPE of* $\mathcal{J}_2$ *in which player 2 receives a relative payoff with respect to* $\mathcal{J}_2$ *of no more than* $1 - \delta + \delta^2$.

Proof. Suppose there exists a SPE e of $\mathcal{J}_2$ in which player 2 receives a relative payoff with respect to $\mathcal{J}_2$ of more than $1 - \delta + \delta^2$. The following argument constructs a SPE f of $\mathcal{J}_2$ from e which satisfies the claim of the lemma.

Suppose that the initial proposition of player 2 in $\mathcal{J}_2$ under e is a unilateral demand to player 1. Should player 1 refuse in the resulting subgame and make a unilateral demand of $\sigma_1 < 1 - \delta$ then it is accepted in any SPE by lemma 5.7. Hence player 1 receives a relative payoff with respect to $\mathcal{J}_2$ of at least $1 - \delta$ under e and therefore player 2 cannot achieve the payoff claimed.

Suppose e specifies that the initial proposition of player 2 in $\mathcal{J}_2$ is $[\sigma_2, (1,3)]$ for some σ_2. Let $\mathcal{J}_1$ be the resulting subgame. Suppose player 1 refuses in $\mathcal{J}_1$ and makes the proposition $[\sigma_1, (2)]$ where $\sigma_1 < 1 - \delta$. Let $\mathcal{J}_3$ be the resulting subgame. By lemma 5.7, in any SPE of $\mathcal{J}_3$ player 2 accepts a lowest demand in any post-proposal subgame of $\mathcal{J}_3$ in which he is responder and σ_1 is available. In $\mathcal{J}_3$, if player 3 makes a proposition of $[\sigma_3, (4)]$ where $\sigma_3 < 1 - \delta$ then it is accepted in any SPE by lemma 5.7. Hence in $\mathcal{J}_3$ under e the exchange 23 does not form and player 1 receives a relative payoff with respect to $\mathcal{J}_3$ of σ_1. Hence $\underline{\pi}_1(\mathcal{J}_1) \geq 1 - \delta$ which is a contradiction.

The remaining case is that the initial propostion of player 2 is $[\sigma_2, (3,1)]$ or $[\sigma_2, (3)]$ for some σ_2. Let $\mathcal{K}_3$ be the resulting subgame. Should player 3 refuse in $\mathcal{K}_3$ and make a proposition of $[\sigma_3, (4)]$ where $\sigma_3 < 1 - \delta$ then it is accepted in any SPE by lemma 5.7. Thus $\underline{\pi}_3(\mathcal{K}_3) \geq 1 - \delta$ and the exchange 23 cannot form in e.

Let $\mathcal{L}_3$ be the last pre-proposal subgame of $\mathcal{K}_3$ with proposer 3 which is reached in the play of e before player 1 acts. Suppose e specifies that player 3 makes a multilateral demand of $\sigma_3 < 1 - \delta$ in $\mathcal{L}_3$. By lemma 5.7, under e player 2 or 4 accepts a demand in any resulting post-proposal subgame in which they are responder. Thus under e player 3 receives a relative payoff with respect to $\mathcal{J}_2$ of σ_3 or zero. However this is a contradiction as $\underline{\pi}_3(\mathcal{K}_3) \geq 1 - \delta$.

Suppose e specifies that player 3 makes a proposition of $[\sigma_3, (2)]$ in $\mathcal{L}_3$ where $\sigma_3 < 1 - \delta$. Let $\mathcal{L}_1$ be the resulting subgame. Then by lemma 5.7 in SPE player 2 must accept a demand when he is next proposer. Since the exchange 12 forms, it must be the case that in $\mathcal{L}_1$ under e player 1 either accepts the demand of player 2 or makes a demand which is accepted by player 2. In the resulting subgame under e player 3 receives a relative payoff with respect to $\mathcal{J}_2$ of no more than $1 - \delta$ by corollary 5.9. Construct a new strategy profile f by altering e such that player 3 makes the proposition $[1 - \delta, (4)]$ in $\mathcal{K}_3$ and player 4 accepts this proposal. This must also be a SPE. If the case at the start of this paragraph does not apply, let $f = e$.

Under f, a post-proposal subgame of $\mathcal{J}_2$ is reached with proposer 1 such that the most recent demand of player 3 is at least $1 - \delta$. Suppose player 1 refuses in $\mathcal{J}_2$ and demands $\sigma_1 < 1 - \delta$. Then by lemma 5.7, player 2 accepts. Thus f satisfies the conditions of the lemma. □

The following lemma describes some conditions in which a proposer cannot receive a positive payoff.

Lemma 5.15. *Under either of the following conditions on a pre-proposal subgame $\mathcal{M}$ of $\mathcal{A}_2$, the proposer receives payoff zero in any SPE.*

1) $\mathcal{M}$ has proposer 3, V_2 contains 1 and V_4 contains 5.

2) $\mathcal{M}$ has proposer 1, V_2 contains 3, V_4 contains 5 and if V_4 contains 3 then the most recent demand of player 4 is 1.

Proof. In case 1) let $(i, j, k) = (3, 1, 5)$. In case 2) let $(i, j, k) = (1, 3, 5)$. Suppose there is a SPE e of $\mathcal{M}$ in which player i receives a non-zero payoff. The exchange ix must form for $x = 2$ or 4. Consider the first proposition $[\sigma_i, V_i]$ made by player i in the play of e such that V_i contains x. It must be the case that $\sigma_i > 0$, otherwise player x would receive a share of 1 in e. Let $\mathcal{M}_u$ be the first post-proposal subgame of $\mathcal{M}$

under e with responder u where u is the neighbour of x other than i. Suppose in $\mathcal{M}_u$ player u refuses and makes a proposition $[\sigma_u, (x)]$ where $0 < \sigma_u < \min[\sigma_i, (1 - \delta)]$. By lemma 5.7 player x accepts this under e. Thus player u receives a non-zero payoff from e in $\mathcal{M}_u$. This finishes the proof for the case 1).

For case 2) it has been shown that player 3 receives a non-zero payoff in $\mathcal{M}_3$ under e. Thus if V_4 contains 3 in $\mathcal{M}$ then player 3 must refuse at least once under e. Suppose the exchange 34 forms in e. Consider the first proposition $[\lambda_3, V_3]$ made by player 3 in the play of e such that V_3 contains 4. It must be the case that $\lambda_3 > 0$, otherwise player 4 would receive a share of 1 in e contradicting the non-zero payoff of player 3. Let $\mathcal{M}_5$ be the first post-proposal subgame in the play of e with responder 5. Suppose in $\mathcal{M}_5$ player 5 refuses and makes a proposition $[\lambda_5, (4)]$ where $0 < \lambda_5 < \min[\lambda_3, 1 - \delta]$. By lemma 5.7 player 4 accepts this in e. Thus the exchange 34 cannot form in e. Instead the exchange 23 must form and player 1 receives payoff zero which is the required contradiction. □

The following lemma shows that an SPE of a subgame $\mathcal{P}$ of $\mathcal{A}_3$ in which player 1 has no offer on the table is still valid if the game is altered so that player 1 has an offer of $\bar{n}$ on the table.

Lemma 5.16. *Let $\mathcal{P}$ and $\mathcal{Q}$ be subgames of $\mathcal{A}_3$ satisfying the following conditions. Either both are pre-proposal subgames with responder $i > 1$ or both are post-proposal subgames with responder $i > 1$. All players are active in both. For $j > 1$, V_j has the same value in both subgames. For $j > 1$ such that $V_j \neq \emptyset$, the most recent demand of player j in both subgames is the same. In $\mathcal{P}$, $V_j = \emptyset$. In $\mathcal{Q}$, $V_j = (2)$ and the most recent demand of player 1 is $\bar{n}$.*

Let e be a SPE of $\mathcal{P}$. There is an SPE e' of $\mathcal{Q}$ such that following any sequence of actions in $\mathcal{Q}$ which is also permissable in $\mathcal{P}$ the action specified by e is taken.

Proof. The construction at the end of the statement of the lemma can be extended to produce a strategy profile e' of $\mathcal{Q}$ by describing the actions specified following all sequences of actions not permissable in $\mathcal{P}$. Construct e' by letting these actions be the same as in any fixed SPE f of $\mathcal{P}$.

Let $\mathcal{R}$ be any subgame of $\mathcal{Q}$. In the case that $\mathcal{R}$ is produced by a sequence of actions not permissable in $\mathcal{P}$ then a player cannot profitably deviate from e' because this would imply that f was not a SPE. Consider the case that $\mathcal{R}$ is produced by a sequence of actions which is permissable in $\mathcal{P}$. A player cannot profitably deviate from e' to a strategy producing a sequence of actions permissable in $\mathcal{P}$ because this would imply that e was not a SPE. The only other possible deviation is for player 2 to use a strategy producing acceptance of the demand of $\bar{n}$ of player 1. Let $\mathcal{S}$ be a post-proposal subgame of $\mathcal{R}$ in which player 2 is responder and has this option. By lemma 5.5, under e' player 2 receives a relative payoff of at least $\delta\bar{n}$ in $\mathcal{S}$. So the deviation described does not increase the payoff of player 2. Thus e' is a SPE of $\mathcal{Q}$, as required. □

5.4.7 The Endogenous Ordering Model on L_5

In this section it is shown that $\mathcal{A}_3 = F(L_5, \Delta_\delta, 3)$ has multiple SPE payoffs. In particular in it is proved that there exist SPEs of $\mathcal{A}_3$ in which player 3 receives payoffs $\bar{n}$ and $\gamma = 1 - \delta + \delta^2 - \delta^3$ respectively.

Low payoff equilibrium

Suppose player 3 makes an initial unilateral demand of $\sigma_3 \leq \gamma$ in $\mathcal{A}_3$. Lemma 5.14 and symmetry shows that there exists a SPE of the resulting subgame in which the responder accepts this demand. It is shown that following any other initial proposition in $\mathcal{A}_3$ there is a SPE of the resulting subgame in which player 3 earns a payoff of no more than γ. The result then follows by lemma 5.8.

Suppose player 3 makes an initial demand $\sigma_3 > \gamma$ unilaterally in $\mathcal{A}_3$. Without loss of generality suppose the responder is player 2. Let $\mathcal{A}_2$ be the resulting subgame. By lemma 5.14 there is a SPE f of $\mathcal{A}_2$ in which player 2 receives a SPE payoff of no more than $1 - \delta + \delta^2$. Alter this SPE so that in $\mathcal{A}'_2$ player 2 refuses and makes the proposition $[1 - \delta + \delta^2, (3, 1)]$. Let $\mathcal{B}_3$ be the resulting subgame. By lemma 5.13 there is a SPE e of $\mathcal{B}_3$ in which player 3 accepts. Further alter f by specifying that e is played in $\mathcal{B}_3$. This results in a SPE of $\mathcal{A}_2$ in which player 3 earns a payoff of $\delta - \delta^2 < \gamma$.

Suppose player 3 makes an initial demand $\sigma_3 \leq \gamma$ multilaterally in $\mathcal{A}_3$. Then in any SPE of the resulting subgame either this demand is accepted or both of players 2 and 4 receive payoffs of at least $1 - \sigma_3$. In either case player 3 receives a payoff of no more than σ_3.

Suppose player 3 makes an initial demand $\sigma_3 > \gamma$ multilaterally in $\mathcal{A}_3$. Note that thus $\sigma_3 > 1 - \delta$. Without loss of generality suppose that the initial proposition is $[\sigma_3, (2, 4)]$. Let $\mathcal{C}_2$ be the resulting post-proposal subgame with responder 2. Suppose player 2 refuses and makes the proposition $[1, (3, 1)]$. Let $\mathcal{C}_4$ be the resulting post-proposal subgame with responder 4. Suppose player 4 refuses and makes the proposition $[1, (5, 3)]$. Let $\mathcal{C}_5$ be the resulting post-proposal subgame with responder 5. By lemma 5.15 (and symmetry), if player 5 refuses in $\mathcal{C}_5$ then she receives payoff zero in any SPE. Thus there is a SPE of $\mathcal{C}_5$ in which player 5 accepts. Let $\mathcal{C}_3$ be the resulting subgame. There is a SPE of $\mathcal{C}_3$ in which player 3 receives a SPE payoff of zero by corollary 5.9. Thus by lemma 5.8, there is a SPE of $\mathcal{C}_3$ in which player 3 accepts. Thus there is a SPE of $\mathcal{C}_2$ in which player 2 receives payoff δ and player 3 receives payoff zero.

High payoff equilibrium

Suppose player 3 makes an initial proposition of $[\bar{n}, (2, 4)]$ in $\mathcal{A}_3$. Let $\mathcal{D}_2$ be the resulting post-proposal subgame with responder 2. Let $\mathcal{D}'_2$ be the subgame that results if player 2 then refuses. It is shown that following any action of player 2 in $\mathcal{D}'_2$ other than a unilateral demand to player 1, there is a SPE of the resulting subgame in which player 2 receives a relative payoff with respect to $\mathcal{D}'_2$ of no more than $\bar{n}$. Lemmas 5.8 and 5.10 then show that there is a SPE of $\mathcal{D}_2$ in which player 2 accepts.

Suppose in $\mathcal{D}_2'$ player 2 makes the proposition $[\sigma_2, (3)]$ or $[\sigma_2, (3, 1)]$ where $\sigma_2 \leq \bar{n}$. Let $\mathcal{D}_4$ be the resulting post-proposal subgame with responder 4. In any SPE of $\mathcal{D}_4$ player 3 must receive a payoff of at least $1 - \sigma_2$ as she can accept the demand of player 2. By lemma 5.5, player 4 is guaranteed a payoff of at least $\bar{n}$ in any SPE of $\mathcal{D}_4$. Thus it must be the case that the exchanges 23 and 45 form in any SPE. Hence in any SPE of $\mathcal{D}_4$ player 2 can receive a relative payoff with respect to $\mathcal{D}_2'$ of at most σ_2.

Suppose in $\mathcal{D}_2'$ player 2 makes the proposition $[\sigma_2, (1, 3)]$. Let $\mathcal{M}_1$ be the resulting subgame. If $\sigma_2 < \bar{n}$ then in any SPE of $\mathcal{M}_1$ player 1 accepts by lemma 5.5. Otherwise suppose player 1 refuses and demands $\bar{n}$. By lemma 5.16 there is a SPE of the resulting subgame $\mathcal{M}_4$ which produces the same sequence of actions as the SPE described below for either $\mathcal{E}_4$ and$\mathcal{F}_4$ (depending on the value of σ_2). Both result in the exchange 34 forming before player 2 acts. There is a SPE of the resulting subgame with active players 1 and 2 in which player 2 accepts the demand $\bar{n}$. Thus by lemma 5.8 there is a SPE of $\mathcal{M}_4$ in which player 2 accepts the demand $\bar{n}$. Hence there is a SPE of $\mathcal{M}_1$ in which player 1 takes the action described above since player 1 cannot receive a higher SPE payoff by lemma 5.5. In this SPE player 2 receives payoff $\delta\bar{n}$ relative to $\mathcal{D}_2'$, as required.

Suppose player 2 makes the proposition $[\sigma_2, (3)]$ in $\mathcal{D}_2'$ where $1 - \delta + \delta^2 < \sigma_2$. Let $\mathcal{E}_4$ be the resulting post-proposal subgame with responder 4. By lemma 5.14 (and symmetry) there is a SPE f of $\mathcal{E}_4$ in which player 4 receives a SPE payoff of no more than $\delta(1 - \delta + \delta^2)$. Alter this strategy profile so that in $\mathcal{E}_4$ player 4 refuses and makes the proposition $[1 - \delta + \delta^2, (3, 5)]$. Let $\mathcal{E}_3$ be the resulting subgame. By lemma 5.13 there is a SPE e of $\mathcal{E}_3$ in which player 3 accepts. Further alter f by specifying that e is played in $\mathcal{E}_3$. Thus under f a pre-proposal subgame of $\mathcal{E}_4$ is reached with proposer 2 and only other active player 1. The payoff to player 2 under f relative to $\mathcal{D}_2'$ is thus $\bar{n}$ as required.

Suppose player 2 makes the proposition $[\sigma_2, (3)]$ in $\mathcal{D}_2'$ where $\bar{n} < \sigma_2 \leq 1-\delta+\delta^2$. Let $\mathcal{F}_4$ be the resulting subgame and $\mathcal{F}_4'$ be the subgame resulting from refusal by player 4. Suppose player 4 makes a proposition of $[\sigma_4, (3, 5)]$ in $\mathcal{F}_4'$ for any σ_4. Let $\mathcal{F}_3$ be the resulting post-proposal subgame with responder 3. By lemma 5.13, there is a SPE of $\mathcal{F}_3$ in which player 3 accepts. The remainder of the argument for this case can be done in tandem with the next case.

Suppose player 2 makes the proposition $[\sigma_2, (3, 1)]$ in $\mathcal{D}_2'$ where $\bar{n} < \sigma_2$. Let $\mathcal{G}_4$ be the resulting subgame and $\mathcal{G}_4'$ be the subgame resulting from refusal by player 4. Suppose player 4 makes a propostion of $[\sigma_4, (3, 5)]$ in $\mathcal{G}_4'$ for any σ_4. Let $\mathcal{G}_3$ be the resulting post-proposal subgame with responder 3. Should player 3 refuse in $\mathcal{G}_3$ then by lemma 5.15, player 3 receives zero payoff in any SPE. Thus there is a SPE of $\mathcal{G}_3$ in which player 3 accepts.

The above arguments show if player 4 makes a proposition of $[\sigma_2, (3, 1)]$ in $\mathcal{F}_4'$ and $\mathcal{G}_4'$ then there is a SPE in which it is accepted by player 3. The resulting game is essentially an alternating offers game between player 1 and 2. Hence in any SPE of this subgame player 2 receives a relative payoff with respect to $\mathcal{D}_2'$ of no more than $\bar{n}$

Let $\mathcal{H}_4'$ be $\mathcal{F}_4'$ or $\mathcal{G}_4'$. It is shown below that following any action of player 4 in $\mathcal{H}_4'$ other than a unilateral demand to player 5, there is an SPE of the resulting subgame in which player 4 receives a payoff of no more than σ_2 relative to $\mathcal{H}_4'$. Application of lemmas 5.8 and 5.10 then shows that there is a SPE of $\mathcal{H}_4$ in which player 4 refuses and makes the proposition described above, and player 2 receives a relative payoff with respect to $\mathcal{D}_2'$ of no more than $\bar{n}$, as required.

Suppose player 4 makes a proposition $[\sigma_4, (3, 5)]$ in $\mathcal{H}_4'$ for any value of σ_4. Let $\mathcal{H}_3$ be the resulting subgame. By the arguments above, there is a SPE of this subgame in which player 3 accepts the lowest demand available. Thus in the case that $\sigma_4 \leq \sigma_2$ there is a SPE in which the demand σ_4 is accepted. In the case that $\sigma_4 > \sigma_2$ there is a SPE in which player 3 accepts the demand of player 2, as required. The resulting subgame is essentially an alternating offers game between players 4 and 5. Hence in any SPE of this subgame player 4 receives a relative payoff with respect to $\mathcal{H}_4'$ of no more than $\bar{n}$, as required.

Suppose in $\mathcal{H}_4'$ player 4 makes a proposition of $[\sigma_4, (3)]$ for any σ_4. Let $\mathcal{J}_3$ be the resulting post-proposal subgame with responder 3. In the case that $V_2 = (3, 1)$ in $\mathcal{J}_3$ then lemma 5.13 can be applied, and the arguments made above for $\mathcal{H}_3$ hold. For the case that $V_2 = (3)$ in $\mathcal{J}_3$, let $\mathcal{J}_3'$ be the subgame generated if player 3 refuses in $\mathcal{J}_3$. Observe that $\mathcal{J}_3'$ is equivalent up to discounting to $\mathcal{A}_3$. Recall the low payoff equilibrium found above for $\mathcal{A}_3$. By lemma 5.3 there is a SPE of $\mathcal{J}_3'$ in which player 3 makes a demand to player 2 and it is accepted. In the resulting alternating offers game with player 5, player 4 receives a SPE relative payoff with respect to $\mathcal{H}_4'$ of no more than $\bar{n}$, as required.

Suppose player 4 makes the proposition $[\sigma_4, (5, 3)]$ in $\mathcal{H}_4'$. Let $\mathcal{K}_5$ be the resulting post-proposal subgame with responder 5. In the case $\sigma_4 \leq \sigma_2$, player 5 is guaranteed a share of σ_4 in any SPE of $\mathcal{K}_5$ and thus player 4 receives a relative payoff with respect to $\mathcal{H}_4'$ of no more than σ_2 in any SPE of $\mathcal{K}_5$, as required. Suppose $\sigma_4 > \sigma_2$. In case that $V_2 = (3, 1)$ in $\mathcal{K}_5$ and $\sigma_2 \leq 1 - \delta + \delta^2$, suppose player 5 refuses in $\mathcal{K}_5$ and makes the unilateral demand $\bar{n}$ to player 4. In some SPE of the resulting subgame, player 3 accepts the demand of player 2 by lemma 5.13 and player 4 accepts in the remaining alternating offers game with player 5. Thus there is a SPE of $\mathcal{K}_5$ in which player 4 receives a relative payoff with respect to $\mathcal{H}_4'$ of no more than $\bar{n}$ as required. In the case where instead $\sigma_2 > 1 - \delta + \delta^2$, suppose player 5 refuses in $\mathcal{K}_5$ and demands σ_5 such that $1 - \sigma_4 < \delta\sigma_5 < 1 - \sigma_2$. Let $\mathcal{K}_3$ be the resulting subgame. If player 3 refuses then player 4 is guaranteed a share of $1 - \sigma_5 > 1 - \delta^{-1}(1 - \sigma_4)$ in SPE. If the exchange 34 forms in such a SPE then player 3 receives a payoff of less than $1 - \sigma_4$ relative to $\mathcal{K}_3$. As this is worse than accepting the offer of player 2, the exchange 34 does not form in any SPE of $\mathcal{K}_3$. Note that $\sigma_5 < 1 - \delta$, so this demand is accepted by player 4 in SPE of $\mathcal{K}_3$. Hence player 5 earns a relative payoff of at least $1 - \sigma_2$ in any SPE of $\mathcal{K}_5$ and player 4 receive a relative payoff of no more than σ_2 as required.

Consider the case that $V_2 = (3)$ in $\mathcal{K}_5$. Suppose player 5 refuses in $\mathcal{K}_5$ and makes a unilateral demand of $\bar{n}$ to player 4. Let $\mathcal{L}_3$ be the resulting subgame and let $\mathcal{L}'_3$ be the subgame that results if player 3 then refuses. By lemma 5.16 there is a SPE of $\mathcal{L}'_3$ which produces the same sequence of actions as the low payoff equilibrium of $\mathcal{A}_3$ described above. Thus by lemma 5.8 there is a SPE of $\mathcal{L}_3$ in which player 3 either accepts the demand of player 2 or refuses and makes a demand to player 2 which is accepted, and in the resulting subgame player 4 accepts the demand of player 5. The latter action is SPE as in this subgame only players 4 and 5 are active. This shows that player 5 is guaranteed a relative payoff of $\bar{n}$ in $\mathcal{K}_5$ so player 4 receives only $\delta\bar{n}$, as required.

Chapter 6

Simulation: Background and Overview

One approach to modelling bargaining situations is through the framework of evolutionary game theory, as briefly described in section 3.3.5. This requires an underlying game with relatively simple strategy sets. Evolutionary game theory models based on the *Nash demand game* have enjoyed much recent success. This game, originally proposed by Nash in [53], models 2 player bargaining. Both players must simultaneously name a demand. If this demand pair lies within the utility cake then each player receives a payoff equal to their demand. Otherwise both receive nothing. In contrast to the alternating offers game, this approach abstracts away most of the details of the bargaining process. In particular, strategies are simply demand values. The Nash demand game supports all Pareto optimal outcomes with strict (pure strategy) Nash equilibria, so there is a significant equilibrium selection problem. However, the simple strategy structure of the game makes it amenable to evolutionary methods, providing a potential method of equilibrium selection.

This chapter gives an overview of an evolutionary model based on an extension of the Nash demand game to general bilateral exchange networks with a view to implementing the model as a computer simulation. The relation of this model to the literature on similar models is also discussed. Chapter 7 explores the details of the evolutionary model, and chapter 8 presents the simulation results. Many unrealistic assumptions are introduced in the evolutionary model, so these results are not intended to be used as quantitative predictions. Instead they are viewed as providing a useful qualitative tool to investigate evolutionary pressures which may determine the outcome of bargaining.

Section 6.1 reviews the literature on similar evolutionary models. Such models can often be represented mathematically as Markov processes, so a summary of relevant material on this topic is included. Section 6.2 gives an overview of the proposed evolutionary model of bargaining. Section 6.3 describes how this model can be represented by a Markov process.

6.1 Literature Review

A general introduction to the approach of evolutionary game theory is given in section 3.3.5. Section 6.1.1 gives a description of the features of some evolutionary models relevant to this chapter. These models can be represented mathematically as perturbed discrete Markov processes. Section 6.1.2 briefly defines these and summarizes some relevant results. Section 6.1.3 describes some particular evolutionary models of bargaining based on the Nash demand game.

6.1.1 General Features

A pioneering evolutionary model is that of Kandori Mailath and Rob (KMR) [37]. Their approach produces a more tractable model than previous similar work (e.g. see Foster and Young [26] and Fudenberg and Harris [28]). The KMR model is a dynamic model for a 2×2 symmetric game (as defined in section 3.3.2). The game is played by members of a single large population. I refer to these members as *agents*[1].

[1] KMR refer to the members of this population as 'players'. The main part of this chapter is concerned with games which are not symmetric. Thus I reserve term 'player' to distinguish between

The model is a discrete time model with time values in $\mathbb{N}^+$. I refer to each time value as representing a *round* of the model. In a given round each agent has a fixed pure strategy of the underlying game. In each round all agents are repeatedly matched to play the game in pairs using their fixed strategies. Agents' strategies in the next round depend on the payoffs received. I refer to the particular method that agents use to update their strategies as their *learning rule.* Whenever an agent updates their strategy, they have a small probability $\epsilon > 0$ of *mutating* to either strategy with equal probability rather than using their learning rule.

KMR study the case where the underlying symmetric game has two strict Nash equilibria in which both players play the same strategy. Under the particular learning rules they consider, based on agents being more likely to update to the strategy with better average payoff, the model has two stable patterns of behaviour in the case $\epsilon = 0$. These are the two population states in which all agents play the same strategy. However, in the case $\epsilon > 0$, mutations occasionally disrupt the stability of these states; if enough agents mutate then the population can eventually settle in the other state. This enables selection between the two pure Nash equilibria. KMR prove that in most cases the probability that in the long-run the population is at one of these states[2] tends to 1 in the limit $\epsilon \to 0$. The exception is that for correctly balanced payoff values in the underlying game, this probability tends to $\frac{1}{2}$ for both states. The meaning of this 'probability in the long-run' is made precise in section 6.1.2 below.

The framework described above has been generalized by many researchers and is the basis of the evolutionary model used in this chapter. The main differences are that the model used here has multiple populations of agents, uses an underlying game with more than 2 players which is not symmetric, and matches only one agent from each population to play this game in each round. A model similar to that of KMR investigating asymmetric games by using multiple populations is given in

player positions in games.

[2]That corresponding to the 'risk-dominant' Nash equilibrium.

Binmore et al [9] and discussed below in section 6.1.3. Some other extensions to the KMR model in the literature include looking at large but finite populations (e.g. Seymour [61]), studying methods of agent matching that rely on structured populations (e.g. Ellis [24]), the effect of state-specific mutation rates (e.g. Blume [14]), or populations which are heterogeneous in either preferences (e.g. Young [76]) or learning rules (e.g. Matros [47]).

The attraction of models using the KMR framework is that they enable progress to be made on selection between strict Nash equilibria such as those found in the Nash demand game. Methods to study equilibrium selection are discussed in section 6.1.2 below. As mentioned above these are 'long-run' selections. The reason for this is that mutation events large enough to move the population between stable states occur rarely. This raises the question of whether the timescale for which these long-run predictions are accurate is relevant to the setting being modelled. It is difficult to answer this question in general as many factors specific to the setting may affect this timescale. For example, mechanisms to reduce the long-run timescale could be provided by correlated mutations, matching based on structured populations (e.g. Ellis [24]) or noisy learning (e.g. Binmore and Samuelson [7]). The question of whether the long-run timescale is relevant for the model of this chapter is discussed in section 9.2 of the conclusion.

The remainder of this subsection briefly discusses some of the relevant features of the KMR model. The assumption of a large population of agents who interact repeatedly is natural for many biological[3] and social settings. Examples are landlords and tenants choosing contracts or two populations representing predator and prey each containing various subpopulations with different behaviours. The use of large populations does not seem so reasonable for an underlying game with a large number of players. This case is discussed in section 7.2.4. In a large population, the random

[3]For biological models the 'learning rule' should be replaced with rules modelling the rates of birth and death of agents based on their payoffs. See Seymour [61] for example.

matching process has the advantage that repeated matchings between agents are rare so that issues such as repeated game effects[4] can be neglected. Another advantage is that some variables in the model can be approximated by expected values. For example, KMR mention a setting where agents are matched a large number of times in each round and their average payoffs, used by the learning rules, are approximated by their expected payoffs. Another example is in the proof that the 'aspiration and imitation' model, an evolutionary model using the framework described above, can be approximated over finite time periods by deterministic equations. This model and the proof are given in section 3.1 of Samuelson [57].

Mutation plays the crucial role of introducing noise into the model which can occasionally disrupt patterns of play based on pure Nash equilibria. This specification of noise has the advantage of being introduced at an agent level, and produces a more tractable model than some alternatives such as Foster and Young [26]. Some possible alternative specifications of noise are mentioned in section 7.2.6.

The use of short-sighted learning rules in evolutionary models is discussed in section 3.3.5. KMR argue that they are especially relevant when the model displays what they refer to as *inertia*. This is the case where the rate at which agents change their strategies is slow compared to how often they play the game. Thus the expected payoff of a strategy changes only slowly and a good short-sighted choice of strategy also does well in the near future.

[4] See proposition 10.2 of Muthoo [51] for an illustration of how a repeated 2 player bargaining games can support a wide array of SPE outcomes.

6.1.2 Perturbed Discrete Markov Processes

A *discrete Markov process* is a family of random variables X_t with the same finite or countable state space Z for time values $t \in \mathbb{N}$ with the property

$$\begin{aligned} P(X_{t+1} = z'|X_t = z) =& P(X_{t+1} = z'| \\ & X_t = z, X_{t-1} = z_{t-1}, X_{t-2} = z_{t-2}, \dots) \forall z_{t-1}, z_{t-2} \\ =& P_{zz'}(t) \end{aligned}$$

for some values $P_{zz'}(t)$ such that $\sum_{z'} P_{zz'}(t) = 1$ for each z and t. The value of X_t is referred to as the state of the system at time t. The values $P_{zz'}(t)$ are called *transition probabilities.* In the case that $P_{zz'}$ is constant over t the process is called *time homogeneous.* The Markov processes discussed in this thesis are discrete, time homogeneous and have finite state spaces.

A state $z' \in Z$ is *accessible* from $z \in Z$ if there is a positive probability of the state changing from z to z' in a finite number of transitions. This is defined[5] to include the case $z = z'$. A state $z \in Z$ is said to *communicate* with $z' \in Z$ if they are accessible from each other. This is an equivalence relation and partitions the state space into equivalence classes referred to as *communication classes.* A *recurrent class* of Z is a communication class such that no state outside the class is accessible from any state inside it. It is straightforward that every Markov process with a finite set of states has at least one recurrent class[6]. States not contained in a recurrent class are called *transient.* A Markov process with exactly one recurrent class containing all states is referred to as *irreducible.*

[5]This material can be presented slightly differently and sometimes 'accessible' is defined so that it does not always cover the case $z = z'$. The presentation used in this section is based on section 3.3 of Young [78].

[6] Otherwise there is always a positive probability of leaving a communication class. Since there can be only a finite number of communication classes, there must be a positive probability of leaving one and returning to it. But this implies the existence of states outside this communication class which communicate with its elements.

Assume Z is finite and index its elements as $\{z_1, z_2, \dots, z_n\}$. Let μ be a probability distribution over Z. The probability distribution in the following time period is given by μP where P is the $n \times n$ matrix $(P_{z_i z_j})$ and μ is written as a row vector. A *stationary distribution* satisfies

$$\mu = \mu P \tag{6.1}$$

An important case is where

$$P_{zz'} = \begin{cases} \lambda Q_{zz'} & \text{for } z \neq z' \\ 1 - \lambda + \lambda Q_{zz} & \text{for } z = z' \end{cases}$$

and $\sum_{z'} Q_{zz'} = 1$. That is, λ is the probability that a transition occurs in any time period and $Q_{zz'}$ is the conditional probability that the state changes from z to z' given that a transition occurs. Then $P = (1-\lambda)I + \lambda Q$ where I is an identity matrix and Q is the $n \times n$ matrix $(Q_{z_i z_j})$. Under these conditions, equation (6.1) becomes

$$\mu = \mu Q \tag{6.2}$$

This shows that stationary distributions depend only on the conditional transition probabilities to other states given that a transition takes place.

It is a well known result[7] that any Markov process with a unique recurrent class has a unique stationary distribution μ^*. Furthermore μ^* describes the time-average asymptotic behaviour of the process *independently* of the initial state μ_0 i.e.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{t=N} \mu_0 P^t = \mu^* \tag{6.3}$$

For each state $z \in Z$ let N_z be the set of integers $n \geq 0$ such that there is a positive probability of the state moving from z to z in exactly n periods. An *aperiodic* recurrent class C is one in which the greatest common divisor of N_z is 1 for all $z \in C$. It can be shown that the greatest common divisor of N_z is the same for any $z \in C$ so 'any' can be substituted for 'all' in the previous definition. If a

[7] A reference for proofs of the results on Markov processes in this section is Chung [20].

Markov process has a unique aperiodic recurrent class then the result of equation (6.3) can be strengthened to the following ergodicity result:

$$\lim_{t \to \infty} \mu_0 P^t = \mu^* \tag{6.4}$$

That is, μ^* describes the long-run expected state of the process independently of the initial state.

The ergodic distribution μ^* can be found by solving (6.1) directly, but the computational cost of this is often prohibitive for a large state space. Kandori Mailath and Rob in [37] present the following alternative method[8]. Recall the definition of a directed tree from section 3.2. A tree rooted at $x \in Z$ is a directed tree with vertices Z and root x. Given a directed tree T, let $P(T)$ denote the the product of the transition probabilities P_{xy} for all directed edges (x, y) of T. Let $v(x) = \sum P(T)$ where the sum is taken over all trees rooted at x. The Markov chain tree theorem states that for an irreducible Markov process $v(x)$ is proportional to μ^*_x. That is

$$\mu^*_x = \frac{v(x)}{\sum_{y \in Z} v(y)}$$

A proof of the Markov chain tree theorem is given in Young [78] (lemma 3.1). Note that this theorem can easily be applied to Markov processes which are not irreducible but contain a unique recurrent class by removing the transient states from the state space, as it is straightforward to prove that transient states receive zero weight in the stationary distribution.

The following methods apply for *perturbed Markov processes.* See section 3.4 of Young [78] for a general definition of these. For the purpose of this thesis, it suffices to note that this definition encompasses Markov processes whose transition probabilities involve a possibility ϵ of mutation as described in the framework of section 6.1.1 and such that for all $\epsilon > 0$ there is a unique recurrent class. The Markov process in the case $\epsilon = 0$ is called the *unperturbed process.*

[8]This method is a discrete version of techniques given in Friedlin and Wentzell [27].

For a perturbed Markov process, taking the low mutation limit of the stationary distribution, $\lim_{\epsilon \to 0} \mu^*$, allows a simplification of the tree analysis. A state is said to be *stochastically stable* if it receives positive probability in this limiting distribution. Given any two states $z, z' \in Z$, define the *one-step resistance* from z to z' to be the minimum number of mutations needed to move from z to z' in a single round. If no such transition is possible define the one-step resistance to be ∞. Define the *resistance* of a directed tree to be the sum of the one-step resistances. A *minimal tree* is a directed tree with minimum resistance. The Markov chain tree theorem implies that a state is stochastically stable if and only if it is the root of a minimal tree. A proof of this result is contained in Young [77] (it is theorem 4 of this paper). The idea behind this result is that as ϵ tends to 0, mutations become extremely rare, so the probability of moving between two states depends principally on how many mutations it involves. A survey of methods which further refine this technique is contained in Binmore et al [9]. Section 7.6 proves a theorem on the evolutionary model of this chapter based on these tree methods.

6.1.3 Evolutionary Models of Bargaining

Young ([76] and chapter 8 of [78]) applies the methods of the previous section to a model of the Nash demand game. Young's model has two populations; one for each player in the game. In each round a single pair of agents are matched to play the Nash demand game once. Agents use their learning rules to choose strategies immediately *prior* to playing the underlying game, and this choice is determined by play in the n most recent rounds. Hence the fixed demands of the agents become irrelevant and the state of the system can be given by the outcome of the n most recent rounds. Young uses a learning rule in which agents play a best reply to the mixed strategy given by the frequencies of demands in a random sample of m of the n most recent rounds. The value of m depends on which population the agent belongs to. Young proves that under some mild conditions, the stochastically stable states of the model correspond to an asymmetric Nash bargaining solution with bargaining powers corresponding to the values of $\frac{m}{n}$ for each population.

Binmore, Samuelson and Young in [9] present another model based on the KMR framework and apply it to the Nash demand game and some closely related variant games. This model also has a population for each player position. Both these populations contain M agents. The paper studies two dynamics which are based on learning involving agents switching to a best reply to the mixed strategy profile given by the frequencies of strategies amongst the agents of the other population[9].

The two dynamics are interpreted as representing the same process under different assumptions about limits on M and ϵ and the occurrence of mutations. Consider a process in which in every round every agent updates to the best reply mentioned above with probability λ if they do not mutate. The first model of [9], *stochastic best response dynamics*, applies the usual model of mutation to this process and investigates the behaviour in the limiting case $\epsilon \rightarrow 0$. The second model, *deterministic best response dynamics*, can be interpreted as representing the case where there is a probability λ in each round of a 'mutation event'. If this occurs then each agent has a probability ϵ of mutating in that round. The deteministic best response dynamics capture the limiting case where the limits are taken in the order of $\lambda \rightarrow 0$ then $M \rightarrow \infty$ and finally $\epsilon \rightarrow 0$. That is, mutations occur very infrequently but many may appear in the same round, and in between such rounds the dynamics follow a deterministic path. Thus after each mutation event the dynamics reach some recurrent class of the unperturbed dynamics with probability 1 before any more mutations occur. Thus setup simplifies the analysis of the case by tree methods. Under an ordinary mutation scheme, a least resistant path between recurrent classes might take a complicated form involving a burst of mutations, followed by learning, then more mutations[10].

[9]Note that under these learning rules it is not necessary for the agents to be matched to play the game each round.

[10]A concrete example of this is given in footnote 11 of [9]

Binmore et al prove that both these dynamics support the symmetric Nash bargaining solution for the Nash demand game. However for the *contract game* the stochastic dynamics support the Kalai-Smorodinsky solution of [36] (discussed briefly in section 4.1). This game is a modification of the Nash bargaining game in which the players receive payoff zero unless their demand pair lies on the Pareto boundary of their utility cake. In the latter case players receive payoffs equal to their demand. The deterministic dynamics support the symmetric Nash bargaining solution for the contract game. This result shows that the relative values of M and ϵ may qualitatively influence the behaviour of an evolutionary bargaining model in the KMR framework.

6.2 An Overview of the Evolutionary Model

This section presents the main algorithm for the evolutionary model of bargaining proposed in this chapter. Outlines of the various steps are sketched here and are discussed in chapter 7. The details of the steps are completed in section 7.5.

A bilateral exchange network $N = (P, E, \mathcal{K})$ is under investigation. For each $i \in P$ there is a *population* A^i which is a set of *agents*. Each population is assumed to contain the same number of agents, M. Each agent in A^i is endowed with an individual state composed of a *demand* in D_i and an *informational state* in I_i. The demand set D_i is a finite set of non-negative reals. Let $D = (D_i)_{i \in P}$. The form of the set I_i varies and is discussed below. It is always a finite set. An initial individual state for each agent must be given[11].

There is an underlying game $\mathcal{N}(N, D)$, defined shortly, with player set P and strategy set D_i for player i. The evolutionary model is as follows.

[11]In the computer implementation it suffices to give the number of agents in each initial state for each population. See the discussion on the computer representation of the aggregate state of the model in section 6.3.

Algorithm (Main loop).

1. Randomly sample an agent from each population using a uniform probability distribution. These are referred to as the *active* agents.

2. Call a subroutine to play $\mathcal{N}$ with the demands of the active agents. Assign the realised payoffs of the game to the active agents[12].

3. For each active agent in turn, call the updating subroutine to determine their new individual state.

4. Alter the population state accordingly.

5. Return to step 1.

An execution of steps 1 to 5 is referred to as a *round.* The number of rounds performed measures how much time has been simulated by the model. On the other hand the length of time spent by a computer program running the simulation is referred to as the *run-time.* When an agent is referred to as active it is usually with reference to a particular round.

The game $\mathcal{N}(N, D)$ is the following extension of the Nash demand game. Let $d = (d_i)_{i \in P}$ be a strategy profile. This is also referred to as a *demand profile.* Let $\Gamma(d)$ be the set of all *maximal consistent outcomes* of N given d. These are defined shortly. A uniform random distribution is used to choose an element of this set. Players' payoffs are given by their shares in this outcome. Note that in the terms of section 3.3, $\mathcal{N}$ is a game in which all player make an action in the first period and then a random move decides the payoffs. The algorithm outlined above could also be used with other underlying games.

Recall that definition 3.8 defines a feasible outcome of a network N. Such an outcome gives a share for each player and a set of realised exchanges. These must satisfy various feasibility constraints generated from the network N.

[12]These payoffs are used in the updating subroutine.

Definition 6.1. A *consistent outcome* of $N = (P, E, \mathcal{K})$ given demand profile $d = (d_i)_{i \in P}$ is a feasible outcome such that condition i) below holds. If condition ii) also holds then it is called a *maximal consistent outcome*

i) If player i is involved in a realised exchange then their share equals d_i.

ii) If $(d_i, d_j) \in \mathcal{K}^{ij}$ then at least one of i and j is involved in a realised exchange.

The second condition means that in a consistent outcome it is not possible that two neighbouring players do not exchange but have a demand pair which lies in the corresponding utility cake. Given a consistent outcome which is not maximal, realised exchanges can be added until a maximal consistent outcome is generated.

Agents use *learning rules* to update their demands. As discussed in section 3.3.5, these are relatively simple shortsighted heuristic rules. It is assumed that learning rules are deterministic[13] and all agents use the same learning rule. The role of agents' informational states is to provide the only additional input for learning rules beyond the demands and payoffs of the active agents. An informational state might encapsulate information that the agent has learned from recent play (e.g. the average of the realised payoffs for each demand), or it could contain information about the agent's recent play (e.g. average payoff received) which other agents sometimes observe. Thus the specifications of I_i and the updating subroutine depend upon which learning rule is used.

There is assumed to be a possibility of mutation. Upon updating in step 3, there is a probability ϵ that an agent may switch to a random demand rather than using their learning rule. The random demand is chosen by a uniform distribution on the corresponding demand set. The updating subroutine must describe this possibility as well as the learning rule and the details of how the informational state is updated.

Note that some learning rules[14] do not depend on the payoffs agents receive. In

[13] This restriction is not essential. It is made because it simplifies some later exposition and because in practice only deterministic learning rules were used in the course of this research.

[14] For example, the sampled best reply learning rule described in section 7.1.2.

this case, step 2 of the algorithm can be removed to reduce the computational cost. For the results of the model to be based on the underlying game, such learning rules must use $\mathcal{N}$ directly.

For the learning rules considered below, either I_i is a singleton – if agents use only the most recent realised payoffs and demands of active agents to update their demands – or $I_i = D_i \cup \{0\}$ and an agent's individual state represents their most recent payoff[15]. An example of the latter case is if the learning rule involves sampling another agent and deciding whether to imitate them based on their most recent payoff.

6.3 The Model as a Markov Process

The evolutionary model of the previous section can be described as a discrete time homogeneous Markov process. The time value corresponds to the number of rounds played. A state of the model can be represented by a specification of the individual state of each agent. The state of the model is often referred to below as the *aggregate state* to differentiate it from the individual state of an agent.

Note that at no point in the model does the identity of an agent play a part, only the population to which they belong and their individual state. This suggests an alternative representation of aggregate state. In a given round, each population can be partitioned into subpopulations which are homogeneous with respect to agents' individual states. The vector of sizes of each of these subpopulations acts as a representation of aggregate state. The aggregate state space in this case is the set of all such vectors which conserve the initial total population sizes.

The latter representation is used in the computer implementation of the model, as it makes less demands on storage[16]. However, the former representation is used

[15] In a variation where agents update their strategies *before* playing the one-shot game, $I_i = \{0, 1\}$ can serve this purpose. 0 can represent a most recent payoff of zero and 1 can represent most recent payoff equal to the agent's demand.

[16] In the case where M is small compared to the number of possible individual states in a typical

for the remainder of this chapter. Arguments expressed using this representation are easier to follow. This is due to the fact that agents are given individual identities. It is thus straightforward to refer to a particular agent, such as one which mutates, over several following rounds. These references could easily be translated to accommodate the other representation of aggregate state but would be much less concise.

Under either representation of aggregate state, the transition probabilities can be calculated directly from the description of the model. This completes the definition of a Markov process. In a slight abuse of terminology, for the remainder of this chapter I sometimes refer to features of the model when I strictly mean features of this Markov process.

Definition 6.2. A state is said to *support* demand d_i for population A^i if there exists an agent in that population with that demand. A state s is said to support a demand profile $(d_i)_{i \in P}$ if for all $i \in P$ it supports d_i for population A^i. A set of states S is said to support a demand profile d if some state $s \in S$ supports d.

Given an aggregate state, a vector giving the number of agents in each population playing each demand can be calculated. Such a vector is referred to as the *aggregate demand state.*

Lemma 6.1. *The following conditions are sufficient for the Markov process just defined to have a unique aperiodic recurrent class.*

1. $\epsilon > 0$

2. *There exists an n such that the following is true. If at least n rounds have been played then the learning rule selects a new informational state dependent only on the demands and payoffs of the active agents and the aggregate demand state in the most recent n rounds*

population, the former representation might become more efficient. However, for the simulations of chapter 8 this is not the case.

Condition 2 effectively limits the informational content of informational states to events in the most recent n rounds[17].

Proof. Fix N, D, $\epsilon > 0$, a demand profile d of $\mathcal{N}(N, D)$ and a learning rule satisfying condition 2. Choose some n as described in condition 2.

Let π be a possible payoff profile of $\mathcal{N}(N, D)$ given demand profile d. Let $\theta_i \in I_i$ be the informational state chosen by the learning rule for an agent in population A^i if for the previous n rounds: the sampled demand profile is d, the resulting payoff profile is π and for all $j \in P$ the demand of every agent in population A^j is d_j.

Let s_0 be any aggregate state. In any round there is a positive probability that the active agent in population A^i mutates to demand d_i. Assume this takes place for all $i \in P$ throughout the following description. Each round there is an active agent from some population A^i who does not use demand d_i until no more such agents exist. Then n further rounds take place. In each following round the realised payoff profile is π and there is an active agent from some population A^i who does not have informational state θ_i. This results in the state s in which for all $i \in P$ the agents in population A^i have demand d_i and informational state θ_i.

The preceding description is constructed to have positive probability. Thus s is accessible from any choice of s_0, so no more than one recurrent class can exist. From state s there is a positive probability that in the following round all sampled agents mutate so that their demands are unchanged and the realised payoff is π. The resulting state must be s. Hence the greatest common divisor of N_s is 1 as required. □

An example of where condition 2 fails is in an implementation of fictitious play[18].

[17] An alternative would be to require that each agent has been active at least n times and the learning rule selects a new informational state depending on the demands and payoffs of the active agents and the aggregate demand state in the most recent n rounds in which the updating agent was active. The proof for this case is similar.

[18] Fictitious play is a learning rule in which a mixed strategy for each population A_i is constructed from the relative frequencies with which each action has been played over all preceding rounds.

In this case, informational states must contain information on the entire past history. Another example is if an agent's informational state depends in part on their initial informational state. In this case, initial informational states can effectively classify the agents into different types which cannot be changed by learning.

Note that for the case where an informational state represents the most recent payoff, there usually also exist some transient states. For example, an aggregate state in which all agents have an informational state representing receiving the maximum payoff is not accessible from any other state, except in some trivial networks, as it would require all active agents in the previous round to receive their maximum payoff. This is why some results of section 6.1.2 are quoted for Markov processes with a unique recurrent class, rather than for irreducible processes.

Agents then update to best replies to the resulting mixed strategy profile. See chapter 2 of Fudenberg and Tirole [29] for example.

Chapter 7

Simulation: Details

The chapter explores the details of the steps of the evolutionary model defined in chapter 6. Section 7.1 discusses various learning rules which agents may employ in the model and proposes several candidates for use. Section 7.2 explains the reasons for the modelling choices made, and discusses various alternatives. Section 7.3 contains predictions of the behaviour of the model under the candidate learning rules. Predictions of general behaviour are developed based on the preceding discussion of the candidate learning rules and the material of chapter 6. There is also discussion of a theorem predicting the outcome for some positions in unit cake networks under certain learning rules. Section 7.4 develops methods of reporting the results of the simulation based on the predictions of general behaviour. Section 7.5 completes the description of the model by giving details of various steps, paying particular attention to how these steps are implemented in a computer simulation. Finally, section 7.6 is an appendix which states and proves the theorem mentioned above.

7.1 Learning Rules

Sections 7.1.1 – 7.1.3 describe various classes of learning rules adapted from both the sociology and economic theory literatures. The theoretical results from chapter

6 are used to predict their behaviour in the model and assess their likely usefulness. Three learning rules are identified as candidates for use in simulation: *imitate better*, *proportional imitation* and *sampled best reply*. Some variations on these are also defined. Various predictions of the behaviour of the model under the learning rules are made in this section. These are summarized and developed further in section 7.3.1.

Note that a major criterion for selection of learning rules for use in computer simulation is that they must be reasonably computationally efficient. The Markov process structure of the evolutionary model of chapter 6 means that a large number of rounds must be used to gain an insight into its behaviour. In this situation, every part of the main loop described in section 6.2 is used a large number of times. Slow subroutines can therefore have a large effect on the computational speed and reduce the maximum size of networks which can be studied in a reasonable runtime. In particular, this means that learning rules must typically be based on information gathered from a few agents rather than on information aggregated from all agents[1].

Recall that one aim of this thesis is to find a method which is capable of investigating reasonably large networks. This means that memory limits become an issue[2]. Recall from section 6.3 that the computer implementation stores the state of the model as a vector of sizes of subpopulations which are homogeneous in individual state. It is important that the informational state space remains relatively small to prevent the number of subpopulations from becoming too large for large networks.

7.1.1 Imitative Learning Rules

Imitative learning rules operate by agents sampling others in the same population and sometimes switching to their demands. A simple candidate imitative learning

[1]Some aggregated information is readily available, such as the number of agents in each individual state. However this alone is typically not enough for a learning rule.

[2]These are mainly the limits imposed on the size of arrays by the programming language rather than the available system memory.

rule is *imitate better*. Under this rule an updating agent samples another in the same population with a uniform probability distribution and switches if the sampled agent has a higher most recent payoff. A variation is for the agent to switch with a probability proportional to the payoff advantage of the sampled agent, or probability zero if the sampled agent has a worse payoff. This yields a *proportional imitation* learning rule[3]. A family of such rules exist depending on the *factor of proportionality* used. The factor of proportionality must be selected such that the probability of switching demand is no more than 1. These two simple imitative learning rules are used in many simulations of the next chapter. For both, the informational state of an agent represents their most recent payoff and so $I_i = D_i \cup \{0\}$.

The following variations of these rules are also sometimes used. These are called imitate better and proportional imitation *with sample size n*. Under these the updating agent samples n agents from the same population with a uniform probability distribution and without replacement[4]. For each demand d of a sampled agent, the average most recent payoff of agents in the sample playing d, $a(d)$, is calculated. A demand d^* maximising $a(d)$ is chosen using a uniform distribution on all such demands. The learning rule now proceeds as if the demand d^* and payoff $a(d^*)$ had been sampled by an agent using the ordinary imitate better or proportional imitation rule. This learning rule allows the updating agent to base their new demand on an imperfectly observed picture of the state of their whole population. As n increases the observation of the state becomes more accurate.

Some other possible variations of these rules include introducing a rule to decide whether to update at all (e.g. only if payoff falls below an aspiration level), using a different rule to decide whether to switch the the observed demand, or increasing the amount of information observed on sampled players (e.g. the most recent n payoffs could be observed).

[3]As proposed by Schlag [58].

[4]Sampling is without replacement to minimise the computational cost. For a large population this is a good approximation of sampling with replacement.

The remainder of this section discusses the behaviour of the model under the imitate better and proportional imitation rules, and to some extent under any imitative learning rule. Consider the unperturbed model under an imitative learning rule. In this model, new demands cannot be introduced into populations. Consider an aggregate demand state such that every population is demand homogeneous. The set of all aggregate states corresponding to this aggregate demand state must contain a recurrent class. Under the imitate better and proportional imitation learning rules agents never imitate others with equal payoffs. Thus there may also be recurrent classes in which some populations contain agents playing different demands but all receiving payoff zero. This result can be summarized as follows. Every *B set* contains a recurrent class.

Definition 7.1. An *B set* is a maximal set S of states such that for each $i \in P$ one of the following conditions holds.

i) All agents in A^i have the same demand in all states of S.

ii) If the game $\mathcal{N}(N, D)$ is played using any demand profile supported[5] by S then player i receives an expected payoff of zero.

An example of a B set in which condition ii) holds for a population is the following. Let N be a 3 player line network in which $\mathcal{K}^{12}$ is a 2-unit cake and $\mathcal{K}^{23}$ is a unit cake. There exists a B set in which all agents in population 1 make demand $\frac{1}{2}$, all agents in population 2 make demand $\frac{3}{2}$ and agents in population 3 make various demands and receive payoff zero.

It is shown[6] in lemma 7.4 of the appendix to this chapter that for imitate better and proportional imitation there are no recurrent classes of the unperturbed model which are not contained in B sets. A sketch of the argument is given here as it is

[5] Recall the definition of support from definition 6.2.

[6] In order to apply the lemma to get the result described it is also necessary to observe that, in the notation of appendix 7.6, any $B \in \mathbb{B}$ is a subset of a B set.

persuasive for many other imitative learning rules, although a complete proof may be not be straightforward. In the unperturbed model, for populations other than A^i there is a positive probability of the same agents becoming active each round and not changing their demands due to not sampling better alternatives (e.g. they might sample themselves). Thus there is a positive probability that every agent in population A^i becomes active in turn and makes a similar sample of the population. If they all sample an agent a in population A^i with highest payoff, then there is a positive probability that every agent with a lower payoff will switch to the demand of agent a. A feature of the underlying game is that any agent receiving the same payoff as agent a must already be playing the same demand, unless the payoff of agent a is zero. If this process is repeated, a state in a B set is eventually reached.

The following argument suggests that under imitate better or proportional imitation in the long term the model is more likely to select B sets supporting demand profiles which are plausible as rational solutions of the underlying game such as strict Nash equilibria. Consider a state in a recurrent class of the unperturbed model. This state must lie in a B set. For a population A^i such that condition i) of definition 7.1 holds, there is a positive probability that a state can be reached without mutation in which every agent in that population has the same most recent payoff. Then it requires only a single agent in population A^i to mutate to a different demand and receive a higher payoff for the state to leave this recurrent class of the unperturbed model. However this argument can also be applied to some B sets which support a unique demand profile which is a strict Nash equilibrium if there are multiple realisable payoffs to this profile. For example, in a network with an odd number of players one must always be excluded from exchange. Nonetheless this argument is useful in proving the prediction of theorem 7.3 in the appendix.

The multiple strict Nash equilibria of $\mathcal{N}$ for a 2 player network all have unique realisable payoffs. Define the *outward resistance* of a recurrent class of the unperturbed model to be the minimum number of mutations required for the state to reach another recurrent class of the unperturbed model. For the 2 player case, it is easy to see that recurrent classes of the unperturbed model within B sets supporting demand profiles which are strict Nash equilibria all have outward resistance of more than 1. The following argument shows that in fact all such recurrent classes have outward resistance of 2. Suppose an agent in each population mutates, their new demands form a different strict Nash equilibrium, and in the next round these agents become active again. One mutant agent must earn a higher payoff than the non-mutants in the same population. There is thus a positive probability that this mutant is imitated by all the non-mutants. It seems plausible from the above argument that under these learning rules selection between the multiple strict Nash equilibria of a 2 player network may not be possible by the resistance arguments described in section 6.1.2 alone. To make any progress theoretically it may be necessary to apply the Markov chain tree theorem directly.

Note that imitate better[7] relies only on the *ordinal* properties of demands. Given a path between two states, if all demands involved in the path are relabelled so that ordinal relations are preserved then the resulting path has the same probability. Thus is seems likely that the results for this network under the imitate better learning rule are not robust[8] to the choice of D. Since demand sets are an exogenously chosen part of the model this raises the possibility that there are many valid choices and so the solution is indeterminate. However, in a bargaining situation players can typically attach a great number of nuances and conditions to their offers, effectively allowing an interval of utility values to be attained. This suggests using evenly spaced demand sets[9] across the range of players' feasible demands and taking the

[7]These comments do not apply for imitate better with larger sample sizes. For these learning rules, the averaging of payoffs means that cardinal properties are also used.

[8]That is, the results depend only on the ordinal structure of the outcomes of the game $\mathcal{N}$. Note that this is not equivalent to the results being dependent on the ordinal structure of the utility cake. As mentioned in section 3.1.2 this would imply an indeterminate solution.

[9]Recall that von Neumann-Morgenstern utilities are used. These are unique up to positive affine transformations so 'evenly spaced' demand sets are well defined.

limit as the space between demands tends to zero. However, it may be possible to make a case for other choices of demand set to be more reasonable. A possible example in a two player bargaining situation are demand sets such that the Pareto optimal outcomes of $\mathcal{N}$ are evenly spaced[10]. Thus results that are dependent on the choice of D are hard to defend. Proportional imitation also involves cardinal properties of demands and so may be more robust.

Imitate better makes robust predictions in other networks. Theorem 7.3 in the appendix proves that under certain conditions, some positions in unit cake networks receive payoffs of approximately 1 and 0 in any stochastically stable state for both imitate better and proportional imitation. A short version of this result is given in section 7.3 below. A prediction of this form can be interpreted as especially powerful since it involves only mechanisms based on the ordinal properties of the payoffs. This is a reason to pursue simulations using imitate better.

7.1.2 Best Reply Learning Rules

This class of learning rules groups together all those which involve changing to a best reply. As described in section 6.1.3, the best reply dynamics used in Binmore et al [9] proved to be a powerful selection mechanism in the 2 player Nash demand game. One motivation for the use of best reply learning rules here is to see whether similar results hold. The best reply dynamics of [9] involved using a best reply to the aggregate mixed strategy given by the frequency of strategies in the other population. This is computationally costly as it involves calculating the expected payoff of many pure strategy demand profiles and taking an appropriate weighted

[10]An argument that this is a reasonable choice of demand sets is as follows. The only serious candidates for recurrent classes of the unperturbed model which are stable for $\epsilon > 0$ are those contained in B sets supporting strict Nash equilibria. Therefore the results of the model are likely to depend only on the ordinal structure of these outcomes. If they are unevenly spaced, this biases the outcome.

sum[11]. For use in the model of chapter 6, an alternative best reply rule is sought based on taking the best reply to a smaller sample of demands.

In the *sampled best reply* learning rule with sample size m, the updating agent samples m agents from each other population with replacement[12]. A mixed strategy is constructed for each other population from the relative demand frequencies in these samples. The updating agent then chooses a new demand which is a best reply to these mixed strategies. The minimum demand which is a best reply is selected. In this learning rule informational states are not required, so each set of informational states is taken to be a singleton.

Many alternative best reply learning rules exist. One source of variation is what information is used to construct the strategies to which a best reply is taken. Some choices are to use a mixed strategy profile for each population given by the relative frequencies of agents' demands in: the most recent m rounds played, the most recent m rounds in which the updating agent was active, the entire history (i.e. fictitious play), a random sample of m_1 of the most recent m_2 rounds, or the entire population. One example of another possible variation in a best reply learning rule is that agents could use a rule to decide whether to change demand at all.

A simple alternative best reply learning rule to sampled best reply is one in which active agents update to best replies given the demands played by the other active agents. The following example illustrates that the model under this rule can exhibit problematic behaviour. Consider a two player unit cake network. Choose demand sets such that $D_2 = \{1 - x | x \in D_1\}$. Select some $x_1 \in D_1$ and let $x_2 = 1 - x_1 \in D_2$. Let $\theta_1(t)$ and $\theta_2(t)$ be the number of agents in populations 1 and 2

[11]One possible method is to calculate and store the expected payoff of every pure strategy demand profile at the start of the simulation. Alternative code using this method still proved to be prohibitively slow for large networks and also ran into memory limitation problems.

[12]Sampling is with replacement to minimise computational costs. For this learning rule, the values of m used in practice are 1 and 2 so there should be little difference between sampling with and without replacement.

playing demands x_1 and x_2 at the start of round t. Let a_1 and a_2 be the active agents in a particular round. In the unperturbed model if exactly one agent a_1 or a_2 plays the corresponding demand x_1 or x_2 then in the next round one of θ_1 and θ_2 is reduced by 1 and the other is increased by 1. Otherwise both are unchanged. Thus $\theta_1(t) + \theta_2(t)$ is constant. Since this property of the aggregate state is conserved, the learning rule can only have very weak selective power. The problem is that demand updating occurs bilaterally; $a_1 \in A^1$ updates based on $a_2 \in A^2$ if and only if $a_2 \in A^2$ updates based on $a_1 \in A^1$. Sampled best reply avoids this relationship.

All best reply learning rules must deal with the case where more than one best reply exists. From the point of view of ease of implementation in a computer simulation two appealing methods are to give all best replies an equal chance or to make a decision based on a lexicographic ordering. As mentioned above, the latter is used in the sampled best reply rule: the minimum demand which is a best reply is selected..

B sets, as defined in definition 7.1, are simpler for a sampled best reply learning rule than for imitative learning rule since they no longer need cope with multiple informational states. A recurrent class of the unperturbed model contained in a B set must be contained in one in which condition i) of definition 7.1 holds for all populations. That is, it must be contained in a B set in which all populations are demand homogeneous. Such a B set contains only a single state. In a B set for which condition ii) holds for a population, agents in that population have multiple best replies and there is a positive probability that they all update to the minimal best reply. A B set which supports a strict Nash equilibrium and in which condition i) of definition 7.1 holds for all populations is a recurrent class of the unperturbed model. Clearly a B set which does not support a Nash equilibrium cannot contain a recurrent class of the unperturbed model.

It is not obvious that all recurrent classes of the unperturbed model are subsets of B sets. For example, this could not be the case if an underlying game were used with no pure Nash equilibria. However, it is possible to show that from any state there is a positive probability in the unperturbed process of reaching a state s in which every population is almost demand homogeneous using an argument similar to that described for imitative learning rules in section 7.1.1[13]. Let d be the demand profile which almost all agents play according to in state s. It can be shown that from s there is a positive probability that almost all agents in any single population now update to a best reply to d, while the agents in the other populations do not change their demands. For some networks this method shows that a demand homogeneous[14] state supporting a strict Nash equilibrium can be reached with positive probability[15]. For the general case this conclusion is not so straightforward. However, it shows that any recurrent class of the unperturbed model which is not contained in a B set would be very large, and is therefore intuitively unlikely to be very stable.

For the model under sampled best reply on a 2 player network, it is possible for the state to move between two B sets supporting different strict Nash equilibria with a single mutation. An informal description of how such a move may occur is as follows. An agent in population $\mathcal{A}^i$ mutates to new demand α. Every agent in the other population, $\mathcal{J}^i$ in turn becomes active, samples this mutant m times and switches to a best reply, β. Meanwhile, the mutant does not become active and thus does not change demand. Once population $\mathcal{A}^j$ has become demand homogeneous,

[13]The argument is as follows. Fix an agent from each population. Label this set of agents C. Suppose over the following rounds every other agent becomes active, and samples only the agents in C. Then all agents outside C update to the same demand.

[14]Suppose all but one agent in each population play according to a strict Nash equilibrium e. Let the set of such agents be C. There is a positive probability that from this state all agents in C become active and sample agents outside C. In the resulting state all populations are demand homogeneous, and only demand profile e is supported.

[15]For example, consider L_3 and suppose $D_1 = D_3 = \{1 - x | x \in D_2\}$. It has been shown that there is a positive probability of reaching a state in which almost all agents use a demand profile (x, y, z) where $y < 1$. There is then a positive probability of reaching a state in which all agents play according to the strict Nash equilibrium $(1 - y, y, 1 - y)$.

its agents continue sampling the mutant and so do not change demand again. In population $\mathcal{A}^i$ every non-mutant agent in turn switches to a best reply to β. In the case that α is the unique best reply to β this completes the argument. The remaining details of the more general case are of little interest here. It can be shown[16] that the preceding argument implies that every strict Nash equilibrium is supported by a stochastically stable state. Sampled best reply is therefore not guaranteed to provide a clear prediction for this network, and it is an interesting setting for the simulation.

Best reply rules are fundamentally more abstract than those of sections 7.1.1 and 7.1.3. The detail of how agents adapt to the outcomes of the underlying game are abstracted away into a procedure of 'taking the best reply' without a description of how this is performed. Indeed in most best reply rules it is not even necessary for agents to play $\mathcal{N}$ in the simulation: only the updating subroutine is required. This introduces a conceptual problem of how a best reply is arrived at. For agents to compute it directly requires a lot of information. Firstly, knowledge of other agents' demands is required[17]. This may not be easy to acquire from agents in distant network positions. Secondly, knowledge about the utility cakes and hence the preferences of other players is required. If agents learn the best reply by trial and error methods, then this prompts the question of what the details of these methods are and whether they can be implemented directly as learning rules. Nonetheless, best reply rules are pursued in simulation to find out whether they offer a means of equilibrium selection.

The main problem of using best reply learning rules is the computational cost. Finding a best reply requires finding the expected payoff of each possible demand.

[16]The techniques of section 7.6 can be used to prove this. Consider a minimal tree T on Z with root z which does not correspond to a strict Nash equilibrium. The argument in the main text shows that there is a path ϕ of resistance zero (in Ω) from z to a state z^* which corresponds to a strict Nash equilibrium. Delete the edges of all states in ϕ from T, and add the edges of ϕ. The resulting graph T' is clearly a tree rooted at z^*. Since the resistance of any outward edge from z^* is non-zero, this new tree has a lower resistance. This produces the required contradiction.

[17]In some circumstances a best reply may be independent of some players' demands.

Performing these calculations every round is a significant processing cost, especially if the best reply to a mixed strategy profile must be found. Costs are likely to rise quickly with the number of players, as this increases the time required to calculate the expected payoff under one demand profile. One option to reduce this cost is to cache[18] the results of these calculations. This seems especially useful if the state spends most of its time in or near B sets, as a small set of the best reply calculations will then be repeated very often. However, the processing costs during transits between B sets could still be large.

7.1.3 Learning Rules Independent of Other Agents

There are a lot of possibilities for learning rules where the updating agent's new individual state depends only on their payoff and current individual state. The principal example I have in mind for this class is the following heuristic which is widely used in the sociology literature[19]. When possible, agents who are excluded from exchange lower their demands, and agents who are included raise their demands. Unlike the learning rules considered above, this directly exploits the structure of the bargaining game[20]; lower demands are more likely to be included.

This heuristic could be adapted to a learning rule for the model described in chapter 6 as follows. An agent who receives a positive payoff updates their demand to the next highest in their demand set, or leaves it unchanged if no higher demand exists. An agent who receives payoff of zero updates their demand to the next lowest in their demand set, or leaves it unchanged if no lower demand exists. Under this learning rule all sets of informational states are singletons. Unlike the other learning rules considered so far, under this learning rule the unperturbed model does not

[18]That is, store the results of a fixed number of the most recently performed best reply calculations to reuse if they are required again.

[19]For example it is a 'scope condition' for the theory of Markovsky et al [45].

[20]It seems reasonable that learning rules specially adapted to social exchange exist, given how commonly such situations arise.

possess recurrent classes contained in B sets, except in some trivial networks. This is because in any demand profile supported by such a recurrent class, all agents who exchange in any outcome would have to receive their maximum payoff, which is not possible.

Instead a recurrent class of the unperturbed model would allow considerable variations of aggregate demand state. Such recurrent classes do not provide a mechanism to directly select between the multiple strict Nash equilibria of the underlying game $\mathcal{N}$ in the low ϵ limit as the previous learning rules do[21]. Without the prediction that the state is usually at a B set, there is no simple mechanism to hand for summarizing the state of the model. Thus interpreting the data from such a model is a considerable task and it may be often be hard to make a case that the model supports any particular demand profile as a solution. This learning rule is not used due to these difficulties.

Nonetheless, this learning rule may sometimes make reasonably clear predictions of behaviour. For example consider the network L_3. In a round where an exchange forms, the demand of one agent in population A^1 or A^3 is increased and the demand of another in the other population is decreased, unless these agents already make maximal or minimal demands respectively. In a round where no exchanges form, the demands of two agents in these populations are decreased unless already minimal. Thus there appears to be considerable downward pressure on the average demand in these two populations and it seems likely that the model provides support for a solution in which player 2 receives most of the available payoff.

A slightly modified version of this learning rule, as used by Bonacich in [15], does allow recurrent classes of the unperturbed model contained in B sets. The modification made to the learning rule is that agents who exchange leave their demands unaltered. However, now any state in which all agents exchange forms a

[21]Indeed, it is not obvious that the mutation mechanism plays a significant role in the model under this learning rule.

recurrent class of the unperturbed model. Also from most of these classes many single mutations can result in another recurrent class of the unperturbed model being reached. This appears to leave little scope for evolutionary pressure to select a solution with much precision. Also, since recurrent classes of the unperturbed model can exist which are not contained in B sets, the problem of interpreting the data applies to this case also.

The learning rules discussed in this section illustrate the problem with using this class of learning rules in the model of chapter 6. The choice of learning rule prescribes too closely which demands could be stable, independently of the structure of the network. The only options which avoid this are to allow either no stable demands or a very wide range of stable demands. Either choice does not appear to allow a precise solution and may result in difficulty in interpreting the data.

7.2 Modelling Choices

There are many reasonable alternatives to various features of the evolutionary model outlined in section 6.2. This section discusses the theoretical reasons for the modelling choices made and highlights approaches which seem to be valid alternatives or extensions. Sections 7.2.1 – 7.2.3 discuss variations in the underlying game $\mathcal{N}$ whereas sections 7.2.4 and 7.2.5 discuss variations to the evolutionary process. The final section, section 7.2.6, contains miscellaneous variations of both types.

7.2.1 Matching Rules

The matching rule is the part of the underlying game that determines which exchanges form given a demand profile d. In $\mathcal{N}$ the matching rule used is to choose from the set of all maximal consistent outcomes[22], $\Gamma(d)$, using a uniform probability distribution.

[22]Recall definition 6.1 of consistent and maximal consistent outcomes.

A simple possible alteration is to select from a different set. One alternative is the set $\Gamma'(d)$ of all Pareto optimal consistent outcomes. That is, those consistent outcomes given d whose share vectors are Pareto optimal. Observe that $\Gamma'(d) \subseteq \Gamma(d)$. To illustrate that $\Gamma'(d) \not\equiv \Gamma(d)$ consider the demand vector $d = (x, 1-x, x, 1-x)$ for network L_4. Then $\Gamma(d)$ contains an outcome with share vector $(0, 1-x, x, 0)$ but $\Gamma'(d)$ does not. An interpretation of these alternatives is that using Γ represents a local matching procedure, while using Γ' represents a procedure which allows participants to propose alternative global matching arrangements until none can suggest an improvement. Under this interpretation, Γ seems more relevant, especially for larger networks.

Another possible alteration is to adjust the probabilities by which an element of Γ is chosen. For example, this could be done by defining a specific local matching procedure. In the model of chapter 6, equal probabilities have been assigned to each element of $\Gamma(d)$ for simplicity.

Another alternative matching rule is to require players to specify a unique target for their demand. That is, players must make *directed demands.* In this case, a pair of neighbouring players exchange if they have selected each other as bargaining partners and their demands lie in the corresponding utility cake. This is perhaps the simplest endogenised matching process. Such a process can avoid the necessity of choosing between multiple realisable outcomes[23].

As an example of the limitations of directed demands, consider the 3 player ring network with unit cakes, R_3. The following argument shows, informally, that under the three candidate matching rules of section 7.1, the unperturbed model cannot support a recurrent class corresponding to a solution in which all 3 exchanges sometimes form. If such a recurrent class existed, it must contain a state in which all populations contain agents who direct demands to both other populations. Under

[23] Also note that many of the solutions described in sections 5.1.2 and 5.2.4 on extensions of the alternating offers game rely on players directing their offers correctly, so it seems unlikely that they can be captured using the game $\mathcal{N}$.

imitate better or proportional imitation, this arrangement is unstable for the following reason. In any realised outcome one player does not exchange and receives payoff zero. There is a positive probability this happens in turn to all but one agent of a population and they all imitate the remaining agent. Under the sampled best reply learning rule there is a positive probability that a state is reached in which all populations are almost demand homogeneous, as described in section 7.1.2. Under the demand profile which most agents play according to in this state, one player must receive payoff zero. Furthermore, any strategy of this player is a best reply. Under most rules for choosing between best replies, this allows new demand values to be introduced to this population. Thus the recurrent class must represent a broader range of possibilities than simply the solution mentioned.

In this example the model cannot capture stable bargaining behaviour in which a small number of multiple outcomes are possible[24]. However such behaviour does seem a likely possibility, especially as it is supported by the market bargaining game (see theorem 4.3). The problem is that the evolutionary model and learning rules under consideration cannot easily support stable behaviour involving more than one strategy in each population[25]. Such behaviour requires agents to be roughly indifferent between these strategies. However, since agents update their behaviour based on only a relatively small sample of randomly chosen other agents this allows small random variations in the numbers of agent playing each strategy. This can easily destabilize the behaviour. The advantage of an exogenous matching rule is that it allows multiple outcomes of the game based on strategy homogeneous populations. This is not to say that a matching rule with some endogenous feature

[24]For some networks it can capture such behaviour. Consider the network L_3 for example and the sampled best reply learning rule. Suppose $\min D_1 = \min D_3 = \delta$ and $\max D_2 = 1 - \delta$. There is a recurrent class of the unperturbed model in which all agents in populations A^1 and A^3 make the demand δ to player 2 and all agents in population A^2 make the demand $1 - \delta$, with some agents in this population directing their demand to either neighbour.

[25]In is often difficult for evolutionary models to support solutions to games involving mixed strategies. See proposition 5.14 of Weibull [70] for example.

may not be useful. Sections 9.1.3 and 9.3 of the conclusion take up this point.

7.2.2 Payoff Rules

The payoff rule is the part of the underlying game that decides what payoffs two players receive conditional on exchanging with each other. In $\mathcal{N}$ the payoff rule used is simply to award players their demands. The matching rule ensures that this produces a feasible outcome.

One variation is to use a *split surplus* rule. Suppose neighbouring players x and y make demands d_x and d_y and exchange with each other. A split surplus rule specifies that if (d_x, d_y) is not on the Pareto boundary of $\mathcal{K}^{xy}$ then they receive some payoffs $(\lambda_x, \lambda_y) \in \mathcal{K}^{xy}$ such that $\lambda_x \geq d_x$, $\lambda_y \geq d_y$ with strict inequality in at least one of these relations. If the demand pair is on the Pareto boundary then both players receive their demand. Under a split surplus rule, changing to a lower demand effectively offers potential exchange partners a higher payoff. There are many possible split surplus rules. One example is used in the 'cushioned demand game' of Binmore et al [9] in which $(\lambda_x, \lambda_y) = \theta(d_x, f^{x,y}(d_x)) + (1-\theta)(f^{y,x}(d_y), d_y)$ for some fixed parameter $\theta \in [0, 1]$. It would be an interesting extension to the simulations carried out in chapter 8 to investigate whether a split surplus rule produces qualitatively different results.

The 'contract game' of Binmore et al [9] employs another payoff rule. This two player bargaining game gives both players payoff zero for any demand pair which is not Pareto optimal. Pareto optimal demand pairs yield payoffs equal to the demands, as usual. The contract game is thus a coordination game in which players must propose the same 'contract' to receive any payoff. If this payoff rule were used in $\mathcal{N}$ then it would allow players to propose contracts between which they are indifferent to multiple neighbours. As mentioned in section 6.1.3, the contract game supports the Kalai-Smorodinsky bargaining solution in some models of [9], illustrating the qualitative effect that an alternative payoff rule can produce. This rule is unattractive for simulation purposes in the evolutionary model of this chapter under imitative learning rule. This is because it would be much more difficult for the state to move between the recurrent classes of the unperturbed model, since any unilaterally deviating player receives a payoff of zero. However, simulation with the sampled best reply learning rule may be possible.

7.2.3 Stochastic Payoffs

The fact that a demand profile can sometimes produce multiple realised payoffs is a source of stochasticity in the model described in section 6.2. For example, suppose the demand profile $(x, 1-x, x)$ is played in $\mathcal{N}$ on the network L_3. The payoff profiles $(x, 1-x, 0)$ and $(0, 1-x, x)$ can both be realised. Under imitative learning rules this can lead to some recurrent classes of the unperturbed model having low outgoing resistances. For example, in the network L_3 under the imitate better learning rule, one of players 1 and 3 receives payoff zero in any realised outcome of $\mathcal{N}$. In a B set it is possible for all agents in one of the corresponding populations to receive payoff zero if the same payoff profile is repeatedly realised. These agents will then imitate any demand which receives a positive payoff. Thus it is sometimes possible for a single mutation to be imitated by an entire population. This argument is a crucial part of the proof of theorem 7.3.

An alternative is to use expected payoffs in the model, eliminating this stochastic element. This feature is used in the KMR model described in section 6.1.1. Expected payoffs can be seen as representing a situation in which active agents play the game for an infinite number of times so that their average payoff equals the expectation or a situation where agents can calculate their expected payoffs. The latter case seems to involve an unreasonably heavy informational requirement on agents. A model which preserves some stochastic element in payoffs seems more natural. However, it would be interesting to see whether using expected payoffs has any qualitative effect on results as an extension to the simulations of chapter 8.

In the common case where agents' informational states represent their most recent payoffs, there is a practical reason for concentrating on the case of stochastic payoffs: using expected payoffs increases the size of the set of individual informational states. This increase can be problematic in larger networks where it represents a significant increase in memory requirements.

7.2.4 Multi-Agent Populations

A central assumption of the model of chapter 6 is that for each player position in the network there is a large population of agents, and samples of one agent from each population are repeatedly taken to play the underlying game $\mathcal{N}$. For small networks this might reasonably model a situation under which a few classes of individuals repeatedly interact with one another on similar terms (e.g. landlords and tenants, employers and several classes of employees). However, large networks instead mainly capture specific social or economic networks of individuals. In a large network it is hard to imagine a situation where agents repeatedly play in one position of a fixed network with the other positions filled by randomly chosen agents from the other populations which differ each time the game is played. It seems much more likely that bargaining situations faced by agents repeatedly will, at least in the short term, have fixed agents in the other positions. So it would be more natural to have a model with a single agent associated with each network position. However, note that one interpretation of the model of section 6.2 is that each population represents a mixed strategy for a single agent at the corresponding position. Learning rules then represent a process by which the agents make small changes to their mixed strategy each round.

Even if not interpreted as a literal description of a bargaining situation, multi-agent populations can still provide a useful tool for qualitatively exploring evolutionary pressures. This is because, as argued in section 7.1, under the 3 candidate learning rules of that section the model spends most of its time near to B sets and

B sets matching rationally plausible demand profiles are likely to be selected most often. This suggests that the results of a simulation on the model of this chapter will be relatively easy to interpret, as discussed in more depth in section 7.3. It also allows theoretical results such as that of section 7.6.

Models involving only single agents in each position offer a starting point for alternative methods[26]. Without pursuing these methods it is not obvious whether they allows the interpretation of simulation results as easily. Also such methods must avoid solutions which are over-dependent on repeated game interactions between agents[27] and thus do not capture the influence of network positions.

7.2.5 Single Agent Updating

In the outline of the model in section 6.2, in each round a single agent from each population becomes active. These agents play the underlying game and have a chance to update their demand. I refer to such a scheme as *single agent updating*. In contrast, in many models discussed in section 7.1, such as the KMR model, in each round all agents are matched to play the game and then have a chance to update. I refer to such a scheme as *all agent updating*.

The principal reason that the model uses single agent updating is computational efficiency. For example, consider the case where in each round samples of one agent from each population must be repeatedly drawn without replacement until no agents remain. Using selection without replacement repeatedly would significantly slow the simulation. The fastest option appears to be single agent updating.

An advantage of single agent updating is that it allows some opportunities for the state to move easily between recurrent classes of the unperturbed model which do not exist for all agent updating. This may allow interesting features of the bargaining

[26] See Tesfatsion [67] for a model of bargaining in networks which uses a single agent in each position of the network. This paper uses different assumptions about what outcomes are available from bargaining to those used here.

[27] See footnote 4 of chapter 6.

situation to be found from simulations with shorter runtimes. In a learning rule which involves agents sampling others in the same population, under single agent updating it is possible for a mutant to be sampled many times before it faces the possibility of updating to another strategy. Thus it is easier for mutants to gain a foothold in populations. Even in learning rules in which agents can only affect others by being selected to play the game and update, the fact the some agents may be more frequently selected than others may well allow rapid change. This feature could be interpreted as allowing short-term variations in the learning rate of agents, which seems realistic.

7.2.6 Miscellaneous

A possible variation to the model is for agents to use a learning rule to choose a demand before the underlying game is played. This has more of a flavour of social learning; agents make decisions when faced with a problem, rather than deciding upon a fixed strategy in advance. Under the candidate learning rules, the behaviour in the case where the state of the model is near a B set except for a few mutants seems likely to be little different under this variation. Thus it does not seems that this variation would produce any qualitative differences to the long-run behaviour.

Another alternative is to use Young's model of [76] described in section 6.1.3. This uses singe agent updating, but does not require large populations. Instead learning rules act on the most recent m plays of the underlying game for some fixed m. This requires a considerable assumption of public information. The case of restricted information embodied by the model in this chapter seems more general.

The underlying game could easily be extended to bargaining situations other than bilateral exchange networks. For example, the case of a 3 player bargaining situation in which players can split a payoff of one unit if they all agree is quite straightforward. However, the general case of multiplayer bargaining is complicated, as the number of possible outcomes given a particular demand profile can become very large. For this case, the approach of assigning equal probabilities to any maximal consistent outcome seems too ad-hoc to be natural and also becomes computationally cumbersome.

In the description given in section 6.2, noise is introduced to the model at an agent level by using a probability ϵ of mutating whenever they update their demands. This is a typical feature of models in the literature and has the advantage of being straightforward to implement in a computer program. Also it is uniform in the sense that mutation probabilities are state independent[28]. Many other possible specifications of noise exist. Non-uniform features could be introduced into the specification of whether an agent mutates, such as payoff dependence (e.g. agents with high payoffs are less likely to mutate), or correlated mutation probabilities. Also, the probability distribution by which the demands of mutants are chosen could be changed. For example small changes in demand could be made most likely. Other sources of noise in the model could also be introduced. For example, information could be observed noisily, utility cakes could vary slightly each round (e.g. as in the smoothed Nash demand game of Nash [53]), learning could be noisy in the sense of Binmore and Samuelson [7], or realised payoffs could have a small random variation.

An alternative to using a finite demand set is to allow any demand in an interval. This goes beyond the discrete framework of the theory in section 6.1.2. Also note that a computer simulation would not allow a truly infinite demand set due to the restrictions of floating point arithmetic.

[28]However, it may not be uniform in another sense. If agent $a \in A^i$ mutates then there is an equal chance that it mutates to each strategy in D_i. Recall D_i is meant to model an interval of demands. If the elements of D_i are not evenly spaced then the mutation probabilities will not be evenly distributed in this interval. However, in most simulations of chapter 8 the elements of D_i are reasonably evenly spaced so this effect is ignored. The exceptions are some simulations which are mainly used to point out that certain results are not robust to the choice of D_i.

7.3 Predictions

This section makes predictions of the behaviour of the model of chapter 6 under the candidate learning rules discussed above. One motivation for this is that it allows the development of useful methods of reporting simulation results, as described in section 7.4.

7.3.1 General Behaviour

This section begins by summarizing the material of sections 6.1.2 and 7.1 on the predicted general behaviour of the model for the three candidate learning rules. Informal arguments are then presented to make the case that for relatively large values of ϵ the general behaviour is similar to that for the limiting case of $\epsilon \to 0$. In particular, it is predicted that the aggregate state may still spend most of its time near B sets and occasionally be driven by mutations to move between them. Furthermore it is argued that the pattern of recurrent classes of the unperturbed model visited most often by the state has some degree of robustness to the choice of ϵ. This is crucial for simulation results to reveal much about the general behaviour of the model.

Under the conditions of lemma 6.1 the model has a unique aperiodic recurrent class. The first condition is simply $\epsilon > 0$ and the second is fulfilled by the 3 candidate learning rules outlined in section 7.1. Section 6.1.2 includes a result that this is a sufficient condition for the model to have a unique stationary probability distribution over its aggregate states which gives both the expected and time average state in the limit $t \to \infty$, independent of initial state. This stationary distribution can be found by a simulation from any starting state of sufficiently long runtime. Of course, depending on the parameter values, the runtime required may be impractically large.

Another result described in section 6.1.2 shows that in the limit $\epsilon \to 0$, all of the weight of this distribution is placed on states referred to as stochastically stable which must lie in recurrent classes of the unperturbed process. Furthermore, the number of mutations required for the state to move between these recurrent classes decides which classes contain stochastically stable states. For the candidate learning rules proposed in section 7.1, it is predicted[29] that these recurrent classes are contained in B sets as defined in definition 7.1. These are sets of states such that each population either is demand homogeneous across the whole set or contains only agents who receive payoff zero given any demand profile supported by the set.

The set of stochastically stable states can be used as a selection mechanism between the multiple Nash equilibria of the underlying game in the long-run. However, reducing ϵ towards zero is not a practical method to generate predictions from a simulation. This is because it also increases the expected number of rounds spent at each recurrent class of the unperturbed model. In a simulation of reasonable run-time, it is quite possible that one such class is reached and the system then remains there[30]. Furthermore, the limiting case of low ϵ represents a case in which agents experiment or are subject to mistakes or other exogenous factors at a much slower rate than they learn. This may not be the most realistic or interesting case.

However, if a relatively large value of ϵ is used then the formal details of the arguments above describing the behaviour of the model begin to break down. For example note that the expected number of mutants in a population of size M is ϵM. Thus for $\epsilon M > 1$, the state is expected to be outside the recurrent classes of the unperturbed model for the majority of rounds. The extreme case is that for sufficiently large ϵ the model becomes mainly driven by mutations and the weight of the stationary distribution is likely to be spread widely so that a precise long-run prediction cannot be made.

The 3 candidate learning rules outlined in section 7.1 provide a mechanism to

[29] This prediction is proved for imitate better and proportional imitation, but not for sampled best reply.

[30] An an alternative simulation aim in this setting would be to find the recurrent class of the unperturbed model that is most commonly reached first. However this does not necessarily correspond to stochastic stability.

stabilize many recurrent classes of the unperturbed process contained in B sets even for relatively large values of ϵM. First observe that in a B set, populations for which condition i) of definition 7.1 holds but condition ii) does not are demand homogeneous and receive non-zero payoffs. Let C be a B set with the following property for all such populations A^i. If a single agent $a \in A^i$ mutates and becomes active again in the following round, then there is a significant probability that the agent will update their demand back to its pre-mutation value. A B set which does not satisfy this property is unlikely to receive much weight[31] in the stationary distribution for any value of ϵ. Now consider a state where only a small number of mutants in a population whose agents receive non-zero payoffs in C deviate from their demand in C. Call an agent whose demand matches that specified in C a *conformist.* There is a significant probability that when non-conformists become active all other active agents are conformists and the non-conformist updates to the conformist demand for the corresponding population. Thus there is a high probability that the non-conformists die out faster than their demands are spread.

There are some cases in which this argument seems especially strong. The first is when imitative learning rules are used. In this case, from many B sets non-conformists must first secure a high payoff by playing the underlying game against other non-conformists before they can be imitated by conformists. A second case is when the learning rule requires samples of more than one agent to be taken and the new demand is based on the frequencies of demands in these samples. An example is the sampled best reply learning rule with sample size greater than 1. In this case, from many B sets conformists must typically sample more than one non-conformist to switch demands. Also note that in either of these cases, a large value of M aids stabilization. To summarize, near many B sets there is a significant probability that a low number of mutations die out even for relatively large values of ϵ.

[31] Except in the case where no B sets satisfy this property. This seems unlikely given that strict Nash equilibria satisfy this property and the underlying games in the model typically possess many.

The conclusion of the argument in the previous paragraph can be characterised as follows. A recurrent class of the unperturbed model lies in a basin of attraction which in turn lies within a basin of likely attraction. For some of these recurrent classes, the 'centres' of these basins of likely attraction are stable in the short-run even for relatively large values of ϵ and contain the state of the model most of the time. Movements of the aggregate state between these centres will be relatively rare, so the model acts roughly like a Markov chain with some B sets as states and probabilities of moving between them as transition probabilities. I refer to a movement of the state between two B sets as a *transit*. The stationary distribution of the system is now determined by the transit probabilities. In fact, as shown by equation (6.2), it is the transit probabilities conditional on any transit occurring that determine the stationary distribution.

This raises the issue of whether there is much similarity between these conditional transit probabilities for different values of ϵ. For example, if two transit probabilities were $\epsilon^2 + \epsilon^3$ and $10^{-6}\epsilon^2 + \epsilon^3$, there would be little similarity in general. For these transit probabilities to exist, it seems necessary for 3 mutations to allow an alternative mechanism of transit between some pair of B sets, rather than just making the 2 mutation mechanism more likely due to higher chance of meetings between mutants. Furthermore, due to the low probability of 3 mutants being selected to interact in some sequence of events, the new mechanism must be much more probable. This argument generalises to other numbers of mutations. If the conditional transit probabilities are similar for different values of ϵ then the behaviour of the model for relatively high values of ϵ gives a rough indication of the results for all values. However, the best that can reasonably be hoped for is qualitative similarity. As mentioned above, as ϵ becomes larger, the results will become less precise until the system is mainly driven by mutation and learning has almost no influence.

The qualitative behaviour of the model may also vary depending on M. As discussed in section 6.1.3 the order in which the limits $\epsilon \to 0$ and $M \to \infty$ are taken can have a qualitative effect on the behaviour of evolutionary models of this sort. Informally speaking, large values of M mean that the law of large numbers causes the effect of the stochastic components of the model to act closer the deterministic approximation given by their expected values. This alters the relative probability of transits between B sets.

One consequence of this behaviour for large values of M is that the expected time between transits is greater. As already noted, in a state close to a B set, a small number of non-conformists are likely to die out. To gain a sustainable foothold in the population, they must enjoy several lucky conversions from non-conformists. For large M the number of such conversions required is much larger and hence much less likely. Also note that since transits between B sets are driven by mutations, the expected time between transits is clearly decreasing in ϵ.

7.3.2 Specific Predictions

The experimental data of section 2.7 and the theoretical results of chapters 4 and 5 on models based on the alternating offers game all produce specific predictions for the outcome of bargaining in particular networks. These are summarized and compared to the results of the simulation in the conclusion, chapter 9.

Section 7.1 predicts that recurrent classes of the unperturbed model containing stochastically stable states correspond to 'rationally plausible' demand profiles for any of the candidate learning rules, in a sense described there. In particular, for the sampled best reply learning rule, recurrent classes of the unperturbed model contained in B sets must correspond to Nash equilibrium profiles. It is not clear whether much selection between strict Nash equilibria takes place in a 2 player network under this learning rule. Investigating this setting is a key first task for the simulation. One prediction that is made for this setting is that the results of imitate better are predicted to depend only on the ordinal structure of the possible outcomes of $\mathcal{N}$.

Note that the main asymmetries in the treatment of populations in the model under the 3 candidate learning rules arise from the choice of network. Many of the networks used in chapter 8 have symmetrical positions. Thus if the simulations for these networks support unique solutions then these are predicted to also be symmetric. Also, note that the model under these learning rules does not provide a natural mechanism for asymmetries corresponding to the 'bargaining powers' of the asymmetric Nash bargaining solution of section 4.1. Note that asymmetries can also arise from the choice of demand sets. Since this is an exogenous modelling choice that does not appear to correspond to a feature of the bargaining situation, I do not consider this to be a 'natural mechanism' for causing asymmetries.

It is possible to make some strong theoretical predictions for the behaviour of the model. Theorem 7.3 is such a prediction. The full statement is lengthy and is in appendix 7.6, the bulk of which contains the proof. The remainder of this section contains a brief overview.

The result is for unit cake networks. Certain conditions are given under which subsets of players can be labelled as W and S, corresponding to weak and strong network positions. Theorem 7.3 applies to the model under the imitate better or proportional imitation learning rules and certain restrictions on D. These restrictions include $\min D_i \setminus \{0\} = \delta$ if i is a W player and $\max D_i = 1 - \delta$ if i is a S player for some value of $\delta > 0$. The theorem states that in any stochastically stable state all agents in populations corresponding to S players make demand $1 - \delta$ and those in populations corresponding to W players make demand δ.

Definition 7.3 gives the conditions for the W and S labelling. It is necessary that every neighbour of a W player is a S player, at least one W player does not exchange in any feasible outcome of N, and that in any feasible outcome of N if an S player does not exchange then at least one of her neighbours must also not exchange and be a W player. The full definition requires a stronger version of the second condition. An example of this definition is for networks L_3 and L_5. In both, odd numbered players can be labelled as W and even numbered players as S. However for longer odd length lines, no players can be labelled as either W or S.

A brief sketch of the proof is as follows. Consider a state not of the form claimed to be stochastically stable. In any round of the model an agent in a W population (i.e. one corresponding to a W player) does not exchange in the realised outcome of the underlying game. This allows a positive probability that over several rounds all agents in a W population receive payoff zero by exclusion from exchange and switch to the demand of a single mutant playing demand δ and receive a positive payoff. In a network as described there is always a positive probability that a demand of δ by a W player is accepted. It can be shown that this process can take place in each W population in turn. If all agents in W populations make the demand δ then S populations can similarly be colonized by single mutants making the demand $1 - \delta$. However, from a state of the form claimed to be stochastically stable at least two mutants are required to colonize a W or S population; a single mutant cannot earn a higher payoff than non-mutants. The proof is completed from these observations by the use of the minimal tree techniques of section 6.1.2.

In the simulations of the following chapter on unit cake networks, every demand set typically includes the demand 1, so theorem 7.3 does not apply. As discussed in section 7.6.4, it is difficult to extend the theorem to this case. However, this section also discusses reasons why the result is intuitively very likely to hold in this setting[32].

7.4 Methods of Result Reporting

This section uses the predictions of the previous section to describe methods of reporting the data produced by a computer simulation implementing this model. As described in section 7.3.1, the model under investigation has a unique stationary

[32]Simulations which are not included in chapter 8 show that excluding the demand 1 from demand sets makes no apparent qualitative difference to results.

probability distribution over states which gives both the expected and time average state of the system in the limit $t \to \infty$, independent of initial state. This suggests result reporting by recording the time average distribution. This is impractical as there are too many states for the data to be interpreted easily. Furthermore this approach does not make it obvious to a user how often transits between B sets take place. This information is very useful because it allows the user to decide quickly whether the current simulation permits enough transits in a reasonable run-time to provide results characteristic of the stationary distribution.

Instead the simulation uses a method of reporting based on B sets. It keeps track of roughly which B set the state of the model is closest to. The word roughly is used because it seems very laborious and unnecessary to keep track of the details of a population for which condition ii) of definition 7.1 holds (i.e. a population in which all agents receive payoff zero). Thus the actual information tracked is equivalent to any demand profile supported by the B set: a player who receives payoff zero under this profile represents a population for which condition ii) of the definition holds.

Recall that it is predicted that the aggregate state is usually near B sets and rarely moves between them. This means that any rough method of finding a demand profile supported by the closest B set should succeed most of the time. The computer implementation uses the method of finding the modal demand in each population. The program reports to the user when the modal demand profile changes. It also records the total number of rounds that each demand profile has been modal and converts this into a proportion of the total rounds played. This gives a rough indication of how strongly B sets are selected by the stationary distribution[33].

The average number of rounds between changes of modal demand, ρ, is also

[33] The proportions for modal demand profiles which are the same except for the demands of players who receive payoff zero could be aggregated to reconstruct B sets. For the simulations of chapter 8 this did not turn out to be necessary. There were almost no cases in which two or more demand profiles representing the same B set were both modal for a significant fraction of all rounds. The exceptions are in section 8.4 and are easy to interpret without making this aggregation.

recorded. This gives a very rough measure of how quickly the simulation explores the possible B sets. Low values of ρ may suggest that B sets are unstable and mutations are too common to reveal much about the typical structure of the stationary distribution. High values of ρ may suggest that the B sets are too stable and the stationary distribution will only be revealed by the simulation over a prohibitively long run-time. It is difficult to record ρ precisely due to the problem of over-reporting. A single transit between B sets may involve the modal demand profile changing many times; usually repeated changes between the origin and destination sets. Also, in large networks there are more opportunities for transits between B sets to take place[34], so straightforward comparison between ρ values for different networks may not be revealing.

A possible pitfall of this method is that it may be possible to partition the B sets into several components such that transits between components are very unlikely relative to transits within components. The method of reporting proposed may only find the stationary distribution restricted to one of these components and ignore the others. This feature seems intuitively unlikely for the underlying game used here. Furthermore, if any plausible demand profile is absent from the list of most commonly modal demand profiles in a simulation run then it can be checked whether it has ever been modal. If not then another simulation can be run in which this demand profile is initially modal to test whether it is contained in a separate component of B sets to those investigated by the previous simulation. For the small networks used in chapter 8 this method did not find any separate components of the form described. However, for large networks this method may not suffice as it is hard to enumerate all possible plausible demand profiles[35].

[34]For example, in a particular B set for each pair of players who exchange there is an opportunity of a transit to a B set in which the terms of their exchange are slightly different.

[35]Another possible test is run many simulations starting from randomly generated states uniformly distributed across the state space to determine whether the results are robust to this choice. The initial state used in the simulations of chapter 8 all agents make the minimal demand and, in

The simulation code also records χ, a measure of how close the state of the process typically is to a B set. It is roughly defined as the average over all agents of the proportion of rounds in which an agent had the demand corresponding to the closest B set. However, agents are not counted when their population satisfies condition ii) of definition 7.1 in the closest B set of the current round. A full definition of χ is given by equation (7.1) below. The value of χ in a simulation with $\epsilon = 1$ gives a baseline value for the case where agents update their demands at random, which can be a useful comparison.

A possible extension is to record a similar measure to χ every L rounds, and plot a histogram of the distribution of these χ_L values. This would hopefully reveal a large region where χ_L is high and the state is close to a B set, and a small region where χ_L takes variable values and a transit between B sets is taking place. This would reveal how close the state is to B set when it is not in transit and also how much time the process spends in transit.

The run-time and number of rounds that have taken place are also recorded. From these the simulation code calculates γ, the average number of rounds performed per unit of run-time. This is a measure of the speed of the simulation[36]. It is a rough measure as the run-time can be affected by external factors such as other processes running on the computer network. Indeed, in practice the measure of runtime generated by FORTRAN seemed to be incorrect by an order of magnitude. Statistics involving run-time thus only seem useful as relative measures.

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the case $I_i = D_i \cup \{0\}$, have the informational state of 0 corresponding to a most recent payoff of zero. This provides some degree of random behaviour early in the model, but it is conceivable that if multiple components of B sets of the form described above exist then this initial state could be biased towards one particular component.

[36]The internal setup of the code (e.g. details of caching) is kept constant over the simulations of chapter 8 and so does not affect the value of γ.

7.5 Details of the Model

This section completes the description of the model begun in section 6.2 by describing the details of the steps in the main algorithm given there. Particular attention is paid to the implementation of the model as a computer simulation. Thus the steps are described as algorithms.

Section 7.5.1 describes the subroutine which plays the underlying game $\mathcal{N}$. An especially important part of this game is finding the different possible sets of realised exchanges. Section 7.5.2 gives an algorithm for this step. Sections 7.5.3 and 7.5.4 describe the updating subroutines for the candidate learning rules described in section 7.1. Section 7.5.5 describes how the result reporting described in section 7.4 is performed. Finally section 7.5.6 discusses the parameters which can be adjusted by the user in the computer implementation.

The computer implementation is written in FORTRAN 95 and compiled using the NAGware f95 compiler. This language was chosen due to its inbuilt operations for handling arrays. The code itself is available on request from the author[37].

7.5.1 The Underlying Game

The algorithm of this section requires the following definition.

Definition 7.2. A subgraph (V, K') of (V, K) is a *consistent* subgraph of (V, K) if no vertex of (V, K') has degree greater than 1. A *maximal* consistent subgraph (V, K') of (V, K) is one such that the following property holds. If $ab \in K$ then there exists some $x \in V$ such that either $ax \in K'$ or $bx \in K'$.

The following algorithm implements the game $\mathcal{N}$:

Algorithm ($\mathcal{N}$ subroutine).

Input: A network $(P, E, \mathcal{K})$ and a demand vector $d = (d_i)_{i \in P}$

[37] Note that the version at the time of writing is not very user friendly!

Output: A payoff vector $\pi = (\pi_i)_{i \in P}$

1. Determine the set $J(d) = \{ab \in E \mid (d_a, d_b) \in \mathcal{K}^{ab}\}$.

2. Call a subroutine to determine the set S of all maximal consistent subgraphs of (P, J).

3. Select (P, X) from S using a uniform probability distribution.

4. For all $ab \in X$ let $\pi_a = d_a$ and $\pi_b = d_b$.

5. Assign all remaining players payoff zero.

Recall definition 6.1 of a maximal consistent outcome. It is straightforward from this and definition 7.2 that an outcome is a maximal consistent outcome of $N = (P, E, \mathcal{K})$ given d if and only if its exchanges are a maximal consistent subgraph of $(P, J(d))$, exchanging players receive shares equal to their demands and non-exchanging players receive shares of zero. Thus the above algorithm does implement the game $\mathcal{N}$ as claimed.

For each exchange $ab \in E$, the computer implementation of the model stores the following representation of $\mathcal{K}^{ab}$. One player in each exchange is selected. Suppose a is that player in the exchange ab. For each $x_a \in D_a$ a value representing $f^{a,b}(x_a)$ is stored. In step 1, determining whether $(d_a, d_b) \in \mathcal{K}^{ab}$ is true is done by testing whether the condition $d_b \geq f^{a,b}(d_a)$ holds[38].

Note that this algorithm can easily be modified to give the expected payoff of each player. This is required later to calculate best replies. All that must be changed is to remove step 3, perform steps 4 and 5 for each $(P, X) \in S$, and take the average of the resulting payoff vectors.

[38]The step proved to be very problematic in practice. Trying to set the recorded value of $f^{a,b}(x_a)$ equal to a particular demand value in D_b often resulted in a slightly different value being recorded due to floating point errors. This led to demand pairs being incorrectly found infeasible. This problem was resolved by treating differences below a certain threshold as equality.

One simulation in chapter 8 involves granting players outside options. To enable this, some players are given an extra action in their demand set, representing accepting their outside option, and the following modifications are made to this algorithm. In step 1, the definition of J is extended so that $ab \notin J$ if the action of player a or b is to accept the outside option. In step 5, a player whose action is to accept their outside option receives a corresponding fixed payoff.

7.5.2 The Calculation of all Maximal Consistent Subgraphs

The following algorithm is highly recursive. This is manifested in the fact that it repeatedly calls part of itself as a subroutine. The instruction "end subroutine" should be read as referring to the most recently called subroutine. In a computer implementation of this algorithm it is important that care is paid to not confusing variables from different levels of recursion. The global variables have only one value at any point in the subroutine. The subroutine level variable v takes a different value for every level of the subroutine; the value defined in the current level of the subroutine must always be used. Fortunately, much of this can be taken care of by the FORTRAN attribute RECURSIVE.

Algorithm (Subroutine to find maximal consistent subgraphs).

Input: A graph (V,E).

Output: The set S of all $E' \subseteq E$ such that (V, E') is a maximal consistent subgraph of (V, E).

Global variables: A set of edges C, and a vector $z = (z_i)_{i \in V}$ such that $z_i \in \{0, 1\}$.

Subroutine level variable: A vertex v.

Notation: Let $Q(C, z) = \{a \in V \mid z_a = 0 \text{ and } ax \notin C \text{ for all } x \in P\}$.

1. Let $C = \emptyset$, $S = \emptyset$, and $z_i = 0$ for all $i \in V$. Go to step 2 as a subroutine and terminate the algorithm on return.

2. If $Q(C, z)$ is empty, add C to S and end subroutine.

3. A vertex $v \in Q(C, z)$ is picked by an unspecified deterministic method.

4. Loop over all w in $Q(C, z)$ neighbouring v.

5. Add vw to C.

6. Go to 2 as a subroutine.

7. Remove vw from C.

8. End of loop.

9. If $z_x = 0$ for all x neighbouring v in (V, E) then:

10. Set $z_v = 1$.

11. Go to 2 as a subroutine.

12. Set $z_v = 0$.

13. End of if statement.

14. End subroutine.

It is not necessary to use a deterministic method to choose v in step 3. This requirement is given simply because a deterministic method is used in the computer implementation (the value of v with lowest index is selected) and because it makes an argument later in this section slightly simpler.

Lemma 7.1. *The output of the above algorithm is as described.*

This is easy to see from the following sketch of how the algorithm operates. Suppose C^0 is an edge set such that no player occurs in more than one edge. Suppose

z^0 is a value of vector z such that if $xy \in C^0$ then $z_x^0 = z_y^0 = 0$. Then calling step 2 as a subroutine finds all maximal consistent subgraphs containing edges C^0 such that any player for which $z_x^0 = 1$ does not exchange. This is done as follows. The set $Q(C^0, z^0)$ contains all players which are neither involved in an exchange in C^0 nor specified as not exchanging by z^0. If $Q(C^0, z^0) = \emptyset$ then C^0 is already the edge set of a maximal consistent subgraph. Otherwise a vertex $v \in Q(C^0, z^0)$ is selected. In turn, the algorithm tries adding vw to C^0 for each $w \in Q(C^0, z^0)$ neighbouring v, and changing the value of z_v^0 to represent v not exchanging, unless v already has a neighbour w who is not exchanging. For each of these cases, the algorithm finds all maximal consistent subgraphs by calling step 2 as a subroutine.

Proof of lemma 7.1. Suppose S is the output of the algorithm and $E' \in S$. If an edge ab can be added to C in step 5 then it must be the case that $a, b \in Q(C, z)$. So no vertex may occur in more than one edge of E'. A vertex a does not occur in any edge of E' if and only if $z_a = 1$ when C is added to S. Therefore given two neighbouring elements of (V, E), step 9 ensures that at least one must occur in an edge of E'. Hence (V, E') is a maximal consistent subgraph of (V, E).

Now suppose (V, E') is any maximal consistent subgraph of (V, E). Define a *history* of the algorithm as a description of what instructions have been performed. Let H^0 be the initial history of the algorithm. A sequence of histories is now defined inductively. Assume H^n has been defined as a history in which the algorithm is at the start of step 2. Let C^n and z^n be the corresponding values of C and z. Let v_n be the first value of v chosen following H^n; if no such vertex exists the sequence terminates. Define w_n such that $v_n w_n \in E'$ or, if no such element exists, let $w_n = 0$. In the case $w_n \neq 0$, it is shown below (a) that there exists a history in which $C = C^n \cup \{v_n w_n\}$, $z_i = z_i^n$ for all $i \in V$ and the algorithm is at the start of step 2. In the case $w_i = 0$, it is shown below (b) that there exists a history in which $C = C^n$, $z_{v_n} = 1$, $z_i = z_i^n$ for all $i \neq v_n$ and the algorithm is at the start of step 2. In either case, let H^{n+1} be the described history. This proves that there exists a history in which $C = E'$, $Q = \emptyset$ and the algorithm is at the start of step 2. In this history C is added to the output S as required.

It remains to prove claims (a) and (b). By construction, $C^n = \{v_k w_k \mid k < n\}$ and if $z_x = 1$ then $x = v_k$ for some $k < n$. Suppose $w_n \neq 0$. Since (V, E') is a maximal consistent subgraph of (V, E), $w_n \in Q(C^n, z^n)$. Inspecting steps 4-8 shows that this proves (a). Suppose $w_n = 0$. Then v_n is not a member of any edge in E'. Consider any $v_n x \in E$. Since (V, E') is a maximal consistent subgraph of (V, E), x must be a member of some edge in E'. Hence $z^n_x = 0$. Inspecting steps 9-13 shows that this proves (b). □

Note also that no set of edges E' is added to S twice. Let $(H^0, H^1, \ldots, H^m)$ and $(\hat{H}^0, \hat{H}^1, \ldots, \hat{H}^{\hat{m}})$ be two sequences of histories as defined in the proof which result in E' be added to F. Define corresponding values of v_i, w_i, $\hat{v}_i$ and $\hat{w}_i$ as in the proof. Suppose $H^n = \hat{H}^n$. Then[39] $v_n = \hat{v}_n$. It must be the case that $w_n = \hat{w}_n$, otherwise the edge sets added to S are different. By definition $H^0 = \hat{H}^0$, so it follows by induction that $H^m = \hat{H}^{\hat{m}}$.

This fact is useful in implementing the above algorithm as code, because it is not necessary to check for repeated entries to S. Any repeated entries would have to be found before a uniform random selection from the elements of S were made, as is required in the algorithm for $\mathcal{N}$.

7.5.3 The Updating Subroutine for Sampled Best Reply

Recall that this learning rule does not use informational states. Hence the informational state space is a singleton and it is only necessary to compute a new demand.

Algorithm (Sampled best reply updating subroutine).

Parameters: Number of agents to sample, m, demand sets, D, and mutation rate ϵ.

[39] This is where the deterministic choice of v in step 3 is required.

Input: The aggregate state of the model and the index i of the population A^i containing the updating agent.

Output: A new demand.

1. Sample a random number $\mu \in [0, 1]$ with uniform distribution. If $\mu \leq \epsilon$ a mutation occurs. In this case choose a new demand from D_i using a uniform probability distribution and terminate the algorithm.

2. Sample the demands of m agents from each population other than A^i without replacement using a uniform probability distribution.

3. Count the values n^j_x: the frequency of demand x in the sample from population A^j.

4. Loop through all values of $d^{-i} = (d_j)_{j \in P \setminus i}$ such that $n^j_{d_j} > 0$ for all $j \in P \setminus i$.

5. For each $d_i \in D_i$, calculate $\pi_i(d_i, d^{-i})$; the expected payoff to player i from demand d_i given the demands of d^{-i} for all other players.

6. Calculate $p(d_i, d^{-i}) = \pi_i(d_i, d^{-i}) \prod_{j \in P \setminus i} n^j_{d_j}$.

7. End loop

8. For each $d_i \in D_i$, calculate $q(d_i) = \sum_{d^{-i}} p(d_i, d^{-i})$.

9. Let $T = \text{argmax}_{d_i \in D_i} q(d_i)$. Let the new demand be $\min T$.

Expected payoffs are calculated in step 5 by a modification of the algorithm in section 7.5.1. The details of the modifications required are mentioned in that section. Finally note that in the computer implementation of this algorithm, the results of steps 3-9 are cached to improve performance.

7.5.4 The Updating Subroutines for Imitate Better and Proportional Imitation

Recall that under these learning rules an agent's informational state is their most recent payoff. Thus a new informational state and a new demand must be specified.

Algorithm (Imitate better/proportional imitation updating subroutine).

Parameters: Demand sets, D, and mutation rate ϵ.

Input: The aggregate state of the system, the realised payoff of the updating agent, p, and the index, i, of the population A^i containing it .

Output: A new individual state for the agent.

Parameters: Number of agents to sample, m, and (for proportional imitation only) a factor of proportionality, λ.

1. Let the new informational state be p.
2. Sample a random number $\mu \in [0, 1]$ with uniform distribution. If $\mu \leq \epsilon$ a mutation occurs. In this case choose a new demand from D_i using a uniform probability distribution and terminate the algorithm.
3. Sample the demands and most recent payoffs of m agents from A^i without replacement using a uniform probability distribution.
4. Calculate the average most recent payoff $a(d)$ earned for each demand d in the sample. Let S be the set of demands achieving the maximum value of $a(d)$. Let q be this value and let $d = \min S$.
5. IMITATE BETTER: If $q > p$ then let d be the new demand. Otherwise leave the demand unchanged.

PROPORTIONAL IMITIATION: If $q > p$ then let d be the new demand with probability[40] $\lambda(q-p)$. Otherwise leave the demand unchanged.

7.5.5 Reporting

This section first describes what is reported to the user and then discusses the parts of the implementation which involve some minor complications. Various types of reporting are used. The most straightforward but also most cumbersome method is to display the entire current aggregate state as stored by the computer program. That is, to display for each population the number of agents in each individual state in the current round. Typically, this display is only useful when the simulation is displaying unexpected behaviour. Otherwise it is set to appear very rarely as a check that the state of the simulation is as surmised from other more concise reporting methods.

At regular intervals the program display the values of χ, ρ and γ as defined in section 7.4, as well as the number of rounds played and total run-time so far. This allows the user to follow the general behaviour of the model. The calculation of χ is discussed shortly.

The main methods of reporting used involve tracking the modal demand profile as described in section 7.4. The user is notified when this changes and told the identity of the new modal demand profile. Also the code roughly[41] calculates the number of rounds that each demand profile is modal. These values are periodically

[40]The program specifies that in the case where this value is more than 1 probability 1 is used. However, in chapter 8 λ and D are chosen so that this case never occurs.

[41]For large networks the total number of demand profiles is too large to do this precisely. Instead, this data is stored for a large fixed number of demand profiles. When the modal demand profile is not in this list, it is added in place of the profile with the lowest total number of rounds. Under the assumption that most of the time is spent near a relatively small number of B sets this method should not be problematic. It is possible to check the accuracy of this method by calculating the difference between the sum of rounds in all these stored records and the total number of rounds played. In all the simulations of chapter 8 this difference was negligible.

reported to the user, as well as the corresponding proportions of the total number of rounds played.

As mentioned in section 7.4, this method is over-sensitive. The modal demand may change several times during a transit between B sets and reporting each of these is not useful. Instead the program reports to the user whenever a modal demand changes and there are no more changes in modal demand for a fixed number of rounds, given by a user chosen program parameter.

The value of χ, roughly defined in section 7.4, is calculated as follows. Let $\theta_i(t)$ be the proportion of agents in population A^i playing the modal demand in round t. Define $\phi_i(t)$ to equal 0 in the case that player i receives an expected payoff of zero from the modal demand profile and 1 otherwise. The former case is meant to capture rounds such that in the closest B set condition ii) of definition 7.1 holds for population A^i. This may occasionally fail during transits between B sets, but since these are predicted to be rare, it should have little effect on the calculated value of χ. This is given by:

$$\chi(t) = \frac{\sum_{i\in P}\sum_{s\leq t}\phi_i(s)\theta_i(s)}{\sum_{i\in P}\sum_{s\leq t}\phi_i(s)} \tag{7.1}$$

The code stores the current values of the sums of the numerator and denominator. The calculation of χ becomes problematic when the denominator becomes so large that adding extra terms does not change its floating point representation. From then on, floating point errors inflate the value of $\chi(t)$. This occurred in a few simulations of the following chapter, especially in networks with a large numbers of players. The values of χ given for these networks are taken for values of t before this problem occurs.

7.5.6 Parameters

Various parameters set by the user define a particular run of the simulation program. Two major choices are which updating subroutine is used in step 3 of the main algorithm and the choice of game rules i.e. the payoff and matching rules[42]. An initialisation file is used to contain the remaining parameters. These include the probability of mutation, ϵ, the total number of rounds to be performed and the initial state of the model. This file also includes the parameters controlling reporting to the user, parameters describing the availability and values of outside options, and the parameters used in the various learning rules, such as the number of demands to sample. Details of the bilateral exchange network under investigation are defined here as well. The definition of the players and edges is straightforward. As mentioned in section 7.5.1, utility cakes are described as follows. For each exchange $ab \in E$ one player is specified. Suppose it is player a. The utility cake for the exchange ab is defined by giving $f^{a,b}(d_a)$ for all $d_a \in D_a$.

Networks with simple (e.g. unit) cakes are straightforward to define directly. For more complicated networks there is an initialisation routine in which the user need only specify a set of interpolation points on the outer boundary[43] of each cake and the demand sets D. The routine defines the cake for an exchange ab by taking the demand set of one player, say player a, and finding the image of each demand in D_a under $f^{a,b}$ assuming that the outer boundary of the cake is piecewise linear and passes through all the interpolation points. Using piecewise linear interpolation is not particularly restrictive as the outer boundary of any cake in this computer implementation is represented by a finite number of discrete points. However, it would be straightforward to alter the type of interpolation used.

7.6 Appendix: Theorem 7.3 and Proof

7.6.1 Statement of Theorem 7.3

Fix a network with unit cakes $N = (P, E, K_{\text{unit}})$.

[42]Only trial runs were done varying the game rules and these are not included in chapter 8.

[43]See definition 3.6.

Definition 7.3. A *SW labelling* on N is a function $l : P \to \{S, W, 0\}$ such that:

i) Let $P_W = \{p|l(p) = W\}$. For any $P' \subset P_W$, let H be the set of consistent subgraphs[44] (P, F) of (P, E) satisfying the following property. Every $p \in P'$ is either included in an edge of F or all of their neighbours are. Then for every $(P, F) \in H$ there exists some $p \in P_W \setminus P'$ who is not included in any edge of F.

ii) Any $p \in P$ such that $l(p) = W$ has at least one neighbour, and any neighbour q of p satisfies $l(q) = S$.

iii) Consider any consistent subgraph (P, F) of (P, E). Let C be the set of players who are not involved in any edge in F. If there exists $p \in C$ such that $l(p) = S$ then there exists $q \in C$ such that $l(q) = W$ and $pq \in E$.

If in at least one SW labelling l on N, $l(p) = S$ then p is called a S player. A W player is similarly defined. These labels correspond to 'strong' and 'weak' positions. The sense in which this is meant is made clear by theorem 7.3 below. Note that a population A^i of the evolutionary model is sometimes referred to as a S or W population when i is respectively a S or W player.

Lemma 7.2. *A player cannot be both a W and S player of N.*

Proof. Suppose such a player p existed. Let l and l' be SW labellings on N such that $l(p) = W$ and $l'(p) = S$. Let $F_0 = \emptyset$. Construct a sequence of edge sets by the following inductive step. Let $F_{n+1} = F_n \cup \{qx\}$ where $qx \in E \setminus F_n$, $l(q) = W$ and $q \neq p$. If no such qx exists then terminate the sequence and let H_0 be the final edge set produced. Construct another sequence of edge sets by the following inductive step. Let $H_{n+1} = H_n \cup \{qx\}$ where $qx \in E \setminus H_n$ and $l'(q) = S$. If no such qx exists then terminate the sequence and let H be the final edge set produced.

The graph (P, H) is a consistent subgraph of (P, E). By construction, every player in $\{q \in P \mid l(q) = W\} \setminus \{p\}$ is either included in an edge of H or all their

[44]See definition 7.2.

neighbours are. Thus by condition i), p is not included in any edge of H. Thus by condition iii) there exists some $px \in E \setminus H$ which is a contradiction. □

Fix finite demand sets D and $0 < \delta < 1$ such that $\min D_w \setminus \{0\} = \delta$ for any W player w and $\max D_s = 1 - \delta$ for any S player s. Let the learning rule be either the imitate better or a proportional imitation rule. Choose any population size $M > 1$. The choice of N, D, ϵ, M and a learning rule fully defines the model of section 6.2 and a corresponding Markov process S_ϵ as described in section 6.1.2.

Theorem 7.3. *In all stochastically stable states of the perturbed Markov process just described, agents in population p make demand δ if p is a W player and $1 - \delta$ if p is a S player.*

Sections 7.6.2 and 7.6.3 prove this result. Recall that a brief sketch is given in section 7.3.2. The proof is based on the minimal tree techniques mentioned in section 6.1.2. See Binmore et al [9] for further discussion of these methods. Section 7.6.4 discusses extensions to the result.

7.6.2 Notation and Supporting Lemmas

The proof of theorem 7.3 is complicated by the fact that unlike in [9], the recurrent classes of the unperturbed process are not singleton sets. Instead the sets described by the following definitions are required.

Definition 7.4. Let B^* be the set of quadruples (Q, Q', d, E) which satisfy the following conditions

i) Q and Q' are a disjoint partition of P.

ii) $f = (f_i)_{i \in Q}$ where $f_i \in D_i$

iii) $G = (G_i)_{i \in Q'}$ where $G_i \subset D_i$, and $|G_i| > 1$

iv) Given any demand profile $d = (d_i)_{i \in P}$ where $d_i = f_i$ for $i \in Q$ and $d_i \in G_i$ for $i \in Q'$, then for $i \in Q'$, the only realisable payoffs to player i in $\mathcal{N}(N, D)$ from demand profile d is zero.

Definition 7.5. For $(Q, Q', f, G) \in B^*$, let $B(Q, Q', f, G)$ be the set of states of S_ϵ where:

1) For $i \in Q$, all agents in population A^i make demand f_i.

2) For $i \in Q'$, all agents in population A^i make a demand in G_i.

3) For $i \in Q'$ and $g \in G_i$, some agent in population A^i makes the demand g.

4) For any $i \in P$, the informational state of any agent in population A^i is one of the realisable payoffs to player i of some demand profile d as described in condition iv) of definition 7.4.

Let $\mathbb{B}$ be the collection of all sets as defined in definition 7.5. Note that the sets of $\mathbb{B}$ are disjoint. Let $\mathcal{B}$ be the union of all elements of $\mathbb{B}$. The sets in $\mathbb{B}$ are similar to the B sets of definition 7.1, but are finer as that definition did not have restrictions on informational state.

Note that S_0 is the process corresponding to the unperturbed model. Let Z be the set of states of S_0. Note that Z is also the set of states of S_ϵ for any ϵ. Let Γ be the set of pairs $(z, z') \in Z^2$ such that there is a positive probability of a transition from z to z' in any S_ϵ with $\epsilon > 0$. Then $\Omega = (Z, \Gamma)$ is a directed graph.

Lemma 7.4. *Any $B \in \mathbb{B}$ contains a recurrent class of S_0 and every recurrent class of S_0 is a subset of some $B \in \mathbb{B}$.*

Proof. Fix some $(Q, Q', f, G) \in B^*$. Consider a state $z \in B = B(Q, Q', f, G)$. Suppose there is a positive probability of a transition to $z' \in Z$ in S_0. To prove the first part of the lemma it is sufficient to show that $z' \in B$. New demands cannot enter the populations in the unperturbed dynamics. Hence conditions 1) and 2) of definition 7.5 continue to hold in z'. Consider $i \in Q'$. In a round of play from state z, the active agent in A^i, a_i, samples an agent with informational state corresponding to payoff zero and hence does not imitate her. Thus condition 3) continues to hold in z'. Condition 4) holds by the definition of how the informational states are updated (i.e. step 1 of the algorithm of section 7.5.4).

To prove the second part of the lemma, it is sufficient to additionally show that for any state $z \in Z$, there is an accessible[45] state $b \in \mathcal{B}$. Thus it is sufficient to consider the case $z \notin \mathcal{B}$. Suppose only condition 4) is violated in z. Then there is a positive probability that every agent in turn becomes active, samples themselves and does not change demand. The resulting state must lie in $\mathcal{B}$ as required. So suppose a condition other that 4) is violated in z. Then there must exist some $i \in P$ such that the state z supports[46] more than one strategy for A^i and at least one agent in A^i has non-zero informational state.

Let J be set of all realisable payoffs to player i in $\mathcal{N}(N, D)$ from any strategy profile supported by state z. Let $\bar{j} = \max J$. Note that $\bar{j} > 0$ and that in $\mathcal{N}(N, D)$ a payoff of $\bar{j}$ can only be achieved by making a demand of $\bar{j}$.

By definition of J, there is a positive probability in S_0 that in a round of play from state z the active agent in population A^i, a, receives payoff $\bar{j}$, samples themselves and so does not switch demand from $\bar{j}$. Let such a resulting state be y.

The following has positive probability from state y in S_0. Each agent $b \in A^i \setminus \{a\}$ in turn becomes active and samples agent a. If the realised payoff of b is less than $\bar{j}$ then the active agent switches to the demand $\bar{j}$ of agent a. If the realised payoff of b is $\bar{j}$ then the active agent samples themselves and so does not switch demand from $\bar{j}$. This process results in a state $p(z)$ in which every agent in A^i plays demand $\bar{j}$.

As mentioned above, once all agents in a population play a single demand, no other demand enters this population in the unperturbed dynamics. The state $p(z)$ has at least one more such population than z. Hence there must be some state $p^m(z)$

[45]This term is defined in section 6.1.2.

[46]This term is defined in definition 6.2.

such that the transformation p cannot be performed. This state satisfies conditions 1 - 3 of definition 7.5. There is a positive probability that from $p^m(z)$ every agent becomes active, samples themselves and does not change demand. The resulting state b must also satisfy condition 4 and thus $b \in \mathcal{B}$ as required. □

This result still holds under altered learning rules in which updating agents may not sample themselves but the proof is lengthier. It is necessary to require all agents to become active immediately after state x so that their informational state lies in J. Following this, the agent a cannot sample an agent with a payoff greater than $\bar{j}$ and so does not switch demand.

Note that for $Q' \neq \emptyset$, the set $B(Q, Q', f, G)$ contains many recurrent classes of S_0. This is because for $i \in Q'$, agents in population A^i receive payoff zero and sample other agents with payoff zero and hence never switch demand. Thus any two states in $B(Q, Q', f, G)$ such that population A^i is different are not accessible from each other.

Recall from section 6.1.2 that given $z, z' \in Z$, the one-step resistance of the transition from z to z', $r_{zz'}$, is the minimum number of mutations required for the state to change from z to z' in one round in S_ϵ with $\epsilon > 0$. If this transition has probability 0 then the one-step resistance is defined as ∞.

Given a path[47] $\alpha = (\alpha_i)_{0 \leq i \leq n(\alpha)}$ in Ω, the resistance of α is defined as

$$r(\alpha) = \sum_{i=0}^{n(\alpha)-1} r_{\alpha_i \alpha_{i+1}}$$

Recall that the resistance, $r(T)$, of a directed tree T on Z is the sum of the one-step resistances associated with its edges. Also recall that a minimal tree is a directed tree on Z with minimum resistance. A version of the Markov chain tree theorem mentioned in section 6.1.2 can now be stated for the model described in this section.

Lemma 7.5. *A state $z \in Z$ is stochastically stable in the process S_ϵ if and only if there is a minimal tree rooted at z.*

[47]This term is defined in section 3.2.

Proof. See theorem 4 of Young [77]. □

Lemma 7.6. *For any $z \in Z$ there exists a path in Ω of resistance 0 from z to some z' in a recurrent class of S_0.*

Proof. Suppose there exists $z \in Z$ for which no such path in Ω exists. Then no element of a recurrent class of S_0 can be accessible from z. Let Z' be the states in Z which are accessible from z. Clearly for any $x \in Z'$ then no $y \notin Z'$ can be accessible from x. Thus it is possible to construct a discrete Markov process on Z' such that the probability of transition between $x, x' \in Z'$ is the same as in S_0. There exists a recurrent class A of Z' by the argument of footnote 6 of chapter 6. Clearly in S_0 any $x \in A$ must communicate[48] with every $x' \in A$ and no $y \notin A$ is accessible from x. Thus A is a recurrent class of S_0 and any $a \in A$ is accessible from z, which is a contradiction. □

Lemma 7.7. *If z is a stochastically stable state of S_ϵ then z is in a recurrent class of S_0.*

Proof. Suppose there exists a stochastically stable z for which this claim fails. By theorem 7.5, there exists a minimal tree T rooted at z. By lemmas 7.4 and 7.6 there must exist a path $\alpha = (\alpha_i)_{0 \leq i \leq n(\alpha)}$ in Ω such that $z_0 = z$, $z_{n(\alpha)} \in \mathcal{B}$ and $r(\alpha) = 0$.

Let $T_0 = T$. A sequence of directed graphs is now defined iteratively. Let T_i be the directed graph generated from T_{i-1} by deleting the outgoing edge of z_{i-1} and adding a new edge $z_{i-1}z_i$. Note that $r(T_i) \leq r(T)$ since $r_{z_{i-1}z_i} = 0$. Let λ be minimal such that $z_\lambda \in \mathcal{B}$. Note this is well defined since $z_n \in \mathcal{B}$. Let $T' = T_\lambda$.

Let q be the unique longest path in T' from z. By construction it must be of the form

$$q = (z_0, z_1, z_2, \ldots, z_\lambda = t_0, t_1, t_2, \ldots, t_{n-1}, t_n = z_k, \ldots)$$

where $(t_0, t_1, t_2, \ldots, t_n)$ are the first terms of the unique longest path in T from z_λ

[48] This relation is defined in section 6.1.2.

and t_n is the first element in this path of the form z_i where $i < \lambda$. Such an element exists since T is a tree rooted at z_0,

By construction, $z_\lambda \in \mathcal{B}$ but $z_k \notin \mathcal{B}$. By the definition of a recurrent class, there must be some transition $t_\mu t_{\mu+1}$ with a positive resistance for $\mu < n$. Delete the edge $t_\mu t_{\mu+1}$ from T' and denote the resulting graph T''. Then $r(T'') < r(T)$.

Consider any $w \in Z$. The unique longest path in T'' from w follows the path in T from w until a state z_i is reached for $i \leq \lambda$. Such a state is reached as T is a tree rooted at z_0. The path then continues by following the path q constructed above until it terminates at t_μ. Hence T'' is a tree rooted at some state other than z. This contradicts the assumption that T is a minimal tree. □

Let U be the subset of B^* such that Q contains all S and W players, and

$$f_i = \begin{cases} 1 - \delta & \text{for S player } i \\ \delta & \text{for W player } i \end{cases}$$

Let $V = B^* \setminus U$. Let $\mathbb{U}$ and $\mathbb{V}$ be the sets of all recurrent classes of S_0 in $\bigcup B(Q, Q', f, G)$ where the union is taken across all (Q, Q', f, G) in U and V respectively. Let $\mathbb{W} = \mathbb{U} \cup \mathbb{V}$ be the set of all recurrent classes of S_0. Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be the respective unions of the states in these classes.

Note that for any $W \in \mathbb{W}$ there exists a unique $B \in \mathbb{B}$ such that $W \subseteq B$. This follows by lemma 7.4 and the fact that the elements of $\mathbb{B}$ are disjoint. Thus the following is well-defined:

Definition 7.6. For $W \in \mathbb{W}$, let $B = B(Q, Q', f, G)$ be the unique $B \in \mathbb{B}$ such that $W \subseteq B$. Define $\theta(W) = \sum_{i \in P} \theta_i(W)$ where:

$$\theta_i(W) = \begin{cases} 1 & \text{for W or S player } i \in Q' \\ f_i & \text{for W player } i \in Q \\ 1 - f_i & \text{for S player } i \in Q \\ 0 & \text{otherwise} \end{cases}$$

7.6.3 Main Lemmas

The following lemma describes how certain 'skeleton trees' on Z in which not all states have outward edges can represent directed trees on Z.

Lemma 7.8. *Suppose the directed graph $H = (Z, F)$ satisfies the following properties:*

i) There exists some $x^* \in \mathcal{W}$ such that every state with an outward edge in H has a unique path in H terminating at x^*.

ii) Each $W \in \mathbb{W}$ contains either x^* or at least one state w with an outward edge in H.

Then there exists a tree T on Z rooted at x^* such that $r(T) = r(G)$.

Note that condition i) implies that x^* has no outward edge in H. Hence, all that is required to prove this is to show that it is possible to add outward edges to all other states without creating any cycles. Note that the condition $x^* \in \mathcal{W}$ is not necessary for this result, but simplifies the proof slightly.

Proof. Let W^* be the recurrent class of S_0 containing x^*. Let $\mathbb{W}' = \mathbb{W} \setminus \{W^*\}$. For any $W \in \mathbb{W}'$, let $\lambda(W)$ be an element of W with an outward edge in H. Let $\lambda(W^*) = x^*$.

Let $H_0 = H$ and $F_0 = F$. A sequence of directed graphs on Z all of which support H as a subgraph can be constructed from H_0 by the following iterative step. Choose some $x \in \mathcal{W} \setminus \{x^*\}$ which does not have an outward edge in H_i. Let $W \in \mathbb{W}$ be such that $x \in W$. By the definition of a recurrent class of S_0, there exists a path $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ in Ω of resistance zero from x to $\lambda(W)$. Let j be minimal such that ϕ_j has an outward edge in H_i. This is well defined since $\lambda(W)$ has an outward edge in H and thus also in H_i. Let $F_{i+1} = F_i \cup \{\phi_i\phi_{i+1} \mid i < j\}$, and $H_{i+1} = (Z, F_{i+1})$. Observe that this inductive step preserves the two properties described in the lemma and the property that H is a subgraph. Also observe that $r(H_{i+1}) = r(H_i)$. Let J be the final directed graph produced by this process.

A further sequence of directed graphs on Z all of which support J as a subgraph can be constructed from $J_0 = J$ by the following inductive step. Choose some $x \in Z \setminus \{x^*\}$ which does not have an outward edge in J_i. By lemma 7.6 there exists a path $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ in Ω of resistance zero from x to some $x' \in \mathcal{W}$. Let j be minimal such that ϕ_j has an outward edge in J_i. This is well defined since x' has an outward edge in J and thus J_i. Add the edges $\{\phi_i\phi_{i+1} \mid i < j\}$ to the edge set of J_i to generate J_{i+1}. The final directed graph produced by this process satisfies the properties described for T. □

Lemma 7.9. *Given $W \in \mathbb{V}$, there exists a path $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ in Ω of resistance 1 such that $\phi_0 \in W$ and $\phi_{n(\phi)} \in W'$ for some $W' \in \mathbb{W}$ satisfying $\theta(W') < \theta(W)$.*

Any path $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ in Ω such that $\phi_0 \in \mathcal{U}$, $\phi_{n(\phi)} \in \mathcal{V}$ and $r(\phi) < M$ requires mutations in at least one W and one S population.

Proof. Part I:

Suppose $z \in W$ where $W \in \mathbb{V}$. Let $B = B(Q, Q', f, G)$ be the unique $B \in \mathbb{B}$ such that $W \subseteq B$. It will be shown that from state z the model can generate a sequence of states requiring only a single mutation resulting in a state in $W' \in \mathbb{W}$ such that $\theta(W') < \theta(W)$. A description of such a sequence is referred to in this proof as a *performance* of the model.

The following facts will be useful. If a W player demands δ in $\mathcal{N}(N, D)$ and has any neighbours, then there is a realisable outcome in which that player receives a positive payoff[49]. In a round of S_0, for each population there is a positive probability that the active agent in that population does not change their demand, as they may sample themselves. In each of the performances described below, it is assumed that the event just described takes place in every round for every population other than A^i.

[49] This would no longer be the case if the demand 1 were added to the demand set of S players.

Since $z \in \mathcal{V}$, at least one of the following possibilities must be true in the state z:

1) There exists a W or S player $i \in Q'$.

2) There exists a W player $i \in Q$ such that $f_i > \delta$.

3) There exists a S player $i \in Q$ such that $f_i < 1 - \delta$.

In the case that 1) holds then the required performance is as follows. An agent $a \in A^i$ mutates to demand δ. In the following round a becomes active again and receives a non-zero payoff. In the following rounds, every other agent in A^i becomes active, receives payoff zero, samples agent a and switches to demand δ.

In the case that 3) holds but 1) and 2) do not then the required performance is as follows. An agent $a \in A^i$ mutates to demand $1 - \delta$. In the following round a becomes active again, receives payoff $1 - \delta$, and does not switch demand. Note that all agents in W populations have demand δ in this round so it is possible for agent a to receive payoff $1 - \delta$. In the following rounds, every other agent in A^i becomes active, samples agent a and switches to demand $1 - \delta$.

The remaining case is that 2) holds but 1) does not. Let C be the set of W players i such that $f_i = \delta$. Given a demand profile such that players in C demand δ, the demand of each player in C is feasible to all their neighbours. Hence in any maximal consistent outcome of $\mathcal{N}(N, D)$ from this demand profile, each player in C either exchanges or all their neighbours exchange. Thus by condition i) of a SW labelling, some W player not in C must be excluded from exchange in any such outcome. Let i be such a player who is excluded in one such outcome. Let e be a demand profile supported by z. The required performance is as follows. An agent $a \in A^i$ mutates to demand δ. In the following round a becomes active again, receives a non-zero payoff, and does not switch demand. In the following rounds, every other agent in A^i becomes active and the demand profile of active agents is e. Each rounds the active agent in population A^i receives payoff zero, samples agent a, and switches to demand δ.

Part II:

Suppose there exists a path $\phi = (\phi_i)_{0\leq i\leq n(\phi)}$ in Ω such that $\phi_0 \in \mathcal{U}$ and $\phi_{n(\phi)} \in \mathcal{V}$, $r(\phi) < M$ which is a counterexample to the second claim.

Suppose the path ϕ requires no mutations in population A^j and in ϕ_0 all agents in A^j have the demand f_j. Then in all states ϕ_i, all agents in A^j have the demand f_j since no new demands are added by mutation and active agents who sample an agent with the same demand cannot switch demand.

Consider the case that in ϕ no mutations are required in S populations. Then in all states ϕ_i all agents in S populations demand $1-\delta$. Thus by condition ii) of an SW labelling, in any state ϕ_i any agent in a W population who makes a demand greater than δ receives a payoff of zero. Thus no agent in a W population switches to any demand other than δ except by mutation. Since less than M mutations are required in ϕ, some agent in each W population must have demand δ in state $\phi_{n(\phi)}$. Thus it cannot be the case that $\phi_{n(\phi)} \in \mathcal{V}$ as assumed.

Consider the case that in ϕ no mutations occur in W populations. Then in all states ϕ_i all agents in W populations demand δ. Thus by condition iii) of an SW labelling, in any state ϕ_i any agent in a S population who makes the demand $1-\delta$ receives a payoff of $1-\delta$. Thus no such agent switches demand except by mutation. Since less than M mutations are required in ϕ, some agent in each S population must have demand $1-\delta$ in state $\phi_{n(\phi)}$. Thus it cannot be the case that $\phi_{n(\phi)} \in \mathcal{V}$ as assumed. □

The following lemma can now be proved. Theorem 7.3 follows by application of lemma 7.5.

Lemma 7.10. *Any minimal tree on Z is rooted in an element of $\mathcal{U}$.*

A sketch of the proof is as follows. The proof is by contradiction. By lemma 7.7, the only other possible case is that there exists a minimal tree T on Z rooted in an element of $\mathcal{V}$. A 'skeleton tree' A is constructed from T by removing edges which do not lie on any path in T between recurrent classes of S_0.

The following iterative step is then performed. The root of the current skeleton tree lies in $\mathcal{V}$. By lemma 7.9 a path ρ in Ω of resistance 1 exists from this root to an element of a recurrent class of S_0, W, with a lower θ value than that containing the root. This path is added to the current skeleton tree, erasing any old outward edges of elements of this path. Some element of $f \in W$ is found with an edge leading out of W. This edge is deleted and a path in Ω from f to the end of ρ is added, again erasing old outward edges. This path is chosen to have resistance zero. Such a path exists as W is a recurrent class of S_0.

This iterative step produces a new skeleton tree on Z and with resistance of at most $r(T)$ and whose root lies in a recurrent class of S_0 with a lower θ value than the corresponding θ value in the previous skeleton tree. Eventually a skeleton tree B is reached such that there exists a path ϕ in Ω of resistance 1 from its root b^* to an element $u \in \mathcal{U}$. This concludes part I of the proof.

There must be a path ρ in B from $u \in \mathcal{U}$ to $b^* \in \mathcal{V}$. A segment of this path from an element of $\mathcal{U}$ to an element of $\mathcal{V}$ containing no other elements of $\mathcal{W}$ is considered. By lemma 7.9 this has resistance of at least 2. The vertex corresponding to state in this segment from which the second mutation is required is labelled α. The part of the path ϕ up until it reaches an element of the segment of ρ mentioned prior to α is added to B, erasing old outward edges, to produce a directed graph C.

This graph satisfies $r(C) \leq r(T) + 1$ and has a unique cycle including α and b^*. The cycle is broken by adding to C a path in Ω of resistance zero from α to an element w^* of a recurrent class W^* of S_0, erasing old outward edges. Such a path exists by lemma 7.6. Let the resulting directed graph be C'. Note that C has been constructed so that there is a path from every recurrent class to α. Hence in C there exists an element of W^* with an outward edge leading outside W^*. A path in Ω of resistance zero from this element to w^* is added to C', erasing old outward edges. It is shown that the resistance of the resulting graph is no more than $r(T)-1$ and that it satisfies the conditions of lemma 7.8. This allows the construction of a minimal tree on Z with resistance less than $r(T)$, producing the desired contradiction.

The following operation is used repeatedly in the proof.

Definition 7.7. Let $X = (Z, F)$ be a directed graph. Let $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ be a path in Ω. Then $X \oplus \phi = (Z, F'')$ where F'' is constructed as follows. Delete from F all outward edges of elements of ϕ and name the remaining set F'. Let

$$F'' = F' \cup \{(\phi_i, \phi_{i+1}) \mid 0 \leq i < n(\sigma)\}$$

Proof of lemma 7.10. Suppose the claim is false. Then by lemma 7.7 there exists a minimal tree T rooted at some $a^* \in \mathcal{V}$.

Part I of the proof constructs a directed graph $B = (Z, H)$ satisfying the properties of lemma 7.8. The value of x^* in that lemma is taken by a state labelled b^* satisfying $b^* \in \mathcal{V}$. The graph B satisfies $r(B) \leq r(T)$. Also, there exists a path in Ω of resistance 1 from b^* to some $u \in \mathcal{U}$. Part II proves the lemma from these facts. Note that many of the symbols used in the notation of part I are reused with different meanings in part II.

Part I

First, some notation is defined. For $z \in Z$ let $p^z = (p_i^z)_{0 \leq i \leq n(p^z)}$ be the unique path in T from state z to a^*. By lemma 7.9 for any $w \in W$ such that $W \in \mathbb{V}$, there exists a path in Ω of resistance 1 from w to an element of W' where $W' \in \mathbb{W}$ and $\theta(W') < \theta(W)$. Let $\phi^w = (\phi_i^w)_{0 \leq i \leq n(\phi^w)}$ denote such a path.

Let A_0 be the directed graph $(Z, \emptyset)$. Construct a sequence of directed subgraphs of T by the following inductive step. Choose some $W \in \mathbb{W}$ such that no $w \in W$ has an outward edge in A_k. If no such W exists then terminate the sequence. Choose any $w \in W$. Let λ be minimal such that either p_λ^w has an outward edge in A_k or $\lambda = n(p^w)$. Define A_{k+1} from A_k by adding the edges of $(p_i^w)_{0 \leq i \leq \lambda}$.

Let A be the final graph in this sequence. Since A is a subgraph of T, $r(A) \leq$

$r(T)$.

Let $B_0 = A$ and $b_0 = a^*$. Suppose that directed graph B_k on Z and $b_k \in Z$ satisfy the properties i) $b_k \in \mathcal{V}$, ii) b_k has no outward edge in B_k, iii) from every $z \in Z$ with an outward edge in B_k there is a unique path in B_k to b_k, and iv) every $W \in \mathbb{W}$ either contains b_k or a vertex with an outward edge in B_k. Construct B_{k+1} as follows. For less cluttered notation, let $b = b_k$, and $c = \varphi^b_{n(\phi^b)}$. Let X be the set in $\mathbb{W}$ containing c. Select some $d \in X$ with an outward edge in B_k to a vertex $d' \notin X$. Such an edge must exist by the properties iii) and iv) above and the fact $b \notin X$. Note that the transition dd' must have one-step resistance of at least one since X is a recurrent class of S_0. By definition of a recurrent class there exists a path in Ω, $\psi = (\psi_i)_{0 \leq i \leq n(\psi)}$ of resistance zero from d to c. Let $B_{k+1} = (B_k \oplus \phi^b) \oplus \psi$. Note that in this operation, adding the edges of φ^b and ψ increases the resistance by 1, and removing the outward edge of d reduces the resistance by at least 1. Hence $r(B_{k+1}) \leq r(B_k)$.

It is now shown that in B_{k+1} every $z \in Z$ with an outward edge has a unique path in B_{k+1} to c. In the case that z is an element of ϕ^b, the required path in B_{k+1} is made up of the elements of ϕ^b from z onwards. In the case that z is an element of ψ, the required path in B_{k+1} is made up of the elements of ψ from z until an element z' of ϕ^b is reached, followed by the elements of the path ϕ^b from z' to c in B_{k+1}. The remaining case is that z has an outward edge in B_k. In this case there is a path ζ in B_k from z to b. The required path in B_{k+1} is made up of the elements of ζ until an element z' of ψ or ϕ^b is reached, followed by the elements of the path in B_{k+1} from z' to c.

Note that any vertex with an outward edge in B_k other than c has an outward edge in B_{k+1}. This shows that B_{k+1} and $b_{k+1} = c$ satisfy properties ii), iii) and iv) above. It also shows that if B_k satisfies property ii) of lemma 7.8 then so does B_{k+1} with $x^* = c$.

This construction can be used to generate a sequence of directed graphs B_k on Z

and vertices b_k. Let W_k be the element of $\mathbb{W}$ containing b_k. By construction $\theta(W_k)$ is strictly decreasing in k. Thus for some k^*, $b_{k^*} \in \mathbb{U}$ and the sequence terminates. The graph B_{k^*-1} and the vertices b_{k^*-1} and b_{k^*} have the properties described above for the graph B and vertices b^* and u.

Part II

Let $\rho = (\rho_i)_{0 \leq i \leq n(\rho)}$ be the unique path in B from u to b^*. Let λ be minimal such that $\rho_\lambda \in \mathcal{V}$. This is well defined since $b^* \in \mathcal{V}$. Let λ' be maximal such that $\lambda' < \lambda$ and $\rho_{\lambda'} \in \mathcal{U}$. By lemma 7.9, the path $(\rho_i)_{\lambda' \leq i \leq \lambda}$ has resistance of at least two. Let $a_1 \in [\lambda', \lambda)$ be minimal such that the transition $\rho_{a_1}p^{u}_{a_1+1}$ has one-step resistance of at least 1. If the one-step resistance of this transition is at least 2 then let $a = a_1$. Otherwise, let a be minimal in (a_1, λ) such that the transition $\rho_a \rho_{a+1}$ has one-step resistance of at least 1. Let $\alpha = \rho_a$.

Recall that there exists a path in Ω of resistance 1 from b^* to u. Let $\phi = (\phi_i)_{0 \leq i \leq n(\phi)}$ be such a path. Let b be minimal such that $\phi_b = \rho_i$ for some $i \leq a$. This is well defined since $b^* = \rho_0$.

Define C as equal to $B \oplus (\phi_i)_{0 \leq i \leq b}$ except that ϕ_b has an outward edge to the same vertex as in B. Note that $r(C) \leq r(B) + 1 \leq r(T) + 1$. Also any state $z \in Z \setminus \{\alpha\}$ with an outward edge in C has a unique path in C to α, as follows. Fix c such that $\phi_b = \rho_c$. If $z = \rho_i$ for some $a > i \geq c$ then the required path is given by the elements of ρ from ρ_i to α. If $z = \phi_i$ for some $i < b$ then the required path is given by the elements of ϕ from ϕ_i to ϕ_b, followed by the elements of the path in C from $\phi_b = \rho_c$ to α. The remaining case is that z has an outward edge in B. In this case there is a path ζ in B from z to b^*. The required path starts with the elements of ζ up to the first element z' such that $z' = \phi_i$ with $i < b$ or $z' = \rho_i$ with $a > i \geq c$. The remaining element are those of the path in C from z' to α.

By lemma 7.6 there exists a path ψ in Ω of resistance zero from α to some $w^* \in \mathcal{W}$. Let $W^* \in \mathbb{W}$ be the set such that $w^* \in W^*$. Select some $\beta \in W^*$ with an outward edge in C to β' such that the transition $\beta\beta'$ has a one-step resistance of at least 1. Some such β must exist for the following reason. If W^* does not contain b^* then W^* contains a element with an outward edge in B since B satisfies condition ii) of lemma 7.8 (with $x^* = b^*$). Therefore this element also has an outward edge in C. If W^* does contain b^* then observe that b^* has an outward edge in C. In either case, there is therefore a path in C from an element of W^* to α. Since α lies outside the recurrent class W^* of the unperturbed dynamics S_0, this path must have positive resistance.

By definition of a recurrent class there exists a path χ in Ω of resistance zero from β to w^*. Let $D = (C \oplus \psi) \oplus \chi$.

It is now shown that any state z with an outward edge in D has a unique path in D to w^*. If z is in χ then the required path is given by the elements of χ from z to w^*. If z is in ψ then the required path starts with the elements of ψ from z to the first z' equal to or following z in ψ which is also in χ. The remaining elements are those of the path in D from z' to w^*. The remaining case is that z has an outward edge in C and there is a path ζ in C from z to α. The required path in D is made up of the elements of ζ until an element z' of ψ or χ is reached, followed by the elements of the path in D from z' to w^*. Such an element z' must exist since ψ begins at α.

Note that with the exception of w^*, all states with an outward edge in B have an outward edge in D. Thus D satisfies the second property of lemma 7.8 with w^* taking the role of x^*. Observe that in the construction of D from C no edges which have been added correspond to transitions with positive one-step resistances. Also, the outward edges of α and β have been deleted, both corresponding to transitions of positive one-step resistance. In the case that $\alpha = \beta$ then it must be the case that $a = a_1$ (see the first paragraph of part II) and the outward edge of α corresponds to a transition with one-step resistance of at least 2. Thus $r(D) \leq R(C) - 2 \leq r(T) - 1$ and by lemma 7.8 a minimal tree exists with resistance $r(T) - 1$ which is a contradiction. □

7.6.4 Other Conditions

This subsection briefly discusses whether theorem 7.3 holds under slightly different conditions.

Suppose demand 1 were included in the demand sets for S players. This creates some difficulties in the proof. If all the neighbours of a W player i make the demand 1 then i receives payoff zero whatever demand they make. This hinders the proof of lemma 7.9 (see footnote 49 of this chapter). However, it seem very likely that the result of the theorem also holds for this case. Observe that no agent ever switches to demand zero under the imitate better or proportional imitation learning rules except by mutation. Thus it must be very rare for demand zero to be made in the long run and so agents making demand 1 usually receive payoff zero.

Under the imitate better or proportional imitation learning rules, it is possible that the updating agent in any round may sample themselves and thus does not switch demand. The proof of lemma 7.9 relies heavily on this property. Under modifications of the imitate better and proportional imitation learning rules in which agents only sample others, the result still holds under the condition:

$$M > \max_{i \in P} |D_i|$$

Two modifications to the proof of lemma 7.9 are required. Recall that the proof concentrates on agents in one particular population, A^i. The first modification is that in the performances described in the proof agents in any population A^j where $j \neq i$ only become active if there is another agent with the same demand in population A^j. The condition on M above ensures that there are always at least 2 agents in any population with the same demand. The second modification is that the active agent in population A^j where $j \neq i$ always samples another agent with the same demand and hence does not changed demand.

Note that the proof of theorem 7.3 does not apply for the sampled best reply learning rule with sample size m. For example, consider the network L_3 and consider a state z such that the only supported demand profile is $(\delta, 1-\delta, \delta)$. Suppose an agent in population A^2 mutates to a demand $\sigma_2 < 1 - \delta$. There is a positive probability in the unperturbed process that in the following rounds each agent in populations A^1 and A^3 in turn sample this mutant and switch to demand $1 - \sigma_2$ if $\frac{1}{m}(1 - \sigma_2) > \delta$. If the demand sets contain sufficiently small demands then this will be true. During this process the mutant agent in population A^2 does not become active again. Now there is a positive probability that every non-mutant agent in A^2 in turn becomes active and samples agents in populations A^1 and A^3 with demands $1 - \sigma_2$, while the active agents in populations A^1 and A^3 continue to sample the mutant agent in population A^2. Thus it is possible for the state to move from z to another recurrent class of the unperturbed process with only a single mutation. Thus lemma 7.9 does not hold.

Chapter 8

Simulation: Results

This chapter contains the results of the simulations using the evolutionary model described in chapters 6 and 7. The aims of the simulations are based on the general aims outlined in chapter 1. In terms of simulation, the three points become:

a) Run simulations using all candidate learning rules on small networks to check whether they produce results consistent with experiment and satisfy other reasonable properties.

b) Run simulations on large networks and characterise the general outcomes.

c) Attempt to find relationships between network parameters and outcomes.

In practice, this section concentrates on aim a). However, aim c) is reflected in the fact that the focus in this section is on the effect of network structure, rather than the effects of varying the parameters and rules of the model, other than the learning rule.

The networks investigated fall into three overlapping categories. The first category is those networks for which experimental data and theoretical predictions exist, allowing a comparison with the results of the simulation as part of aim a). The second category is line networks with unit cakes. These provide a simple setting to investigate the effects of increasing the size of a network as part of aim b). The third category is networks which produce particularly striking simulation results. These include the two 4 player ring networks of sections 8.8 and 8.9.

Section 8.1 investigates which values of M and ϵ are appropriate to use in the simulation. Sections 8.2 to 8.11 each study one network by simulation. Each of these sections begins with a discussion of the motivation for the choice of network. A results subsection follows, describing several simulations under various learning rules and containing the relevant data. The results are summarized and discussed in a final subsection. General conclusions are postponed to the conclusion, chapter 9.

For each simulation, the values of M and ϵ used are selected to illustrate interesting features of the simulation results. The initial state of each simulation is as follows. All agents initially have the lowest available demand. In the case that $I_i = D_i \cup \{0\}$ all agents have an initial informational state of 0, corresponding to a most recent payoff of zero. The other case is that I_i is a singleton and there is only one possible choice of initial informational state. The motivation for this initial state is simply to avoid starting in a recurrent class of the unperturbed dynamics which is stable in the perturbed dynamics.

Each results subsection begins by referring to an inital table (or tables) giving details of the setup of each simulation, including the values of M and ϵ, the number of rounds used, the learning rules and associated parameters, and other relevant information for the particular network. Note that in this and other results tables, the names of learning rules are usually written as initials. Thus imitate better is IB, proportional imitation is PI and sampled best reply is SBR. This initial table also includes the final values of the statistics[1] χ, ρ and γ as defined in sections 7.4 and 7.5.5 to 2 significant figures. Finally, this table includes a 'minimum proportion

[1]Recall that χ is, roughly speaking, the average over all agents of the proportion of rounds in which a given agent had the demand corresponding to the closest B set. ρ is the average number of rounds between changes of modal demand, and γ is the average number of rounds performed per unit of run-time.

displayed' row, the use of which is described shortly.

The main results comprise a table for each simulation which lists the most common modal demand profiles and the proportion of all rounds in which each was modal. The demand profiles displayed are those which were modal for a proportion of rounds of at least the 'minimum proportion displayed' value. The proportions are given to 3 significant figures, as there are sometimes interesting profiles which are modal for a proportion of rounds which can only be shown by this level of precision. Sometimes it is also useful to give the ordinal positions of each demand in the corresponding demand set. These are included in brackets after the demand values. As discussed in section 7.3.1, this list is expected to give a rough indication of the stationary distribution of the model. Results are only given for a single run of the simulation. This is because in practice, as predicted in section 7.3.1, there was little significant difference in results in different runs of the same simulations.

8.1 General Properties and Choice of Parameters

This section investigates the values of χ and ρ as M and ϵ vary for different learning rules. These indicate how stable the B sets of the model are. Also investigated is ρ/γ; the average run time per modal demand change. The aim is to find values of M and ϵ representing a middle ground between over-stable cases where an impractical run-time is required to build up a reasonable picture of the stationary state, and under-stable cases where the structure of the stationary state is eroded by mutation.

These questions are explored in a 2 player network to minimise the required run-time. The results are used to provide a rough indication of which values of M and ϵ provide interesting results for other networks. The simulations of this section use $D_1 = D_2 = \{0, 0.1, 0.2, \ldots, 1\}$. The number of rounds played in each simulation is 2×10^6. The statistics in all the tables are given to 2 significant figures.

Table 8.1 is for the case of $\epsilon = 1$. This produces a model entirely driven by mutation. The results provide a baseline for comparison with the other tables of this section. The value of γ is also included in this table as it can serve as a baseline for all other values of γ in this chapter.

M	20	40	60	100	200
χ	0.21	0.17	0.15	0.14	0.13
ρ	5.6	8.1	10	13	19
ρ/γ	3.1×10^{-5}	4.3×10^{-5}	5.4×10^{-5}	7.5×10^{-5}	1.2×10^{-4}
γ	1.8×10^{5}	1.9×10^{5}	1.9×10^{5}	1.8×10^{5}	1.6×10^{5}

Table 8.1: Statistics for $\epsilon = 1$

Tables 8.2 – 8.5 are for the learning rules imitate better, proportional imitation with factor of proportionality 1, and sampled best reply with sample sizes 1 and 2. Note that starred values of ρ in these tables indicate that a modal demand profile was quickly reached which did not change for the rest of the simulation.

M	ϵ	χ	ρ	ρ/γ
20	0.15	0.82	1.2×10^{4}	7.5×10^{-1}
20	0.19	0.72	4.8×10^{2}	2.9×10^{-2}
20	0.23	0.54	6.5×10^{1}	3.8×10^{-3}
40	0.19	0.74	3.5×10^{4}	2.0
40	0.23	0.58	4.6×10^{2}	2.5×10^{-2}
40	0.27	0.38	6.1×10^{1}	3.4×10^{-3}
60	0.19	0.68	3.8×10^{4} *	2.1
60	0.23	0.55	3.9×10^{3}	2.1×10^{-1}
60	0.27	0.35	1.0×10^{2}	5.7×10^{-4}

Table 8.2: Statistics for IB

These results illustrate that there are a wide range of values of M and ϵ for which the values of χ and ρ are significantly different to their values in the case $\epsilon = 1$. This

M	ϵ	χ	ρ	ρ/γ
20	0.04	0.88	5.0×10^4	2.9
20	0.08	0.68	5.2×10^2	3.0×10^{-2}
20	0.12	0.43	6.5×10^1	4.0×10^{-3}
40	0.04	0.88	7.1×10^4 *	4.1
40	0.08	0.71	8.1×10^3	4.5×10^{-1}
40	0.12	0.41	1.4×10^2	8.3×10^{-3}
60	0.08	0.66	1.9×10^1	1.1
60	0.12	0.37	2.6×10^2	1.5×10^{-2}
60	0.16	0.24	7.4×10^1	4.7×10^{-3}

Table 8.3: Statistics for PI

M	ϵ	χ	ρ	ρ/γ
20	10^{-4}	1.00	4.0×10^3	6.1×10^{-2}
20	10^{-3}	0.98	5.6×10^2	8.8×10^{-3}
20	10^{-2}	0.85	5.8×10^1	1.0×10^{-3}
100	10^{-4}	0.98	2.2×10^3	3.3×10^{-2}
100	10^{-3}	0.92	6.6×10^2	1.1×10^{-2}
100	10^{-2}	0.57	8.4×10^1	1.4×10^{-3}
200	10^{-4}	0.96	1.5×10^3 *	2.5×10^{-2}
200	10^{-3}	0.82	4.6×10^2	7.5×10^{-3}
200	10^{-2}	0.46	1.3×10^2	2.2×10^{-3}

Table 8.4: Statistics for SBR with sample size 1

M	ϵ	χ	ρ	ρ/γ
20	0.02	0.98	3.2×10^3	1.0×10^{-1}
20	0.06	0.89	2.6×10^2	1.0×10^{-2}
20	0.1	0.77	7.8×10^1	6.1×10^{-3}
40	0.02	0.99	7.1×10^4 *	1.7
40	0.06	0.91	4.5×10^3	1.2×10^{-1}
40	0.1	0.80	4.6×10^2	2.1×10^{-2}
60	0.06	0.85	4.4×10^4 *	1.1
60	0.1	0.75	2.1×10^3	8.1×10^{-2}
60	0.14	0.63	3.2×10^2	2.3×10^{-2}

Table 8.5: Statistics for SBR with sample size 2

provides some support for the prediction of section 7.3.1 that the state is usually near a B set even for relatively large values of ϵ. However note that, except under sampled best reply, it does not seem possible to have a value of χ above 0.9 and a value of ρ sufficiently low for much to be revealed about the stationary distribution in a reasonable runtime. This suggests that it is rare for the state to be inside a B set for the typical values of M and ϵ used in this section.

Also, as predicted in section 7.3.1, the value of ρ is decreasing in ϵ and increasing in M, and the value of χ is decreasing in ϵ. Under imitate better the value of χ appears to be increasing in M. For the other learning rules there appears to be a more complicated relationship which cannot be characterised given the limited results available. Also note that rate of change of ρ with ϵ seems independent of the value of M. Thus the range of values of ϵ for which interesting behaviour can be found is of roughly the same size for any value of M.

The values of M used in the simulations of the remainder of this chapter are near the lower end of the ranges used in this section. This is to allow a lower value of ϵ to be used, and, for imitate better, to minimise the value of χ. These features should result in the state staying closer to B sets, and provide the resulting stationary distribution with some degree of protection from being too driven by mutation rather than the probabilities of transits between B sets.

8.2 The 2 Player Unit Cake Network

This section is on the simplest bilateral exchange network: the 2 player unit cake network. Recall that section 7.1.1 contains a prediction that the results for the imitate better learning rule are sensitive to the choice of D. To test this prediction and the robustness of the other learning rules to variations in D, for each learning rule two simulations are carried out in which two choices of D are used. In the first, B, demands are evenly spaced: $B_1 = B_2 = \{0, 0.1, 0.2, \ldots, 1\}$. The second, C, uses very unevenly spaced demands:

$$C_1 = \{0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.15, 0.2, 0.5, 0.7, 1\}$$

$$C_2 = \{1 - x | x \in C_1\}$$

Results

Tables 8.6 and 8.7 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. Tables 8.8 – 8.15 contain the data. Recall that in the latter tables, the figures in brackets after demands are the ordinal positions of the demands in the corresponding demand sets.

Summary and Discussion

For all of the learning rules used here, these results are evidence in favour of the prediction of section 7.3.1 that the model concentrates on strict Nash equilibrium outcomes. Note that this prediction was made under the assumption of low ϵ. Thus

Table	8.8	8.9	8.10	8.11
Learning rule	IB	IB	IB, $m = 12$	PI
Demand sets	B	C	B	B
M	25	25	35	25
ϵ	0.21	0.21	0.23	0.08
ρ	3.3×10^2	1.8×10^4	1.2×10^2	1.0×10^3
χ	0.56	0.56	0.69	0.59
γ	8.4×10^3	3.4×10^3	6.7×10^3	1.8×10^4
Rounds played	5×10^6	5×10^6	5×10^6	5×10^6
Minimum proportion displayed	0.01	0.01	0.01	0.01

Table 8.6: Guide to the simulations of section 8.2 (1)

Table	8.12	8.13	8.14	8.15
Learning rule	PI	SBR,	SBR,	SBR,
		$m = 1$	$m = 2$	$m = 2$
Demand sets	C	B	B	C
M	25	100	50	50
ϵ	0.1	0.001	0.1	0.15
ρ	2.9×10^2	4.5×10^2	9.4×10^2	1.5×10^3
χ	0.60	0.94	0.67	0.76
γ	6.3×10^3	5.9×10^4	2.3×10^4	2.6×10^4
Rounds played	5×10^6	10^7	10^7	10^7
Minimum proportion displayed	0.01	0.01	0.001	0.001

Table 8.7: Guide to the simulations of section 8.2 (2)

Modal strategy		
1	2	Proportion
0.5	0.5	0.665
0.4	0.6	0.147
0.6	0.4	0.134
Others		0.054

Table 8.8: IB on a 2 player unit cake network with demands B

Modal strategy		
1	2	Proportion
0.05 (6)	0.95 (6)	0.674
0.15 (7)	0.85 (5)	0.144
0.04 (5)	0.96 (7)	0.133
Others		0.049

Table 8.9: IB on a 2 player unit cake network with demands C

Modal strategy		
1	2	Proportion
0.5 (9)	0.5 (3)	0.541
0.2 (8)	0.4 (4)	0.235
0.15 (7)	0.8 (4)	0.101
0.15 (7)	0.85 (5)	0.099
Others		0.024

Table 8.10: IB on a 2 player unit cake network results with demands C and sample size 12

Modal strategy		
1	2	Proportion
0.5	0.5	0.812
0.4	0.6	0.110
0.6	0.4	0.028
0.4	0.5	0.025
0.5	0.4	0.018
Others		0.007

Table 8.11: PI on a 2 player unit cake network with demands B

Modal strategy		
1	2	Proportion
0.5 (9)	0.5 (3)	0.814
0.2 (8)	0.8 (4)	0.115
0.15 (7)	0.8 (4)	0.021
0.5 (9)	0.8 (4)	0.018
Others		0.032

Table 8.12: PI on a 2 player unit cake network with demands C

Modal strategy		
1	2	Time
0.2	0.8	0.185
0.3	0.7	0.138
0.6	0.4	0.117
0.9	0.1	0.113
0.4	0.6	0.099
0.7	0.3	0.092
0.5	0.5	0.084
0.1	0.9	0.082
0.8	0.2	0.075
Others		0.015

Table 8.13: SBR on a 2 player unit cake network with demands B and sample size 1

Modal strategy		
1	2	Proportion
0.5	0.5	0.614
0.6	0.4	0.191
0.4	0.6	0.181
0.4	0.5	0.004
0.5	0.4	0.004
0.7	0.3	0.001
Others		0.005

Table 8.14: SBR on a 2 player unit cake network with demands B and sample size 2

Modal strategy		
1	2	Proportion
0.5	0.5	0.985
0.7	0.3	0.007
0.5	0.3	0.006
Others		0.002

Table 8.15: SBR on a 2 player unit cake network with demands C and sample size 2

the experimental results provide some evidence that the qualitative features of the model are robust to variations in ϵ as predicted of section 7.3.1.

The demand profile $(0.5, 0.5)$ receives strong support under all learning rules. However, as predicted in section 7.1.1, the results of the imitate better learning rule are highly sensitive to the choice of D. This makes it problematic to use the results of this learning rule, as discussed in section 7.1.1. Nonetheless, the results do concentrate on a small number of outcomes: those close to the median of the Pareto optimal outcomes of the game if they are ordered according to the payoff to either player[2]. This suggests that if demand sets are evenly spaced then the model under this learning rule might support a bargaining solution for the 2 player problem in which the outcome is halfway along the outer boundary[3] of the utility cake. However, more simulations would be required to test this hypothesis.

Imitate better with a larger sample size is more robust to the choice of D. However, the sample size needs to be quite large; the simulation above required a sample size of 12 from a total population of 35. Simulations with sample sizes close to 1 have very similar results to those with sample size 1. These results are not included as they exhibit no novel features. This learning rule is not used often in

[2]This is well defined since for the choices of D made in this section, the Pareto optimal outcomes of the game are of the form $(a, 1 - a)$.

[3]Recall definition 3.6.

the remainder of this chapter as it becomes very computationally expensive in larger networks.

The results for sampled best response learning rule with sample size 1 show that it only selects very weakly, or perhaps even not at all, between strict Nash equilibrium outcomes. A possible explanation for this is that in states close to B sets there is no mechanism for a cardinal comparison of the utilities to be gained from playing a demand against a mutant and a non-mutant. Suppose all mutant agents have a superior payoff. Then an updating agent switches away from the B set demand if and only if they sample a mutant. Thus the probability of a transit cannot involve utility comparisons and most transits between strict Nash equilibria have roughly equal probability. With a larger sample sizes, updating agents will sample both mutants and non-mutants more often than they sample only mutants. Averaging calculations then allow cardinal utility effects.

The proportional imitation learning rule supports the solution $(0.5, 0.5)$ for both the evenly spaced demand sets of B and the very unevenly spaced sets of C. In the latter case there is also some support for $(0.2, 0.8)$. This may be because the demands vary by only one ordinal position from the main solution and are closer to the median demands, echoing the support of imitate better for median demands. A possible explanation for the success of non-Pareto optimal strategy profiles such as $(0.15, 0.8)$ is that the probability of agents with positive payoffs increasing their demands to Pareto optimal demands is often low. A possible explanation for the success of non-feasible strategy profiles such as $(0.5, 0.8)$ is that these occur when multiple demands coexist in populations during the transit between B sets. For example consider a state which supports only the demands 0.2 and 0.5 in population A^1 and only the demands 0.5 and 0.8 in population A^2. As agents are more likely to switch to demands earning higher payoffs, the demand pair $(0.5, 0.8)$ could become modal for some rounds even though it is not a feasible pair. This explanation would mean that the apparent success of such profiles is only due to the method of reporting used[4].

8.3 A 2 Player non-Unit Cake Network

This section investigates the behaviour of the simulation for a particular 2 player network whose utility cake is defined by a non-linear boundary function. The cake is chosen to investigate whether the axiom of independence of irrelevant alternatives used in the axiomatic definition of the Nash bargaining solution (see definition 4.2) holds. The cake chosen is a subset of the unit cake containing the point (0.5, 0.5). If the axiom holds, the solution should be (0.5, 0.5). The cake is defined by the boundary function

$$f^{1,2}(x) = \begin{cases} 0.8 - 0.6x & \text{for } x \leq 0.5 \\ 1 - x - 10(x - 0.5)^2 & \text{for } x > 0.5 \end{cases}$$

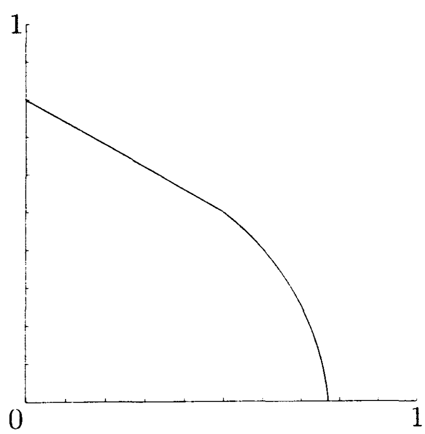

Figure 8.1: $f^{1,2}(x)$ for section 8.3

This is a concave function over $[0, 1]$. One reason for its choice is that it is asymmetric.

[4]A possible alternative is as follows. Rather that count the number of rounds on which each demand profile is modal, count the total contribution to χ on rounds in which each demand profile is modal. This puts a lower weight on those rounds in which the state is far from a B set.

In each simulation of this section, the demand sets are $D_1 = \{0, 0.05, 0.1, \ldots, 0.7\}$ and $D_2 = \{f^{1,2}(d) | d \in D_1\}$. Note that in this section, a demand pair is Pareto optimal if their ordinal positions in these sets, given in brackets in the results tables, sum to 16.

Results

Tables 8.16 and 8.17 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. Tables 8.18 – 8.22 contain the data.

Table	8.18	8.19	8.20
Learning rule	IB	IB	PI
M	25	35	35
ϵ	0.19	0.08	0.08
ρ	1.2×10^3	1.2×10^3	7.6×10^2
χ	0.60	0.93	0.71
γ	1.4×10^4	4.3×10^4	1.4×10^4
Rounds played	5×10^6	2×10^6	2×10^6
Minimum proportion displayed	0.001	0.001	0.005

Table 8.16: Guide to the simulations of section 8.3 (1)

Summary and Discussion

Each of these simulations again concentrates on a few demand profiles. Except under imitate better, these all are close to the outcome $(0.5, 0.5)$. Imitate better again selected the median outcome (in the sense described in section 8.2).

However, the profiles which were most commonly modal under the imitate better and sampled best reply learning rules with sample sizes 12 and 2 respectively were not $(0.5, 0.5)$. In the profiles which were most commonly modal for these learning

Table	8.21	8.22
Learning rule	SBR, $m = 2$	SBR, $m = 5$
M	35	35
ϵ	0.08	0.08
ρ	6.6×10^2	3.6×10^2
χ	0.64	0.63
γ	4.5×10^3	1.6×10^3
Rounds played	2×10^6	2×10^6
Minimum proportion displayed	0.005	0.005

Table 8.17: Guide to the simulations of section 8.3 (2)

Modal strategy		
1	2	Proportion
0.35 (8)	0.59 (8)	0.534
0.4 (9)	0.56 (7)	0.231
0.3 (7)	0.62 (9)	0.163
0.45 (10)	0.53 (6)	0.033
0.25 (6)	0.65 (10)	0.026
0.35 (8)	0.56 (7)	0.002
0.3 (7)	0.59 (8)	0.002
Others		0.009

Table 8.18: IB on a 2 player non-unit cake network

Modal strategy		
1	2	Proportion
0.4 (9)	0.56 (7)	0.597
0.45 (10)	0.53 (6)	0.346
0.4 (9)	0.53 (6)	0.016
0.5 (11)	0.5 (5)	0.016
0.35 (8)	0.59 (8)	0.005
0.45 (10)	0.5 (5)	0.001
0.6 (13)	0.3 (3)	0.001
Others		0.018

Table 8.19: IB on a 2 player non-unit cake network with sample size 12

Modal strategy		
1	2	Proportion
0.5 (11)	0.5 (5)	0.681
0.45 (10)	0.53 (6)	0.173
0.45 (10)	0.5 (5)	0.094
0.4 (9)	0.56 (7)	0.021
0.4 (9)	0.5 (5)	0.009
0.4 (9)	0.53 (6)	0.007
Others		0.015

Table 8.20: PI on a 2 player non-unit cake network

Modal strategy		
1	2	Proportion
0.45 (10)	0.53 (6)	0.686
0.4 (9)	0.56 (7)	0.266
0.4 (9)	0.53 (6)	0.021
0.35 (8)	0.59 (8)	0.005
Others		0.022

Table 8.21: SBR on a 2 player non-unit cake network with sample size 2

Modal strategy		
1	2	Proportion
0.5 (11)	0.5 (5)	0.680
0.45 (10)	0.53 (6)	0.235
0.55 (12)	0.425 (4)	0.035
0.4 (9)	0.56 (7)	0.025
0.45 (10)	0.5 (5)	0.014
Others		0.011

Table 8.22: SBR on a 2 player non-unit cake network with sample size 5

rules, player 2 received a payoff of slightly more than 0.5. However, increasing the sample size of sampled best reply to 5 did yield a most common modal demand profile of (0.5, 0.5).

This indicates that the outcome of these learning rules is not completely robust to variations in the utility cakes which are irrelevant under the axioms of the Nash bargaining solution. However, robustness does appear to increase with the sample size of each learning rule.

With the exception of the imitate better learning rule, the results of this and the previous section certainly do not contradict the Nash bargaining solution. However, further simulations or theoretical results would be required before concluding that they offer strong support for this solution.

8.4 A 2 Player Unit Cake Network with an Outside Option

This section investigates the effects of introducing an outside option for player 2 into a 2 player unit cake network. The details of how the simulation code implements this are given in section 7.5.1. The motivation for this investigation is to discover whether direct outside options have the same effect as the indirect outside options implicit in the possibility of exchanging with another player in networks of more than 2 players. In all the simulations of the section, both demand sets are $\{0, 0.05, 0.1, \ldots, 1\}$.

Results

Tables 8.23 and 8.24 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. The 'proportion exercised' row gives the proportion of all rounds in which the outside outside is exercised. Note that the factor of proportionality for the proportional imitation learning rule is 1. Also, note that a table of data is not given for the simulation corresponding to the entry of table 8.24 with no table number. Such a table would not have been informative because in this simulation no individual strategy profile was modal for more than 0.1 of all rounds and the most common modal demand profiles all involved player 2 taking the outside option. Note that the unusually low value of ρ in the simulation of table 8.26 is probably a consequence of behaviour in population A^1 being mainly driven by random mutations since the outside option is taken so often by agents in population 2. Tables 8.25 – 8.29 contain the data.

Table	8.25	8.26	8.27
Outside option	0.41	0.6	0.41
Learning rule	IB	IB	PI
M	25	25	25
ϵ	0.21	0.21	0.07
Proportion exercised	0.05	0.33	0.16
ρ	9.6×10^3	56	3.0×10^2
χ	0.50	0.49	0.54
γ	9.5×10^3	8.4×10^3	8.6×10^3
Rounds played	5×10^6	5×10^6	5×10^6
Minimum proportion displayed	0.01	0.02	0.01

Table 8.23: Guide to the simulations of section 8.4 (1)

Summary and Discussion

Section 4.2.1 mentions a variation of the alternating offers game incorporating outside options. For a unit cake network with equal discount factors the game predicts that an outside option of less than $\frac{1}{2}$ leaves the outcome unchanged from the game without outside options: both players receive a payoff of $\frac{1}{2}$. For an outside option of more than $\frac{1}{2}$ it predicts that an exchange forms and player 2 receives a payoff equal to the value of the outside option. Note that equal discount factors seem appro-

Table	n/a	8.28	8.29
Outside option	0.6	0.41	0.6
Learning rule	PI	SBR, $m = 2$	SBR, $m = 2$
M	25	100	100
ϵ	0.07	0.1	0.1
Proportion exercised	0.82	0.12	0.33
ρ		1.1×10^3	1.6×10^2
χ		0.64	0.57
γ		6.3×10^3	4.4×10^3
Rounds played	5×10^6	5×10^6	5×10^6
Minimum proportion displayed		0.001	0.001

Table 8.24: Guide to the simulations of section 8.4 (2)

Modal strategy		Proportion
1	2	
0.45	0.55	0.306
0.5	0.5	0.297
0.4	0.6	0.150
0.35	0.65	0.046
0.45	0.5	0.011
0.4	0.55	0.010
Others		0.180

Table 8.25: IB on a 2 player unit cake network with outside option 0.41

Modal strategy		
1	2	Proportion
0.3	0.7	0.108
0.3	option exercised	0.065
0.25	option exercised	0.062
0.35	option exercised	0.060
0.2	option exercised	0.056
0.15	option exercised	0.053
0.005	option exercised	0.044
0.1	option exercised	0.044
0.25	0.75	0.041
0.4	option exercised	0.029
0.45	option exercised	0.021
0.3	0.65	0.020
Others		0.397

Table 8.26: IB on a 2 player unit cake network with outside option 0.6

Modal strategy		
1	2	Proportion
0.45	0.55	0.289
0.4	0.6	0.180
0.5	0.5	0.136
0.4	0.55	0.049
0.45	0.5	0.042
0.45	option exercised	0.036
0.5	option exercised	0.034
0.4	option exercised	0.025
0.35	0.65	0.024
0.55	option exercised	0.021
0.35	option exercised	0.016
0.35	0.6	0.016
0.6	option exercised	0.015
0.4	0.5	0.010
Others		0.107

Table 8.27: PI on a 2 player unit cake network with outside option 0.41

Modal strategy		
1	2	Proportion
0.55	0.45	0.858
0.5	0.5	0.114
0.45	0.55	0.015
0.5	0.4	0.005
0.4	0.6	0.001
0.55	option exercised	0.001
Others		0.006

Table 8.28: SBR on a 2 player unit cake network with outside option 0.41 and sample size 2

Modal strategy		
1	2	Proportion
0.35	0.65	0.839
0.35	option exercised	0.160
Others		0.001

Table 8.29: SBR on a 2 player unit cake network with outside option 0.6 and sample size 2

priate as in the simulation there are no asymmetries between the two populations corresponding to non-equal discount factors.

Most of the results of this section differ from this prediction. For imitate better and proportional imitation, an outside option of less than $\frac{1}{2}$ increases the typical payoff of agents in population 2. This is similar to some experimental results, such as those of Binmore et al [10]. It is intuitively plausible that the existence of an outside option strengthens the bargaining position of player 2, justifying this increased payoff. However, I do not have a candidate mechanism explaining how this takes place in the model.

For sampled best reply with sample size 2, the same outside option reduces the typical payoff of agents in population 2! This is a counter-intuitive result. However, note that in the most common modal demand profile the demand played by agents in population 2 is only one ordinal position lower than 0.5, so there is a possibility that for finer demand sets agents in population 2 receive payoffs very close to 0.5.

Under imitate better an outside option of more than $\frac{1}{2}$ is exercised quite often. When it is not exercised, the typical payoff of agents in population 2 is above that of the outside option under imitate better. Under proportional imitation the outside option is exercised in the majority of rounds.

Under sampled best reply with sample size 2, an outside option of more than $\frac{1}{2}$ is exercised occasionally and the results concentrate on an outcome where player 2 exchanges and receives slightly more than the outside option. This is the only simulation of this section that matches the alternating offers prediction.

An explanation for the tendency of player 2 to accept the outside option of 0.6 under imitative learning rules is that the evolutionary model does not capture 'sensible' behaviour of agents in the face of this outside option. For example consider a state where all agents in population 2 accept the outside option. The 'sensible' response of agents in population 1 is to make low demands in an attempt to make exchanging more attractive than taking the outside option to agents in population

2. However under imitative learning rules, all demands of agent 1 receive payoff zero, so there is no evolutionary pressure to make any response at all. On the other hand, if agents in population 2 were exchanging with a third population and making a demand feasible to population 1, then a low demand by an agent in population 1 would occasionally receive a non-zero payoff, providing evolutionary pressure to reduce demands. Thus in this case, this direct outside option setting does not seem to correspond well to the indirect outside options sometimes available in networks.

8.5 The 3 Player Line Network with Unit Cakes

This section is on the network L_3. The expected outcome here is that player 2 receives a high payoff. In particular, theorem 7.3 proves that the outcome of the evolutionary model in the limit $\epsilon \to 0$ under the imitative learning rules considered here and a few other assumptions is that player 2 receives the maximum possible payoff. One motivation for simulations on this network is to determine how well this prediction holds for general values of ϵ. This prediction is also supported by the market bargaining game of section 4.4. In all simulations of this section, each player has the demand set[5] $\{0, 0.05, 0.1, \ldots, 1\}$.

Results

Tables 8.30 and 8.31 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. Tables 8.32 – 8.36 contain the data.

Summary and Discussion

The strong prediction that player 2 receives the payoff 0.95 is only unambiguously supported by the sampled best reply rule with sample size 1. In other simulations

[5]Note that the inclusion of the demand 1 means the conditions of theorem 7.3 do not hold. However, other simulations without this demand produce very similar results.

Table	8.32	8.33	8.34
Learning rule	IB	IB	PI
M	15	30	40
ϵ	0.14	0.19	0.06
ρ	5.8×10^2	2.7×10^2	5.9×10^2
χ	0.69	0.54	0.61
γ	7.1×10^3	6.9×10^3	6.4×10^3
Rounds played	2×10^6	2×10^6	2×10^6
Minimum proportion displayed	0.002	0.02	0.02

Table 8.30: Guide to the simulations of section 8.5 (1)

Table	8.35	8.36
Learning rule	SBR, $m = 1$	SBR, $m = 2$
M	50	40
ϵ	0.2	0.12
ρ	3.8×10^2	5.0×10^2
χ	0.51	0.71
γ	2.0×10^3	6.0×10^2
Rounds played	2×10^6	10^6
Minimum proportion displayed	0.001	0.01

Table 8.31: Guide to the simulations of section 8.5 (2)

Modal strategy			
1	2	3	Proportion
0.05	0.95	0.05	0.905
0.15	0.85	0.15	0.017
0.1	0.9	0.1	0.004
0.05	0.95	0.05	0.002
0.1	0.9	0.05	0.002
0.05	0.95	0.1	0.002
Others			0.068

Table 8.32: IB on L_3 (1)

Modal strategy			
1	2	3	Proportion
0.1	0.9	0.1	0.473
0.15	0.85	0.15	0.317
0.2	0.8	0.2	0.055
0.05	0.95	0.05	0.043
Others			0.112

Table 8.33: IB on L_3 (2)

Modal strategy			Proportion
1	2	3	
0.3	0.7	0.3	0.533
0.35	0.65	0.35	0.171
0.4	0.6	0.4	0.064
0.25	0.7	0.3	0.041
0.3	0.65	0.35	0.036
0.3	0.7	0.25	0.035
Others			0.120

Table 8.34: PI on L_3

Modal strategy			Proportion
1	2	3	
0.05	0.95	0.05	0.979
0.1	0.95	0.05	0.003
0.05	0.95	0.1	0.003
0.1	0.9	0.1	0.002
0.15	0.95	0.05	0.001
Others			0.012

Table 8.35: SBR on L_3 with sample size 1

Modal strategy			
1	2	3	Proportion
0.2	0.8	0.2	0.842
0.25	0.75	0.25	0.108
0.15	0.85	0.15	0.012
0.35	0.65	0.35	0.016
Others			0.022

Table 8.36: SBR on L_3 with sample size 2

which are not recorded here, this solution is found to be robust to variations in the parameters M and ϵ. The imitate better learning rule provides some support for the same prediction but this solution is not robust to M and ϵ.

Where the strong prediction is not supported, the simulations instead support an outcome in which player 2 makes a demand of between 0.9 and 0.7 and players 1 and 3 make equal demands for the remainder of the unit of payoff. In some cases a single such demand profile is modal most of the time, in others the time is shared between several similar profiles.

The fact that all learning rules support qualitatively similar solutions suggest that the evolutionary mechanism driving this outcome is simpler than in the 2 player case. Even sampled best reply with sample size 1 has some predictive power in contrast to the 2 player case. The fact that imitate better and proportional imitation have qualitatively similar results for L_3 suggests that the evolutionary mechanism mentioned does not depend on cardinal payoff comparisons.

The results for imitative learning rules are qualitatively similar to the prediction of theorem 7.3 but exhibit some differences. This gives some support to the prediction of section 7.3.1 that qualitative results are robust to small variations in ϵ. The prediction that M and ϵ can affect the exact outcome selected is also supported, as illustrated by the difference between the results of tables 8.32 and 8.33. Further simulations, whose results are not recorded here, show that the most commonly modal demand profile also varies depending on the choice of M but the qualitative outcome is robust. This is reflected in the fact that different values of M are used in combination with different values of ϵ to illustrate different possible outcomes.

Note that the typical demand difference between the central and outlying agents is smaller for proportional imitation than for imitate better. A possible explanation for this is that since probabilities of imitating demands are lower under proportional imitation, the evolutionary pressure on agents in populations 1 and 3 to undercut each other, captured by theorem 7.3, is weaker. However it is not obvious what other evolutionary force countervails this.

8.6 A 3 Player Ring Network

This section investigates a 3 player ring network. As discussed in section 4.4, under certain conditions such networks have a unique von Neumann-Morgenstern triple containing three outcomes and the market bargaining game of that section supports all of these outcomes as possible results of bargaining. The interpretation of the solution to the market bargaining game of that section is that one of these outcomes occurs. The network of this section is constructed so that it supports a von Neumann-Morgenstern triple whose outcomes are all those in which two players receive payoff 0.5.

The outer boundary of each cake is constructed from two line segments from the point $(0.5, 0.5)$ to points on each axis. From definition 4.3 it can be seen that such a network supports the required von Neumann-Morgenstern triple. Each cake can be described completely by giving the points at which its boundary function intercepts the axes. These are as follows: for $\mathcal{K}^{12}$ $(0, 1.3)$ and $(0.8, 0)$, for $\mathcal{K}^{23}$ $(0, 1)$ and $(1, 0)$, and for $\mathcal{K}^{31}$ $(0, 0.65)$ and $(1.6, 0)$. Recall that a point $(x, y) \in \mathcal{K}^{ab}$ is written so that x is the payoff to player a and y that to player b.

The demand sets are given by $D_i = \{0, 0.1, 0.2, \ldots, M_i\}$ where M_i is the max-

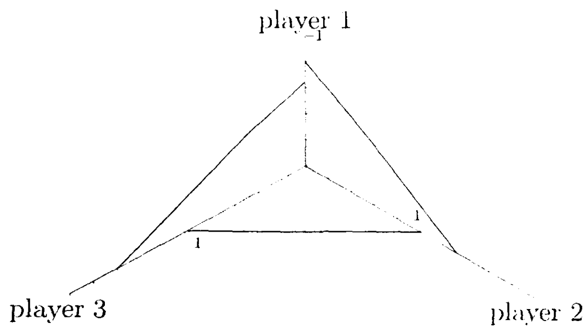

Figure 8.2: The network under investigation in section 8.6

imum feasible payoff to i in any cake. For these cakes $M_1 = 0.8$, $M_2 = 1.3$ and $M_3 = 1.6$. This allows for easy presentation of the results.

It can be argued that this choice of D does not produce outcomes of the game evenly spread along the Pareto boundaries of the cakes and that this factor potentially biases the results. However, for networks of more than 2 players, such an even spread does not seem easy to achieve, except in the most symmetric cases (e.g. unit cakes). The fact that the outcomes are not spread evenly can be viewed as a small test of robustness.

Note that $\frac{1}{2}$ is neither the median demand of D_1 or D_3 nor the payoff either player receives in the symmetric Nash bargaining solution[6] in two player bargaining on $\mathcal{K}^{12}$ or $\mathcal{K}^{31}$. This feature is chosen to prevent the model concentrating on the demand 0.5 for these reasons.

[6]For these cakes, whichever player receives a payoff of $x > 1$ in an intercept with an axis receives a payoff of $\frac{1}{2}x$ in the symmetric Nash bargaining solution of that cake for the following reason. By the scale independence and symmetry axioms, this is the player's Nash bargaining solution payoff in a cake whose outer boundary is formed by extending the line segment between this intercept and $(0.5, 0.5)$. By the axiom of the independence of irrelevant alternatives, the Nash bargaining solution is the same for the cake constructed in the main text.

Results

Tables 8.37 and 8.38 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is $\frac{5}{8}$: the reciprocal of the maximum demand. Note that a relatively small number of rounds are played in the simulation of table 8.41 as the value of γ is low. The large value of ρ is in the same simulation is probably simply due to the perturbations caused by the large value of ϵ rather than illustrating common transits between B sets. Tables 8.39 – 8.41 contain the data.

Table	8.39	8.40
Learning rule	IB	PI
M	50	50
ϵ	0.21	0.05
ρ	3.8×10^3	2.6×10^3
χ	0.52	0.51
γ	9.8×10^3	1.0×10^4
Rounds played	5×10^6	5×10^6
Minimum proportion displayed	0.001	0.001

Table 8.37: Guide to the simulations of section 8.6 (1)

Summary and Discussion

All the learning rules discussed here provide very strong support for the demand profile $(0.5, 0.5, 0.5)$, matching the prediction of the market bargaining game. Other simulations, not included here, show that this support is robust to changes in M and ϵ. Indeed, the reason that such a large value of ϵ is chosen for the simulation of table 8.42 is that for smaller values no other demand profiles were modal for a significant proportion of rounds.

Table	8.41	8.41
Learning rule	SBR, $m = 1$	SBR, $m = 2$
M	25	20
ϵ	0.2	0.38
ρ	2.6×10^2	91
χ	0.53	0.53
γ	6.4×10^3	6.4×10^2
Rounds played	5×10^6	5×10^5
Minimum proportion displayed	0.001	0.01

Table 8.38: Guide to the simulations of section 8.6 (2)

Modal strategy			Proportion
1	2	3	
0.5	0.5	0.5	0.995
0.5	0.5	0.4	0.002
Others			0.003

Table 8.39: IB on a 3 player ring network

Modal strategy			Proportion
1	2	3	
0.5	0.5	0.5	0.950
0.5	0.5	0.4	0.022
0.5	0.4	0.5	0.012
0.4	0.5	0.5	0.005
0.5	0.5	0.3	0.001
Others			0.010

Table 8.40: PI on a 3 player ring network

Modal strategy				
1	2	3	Feasbile exchanges	Proportion
0.5	0.5	0.5	12.23.31	0.983
0.5	0.5	0.7	12	0.005
0.5	0.5	0.9	12	0.004
0.5	0.8	0.5	31	0.002
Others				0.006

Table 8.41: SBR on a 3 player ring network with sample size 1

Modal strategy				
1	2	3	Feasible exchanges	Proportion
0.5	0.5	0.5	12.23.31	0.955
0.4	0.5	0.5	12.23.31	0.020
Others				0.025

Table 8.42: SBR on a 3 player ring network with sample size 2

I do not have a candidate mechanism which explains such strong support for this outcome. It seems unlikely that minimal tree analysis such as that used in section 7.6 can be used. In 3 player ring networks at least one agent is excluded from exchange given any demand profile. Under imitative learning rules these agents may imitate any demand with a positive payoff. It can easily be shown that every recurrent class of the unperturbed model thus has outward resistance of 1, so the resistance based arguments of minimal tree analysis seem to have little power.

8.7 A 4 Player Line Network with Unit Cakes

This section is on the network L_4. One motivation for studying this network is that the experimental data of section 2.7 indicates that players 2 and 3 do better than players 1 and 4, but the alternating offers approach did not capture this result, as shown in section 5.2.4 where the payoff vectors $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2}, 0)$ are offered as the only 'plausible' limiting solutions of a model on this network.

Results

Table 8.43 gives the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. In the simulation of table 8.45 the demand sets were:

$$D_1 = D_3 = \{0, 0.01, 0.04, 0.09, 0.14, 0.2, 0.3, 0.4, 0.5, 0.75, 1\}$$

$$D_2 = D_4 = \{1 - x \mid x \in D_1\}$$

In all the other simulations of this section, all demand sets were $\{0, 0.05, 0.1, \ldots, 1\}$. Note that a relatively number of rounds were played in the simulation of table 8.47 as the final value of γ was so low. Tables 8.44 – 8.47 contain the data. Recall that in the latter tables, the figures in brackets after demands are the ordinal positions of the demands in the corresponding demand sets.

Table	8.44	8.45	8.46	8.47
Learning rule	IB	IB	PI	SBR, $m = 2$
M	25	25	25	50
ϵ	0.18	0.18	0.18	0.12
ρ	2.5×10^2	5.2×10^2	4.0×10^2	4.4×10^2
χ	0.61	0.65	0.62	0.78
γ	5.0×10^3	1.0×10^4	4.8×10^3	2.2×10^2
Rounds played	2×10^6	2×10^6	2×10^6	5×10^5
Minimum proportion displayed	0.02	0.01	0.02	0.01

Table 8.43: Guide to the simulations of section 8.7

Modal strategy				
1	2	3	4	Proportion
0.35	0.65	0.65	0.35	0.164
0.35	0.65	0.6	0.4	0.143
0.4	0.6	0.65	0.35	0.131
0.35	0.65	0.7	0.3	0.084
0.3	0.7	0.65	0.35	0.078
0.35	0.65	0.55	0.45	0.069
0.4	0.6	0.6	0.4	0.043
0.3	0.7	0.7	0.3	0.037
0.4	0.6	0.7	0.3	0.031
0.45	0.55	0.65	0.35	0.022
Others				0.198

Table 8.44: IB on L_4 with evenly spaced demand

Modal strategy				Proportion
1	2	3	4	
0.09 (4)	0.91 (8)	0.3 (7)	0.7 (5)	0.310
0.14 (5)	0.86 (7)	0.3 (7)	0.7 (5)	0.243
0.14 (5)	0.86 (7)	0.4 (8)	0.6 (4)	0.232
0.09 (4)	0.91 (8)	0.4 (8)	0.6 (4)	0.127
0.14 (5)	0.86 (7)	0.5 (9)	0.5 (3)	0.026
0.09 (4)	0.91 (8)	0.2 (6)	0.8 (6)	0.020
Others				0.042

Table 8.45: IB on L_4 with unevenly spaced demands

Modal strategy				Proportion
1	2	3	4	
0.45	0.55	0.55	0.45	0.220
0.4	0.6	0.6	0.4	0.167
0.4	0.6	0.55	0.45	0.165
0.35	0.65	0.55	0.45	0.031
0.45	0.55	0.6	0.4	0.067
0.35	0.65	0.6	0.4	0.040
0.4	0.55	0.55	0.45	0.027
0.4	0.6	0.55	0.4	0.021
Others				0.262

Table 8.46: PI on L_4

Modal strategy				Proportion
1	2	3	4	
0.5	0.5	0.5	0.5	0.799
0.45	0.55	0.45	0.55	0.053
0.45	0.55	0.5	0.5	0.032
0.55	0.45	0.55	0.45	0.015
Others				0.028

Table 8.47: SBR on L_4 with sample size 2

Summary and Discussion

Both imitative rules provide support for the experimental observation of the central players receiving an advantage. The similarity of these results to the outside option results of section 8.4 is discussed in section 9.2.4 of the conclusion. The imitate better rule provides stronger support for the experimental observation, but again fails to be robust to choice of D, as in the 2 player case. Simulations for the other learning rules, which are not recorded here, showed they did not suffer from these robustness problems. The sampled best reply rule supported the demand profile of $(0.5, 0.5, 0.5, 0.5)$.

The results for proportional imitation show only a small advantage for players 2 and 3. In the most commonly modal demand profile they receive a payoff only one ordinal position higher than 0.5. This suggests the possibility that for finer demand sets the advantage is still only that of one ordinal position, and so tends to zero as the demand set size becomes larger. Simulations with finer demand sets show this to be incorrect. These results are not included as for such demand sets a very large number of demand profiles are modal for a significant number of rounds so the results are hard to display using the reporting methods of this chapter.

8.8 A 4 Player Ring Network

This section presents a 4 player ring network which can support a demand profile in which all pairs of demands of neighbouring players lie on the Pareto boundary of the corresponding cakes. This is similar to a von Neumann-Morgenstern triple. The cakes of this network are defined by the boundary functions $f^{1,2} = f^{2,3} = f^{3,4} = f^{4,1} = \frac{1}{4}(7-10x)$ (see figure 8.3). The outcome mentioned is $(0.5, 0.5, 0.5, 0.5)$. Note that this does not coincide with the Nash bargaining solution in any single utility cake. The Nash bargaining solution in the cake $\mathcal{K}^{12}$ is $(\frac{7}{8}, \frac{7}{20})$. In all simulations of this section, all demand sets are $\{0, 0.05, 1, \ldots, 1.75\}$.

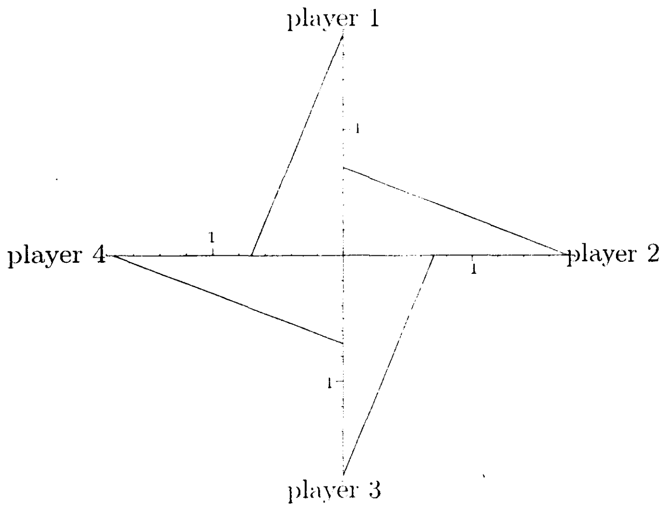

Figure 8.3: The network under investigation in section 8.8

Results

Table 8.48 gives the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is $\frac{1}{7}$: the reciprocal of the maximum demand. A relatively low number of rounds were played in the simulation of table 8.51 since the corresponding value of γ was so low. Tables 8.49 – 8.51 contain the data. In all the demand profiles of these tables, all exchanges are feasible.

Table	8.49	8.50	8.51
Learning rule	IB	PI	SBR, $m = 2$
M	35	40	20
ϵ	0.23	0.035	0.28
ρ	2.8×10^2	1.1×10^3	3.6×10^2
χ	0.55	0.60	0.58
γ	2.7×10^3	1.5×10^3	85
Rounds played	2×10^6	2×10^6	2.5×10^5
Minimum proportion displayed	0.005	0.01	0.001

Table 8.48: Guide to the simulations of section 8.8

Modal strategy				Proportion
1	2	3	4	
0.5	0.5	0.5	0.5	0.897
0.5	0.5	0.5	0.45	0.011
0.5	0.45	0.5	0.5	0.008
0.5	0.5	0.45	0.5	0.008
0.45	0.5	0.5	0.5	0.006
Others				0.070

Table 8.49: IB on a 4 player ring network

Modal strategy				Proportion
1	2	3	4	
0.5	0.5	0.5	0.5	0.693
0.5	0.5	0.5	0.45	0.057
0.5	0.5	0.45	0.5	0.055
0.45	0.5	0.5	0.5	0.051
0.5	0.45	0.5	0.5	0.043
Others				0.101

Table 8.50: PI on a 4 player ring network

Modal strategy				Proportion
1	2	3	4	
0.5	0.5	0.5	0.5	0.979
0.5	0.5	0.5	0.4	0.001
0.4	0.5	0.5	0.5	0.001
0.5	0.5	0.45	0.5	0.001
0.5	0.5	0.5	0.45	0.001
Others				0.017

Table 8.51: SBR on a 4 player ring network with sample size 2

Summary and Discussion

All the simulations of this section provide strong support for the von Neumann-Morgenstern like demand profile (0.5, 0.5, 0.5, 0.5). However, as for results of the 3 player ring network, I do not have a candidate mechanism to explain this outcome. For this case, it might be possible to use minimal tree analysis, as not all outcomes of the network involve a player being excluded from exchange.

8.9 A Second 4 Player Ring Network

This section presents a 4 player ring network which can support an outcome outside the core. The cakes of this network are defined by the boundary functions $f^{1,2} = f^{3,4} = \frac{3}{2}(1-x)$, $f^{2,3} = f^{4,1} = \frac{1}{2}(3-4x)$ (see figure 8.4). In all simulations of this chapter all demand sets are $\{0, 0.1, 0.2, \ldots, 1.5\}$.

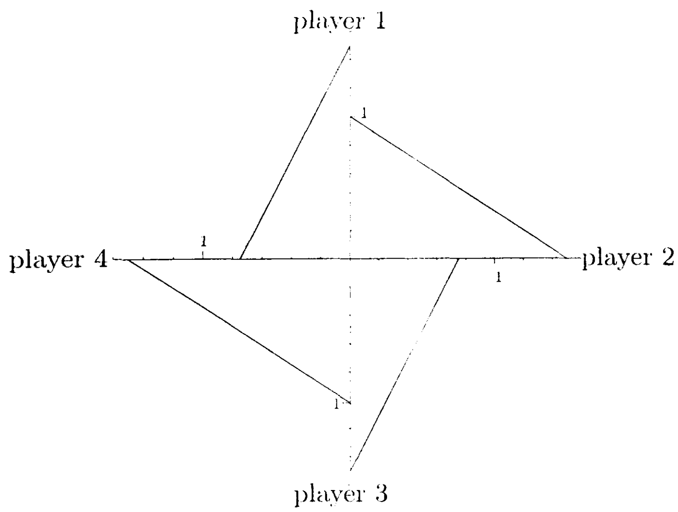

Figure 8.4: The network under investigation in section 8.9

Results

Table 8.52 gives the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is $\frac{2}{3}$: the reciprocal of the maximum demand. A relatively low number of rounds were played in the simulation of table 8.55 since the corresponding value of γ was so low. Note that in the same simulation, although the value of ρ was very low, in practice most changes of modal demand only lasted for a short number of rounds before returning to the most commonly modal demand profile. Tables 8.53 – 8.55 contain the data.

Table	8.53	8.54	8.55
Learning rule	IB	PI	SBR, $m = 2$
M	40	40	40
ϵ	0.06	0.04	0.075
ρ	3.1×10^2	6.2×10^2	25
χ	0.68	0.62	0.69
γ	2.8×10^3	2.7×10^3	2.4×10^2
Rounds played	2×10^6	5×10^6	5×10^5
Minimum proportion displayed	0.05	0.02	0.01

Table 8.52: Guide to the simulations of section 8.9

Modal strategy				Feasible exchanges	Proportion
1	2	3	4		
0.6	0.6	0.6	0.6	12,34	0.502
0.6	0.6	0.55	0.65	12,34	0.073
0.55	0.65	0.6	0.6	12,34	0.072
0.6	0.6	0.5	0.75	12,34	0.068
Others					0.285

Table 8.53: IB on a 4 player ring network

Modal strategy				Feasible exchanges	Proportion
1	2	3	4		
0.5	0.75	0.5	0.75	12.34	0.260
0.6	0.4	0.7	0.45	12,23.34.41	0.112
0.6	0.45	0.6	0.45	12,23.34.41	0.084
0.7	0.45	0.6	0.4	12,23.34.41	0.067
0.5	0.75	0.4	0.9	12.34	0.052
0.6	0.5	0.5	0.45	12,23.34,41	0.025
0.45	0.8	0.5	0.75	12.34	0.021
Others					0.379

Table 8.54: PI on a 4 player ring network

Modal strategy				Feasible exchanges	Proportion
1	2	3	4		
0.6	0.4	0.7	0.4	12,23.34,41	0.419
0.7	0.4	0.7	0.4	12,23.34,41	0.251
0.6	0.4	0.7	0.45	12.23.34.41	0.197
0.7	0.45	0.6	0.4	12,23.34.41	0.070
0.6	0.45	0.6	0.45	12.23.34,41	0.034
Others					0.029

Table 8.55: SBR on a 4 player ring network with sample size 2

Summary and Discussion

Table 8.56 contains all strategy profiles which are modal for at least 0.1 of all rounds in any of the simulation of this section. The demand of player i is given in the column headed σ_i. The four columns on the right illustrate whether these demand profiles lie in the core. A non-core profile satisfies $f^{i,i+1}(\sigma_i) < \sigma_{i+1}$ or $f^{4,1}(\sigma_4) < \sigma_1$.

σ_1	σ_2	σ_3	σ_4	$f^{4,1}(\sigma_4)$	$f^{1,2}(\sigma_1)$	$f^{2,3}(\sigma_2)$	$f^{3,4}(\sigma_3)$
0.6	0.6	0.6	0.6	0.3	0.6	0.3	0.6
0.5	0.75	0.5	0.75	0	0.75	0	0.75
0.6	0.4	0.7	0.45	0.6	0.6	0.7	0.45
0.6	0.4	0.7	0.4	0.7	0.6	0.7	0.45
0.7	0.4	0.7	0.4	0.7	0.45	0.7	0.45
0.6	0.4	0.7	0.45	0.6	0.6	0.7	0.45

Table 8.56: Illustration of whether most common demand profiles lie in the core

This demonstrates that sampled best reply with sample size 2 and proportional imitation offer some support for demand profiles corresponding to non-core solutions to this network. However imitate better selects a demand profile which corresponds to a core solution. Proportional imitation also offer some support to a different such demand profile.

An explanation for non-core profiles receiving significant weight in these models is the matching rule. For example suppose demand profile σ is played, all exchanges are feasible and $f^{2,3}(\sigma_2) < \sigma_3$ but $f^{2,1}(\sigma_2) = \sigma_1$. If player 2 raises his demand slightly then there are two maximal consistent outcomes, with corresponding sets of exchanges $\{34\}$ and $\{23, 14\}$. Player 2 gains in the second case, but this is offset by the possibility of exclusion from exchange in the first case, so that player 2 does not wish to change his demand.

The significance of support for non-core solutions and this explanation for their occurrence is discussed further in section 9.1.3.

8.10 The 5 Player Line Network with Unit Cakes

This section is on the network L_5. Several different predictions for the outcome of this network have been made in the course of this thesis. Theorem 7.3 proves that the outcome of the evolutionary model under imitate better or proportional imitation in the limit $\epsilon \to 0$ and under a few other assumptions is that players 2 and 4 receive the maximum possible payoff. One motivation for simulations on this network is to determine how well this prediction holds for relatively large values of ϵ. On the other hand, the models based on the alternating offers game of chapters 4 and 5 allow a wide range of solutions. An interesting question is whether the evolutionary model selects any of these solutions and how robust this selection is to the values of M and ϵ. In all simulations of this section, each player has demand set[7] $\{0, 0.05, 0.1, \ldots, 1\}$.

Results

Tables 8.57 and 8.58 give the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. Tables 8.59 – 8.64 contain the data.

Summary and Discussion

The prediction of theorem 7.3 of a demand profile of $(0.05, 0.95, 0.05, 0.95, 0.05)$ does not hold in general. The only simulation for which it does hold uses imitate better, and even under this learning rule, the result is not robust to variations in ϵ and M.

Imitate better is the only learning rule to strongly select a single demand profile. For proportional imitation in particular, a large number of profiles are modal for

[7]Note that the inclusion of the demand 1 means the conditions of theorem 7.3 do not hold. However, other simulations without this demand produce very similar results.

Table	8.59	8.60	8.61
Learning rule	IB	IB	PI
M	15	40	15
ϵ	0.09	0.18	0.04
ρ	3.1×10^3	6.1×10^2	2.4×10^2
χ	0.85	0.72	0.70
γ	4.1×10^3	4.2×10^3	3.6×10^3
Rounds played	2×10^6	2×10^6	2×10^6
Minimum proportion displayed	0.01	0.01	0.02

Table 8.57: Guide to the simulations of section 8.10 (1)

Table	8.62	8.63	8.64
Learning rule	PI	SBR, $m = 1$	SBR, $m = 2$
M	75	40	20
ϵ	0.09	0.002	0.0025
ρ	1.8×10^2	1.5×10^2	6.7×10^2
χ	0.41	0.97	0.84
γ	5.4×10^2	7.3×10^3	2.8×10^3
Rounds played	2×10^6	2×10^6	2×10^6
Minimum proportion displayed	0.02	0.05	0.02

Table 8.58: Guide to the simulations of section 8.10 (2)

Modal strategy					Proportion
1	2	3	4	5	
0.05	0.95	0.05	0.95	0.05	0.929
0.15	0.85	0.15	0.9	0.1	0.022
0.05	0.95	0.1	0.9	0.1	0.020
Others					0.029

Table 8.59: IB on L_5 (1)

Modal strategy					Proportion
1	2	3	4	5	
0.25	0.75	0.25	0.75	0.25	0.661
0.2	0.8	0.2	0.8	0.2	0.208
0.25	0.75	0.25	0.8	0.2	0.033
0.2	0.8	0.25	0.75	0.25	0.029
0.25	0.75	0.25	0.85	0.15	0.010
Others					0.059

Table 8.60: IB for L_5 (2)

Modal strategy					Proportion
1	2	3	4	5	
0.35	0.65	0.35	0.65	0.35	0.250
0.35	0.65	0.35	0.6	0.4	0.067
0.35	0.65	0.3	0.7	0.3	0.060
0.4	0.6	0.35	0.65	0.35	0.057
0.35	0.65	0.35	0.65	0.3	0.040
0.3	0.65	0.35	0.65	0.35	0.032
0.4	0.6	0.4	0.6	0.4	0.032
0.35	0.65	0.35	0.7	0.3	0.028
0.35	0.65	0.3	0.65	0.35	0.025
0.3	0.7	0.3	0.65	0.35	0.021
Others					0.388

Table 8.61: PI on L_5 (1)

Modal strategy					Proportion
1	2	3	4	5	
0.4	0.6	0.4	0.6	0.4	0.115
0.45	0.55	0.45	0.55	0.45	0.106
0.4	0.55	0.45	0.55	0.45	0.069
0.45	0.55	0.45	0.55	0.4	0.066
0.4	0.55	0.45	0.55	0.4	0.037
0.35	0.6	0.4	0.6	0.4	0.036
0.45	0.55	0.4	0.6	0.4	0.033
0.4	0.6	0.4	0.6	0.35	0.033
0.4	0.55	0.4	0.6	0.4	0.028
Others					0.477

Table 8.62: PI on L_5 (2)

Modal strategy					Proportion
1	2	3	4	5	
0.15	0.85	0.15	0.85	0.15	0.212
0.2	0.8	0.2	0.8	0.2	0.100
0.2	0.8	0.15	0.85	0.15	0.085
0.15	0.85	0.15	0.9	0.1	0.057
0.1	0.9	0.1	0.9	0.1	0.054
0.2	0.8	0.2	0.75	0.25	0.051
0.25	0.75	0.25	0.75	0.25	0.050
Others					0.391

Table 8.63: SBR on L_5 with sample size 1

Modal strategy					
1	2	3	4	5	Proportion
0.3	0.7	0.3	0.7	0.3	0.495
0.35	0.65	0.35	0.65	0.35	0.213
0.3	0.7	0.35	0.65	0.35	0.043
0.35	0.65	0.35	0.7	0.3	0.043
0.35	0.65	0.35	0.6	0.4	0.039
0.35	0.65	0.3	0.7	0.3	0.028
0.35	0.65	0.4	0.6	0.4	0.025
0.3	0.7	0.3	0.65	0.35	0.023
0.35	0.65	0.35	0.75	0.25	0.021
Others					0.070

Table 8.64: SBR on L_5 with sample size 2

a significant proportion of rounds. Most of the demand profiles selected are of the form $(1-x, x, 1-x, x, 1-x)$ with $x > \frac{1}{2}$, and the others differ only slightly. The value of x is variable, depending on the choice of ϵ and M. However, for each learning rule it is generally lower than the payoff player 2 receives in the simulations on L_3. Note that again the sampled best reply rule with sample size one does have some predictive power, in contrast to the 2 player case.

8.11 The 7 Player Line Network with Unit Cakes

This section is on the network L_7. In contrast to L_3 and L_5, theorem 7.3 does not apply to here. A motivation for study is to investigate whether the patterns observed for L_3 and L_5 hold or break down. More generally, this is an opportunity to investigate whether any qualitative differences are revealed as network size increases. In all simulations of this section, each player has demand set $\{0, 0.05, 0.1, \ldots, 1\}$.

Results

Table 8.65 gives the details of the simulations of this section. Recall that m is the sample size of a learning rule. Note that the factor of proportionality for the proportional imitation learning rule is 1. Tables 8.66 – 8.68 contain the data.

Table	8.66	8.67	8.68
Learning rule	IB	PI	SBR, $m = 2$
M	15	15	20
ϵ	0.09	0.04	0.025
ρ	4.6×10^2	1.5×10^2	3.2×10^2
χ	0.88	0.73	0.98
γ	2.6×10^3	2.4×10^3	4.3×10^2
Rounds played	2×10^6	2×10^6	10^7
Minimum proportion displayed	0.01	0.015	0.01

Table 8.65: Guide to the simulations of section 8.11

Summary and Discussion

The results of these simulations follow several patterns of L_3 and L_5. In the most commonly modal demand profiles, even numbered players make demands above 0.5 and odd number players make demands below 0.5. The typical demand difference between even and odd players is smaller than for L_3 and L_5. Imitate better yields the highest payoff difference. However, unlike L_3 and L_5, I could not find a choice of parameters M and ϵ which selected a demand profile in which even players played the demand 0.95. Demand profiles are less strongly selected than for L_3 and L_5, and indeed most other networks investigated in this chapter, in the sense that more profiles are modal for a significant proportion of rounds. This was especially pronounced for proportional imitation where no demand profile is modal for more than

Modal strategy							
1	2	3	4	5	6	7	Proportion
0.1	0.9	0.15	0.85	0.15	0.9	0.1	0.560
0.3	0.7	0.3	0.7	0.3	0.8	0.2	0.097
0.1	0.9	0.1	0.9	0.1	0.9	0.1	0.089
0.1	0.9	0.2	0.8	0.2	0.8	0.2	0.081
0.1	0.9	0.2	0.8	0.2	0.9	0.1	0.051
0.15	0.85	0.15	0.85	0.15	0.9	0.1	0.025
0.1	0.9	0.15	0.85	0.1	0.9	0.1	0.020
0.1	0.9	0.1	0.85	0.15	0.9	0.1	0.014
0.35	0.65	0.35	0.65	0.35	0.8	0.2	0.012
Others							0.051

Table 8.66: IB on L_7

0.1 of all rounds!

In any maximal consistent subgraph of L_7, as defined in definition 7.2, players 2 and 6 are included in an edge. However there are subgraphs in which player 4 is not. This suggests that player 4 is in a weaker bargaining position than players 2 and 6 and must make more concessions to his neighbours. Nonetheless, the simulations show that the even numbered players usually make equal demands. Only under imitate better is it apparent that there is a slight tendency for player 4 to demand less than the others[8].

Finally, note that the value of ρ tends to decrease as network size increase. In particular, note that for each simulation on L_7 in this section there is a simulation

[8]Given the fact that under proportional imitation a very large number of demand profiles are modal for a significant proportion of rounds, it is not possible to rule out the existence of a slight tendency for player 4 to demand less. This suggests that additional methods of reporting should be used.

Modal strategy							Proportion
1	2	3	4	5	6	7	
0.4	0.6	0.4	0.6	0.4	0.6	0.4	0.064
0.35	0.65	0.35	0.65	0.35	0.65	0.35	0.035
0.4	0.6	0.4	0.6	0.35	0.65	0.35	0.032
0.35	0.65	0.35	0.6	0.4	0.6	0.4	0.029
0.4	0.6	0.4	0.6	0.4	0.6	0.35	0.021
0.4	0.6	0.4	0.55	0.45	0.65	0.35	0.016
0.3	0.7	0.3	0.65	0.35	0.65	0.35	0.016
0.4	0.6	0.4	0.55	0.4	0.6	0.4	0.015
0.3	0.7	0.4	0.6	0.4	0.6	0.4	0.015
0.35	0.65	0.35	0.6	0.35	0.65	0.35	0.015
0.4	0.6	0.4	0.6	0.4	0.65	0.35	0.015
0.35	0.65	0.4	0.6	0.4	0.6	0.4	0.015
Others							0.611

Table 8.67: PI on L_7

Modal strategy							
1	2	3	4	5	6	7	Proportion
0.4	0.6	0.4	0.6	0.4	0.6	0.4	0.517
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.155
0.5	0.5	0.5	0.5	0.5	0.6	0.4	0.103
0.4	0.6	0.4	0.6	0.4	0.7	0.3	0.051
0.4	0.6	0.4	0.5	0.4	0.6	0.4	0.019
0.3	0.7	0.4	0.6	0.4	0.6	0.4	0.015
0.6	0.4	0.5	0.5	0.5	0.5	0.5	0.015
0.6	0.4	0.6	0.4	0.6	0.6	0.4	0.010
Others							0.115

Table 8.68: SBR on L_7 with sample size 2

on L_5 in section 8.11 using the same learning rule and values of M and ϵ. For each simulation in this section, the corresponding simulation in section 8.11 has a smaller value of ρ. This suggests that a lower value of ϵ may be required for larger networks. On the other hand, a possible interpretation is that in a larger network, there is roughly the same possibility of a transit occurring in each realised exchange and more transits take place simply because there are more realised exchanges. There may be networks where transits occurring in any exchange rarely affect the whole state of the model and thus a lower value of ρ may not indicate faster convergence to the stationary distribution of the model. So lower values of ρ in larger networks do not necessarily indicate that ϵ should be reduced.

Chapter 9

Summary and Conclusions

Section 9.1 discusses the suitability of the various models discussed in this thesis to the task of modelling bargaining in bilateral exchange networks. Section 9.2 summarizes the solutions of these models which have been found for various networks. These results are also compared with each other and the experimental data and theoretical predictions of the sociology literature. The final section, 9.3, considers possible extensions to this research. Recall that the notation used to refer to networks, such as L_n, is defined in section 3.1.3.

9.1 The Suitability of the Proposed Models

Section 9.1.1 collects the various desirable properties of a bargaining model which have been mentioned throughout the thesis. Sections 9.1.2 and 9.1.3 discuss how well the proposed models based on the alternating offers and Nash demand games respectively meet these properties.

9.1.1 Desiderata

Consistency with Existing Solutions and Experimental Evidence

The bargaining models in this thesis are extensions of models for bargaining between 2 players. Hence for the special case of a 2 player network, the solutions of these new models should be consistent with at least some parts of the existing analysis in the literature. The same is true for other special cases for which analyses already exist e.g. 3 player ring networks.

The results should also be reasonably consistent with experimental results under appropriate conditions. Sometimes differences between the assumptions made in a model and the design of an experiment may mean that they capture different situations. So direct comparison with experimental results under inappropriate conditions is not useful.

Robustness

The details of a bargaining structure will be imprecisely known to an investigator. Also, they are subject to exogenous perturbations. For example there may be variations in the quality of exchange items or in the preferences of bargainers. For these reasons, bargaining solution concepts should be reasonably robust to small changes in the details of networks and other information on the bargaining situation, otherwise they have little predictive power. An important example of this requirement is used in section 4.4.4 where it is argued that adding empty exchange opportunities should not affect the outcome of a 2 player bargaining situation. After all, a bargainer is always likely to have all sorts of unprofitable alternatives to engaging in bargaining, and it should be possible to neglect the details of these.

Low Computational Cost

This point applies to computer based models. The computational costs involved in using these models should be practical. This is of particular importance in constructing models that can be extended to large networks which is one of the aims mentioned in chapter 1.

Selection of Solutions

Simple bargaining models often support an extremely wide array of solutions. Examples are the Nash demand game and the alternating offers game without time preferences[1]. It is crucial that a model selects the 'interesting' solutions from these. This is of course a subjective decision, but it can also be partly guided by experimental data. For example, the experiments of the sociology literature described in section 2.7 often concentrate on a narrow range of possible outcomes[2]. This strongly suggests that some selection beyond that mentioned for the simple bargaining models is appropriate. Experimental data also exists on two player bargaining which supports similar conclusions for these cases (e.g. Binmore et al [6] and [10]).

In section 3.3.5 the argument is made that evolutionary methods are not currently available for extensions of the alternating offers game and there is no firm intuitive basis to pick an equilibrium refinement. In the absence of such methods of equilibrium selection, it is desirable for bargaining games to have a unique SPE (or at least a set with little variation in outcome). Multiple solutions can then be represented in a bargaining model by exogenising part of the structure which selects between outcomes, as in the market bargaining game of section 4.4.

On the other hand, evolutionary simulations, such as those of chapter 8, entail

[1]For brief discussions of these models see the introduction to chapter 6 and section 4.2.1 respectively.

[2]It is possible to dispute whether this evidence is sufficient. The outcomes may recur simply because the subjects have all been picked from a society which enforces one bargaining convention of many possibilities.

a degree of vagueness in their results due to their stochastic nature and may often place weight on several outcomes. Here a judgement must be made on whether the weight is spread across too many solutions for the results to be useful. For example, it is straightforward that this is the case for the results of table 8.13 which place approximately equal weight on all Pareto optimal outcomes.

Instantly Adaptive Exchange

It seems intuitively likely that in a bargaining situation of sufficient complexity players sometimes form an exchange without realising that an exchange in a distant part of the network formed very recently. As discussed in section 5.3.1, perfect information models of bargaining seem poorly adapted to capture this possibility. Instead whenever one exchange forms, the remaining players typically are able to instantly adapt their behaviours to take account of the reduced network of bargaining opportunities. This property is referred to as instantly adaptive exchange and is discussed in more depth in section 5.3.1. It is desirable that a bargaining model should allow the possibility of exchange which avoids this property.

Realism, Tractability and Concision

These three properties are obviously desirable. They are grouped together since there are trade-offs between achieving them in a bargaining model. Literal realism is often sacrificed to tractability and concision in constructing any mathematical model. Examples of features whose literal realism is doubtful in this thesis include the use of multilateral demands, as discussed in section 4.4.5, and insisting on perfect information. It is more important that the model should capture realistic behaviour rather than include all its details. It is this interpretation of 'realism' that is a desirable feature of a bargaining model. Some examples are that players should be treated reasonably symmetrically except for the differences due to network position and that the level of commitment available to players should be judged correctly.

A particularly important instance of realism which should not be sacrificed is that solutions should not depend on unrealistic limitations that a model places on players' actions. This is because players would not voluntarily submit to bargaining conventions which any of them have a unilateral incentive to break[3]. This condition is based on the argument of Binmore in [3] mentioned in section 4.4.1.

9.1.2 Models Based on the Alternating Offers Game

Chapters 4 and 5 discuss various extensions to the alternating offers game. For the purpose of tractability, all of them retain the feature of perfect information, with the exception of the model of Corominas-Bosch described in section 4.5. As discussed in section 3.3.4, the natural solution concept for perfect information models is SPE. When such models are applied to networks of more than 3 players, the solutions suffer from the limitations associated with instantly adaptive exchange described in section 5.3.1. This problem is put aside as it may often not apply to small networks, and the proposed models are considered in terms of the remainder of the desiderata. Most important is the consideration that for these models to have much predictive power a unique SPE is required (or at least a set of SPE payoffs with little variation).

Section 4.3 introduces the telephoning model of Binmore [3]. Section 4.4.5 uses an argument of Binmore [3] which states that players would wish to unilaterally break the bilateral bargaining convention of this model and that it therefore does not describe the main case of bargaining. This argument also applies to the bilateral bargaining models of Calvó-Armgenol [16, 17, 18] discussed in section 4.5; indeed the papers proposing these models make explicitly assume a setting in which only bilateral bargaining is possible[4].

Section 4.4 discusses the market bargaining game of Binmore [3], which is a

[3]This could be interpreted as an informal evolutionary stability criterion on the bargaining rules.

[4]Another reason for not pursuing the bilateral bargaining approach is that these models already deal with general bilateral exchange networks, although they have the limitation that bargaining stops after the first exchange.

model of bargaining for 3 player ring networks. It is argued that public delays do not allow the model to provide consistency with the 2 player solution, and a modification involving personal delays is proposed. Otherwise, the model meets the desiderata quite well and is the basis of the novel models in chapter 5.

Some other bargaining models from the literature are discussed in section 4.5. The unilateral demand exogenous order models of Binmore [3] allow non-bilateral bargaining, but it is argued that to also treat the players symmetrically, and thus provide consistency with the 2 player case, requires a very complicated order of play. This seems both unrealistic and difficult to generalise. The model of Corominas-Bosch [23] is not pursued mainly because it seems difficult to generalise to non-bipartite networks. Also it supports multiple SPEs for the network 5 player line network L_5.

Chapter 5 proposes two novel extensions of the market bargaining game which can be used on any bilateral exchange network. The first, the exogenous ordering model, is a straightforward extension of the market bargaining game. It requires an exogenously specified ordering over the players which determines the order of play. Lemmas 5.1 and 5.2 illustrate that for the case of the line network L_5, different choices of ordering can permit widely differing SPE outcomes. This does not give the model much predictive power. An interpretation is that too much structure has been exogenised including features crucial to selecting the solution. Furthermore, it appears that the model is not easily tractable as there are a large number of possible choices of exogenous ordering[5] and there is no obvious general method of solution.

The second model of chapter 5, the endogenous ordering model, attempts to resolve these difficulties by endogenising the order of play. As discussed in section 5.2.1, this appears to necessitate the introduction of rules which approach the acceptable boundaries of realism and concision. Also, in section 5.2.4 it is demonstrated that

[5]Indeed, there are some for which I could not solve the resulting bargaining game. See footnote 5 of chapter 5.

this model can support multiple SPE outcomes for L_5 with large qualitative differences. Furthermore, the proof of this theorem is lengthy, indicating that tractability of this model for more general networks is problematic.

To conclude, in addition to the problems of instantly adaptive exchange, it appears that the approach of extending the alternating offers game supports too wide a range of solutions and produces models which are not easily tractable. These problems are likely to increase for larger networks.

9.1.3 Evolutionary Models based on the Nash Demand Game

Chapter 6 introduces a bargaining model for general bilateral exchange networks based on the Nash demand game in which strategies are simply demand values. The simple strategy space of this game means that it is easy to use it as the basis of an evolutionary model which can be implemented as a computer based simulation. The following is a brief recap of the evolutionary model defined in chapter 6. A population of M agents is associated with each network position. Each agent is given an initial demand value. In each round of play, an agent from each population is selected at random. These agents then play the bargaining game and change their demands using simple learning rules based on their payoffs and some other information about the state of the model. Which extra information is used is specific to the learning rule. There is also a small probability $\epsilon > 0$ that agents mutate to a random demand rather than use their learning rule. In order that this model may be implemented as a computer simulation, the strategy space of the underlying game is discretised by nominating a finite demand set for each player in the network under investigation.

Section 7.1 proposes 3 simple candidate learning rules for use in the model: imitate better, proportional imitation and sampled best reply. Imitate better involves updating agents sampling another agent from the same population. If the most recent payoff of the sampled agent is higher than that of the updating agent then the updating agent switches to the demand of the sampled agent. Under proportional imitation the probability of this switch is proportional to the payoff difference. Variations on these learning rules with larger sample sizes are also proposed. Sampled best reply with sample size m involves the updating agent sampling m agents from each other population. A mixed strategy for each population is constructed from the frequencies with which demands occur in the corresponding sample. The updating agent switches to a demand which is a best reply to these mixed strategies. For each of these learning rules, in particular sampled best reply, large sample sizes increase the computational cost.

Standard results of Markov chain theory are used to show that under a wide class of learning rules there is a unique stationary distribution over the possible states of the model, and that this corresponds to the expected state of the model in the long term, independently of the initial state. Furthermore, it is argued in section 7.1 that for the 3 candidate learning rules most of the weight of this stationary distribution is placed close to B sets. These are sets of states in which some populations are demand-homogeneous and the other populations correspond to players who receive payoff zero from any demand profile supported by the set of states. If the stationary distribution places most of its weight on a few B sets then the model effectively selects a bargaining solution. As ϵ increases, this solution becomes more driven by mutation until it has little connection to the bargaining situation. However, it is argued in section 7.3.1 that for small ϵ, the qualitative features of the solution are likely to be relatively robust to variations in ϵ.

The stationary distribution can be investigated by simulation. As discussed in section 7.4, it is sufficient for the simulation to keep track of which demand profile is closest to the state of the model in each round. Chapter 8 presents data from simulations by listing the proportion of rounds that the most common demand profiles are modal.

This evolutionary model has several unrealistic features. One, discussed in section 7.2.4, is that it seems unnatural to have a population of agents associated with each player position, as the main application is to social networks. However, the Markov chain results just mentioned show that this setting may often permit selection of solutions by simulation. In terms of the desiderata, there is a trade-off between tractability and realism.

The results of section 8.2 show that the results of the imitate better learning rule are not robust to the choice of demand sets used in the game. As discussed in section 7.1.1 this is not necessarily a reason to dismiss it. However it is not a convenient feature for simulation. Considerable effort must be put into showing that any solution it supports is robust to various 'reasonable' choices of demand sets. Some possible choices are demand sets which are evenly spaced, or demand sets such that the Pareto optimal outcomes of the game lie evenly spaced along the Pareto boundary. Also, selecting demand sets satisfying certain 'reasonable' properties may become a difficult task for larger networks in which the demand sets of one player can be used in several cakes.

Simulations detailed in section 8.2 show that the other candidate learning rules are reasonably robust to the choice of demand sets. In particular, this includes imitate better with a sufficiently large sample size. However this learning rule becomes too computationally costly to use for larger networks.

The sampled best reply updating rule with sample size 1 is rejected due to its behaviour for 2 player networks. It does not select between strict Nash equilibria of the underlying game, instead placing roughly equal weight on each. This fails the 'selection of solutions' item of the desiderata especially as these results do not match those of experimental studies of 2 player bargaining (an example is Binmore et al [6]). In these experiments bargaining outcomes in which one player takes almost all of the available cake are almost never observed.

For 2 and 3 player networks, the learning rules proportional imitation and sampled best reply with sample size 2 usually match some theoretical and experimental solutions of the corresponding situations. An exception is that sampled best reply with sample size 2 produces an unusual outcome for a 2 player network with an outside option; table 8.28 shows that an outside option can sometimes reduce a player's average payoff. Proportional imitation has the particularly attractive feature of directly describing the process by which agents update their demands based on the outcomes of the underlying game. In contrast, sampled best response requires an undescribed mechanism to generate the best response.

Many of the simulations detailed in chapter 8 with the learning rules just mentioned do concentrate on solutions which are generally in accord with the desiderata. However two problems emerge. As network size increases, the proportions of rounds in which demand profiles are modal are typically spread much more evenly across demand profiles. This is especially pronounced for the proportional imitation learning rule; in table 8.67 no demand profile is modal for more than 0.07 of all rounds! It is possible that the weight given by these proportions is still concentrated close to a small number of demand profiles but spread thinly between many profiles nearby to these. To investigate this possibility it would be necessary to use other methods of interpreting the data. One simple method would be to find for each population the proportion of rounds for which each demand is modal.

The second problem is much more serious. The simulations of section 8.9 are on a particular four player ring network. For proportional imitation and sampled best reply with sample size 2, these simulations spend a significant proportion of rounds with non-core modal demand profiles. In this situation two neighbouring players have a feasible exchange with each other which would improve both their payoffs, but they cannot unilaterally raise their demands to take advantage of it because this causes a risk of exclusion. I interpret this as revealing that the underlying game places unrealistic limitations on strategies and allows artificial solutions to be supported by the evolutionary model. Either of the neighbouring players mentioned would like to be able to make an offer to the other. It seems reasonable that a player can undertake this option while bargaining as usual with their other neighbour[6]. Such an option would allow the non-core outcome to be easily destabilized.

Altering the underlying game to allow such options does not seem a straightforward task. For example, section 7.2.1 discusses the limitations of using directed demands. Also, it would be hard to prove when enough options had been added to prevent artificial solutions being supported. Note that increasing the size of the set of strategies and the complexity of the underlying game is likely to increase the computational cost of simulation.

Another possible resolution of this problem is to adjust the probabilities of the outcomes of the matching rule as discussed in section 7.2.1. This might destabilize the particular non-core solution described. However it does not appear obvious that this prevents the problem occurring for other networks.

In conclusion, the simulation performs well for many small networks. However, simulations reveal that the extension of the Nash demand game used as an underlying game does not allow players options they would realistically use in bargaining and the simulation thus sometimes supports unrealistic solutions. In addition, the precision of the simulation results may be decreasing with network size, in the sense that the most commonly modal demand profiles are modal for a smaller proportion of rounds. which would be problematic for investigating large networks. There is certainly scope for attemping to resolve both these problems by altering the evolutionary model and the method of reporting results.

9.2 A Comparison of all Results and Predictions

This section summarizes and compares the results and predictions about bargaining outcomes contained in this thesis. These comprise theoretical solutions of bargain-

[6]As discussed in section 4.1 in bargaining situations in which making such an offer does interfere with other bargaining opportunities, then there may well be bargaining situations in which non-core solutions are reasonable.

ing models, data from the experiments summarized in section 2.7, data from the simulations of chapter 8, and the predictions of the sociology literature of section 2.5. Section 9.2.1 discusses the extent to which the experimental and simulation results are comparable. Sections 9.2.2 – 9.2.6 each compare the results found for one particular network. Section 9.2.7 collates other miscellaneous results. Section 9.2.8 discusses relations between the results of this thesis and the theoretical predictions of the sociology literature.

9.2.1 Time Scales and Comparability

A general issue is whether the experimental and simulation data investigate comparable time-scales. After all, the experiments contain a maximum of 60 rounds, whereas the simulations are run for at least 10^5 rounds.

This raises the possibly that the experiments capture outcomes which would not be stable over a number of rounds representing the typical timescale in which social exchange takes place. Indeed, most of the sociological experiments do not directly investigate whether their solutions vary over time. On the other hand, there is also the possibility that the simulation data investigates too long a timescale. Over the long run, the network may change as players find new exchange opportunities and the values of exchanges alter. So behaviour in a constant network over too long a timescale may be irrelevant.

There are several reasons why the timescales which the results of simulation and experiment represent may not be as far apart as is it appears simply from the number of rounds. The fact that only one agent per population updates their individual state in each round of the simulation means that a round corresponds to less time than in the experimental setting. As discussed in section 6.1.1, there are many variations to evolutionary models of the sort discussed in chapter 6 which increase the speed at which the stationary distribution becomes relevant and furthermore often increase the realism of the model. Also, the learning rules used in the simulation may be far more simplistic than the behaviour of actual bargainers. It is possible that more sophisticated learning rules, perhaps especially adapted for use in bargaining situations, could guide participants into reaching stable outcomes more quickly.

Even if the long term nature of the simulation results means that they cannot be achieved in a timescale appropriate for application to social exchange, they may still be of some use in indicating the evolutionary pressures that exist. For example, the simulation results for the 3 player ring network in section 8.6 are very strong and suggest that one particular solution is selected even in the short run. On the other hand, the simulation results for the two player unit cake network in section 8.2 suggest that the only outcome which is stable in the long run is an equal split. However, over shorter timescales near-equal splits may also be stable with weak evolutionary pressures present encouraging an eventual shift to the equal split.

Finally, note that since experiments have a single agent at each position rather than a population, the structure of the evolutionary process – many stable solutions in the short run, some of which are selected in the long run – may not carry over. The agreement between repeated experiments in sociology papers seems to support this[7]. Without this property, it is possible that there may not be a major qualitative difference between short and long run solutions of the experiments, so the low number of rounds may not be important.

9.2.2 2 Player Networks

The alternating offers game supports a unique SPE in the 2 player situation. As discussed in section 4.2.1, this matches the asymmetric Nash bargaining solution, as defined axiomatically in section 4.1, with the bargaining powers determined by players' discount factors.

The simulation results of chapter 8 support a unique outcome for all learning

[7]An alternative explanation is that agents are taken from the same society which use one specific convention of play.

rules other than sampled best reply with sample size one. Also, for all these learning rules other than imitate better with sample size one, the solution $(0.5, 0.5)$ for the unit utility cake $\mathcal{K}_{\text{unit}}$ is conserved for a utility cake which is a subset of $\mathcal{K}_{\text{unit}}$ and also contains $(0.5, 0.5)$. This is evidence that the axiom of independence of irrelevant alternatives may hold under these learning rules. This offers some support for the symmetric Nash bargaining solution since the other axioms hold by the design of the model[8].

However, the simulations with an outside option for one player contained in section 8.4 provide some odd results. In particular a simulation for sampled best reply with sample size 2 illustrates a situation in which possessing an outside option worsens a player's payoff! These results certainly do not match the predictions of the alternating offers game with outside options mentioned in section 4.2.1.

9.2.3 3 Player Networks

The market bargaining game is a model for 3 player ring networks and possesses a unique limiting SPE outcome as the delay between demands tends to zero. The results fall into two cases, depending upon whether a von Neumann-Morgenstern triple of outcomes, described in definition 4.3, exists. In the case of existence the unique triple gives all 3 possible outcomes. However, which of these outcomes is selected depends upon the exogenously chosen order in which the players act.

The simulations of section 8.6 investigate a 3 player ring network in which a von Neumann-Morgenstern triple exists. They provide very strong support for the corresponding outcome under all learning rules for at least some parameter choices.

[8] An exception is that scale independence does not hold for the proportional imitation learning rule. Indeed, this learning rule is not even always well defined under rescaling of utilities, as this may sometimes produce a probability of switching demand of more than 1. A possible resolution of this problem is to make the factor of proportionality for population A^i equal to the reciprocal of the maximum feasible demand that player i can receive in any utility cake. In this case scale independence would hold for proportional imitation.

However, I do not have a candidate mechanism which explains such strong support for this outcome. As discussed in section 8.6, it seems unlikely that the minimal tree analysis described in section 6.1.2 can be applied to this case directly.

A 3 player network in which a von Neumann-Morgenstern triple does not exist is L_3. This case can be represented in the market bargaining game by 3 player ring network in which two cakes are unit cakes and the third contains only the utility pair $(0,0)$. The payoff vector associated with the limiting outcome of the market bargaining game in this case is $(0,1,0)$. The generalisations of the market bargaining game considered in chapter 5 support the same limiting solution for L_3 (see lemma 5.4). For the evolutionary model, theorem 7.3 supports this solution for imitate better and proportional imitation under various assumptions. Other bargaining models support a different solution. For example the telephoning game[9] of section 4.3 can support limiting SPEs which are equal to the solution if a particular outlying player were removed[10].

Section 4.4.5 argues that the crucial feature generating the qualitative difference between the SPE outcomes of the telephoning and market bargaining games is that in the telephoning game only bilateral bargaining is allowed, whereas the market bargaining game allows players to break this convention. The existence of this feature is not investigated by the simulations and experiments[11]. The rules of the underlying game of the evolutionary model of chapter 6 implicitly allow non-bilateral bargaining. Similarly, the sociology experiments are computer based, and so pre-specify

[9]These solutions are also supported by most other models with unilateral demands.

[10]That is, the solution to the alternating offers game on a unit cake between players 1 and 2 or player 2 and 3, with the same discount factors as used in the original situation. This is a result of Binmore [3].

[11]It is not obvious whether this feature holds as much importance in a evolutionary setting. For example, consider an evolutionary model for the network L_3 based on an underlying bargaining game using only bilateral bargaining. There still appears to be pressure on players 1 and 3 to undercut each other's demands in order to increase their chance of exchanging with player 2 in future rounds.

the rules governing the bargaining interactions between subjects. Thus typically subjects do not have a choice of whether to bargain bilaterally or not[12] and these results do not allow conclusions to be drawn about whether bargainers choose to use bilateral bargaining.

The theoretical predictions above for this network can be compared with data from simulations and experiments. Note that in these cases the players are treated symmetrically so the appropriate version of the telephoning game is one in which all discount factors are equal and so two players make an equal split in SPE. The data supports outcomes in between the extremes of this outcome and the outcome with payoffs $(0, 1, 0)$ mentioned above. Typically player 2 receives $0.5 < x < 1$ and players 1 and 3 have an equal chance of receiving $1 - x$. In the simulations the value of x varies from 0.7 to 0.95. The value of x is shown to be sensitive to variations in M and ϵ. Section 2.7 only contains data from two experiments on this network and the corresponding values of x are 0.67 and 0.83. In conclusion, the arguments of the market bargaining game and theorem 7.3 appear to have some validity, but there seem to also be countervailing evolutionary forces preventing player 2 from reaching payoff 1.

9.2.4 The 4 Player Line Network

The only theoretical result produced for this network is that described in section 5.2.4. This gives two limiting outcomes for the case of equal discount factors with limiting payoffs $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2}, 0)$. Eliminating the non-core solution leaves a unique prediction. Data from both simulations and experiments supports outcomes with payoff vectors of the form $(1 - x, x, x, 1 - x)$ with $0.55 < x < 0.66$. The simulation data provides some evidence that these payoffs are long-term features of the bargaining situation and do not simply occur in experiments because only a

[12]As noted in section 2.7, it is not always clear what the underlying rules of experiments are, so it is sometimes hard to infer whether they allow non-bilateral bargaining.

short number of rounds are run. However note that simulation with another learning rule supported the outcome with payoff vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

If the payoff vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ were the result of the current bargaining convention then player 2 would be indifferent about exchanging with players 1 and 3. Intuitively, there is therefore an incentive for player 1 to offer player 2 more to guarantee exchange. However, this thesis does not provide any grounds for a theoretical argument to capture this intuition.

Note that the payoffs typically received by position 2 or 3 in a simulation or experiment on this network are similar to some simulation results detailed in section 8.4 for a player in a 2 player unit cake network with an outside option of less than $\frac{1}{2}$. This suggests that the mechanism providing players 2 or 3 with a payoff above $\frac{1}{2}$ is based on the outside options they provide each other.

9.2.5 The 5 Player Line Network

The two models of chapter 5 both support a wide range of SPEs for this network. It is shown (corollary 5.6) that in any SPE the payoff of player i lies in an interval I_i. These are $I_1 = I_3 = I_5 = [0, \frac{1}{2}]$ and $I_2 = I_4 = [\frac{1}{2}, 1]$. Furthermore, in sections 5.1.2 and 5.2.4 SPEs for both models are shown illustrating that some player i may receive a payoff at either bound of I_i.

Experimental and simulation data are within these bounds, but do not seem to make a clear prediction. Simulations give payoffs in the range $[0.65, 0.85]$ to even numbered players, whereas experiments have the range $[0.55, 0.88]$. Note also that simulation results are sensitive to the choice of parameters M and ϵ.

These results show that this network gives players 2 and 4 an advantage but the size of advantage is sensitive to details of the bargaining situation. However, they do not indicate what the relevant details might be.

Theorem 7.3 predicts the solution $(0, 1, 0, 1, 0)$. The lack of other support for this solution suggests that the limiting case of low mutation for which this theorem applies does not represent the general case[13].

9.2.6 4 Player Ring Networks

The simulation results of section 8.8 support a von Neumann-Morgenstern triple like solution for a particular four player ring network. That is, each player is guaranteed a particular utility level and may achieve this by exchanging with either neighbour. However, as for the von Neumann-Morgenstern solution for 3 player ring networks, no theoretical mechanism is proposed to explain the strong support for this outcome in simulations.

9.2.7 Miscellaneous

Section 7.3.1 predicts that the state of the evolutionary model of chapter 6 under the candidate learning rules of section 7.1 spends most of its time near B sets even for relatively large values of ϵ. The simulation results of section 8.1 provide some support for this hypothesis. Section 7.3.1 also contains the prediction that the qualitative results of the model are relatively robust to the choice of ϵ. The results of chapter 8 show this is often correct. Some results, such as those for the 3 player ring network of section 8.6 are very robust to the choice of ϵ. In other networks, such as odd length unit cake line networks, the results are sensitive to the choice of ϵ and M but the same qualitative features are always present.

Theorem 7.3 predicts that certain positions in unit cake networks receive payoffs of 0 or 1. For the networks L_3 and L_5, it predicts that even numbered players receive payoff 1 and odd number players receive payoff 0. This matches the qualitative features of the simulation and experimental results, but predicts more extreme payoff values. The theorem gives the behaviour of the evolutionary model under

[13]Recall that the prediction of theorem 7.3 only applies in the long term. However, it does not seem possible to explain the discrepancy by interpreting all the other results as short term solutions. The simulation data shows that solutions with more extreme payoff values than those most often selected are sometimes reached but are less stable.

the limit $\epsilon \to 0$ and certain other conditions. It is not surprising that the sharpness of its results do not hold for the relatively large values of ϵ used in the simulation. However, the experimental results also only support the qualitative results of this theorem rather than the extreme payoff values. This suggests that the evolutionary pressures embodied in theorem 7.3 do not fully capture behaviour in the corresponding bargaining situations. Other forces exist which prevent extreme payoffs being reached.

Lemma 5.5 predicts that players with only one neighbour in a unit cake network with uniform discount factors receive limiting payoffs of no more than $\frac{1}{2}$. This intuitively obvious[14] result is also supported by experimental and simulation results.

9.2.8 The Theoretical Predictions of the Sociology Literature

The results of theorem 7.3 have some similarities to the 'strong power' predictions of NET outlined in sections 2.5.1 and 2.5.2. For example, many of the heuristics summarizing the properties of strong power in section 2.5.2 hold under theorem 7.3, if "low strong-power position" and "high strong-power position" are interpreted as S players and W players respectively[15]. However the GPI formula of section 2.5.1 identifies many more positions as strong power than theorem 7.3. For example the GPI formula predicts strong power in all odd length line networks, whereas theorem 7.3 only applies to L_3 and L_5.

Section 2.5.2 contains the following quote from Markovsky et al [44]:

"strong power structures exhibit a 'ratcheting' process whereby actors in struc-

[14] This situation can be thought of as a 2 player bargaining situation in which one player has outside options.

[15] "Weak or equal power positions" should be interpreted as those players which are neither S or W players. "Breaks" should be interpreted as exchanges which are never realised. Heuristic iii) is stronger than the result of the theorem. This heuristic predicts that adding an exchange between two weak positions destroys the strong power structure. Under theorem 7.3, some players may remain S or W players in this case.

turally disadvantaged positions serially outbid one another..."

That section also notes that in [44], the threat of exclusion is claimed to be the driving force behind strong power effects. These comments could also serve as an interpretation of the proof of 7.3. The sketch in 7.3.2 describes how the proof is driven by a process in which agents in W populations are excluded and imitate successful low demands. This creates the conditions for agents in other W populations to be excluded and switch to low demands. This is effectively a mechanism where different positions undercut each other and their demands are driven down.

The other results of this thesis do not correspond as closely to sociological theories. For example NET predicts "maximum differentiation" of payoffs for cases of strong power. In the context of unit cake networks this means that players in some positions receive payoffs of 0 and 1. The models based on the alternating offers games of chapters 4 and 5 predict a unique solution with maximun differentiation for the network L_3[16] but not for other networks for which NET predicts maximum differentiation, such as L_5. In the experimental and simulation results of section 2.7 and chapter 8 maximum differentiation is rarely observed. In particular, typical payoffs are often less extreme for larger networks[17]. No support is found for the GPI formula or the weak power theories of NET. Also, no support is found for degree dependence of section 2.5.3 but there has been little investigation of settings in

[16] As noted in section 5.4.2, lemma 5.4, which supports this prediction for L_3, can be extended to other situations where a player has unit cake exchange opportunities with at least two neighbours who have no alternative exchange opportunities. This prediction and lemma 5.4 itself are the only cases in chapter 5 where models based on the alternating offers game unambiguously predict maximum differentiation (i.e. the prediction does not depend on exogenous structure).

[17] An exception is in the "strong4" network of section 2.7. The positions with extreme payoffs in this network match the situation described in footnote 16 of this chapter. This suggests that in some networks the stable bargaining outcome is the same as if some exchange relations are removed and the network is decomposed into several connected components, and it is only in the larger components that payoffs are less extreme.

which it might occur.

9.3 Future Research

The difficulties summarized in section 9.1.2 mean that it does not seem fruitful to pursue modelling bargaining in networks using perfect information extensions of the alternating offers game. However the evolutionary model suggests many possibilities for future research. There are two main directions. The first is to further investigate properties of the model for simple networks in which the problems associated with non-core solutions discussed in section 9.1.3 seem unlikely to arise. The second is to attempt to alter the model so that it overcomes these problems and can be applied to large networks.

There are many small networks other than those considered in chapter 8 on which it would be interesting to perform simulations, such as the stem network of section 2.7. Other subjects which can be investigated by simulation include the degree dependence hypothesis of section 2.5.3, whether results are robust to small variations in utility cakes, and whether a property similar to the axiom of independence of irrelevant alternatives holds for general networks. If this last property held then it would suffice to study networks whose cakes have linear boundary functions, simplifying the analysis of general networks. Also, there are many interesting variations which could be made to the model. For example the underlying game could be altered to model a bargaining problem in which 3 players must split one utility cake. Many other possible variations are given in section 7.2.

It may be possible to obtain theoretical results on the evolutionary model of chapter 6 in addition to theorem 7.3. For example, consider a 2 player bargaining network under the proportional imitation learning rule. In the limit $\epsilon \rightarrow 0$, the state of the model spends most of its time at B sets corresponding to strict Nash equilibria. It is shown in section 7.1.1 that a transit between such sets requires only two mutations. Thus to find the first term in asymptotic expansions in ϵ of each transit probability it suffices to investigate transits where only two mutations occur. During such a transit each population supports only two demands, which may allow approximate or numerical calculations to succeed. This case is of particular interest as it may explain the outside option results of section 8.4 which have implications for network results, as discussed in section 9.2.4.

One method of altering the model to avoid the non-core solution discussed above is to alter the underlying game. As discussed in section 9.1.3 in seems necessary to alter the matching rule, probably by endogenising part of it. Another approach is to alter the evolutionary model. For example a model placing a single agent at each network position seems more realistic and may make non-core solutions less stable. Some methods for constructing such models are mentioned in Tesfatsion [67].

Finally, relaxing the restrictions placed on bargaining outcomes in section 3.1.1 could enable other approaches to be more successful. For example if each player could participate in two exchanges, then it would be easier to adapt the model of Corominas-Bosch [23] from section 4.5 to general networks, as follows. Rounds alternate between those in which everyone simultaneously makes a demand and those in which everyone simultaneously makes an acceptance decision. A player is allowed to exchange once by being accepted and once by accepting. Altering these restrictions would mean that the experimental results of section 2.7 could not be used. However the results could still be interpreted as modelling social exchange. As discussed in section 2.4, restricting each player to a single exchange is only introduced in the first place on grounds of experimental expediency and simplicity rather than on conceptual grounds.

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