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Score Driven Asymmetric Stochastic Volatility Models

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Abstract

In this paper we propose a new class of asymmetric stochastic volatility (SV) models, which specifies the volatility as a function of the score of the distribution of returns conditional on volatilities based on the Generalized Autoregressive Score (GAS) model. Different specifications of the log-volatility are obtained by assuming different return error distributions. In particular, we consider three of the most popular distributions, namely, the Normal, Student-t and Generalized Error Distribution and derive the statistical properties of each of the corresponding score driven SV models. We show that some of the parameters cannot be property identified by the moments usually considered as to describe the stylized facts of financial returns, namely, excess kurtosis, autocorrelations of squares and cross-correlations between returns and future squared returns. The parameters of some restricted score driven SV models can be estimated adequately using a MCMC procedure. Finally, the new proposed models are fitted to financial returns and evaluated in terms of their in-sample and out-of-sample performance.

Keywords: BUGS, Generalized Asymmetric Stochastic Volatility, Leverage effect, MCMC, Score driven models

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I. Introduction

There is a large literature on modelling the second arder dynamics of univariate financia! returns with leverage effect. The main interest is to obtain accurate volatility estimates which are important components of many financia! models. Two main alternative families of models are usually implemented to represent the dynamic evolution of asymmetric volatilities. The first family is based on the Generalised Autoregressive Conditional Heteroscedasticity (GARCH) model of Bollerslev (1986), with the volatility specified as a function of past returns and, consequently, observable one-step ahead; see Engle (1995), Giraitis et al. (2007) and Terasvirta (2009) for comprehensive reviews on GARCH models and Rodríguez and Ruiz (2012) for a review on popular GARCH models with leverage effect. Alternatively, the second family includes Stochastic Volatility (SV) models, which specify the volatility as a latent variable that is not directly observable; see Ghysels et al. (1996) and Cavaliere (2006) for reviews on SV models and their applications and Mao et al. (2013) for the comparison of popular alternative asymmetric SV models.

Besides heteroscedasticity and leverage effect, another important and well documented empirical feature of standardized financia! returns is the fact that they are heavy-tailed distributed; see, for instance, Liesenfeld and Jung (2000), Jacquier et al. (2004) and Chen et al. (2008) among many others. In arder to capture this latter feature, both GARCH and SV models have been extended by assuming fat-tailed return errors. Two examples are the GARCH-t model of Bollerslev (1987) and the asymmetric SV model with Student-t distribution of Asai and McAleer (2011). Nonetheless, these traditional models often specify the asymmetric volatility as being driven by past return errors. Consequently, they can suffer from a potential drawback since a large realisation of the return error, which could be due to the heavy-tailed nature of its distribution, will be attributed to an increase in volatility. Therefore, in the GARCH

context, Creal et al. (2013) and Harvey (2013) have recently proposed models in which the dynamic of volatility is driven by the lagged score of the conditional distribution of returns to automatically correct for infiuential observations. This gives rise to the Generalised Autoregressive Score (GAS) models which are also known as dynamic conditional score (DOS) models.

In this paper, we extend the GAS idea to SV models by specifying the unobserved volatility to be driven by lagged scores. Given that the conditional distribution of returns does not have an analytical expression, the score is computed with respect to the distribution of returns conditional on the volatilities. We show that this type of models lay in the Generalised Asymmetric SV (GASV) family recently proposed by Mao et al. (2013). We denote the new models as GAS-GASV (GAS^2V) and consider three alternative GAS^2V models depending on the assumed distribution of the return errors, namely, Normal, Student-t and Generalised Error Distribution (GED). Closed-form expressions of severa! relevant statistics of these models are derived to analyse their ability to represent the main empirical features often observed in financia! returns, namely, the excess kurtosis, positive autocorrelations of power-transformed absolute returns and negative cross-correlations between returns and future power-transformed absolute returns. lt is important to point out that analytical expressions of these moments ofthe GAS^2V model with Student-t errors can now be derived, in opposition to the traditional specifications of the SV models in which their derivation is hardly possible when the errors are Student-t. Moreover, we show that the GAS^2V model with Student-t errors generates returns with very similar properties as those generated by the GAS^2V model with GED errors as far as the parameters of both distributions are chosen to have the same kurtosis. Therefore, there could be difficulties in identifying the parameters of the GAS^2V model when looking at the moments.

Although SV models are considered competitive alternatives to GARCH models,

their estimation usually limits their empirical implementation due to the intractability of their likelihood function; see Carnero et al. (2004) for the advantages of SV models when compared with GARCH models. In recent decades, many efforts have been done in this direction and considerable advances have been achieved with simulation based procedures being a very popular alternatives; see Broto and Ruiz (2004) for a survey on the estimation of SV models. Examples of procedures based on the Monte Cario likelihood evaluation are the simulated Maximum Likelihood (MCL) procedure of Durbin and Koopman (1997) and the Efficient Importance Sampling (EIS) procedure of Liesenfeld and Richard (2003) and Richard and Zhang (2007); see also Asai and McAleer (2011) for the implementation of the latter procedure for estimating their exponential SV model and Koopman et al. (2014) for an extension. Alternatively, Monte Cario Markov Chain (MCMC) has become an standard estimation method because it is efficient without relying on asymptotic approximations for inference on the parameters; see, for example, Jacquier et al. (1994), Omori et al. (2007), Abanto-Valle et al. (2010), Tsiotas (2012) and Yu (2005, 2012) among others for implementations of MCMC to estímate SV models. Furthermore, MCMC also allows obtaining one-step-ahead densities of the underlying volatilities. In this paper, we consider a MCMC estimator implemented by the user-friendly and freely available software BUGS; see Meyer and Yu (2000) and Yu (2005, 2012) for the reliability of BUGS when estimating the parameters of the SV model. We conduct an intensive Monte Cado study on the finite sample properties of the MCMC estimator implemented using BUGS when estimating the model parameters and find that it is adequate when the model is restricted as to avoid the identification issues detected when looking at the statistical properties of the GAS^{2}V models. Using the MCMC estimator, the three GAS^{2}V models considered in this paper are fitted toa series of daily S&P500 returns and two series of weekly financial returns, namely S&P500 and NIKKEI225. The performance is evaluated both in-sample and out-of sample.

The remainder of this paper is organized as follows. In section 2, the GAS^{2}V model is proposed and its statistical properties are derived when the errors are Gaussian, Student-t and GED. In section 3, we perform a Monte Cario experiment to analyse the finite sample properties of the MCMC estimator of the model parameters. The GAS^{2}V models are fitted to daily and weekly real time series of financia! returns and their in-sample and out-of-sample performances are evaluated in section 4. Finally, we conclude in section 5.

2. Score driven asymmetric SV models

In this section, we propase the GAS^{2}V model and derive its statistical properties when the errors have Normal, Student-t and GED distribution. In particular, we obtain closed-form expressions of the marginal variance, the kurtosis, the autocorrelation function (acf) of power-transformed absolute returns and cross-correlation function (ccf) between returns and future power-transformed absolute returns.

2.1. The GASJ V model

Let *Yt* be the return at time t, $_t$ its volatility and $ht = \log ;$. The family of GAS^{2}V models is defined as follows[1]

$$y_t = \exp(h_t/2)\epsilon_t, \qquad t = 1,2,\cdots,T, \tag{1}$$

$$ht-p, = <\mathrm{P}(\mathrm{ht.-1-\ p,}) + f(Ut\text{-}1) + \text{'TJt-1,} \tag{2}$$

where $'T)t$ is a Gaussian white noise with variance and Et is a strict white noise with variance one which is distributed independently of $'T)t$ for all leads and lags. $p,$ is a scale parameter related with the marginal variance of returns while the parameter $<P$

[1]We only consider one lag of past scores and volatilities as this is the most popular specification in the empírica! implementation of related similar models for volatilities.

is related with the persistence of the volatility shocks. Finally, $f(\cdot)$ is a function of the scaled conditional score of the lagged return, Ut-$_1$, which is defined as follows

$$u_t = C\frac{\partial lnP(y_t|h_t)}{\partial h_t}, \tag{3}$$

where C is any real number introduced to simplify the expression of the score and *P(Ytlht)* is the density of returns conditional on volatilities. Denoting by '1/;(Et) the probability density function (pdf) of ft, the density function of *Yt* conditional on *ht* is given by *P(ytlht)* = exp(-*ht/2)'1/J(Yt* exp(-ht/2)). It follows immediately that

$$u_t = -\frac{C}{2} + \frac{C}{2}\frac{\epsilon_t\psi'(\epsilon_t)}{\psi(\epsilon_t)}, \tag{4}$$

where *'1/J'(Et)* denotes the derivative of '1/;(Et) with respect to ft. Thus, *Ut* depends on *Et* and, consequently, after writing *f(Ut-d* = *f* (−1 + ÍEtJ E))) in equation (2), the GAS^{2}V model in equations (1) and (2) can be obtained as a particular case of the GASV family defined by Mao et al. (2013) and the results on the properties of this family can be directly used. In particular, according to Theorem 2.1 of Mao et al. (2013), when lif>l < 1 and the distribution of *Et* is such that E(exp(f(Et))) < oo, the GAS^{2}V model is stationary. Moreover, for any non-negative integer *e,* if the distribution of *Et* is such that E(exp(0.5cf(Et))) < oo and E(IEt¡c) < oo, both *Yt* and IYtl have finite moments of arder *c.* In particular, the marginal variance and kurtosis of *Yt* are given by

$$\sigma_y^2 = \exp\left(\mu + \frac{\sigma_\eta^2}{2(1-\phi^2)}\right) P(\phi^{i-1}) \tag{5}$$

and

$$\kappa_y = \kappa_\epsilon \exp\left(\frac{\sigma_\eta^2}{1-\phi^2}\right)\frac{P(2\phi^{i-1})}{(P(\phi^{i-1}))^2}, \tag{6}$$

respectively, where $P(bi)$ - $\prod_{1}$ $E(exp(bd(ut-i)))$.

Under the same conditions, Theorem 2.2 of Mao et al. (2013) established that the autocorrelation function of IYtlc is also finite and given by

$$\rho_c(\tau) = \frac{E(|\epsilon_t|^c)E(|\epsilon_t|^c \exp(0.5c\phi^{\tau-1}f(\epsilon_t)))\exp\left(\frac{\phi^\tau c^2\sigma_\eta^2}{4(1-\phi^2)}\right)P(0.5c(1+\phi^\tau)\phi^{i-1})T(\tau,0.5c\phi^{i-1}) - [E(|\epsilon_t|^c)P(0.5c\phi^{i-1})]^2}{E(|\epsilon_t|^{2c})\exp\left(\frac{c^2\sigma_\eta^2}{4(1-\phi^2)}\right)P(c\phi^{i-1}) - [E(|\epsilon_t|^c)P(0.5c\phi^{i-1})]^2},$$

where $T(n,bt)$- $\prod_{i=1}^{n-1} E(exp(bd(ut-i)))$ if $n > 1$ and $T(1,bi) = 1$. Finally, the finiteness of the cross-correlation function between Yt and IYt+rlc, for $\tau = 1, 2, \cdots$, is guaranteed when further E(letl^{2}c) < oo. It is given by

$$Pc1(T) = \frac{E(1\,tl^{o})exp(;(f;;u)E(cotexp(0.5ctf_{i}.-=\frac{1}{1}f(t)))P(0.5(1+\sigma tf_{i}.-)'-^{1})\cdot T_{u}:-;(t;)^{1}}{E(Icol2c)exp(\ _{4}C\ \ i_{2}l\ )P(ctf_{i}i\text{-}1)\text{-}[E(Icotlc)P(0\ .5ctf_{i}'\text{-}l\ )]2}$$

Later in this paper, we obtain closed-form expressions of these moments for particular assumptions on the function $f(\cdot)$ and on the error distribution. In particular, in order to represent the leverage effect often observed when dealing with time series of financial returns, we consider the following specification $off(\cdot)$

$$f(Ut\text{-}1) = ct!(Et\text{-}1 < O) + kut\text{-}1 + k^{*}sign(\text{-}Et\text{-}1)(ut\text{-}1 + 1), \qquad (7)$$

where $I(\cdot)$ is an indicator function that takes value one when the argument is true and zero otherwise. The parameter k represents an ARCH effect while the parameters a and k^{*} represent the leverage effect with a dealing with changes in the scale parameter depending on the sign of past returns and k^{*} with changes in the dynamics involving the score. Note that the last term in (7) is based on the proposal of Harvey (2013) in the context of asymmetric score GARCH models. As pointed out by Harvey (2013), although the statistical validity of the model does not require it, proper restriction

may be imposed on k and k^* in order to ensure that an increase in the absolute value of a standardized observation does not lead to a decrease in volatility.

In order to represent the leverage effect, besides the cross-correlations, Mao et al. (2013) propose the Stochastic News lmpact Surface (SNIS) that relates al with Et-1 and "lt-1. As in Engle and Ng (1993), the lagged volatilities are evaluated at the marginal variance, so that, at time $t-1$, the volatility is equal to an "average" volatility. Consequently, the effect of level shocks, Et-b and volatility shocks, "lt-b on the volatility at time t is given by

$$SNIS_t = \exp((1-\phi)\mu)\sigma_y^{2\phi}\exp(f(u_{t-1}) + \eta_{t-1}), \tag{8}$$

where a is the marginal variance of Yt in (5) and f (Ut-1) is given in (7). 1t is important to note that the score, ut, is different depending on the particular assumption on the error distribution. Severa! distributions of return errors have been proposed in the related literature being the Gaussian distribution the most popular; see, for example, Jacquier et al. (1994) and Harvey and Shephard (1996). When the errors are Gaussian, the score is given by

$$ut\text{-}1 = E\text{-}1 - 1. \tag{9}$$

The corresponding SNIS is plotted in the top panel of Figure 1when the GAS^{2}V model has parameters $\{a, </>, k''', k, a\} = \{0.07, 0.98, 0.08, 0.1, 0.05\}$. The scale parameter, $¡;,$ is chosen so that exp((1- <f>)¡;,)a;cfo =1. It shows that the volatility response is larger when the lagged return is negative than when it is positive. Therefore, this model is able to capture the leverage effect. Moreover, the difference in the response of the volatility to positive and negative Et-1 depends on the log-volatility noise, "lt-1. Stronger leverage effect is observed when 'Tft-1 is positive and large. The News lmpact

Curve (NIC), defined by Engle and Ng (1993), is obtained when 7Jt-l =O, which is also plotted in Figure I. The inclusion of 7Jt-l in the model allows it to be more flexible in representing the leverage effect.

However, the Gaussian distribution does not fully capture the fat tails of financial time series often observed in practice and may suffer from a lack of robustness in the presence of extreme outlying observations. Consequently, several authors consider heavy-tailed distributions such as the Student-t or the GED distributions;[2] see, for example, Chen et al. (2008), Choy et al. (2008) and Wang et al. (2011, 2013). Consider first the GAS^{2}V model when ft has a Student-t distribution with ν_0 degrees offreedom. In this case, the score is given by

$$u_t = (\nu_0 + 1)\frac{\epsilon_t^2}{\nu_0 - 2 + \epsilon_t^2} - 1. \tag{10}$$

The SNIS of the GAS^{2}V model with Student-t errors is plotted in the middle panel of Figure 1 for the same parameters as above and $\nu_0 = 6$. The asymmetric response of volatility to Et-1 is similar to that of the GAS^{2}V model with Gaussian errors.

Finally, when ft is assumed to follow a GED(v) distribution, then the score function is given by

$$u_t = \frac{\nu}{2}\left|\frac{\epsilon_t}{\varphi}\right|^{\nu} - 1, \tag{11}$$

with $r.p = y'2^{-2}/vr(1/v)/r(3/v)$. The SNIS of the GAS^{2}V model with GED errors when $\nu_1 = 1.5$ is plotted in the bottom panel of Figure I. The volatility responds asymmetrically to the positive and negative returns errors. However, no big difference can be observed among the SNISs of all the three GAS^{2}V models.

[2]There are also proposals to include simultaneously leptokurtosis and sk.ewness in the distribution of ε_t, such as the sk.ewed-Normal and skew-Student-t in Nakajima and Omori (2012) and the asymmetric GED in Cappuccio et al. (2004). It is not straightforward to capture the moments of returns when the distribution of ε_t is asymmetric. Consequently, we leave this extension for future research and focus on symmetric distributions.

2.2. Different GA$2V models

In this subsection, we analyze the properties of three GAS^{2}V models corresponding to three different return error distributions.

2.2.1. GA$2V-N

If ε_t follows a Gaussian distribution, then, the scaled score, u_t is given by expression (9) and the specification of the log-volatility with ω defined as in (7) reduces to

$$h_t - \mu = \phi(h_{t-1} - \mu) + \alpha I(\epsilon_{t-1} < 0) + k(\epsilon_{t-1}^2 - 1) + k^* sign(-\epsilon_{t-1})\epsilon_{t-1}^2 + \eta_{t-1}. \quad (12)$$

The resulting model is denoted as GAS^{2}V-N.It is important to note that although the specification of the volatility in (12) is closely related to that in the T-GASV model of Mao et al. (2013), the way in which the leverage is introduced is different in both cases. In (12), the log-volatility depends on squared returns and the leverage effect is introduced in the same fashion as in the TGARCH model of Zakoian (1994). However, the log-volatility in the T-GASV model depends on past absolute returns and the leverage is introduced as in the EGARCH model. Rodríguez and Ruiz (2012) show that the TGARCH and EGARCH models are very similar. Therefore, we expect that, if ε_t is Gaussian, the GAS^{2}V-N and T-GASV models have very similar properties. The analytical expressions of E(:¡exp(bf(t:t))) and E(lt:tlcexp(bf(t:t))) are given in Appendix A.l. Using these expressions we can verify that when I<PI < 1and $k+$ W'l < 1/2, the model is stationary, *Yt* and IYtl have finite moments of order *e* and the acf of IYtlc and ccf between *Yt* and *iYt+hic* are finite when *ck* + lck*l < I.

We first explore the kurtosis of the GAS^{2}V-N model. It is the kurtosis of the ARSV(1) model proposed by Harvey et al. (1994), $k\,exp\,C:\,i_{2>}$. multiplying the factOr $R = \frac{n;lE(^{2}4>'-lf(uH))}{1^{E2(1jJi-1}f(ut-i))}$ A S an "ll 1 US t ra t"IOn, Figure 2 p lO t S R as a fu nc t"IOn O f the leverage parameters *a* and k^* when $k =$ O and 0.1 for three different persistence

parameters, namely, $\phi = 0.5$, 0.9 and 0.98. For these particular parameter values, we can observe that the ratio is always larger than 1. Therefore, the GAS^{2}V-N model generates returns with larger kurtosis than the corresponding basic ARSV(1). Furthermore, the kurtosis increases with α, κ^* and κ. The increment is more prominent when ϕ is larger.

In order to illustrate how the autocorrelations and the cross-correlations depend on the parameters, we have considered a particular GAS^{2}V-N model with parameters $\phi = 0.98$ and $\omega = 0.05$. The leverage parameters α and κ^* take values between 0 and 0.2 and 0 and 0.1, respectively. Figure 3 plots the first order autocorrelations of squares, $\rho_2(1)$ (top left panel), the first order autocorrelations of absolute returns, $\rho_1(1)$ (top right panel), and the first order cross-correlations between returns and future squared returns, $\rho_{21}(1)$ (bottom left panel), and future absolute returns, $\rho_{11}(1)$ (bottom right panel) when $\kappa = 0$. These moments are also plotted in Figure 4 when $\kappa = 0.1$. We can observe that they have very similar patterns as those of the GASV model; see Figure 2 of Mao et al. (2013). First, the first order autocorrelations are positive and the surface is rather flat and it is not affected by the leverage effect parameters κ^* and α. However, the first order autocorrelation of absolute returns is larger than that of the squared returns and increases with the two parameters. Finally, the cross-correlations are negative and decrease with the two leverage effect parameters, α and κ^* linearly. By comparing Figure 3 and Figure 4, we can observe that larger value of κ gives larger first order autocorrelations but negligible difference in cross-correlations.

To illustrate the shape of these moments for different lags, Figure 5 plots the first twenty orders of these moments for a GAS^{2}V-N model with parameters $\mu = 0$, $\phi = 0.98$, $\omega = 0.05$, $\alpha = 0.07$, $\kappa^* = 0.1$ when $\kappa = 0$, while Figure 6 illustrates these moments when $\kappa = 0.1$. The values of the parameters are chosen to be very

similar to those obtained when fitting these models to financial data; see section 4. The figures show that both the acf and absolute ccf decay exponentially towards zero. The absolute values of the moments related with absolute returns are larger than those of the squared returns. Therefore, we can conclude that the model is able to capture the Taylor Effect, phenomenon characterised by the autocorrelations of absolute returns to be larger than those of squares. Moreover, the larger value of k allows the model to capture larger autocorrelations of squared and absolute returns, therefore, volatility clustering.

2.2.2. GA$2 V-T

Alternatively, if ft is distributed as a standardized Student-t distribution with degrees of freedom $v_0 > 2$, pdf 7/Jo(ft) = $\frac{t}{7rJIQ}\left(\frac{}{liO}\right)_{<po}$ (1 + $\frac{Yl_{2)-}}{liO<flo}$ ^{2} with <Po = 'then Ut is given by (10). We denote the model specified by equations (1), (2), (7) and (10) as GAS^{2}V-T. When 14>1 < 1, the model is stationary. Moreover, for sorne non-negative integer e, if $v_0 > e$, then the acf of IYtlc is finite. If further, $v_0 > 2c$, the ccf between Yt and IYt+rlc for a positive integer T is also finite. The expectations needed to obtain the analytical expressions of the moments are derived in Appendix A.2.

Analogously, we illustrate the kurtosis of GAS^{2}V-T by plotting the factor R in Figure 2, for the same parameters chosen for the GAS^{2}V-N model and v_0 = 11.8745. Note that va guarantees Et to have the same kurtosis when it follows a GED distribution with degrees of freedom v = 1.5. We can observe that the ratio of the GAS^{2}V-T model is smaller than that of the GAS^{2}V-N when $cf>$ = 0.98, while they are indistinguishable when $cf>$ is small.

As previously, we illustrate the first order of the acfs and ccfs of GAS^{2}V-T models in Figure 3 and Figure 4 when va = 11.8745. The other parameters are the same

as those chosen for the GAS^{2}V-N model. We observe that the GAS^{2}V-N model generates larger first order autocorrelations for both absolute and squared returns than the corresponding GAS^{2}V-T models. Moreover, the absolute values of the cross-correlations are also larger for the GAS^{2}V-N model than for the GAS^{2}V-T when k = O. However, the absolute cross-correlation between returns and future squared returns are smaller in the case of the GAS^{2}V-N when k = 0.1 and k* approximates 0.1.

We illustrate the first twenty orders of acfs and ccfs in Figure 5 and Figure 6 for the same parameter values used in the illustrations of the GAS^{2}V-N model while considering two different values ofthe degrees offreedom, namely 11.8745 and 19.8387, which guarantee the same kurtoses of Et when Et ''' *GED* with v = 1.5 and v = 1.7, respectively. We observe that the autocorrelations and cross-correlations of the absolute values are smaller than those of GAS^{2}V-N models for the considered parameter values. Moreover, larger degrees of freedom imply larger autocorrelations and larger cross-correlation of absolute values. Therefore, we may conclude that fatter tails of Et imply smaller autocorrelations of both absolute and squared returns, which coincides with the conclusion of Carnero et al. {2004).

2.2.3. GA V-GED

Finally, we assume that Et follows a *GED(v)* distribution with probability density function (pdf) 'lj;(Et) = $\frac{1}{2}$ HI $\frac{1}{\text{"¡pr(l+l/v)}}$ exp (− $\frac{1}{IP}$ lv) with *r.p* = .j2-^{2}/vf(1/v)/f(3/v). Then *ut* is given by (11) where *9t* = ı lv follows a Gamma (2, 1/v) distribution; see Harvey (2013). The model defined by equations (1), (2), (7) and (11) are denoted as GAS^{2}V-G. It is strictly stationary if 1q,1 < 1 and if further k+ lk*l < ;e, *Yt* and lYtl have finite and time-invariant moments of non-negative integer order *c*. Under these conditions, the acfs and ccfs are also finite. The analytical expressions of the two expectations are given in Appendix A.3.

In Figure 2, we also plot the ratio ofthe kurtoses between GAS^{2}V-G and ARSV(l) for the same parameter values specified for the GAS^{2}V-N model while $\nu = 1.5$. Though this GAS^{2}V-G always generates returns with higher kurtosis than the ARSV(l) model, its kurtosis is smaller than that of the corresponding GAS^{2}V-N with similar parameter values. As the Gaussian distribution is a special case of the GED distribution with $\nu = 2$, we might conclude that a fatter tailed GED generates less kurtosis. Moreover, the ratio of GAS^{2}V-G is indistinguishable from that of the GAS^{2}V-T model when the return errors are assumed to have the same kurtosis in both models. Apparently, the kurtosis of the return generated by the GAS^{2}V model depends on the kurtosis of the errors and barely on the type of distribution.

We also analyse the first order acfs and ccfs of the returns generated by the GAS^{2}V-G model when $\nu = 1.5$ in Figure 3 and Figure 4. We find that when the kurtoses of return errors are the same as in GAS^{2}V-T, these moments related with squared returns are indistinguishable for both models. The first order autocorrelation of absolute returns and first order cross-correlation between returns and future absolute returns of GAS^{2}V-T models are larger than those of the GAS^{2}V-G model.

Figure 5 and Figure 6 illustrate the first twenty orders of these moments for two different GAS^{2}V-G models with two different values of the GED parameter, $\nu = 1.5$ and 1.7. As expected, the acfs of $|Y_t|$ and Y_t have both an exponential decay. Furthermore, fatter tails of ε_t imply smaller autocorrelations, but it has very mild influence on the cross-correlations. It verifies again that the acf of squared returns and ccf between returns and future squared return are indistinguishable to those of GAS^{2}V-T model with f_t having the identical kurtosis.

3. MCMC estimation

3.1. Model estimation and comparison method

Due to the lack of observability of the volatilities, it is not possible to obtain an analytical expression of the likelihood function of SV models, which complicates the estimation of the parameters and underlying volatilities; see Broto and Ruiz (2004) for a survey on alternative procedures to estímate SV models. Fortunately, the user-friendly and freely available software BUGS provides a MCMC estimator which uses the Gibbs Sampling algorithm. There are two main versions of BUGS, namely WinBUGS and OpenBUGS. WinBUGS is an established and stable, stand-alone version, which is not further developed; see Meyer and Yu (2000) for a description of WinBUGS and Yu (2012) and Wang et al. (2013) for its application. In this paper, we adopt OpenBUGS that is still being developed and refer to Mao et al. (2013) for the detailed algorithms.

To compare two competitive models, saying M_0 and M_1, we consider the Bayes Factor (BF). The BF, which is defined as the ratio of the marginallikelihood values of two competing models, **:** **,** where $p(yiMk)$ is the marginallikelihood of model k with k = O, **I. If** the prior odds ratio is 1by Bayes' theorem, the posterior odds ratio takes the same value as the BF. Jeffreys (1961) gave a scale for the interpretation of BFs. **If** $ln(BF)$ is less (bigger) than O, there is evidence in favor of (against) M_0. Moreover, **if** $ln(BF)$ E (0, 1), the evidence against M_0 is barely worth mention; if $ln(BF)$ E (1, 3), the evidence against M_0 is positive; if $ln(BF)$ E **(1,** 3) (Or (3, oo)), the evidence against M_0 is strong (or very strong).

3.2. Sampling performance

To check the reliability of this MCMC estimator, we simulate data from the three GAS^{2}V models, GAS^{2}V-N, GAS^{2}V-T and GAS^{2}V-G, with parameter values

$\{JL, e/>, u, a, k^*, \nu, \nu_0\} = \{0, 0.98, 0.05, 0.07, 0.08, 1.5, 11.8745\}$ while imposing the restriction $k = 0$. Recall that, in the previous section, we show that the GAS^{2}V-G and GAS^{2}V-T model generate returns with very similar properties when the parameters of both distributions are chosen to have the same kurtoses. Hence, there could be potential identification problem. Therefore, we consider the restricted GAS^{2}V models in the rest ofthe paper. For each model, $T = 1000$ observations are simulated. The posterior mean and standard deviation of each parameter in the model is obtained by fitting the model to these simulated data using the MCMC estimator. The total number of iterations is 30,000 with the first 10,000 iterations used as burn-in. We replicate the experiment for $r = 200$ times.

Table 1 reports the Monte Cario average of these posterior means and standard deviations together with the standard deviation of these posterior means based on these r replicates for each model. We find that the MCMC estimator is quite reliable for all parameters in all cases.

4. Empirical application

4.1. Estimation results from daily data

In this subsection, we fit the restricted GAS^{2}V models to one daily mean-adjusted return series, namely S&P500, observed from September 1, 1998 till July 25, 2014 with $T = 4000$ observations. The same data is analysed in Mao et al. (2013) and fitted to models of the GASV family.

Table 2 reports the posterior mean, the 95% credible interval for each parameter and the marginallog-likelihood. From the table, several conclusions can be drawn. First, all the parameter estimates are different from zero. The credible intervals of the degrees of freedom in both GAS^{2}V-T and GAS^{2}V-G model exclude the case of the Normal distribution, which implies that the return error follows a fat-tailed

distribution. Regarding the goodness of fit of the models, we observe, analysing the log-likelihood values, that GAS^{2}V-N model is outperformed by the other two models and that the GAS^{2}V-T model fits the data better than the GAS^{2}V-G model. However, the preference is negligible.

4.2. Estimation results from weekly data

In this subsection, we fit the restricted GAS^{2}V models to the series of mean-adjusted S&P500 and NIKKEI225 weekly returns observed from January 13, 1992 to December 27, 2010. The number of observations are $T_1 = 990$ and $T_2 = 986$, respectively. Although the sample size is relatively small, according to our Monte Cario experiments, we can obtain reliable estimation results. For completeness, we also fit two GASV models, namely, T-GASV-G and T-GASV-N, to these two series. Sorne relevant sample statistics are reported in Table 3. We observe that the sample autocorrelations of the squared returns are significantly positive and the cross-correlations between returns and future squared returns are significantly negative, confirming the volatility clustering and leverage effect.

Estimation results are reported in Table 4. According to the log-likelihood, the GAS^{2}V-G model fits the S&P500 returns the best while T-GASV-G provides the best fit to both S&P500 and NIKKEI225 series ofreturns. However, the advantage ofthis model compared to the others is barley worth mention.

4.3. Forecasting results from weekly data

A model good in-sample performance does not necessary imply a model good out-of-sample performance. In this section, we compare the out-of-sample performance of the proposed models using the two weekly return series described above. The three GAS^{2}V models and two GASV models are fitted to the return data and used to obtain one-period-ahead out-of-the-sample forecasts of weekly volatility. We split the weekly

sample into an in-sample estimation period and an out-of-sample forecast evaluation period. For estimation we use the rolling window scheme, where the size of the sample, which is used to estímate the competing models, is fixed at Ti with i= 1and 2. The first forecast is made for the first week of January, 2011. When a new observation is added to the sample, we discard the first observation and re-estímate all the models. The re-estimated models are then used to forecast volatility. This process is repeated until we reach the end of the sample, December 30, 2013. In total, we obtain 157 forecasts from each model.

Two alternative criteria are considered in this paper to compare the out-of-sample performances of these models, namely Mean Absolute Error (MAE) of the volatility forecasts and the Log Predictive Score (LPS), which is computed using the MCMC output. In Table 5, we report the MAE of the volatility forecasts. First, we calculated the weekly realized volatility (RV) obtained from the sum of daily squared returns. Let RVt denote the weekly RV and $p(t, k)$ denote the k-th daily log-price in week t. Then RVt is defined as $\sum_{1}(p(t,k)-p(t,k-1))^2$, where Nt is the number of trading days in week t and $p(t,0) = p(t-1,Nt_{-1})$. We match each volatility forecast with the corresponding realized volatility. Table 5 summarizes the MAE of the volatility forecasts. We can see that all the models perform nearly equally in forecasting the volatility of the S&P500 and NIKKEI225 returns.

On the other hand, LPS is a scoring rule introduced by Good (1952) that examines the model's performance when its implied predictive distribution is compared with observations not used in the inference sample. In this sense, it evaluates the out-of-sample behaviour of different models by mean of their divergence between the actual sampling density and the predictive density. The formula for the LPS is given as follows

$$LPS = \frac{1}{K}\sum_{k=1}^{K} log(f(YT+kiYk, \cdots, YT+k-1)), \tag{13}$$

where K is the total number of forecasts we've obtained. The one-step-ahead LPS are reported in the lower panel of Table 5. Again, it seems that these different models provide very similar forecasts.

5. Conclusion

In this paper, we propose a new score driven stochastic volatility model, named GAS^{2}V, in the spirit of the proposal of Creal et al. (2013) and Harvey (2013). Particularly, three different GAS^{2}V are considered, depending on the return error distribution, named GAS^{2}V-N if the error distribution is Normal, GAS^{2}V-T if it is Student-t and GAS^{2}V-G if the return error follows a GED distribution. The closed-form expressions of the marginal variance, kurtosis, autocorrelations of power-transformed absolute returns and cross-correlations between returns and future power-transformed absolute returns are obtained for these models. The GAS^{2}V models are included in the GASV family and therefore are flexible to represent the empirical features of the data. Moreover, the GAS^{2}V-T and GAS^{2}V-G can represent very similar properties when the kurtosis of the return error is fixed. Therefore, there can be problem of identification of the parameters. Through Monte Carlo studies, we show that the MCMC estimator implemented by BUGS is able to estímate the parameters of sorne restricted GAS^{2}V models adequately. These restricted GAS^{2}V models are fitted to both daily and weekly financia! data and we observe that the GAS^{2}V-T model provides the best fit in-sample for the daily S&P500 return series. Regarding the out-of-sample performance of the models in forecasting the volatility of the weekly financia! returns of the S&P500 and NIKKEI225, all models provide similar mean absolute forecast errors when the volatility forecasts are compared with a consistent measure of volatility, the realized volatility. The same conclusion is obtained when we consider the alternative Log Predictive Score criterion. We leave for future research

the comparison of the GAS^{2}V models with the robust GARCH models such as the Beta-t-EGARCH and Gamma-GED-GASV models of Harvey (2013). Moreover, the analysis of the robustness of our GAS^{2}V models in front of outliers is in our research agenda.

Figures and Tables

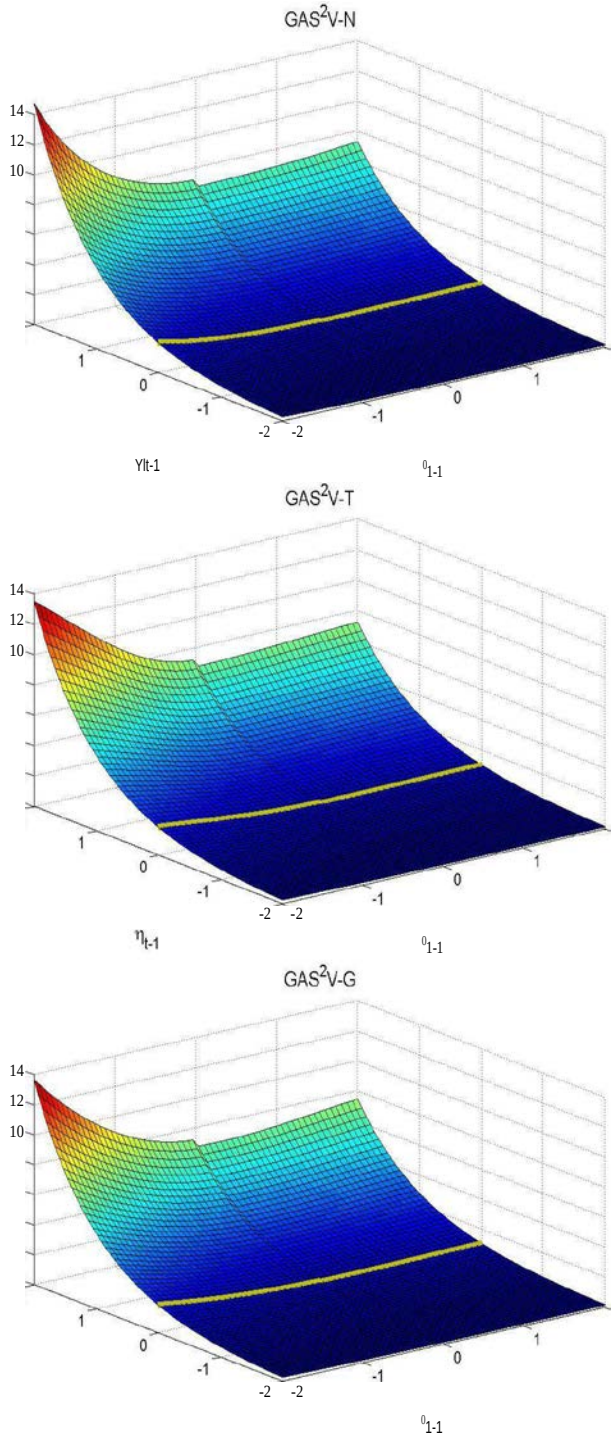

Figure 1: SNIS of GAS^2V-N (top panel) with parameters $(a, cf, k^*, k, u) = (0.07, 0.98, 0.08, 0.1, 0.05)$ and exp((1- cfJ)J.L)u;<> = 1, GAS^2V-T (middle panel) with $v_0 = 6$ and GAS^2V-G (bottom panel) with $v = 1.5$

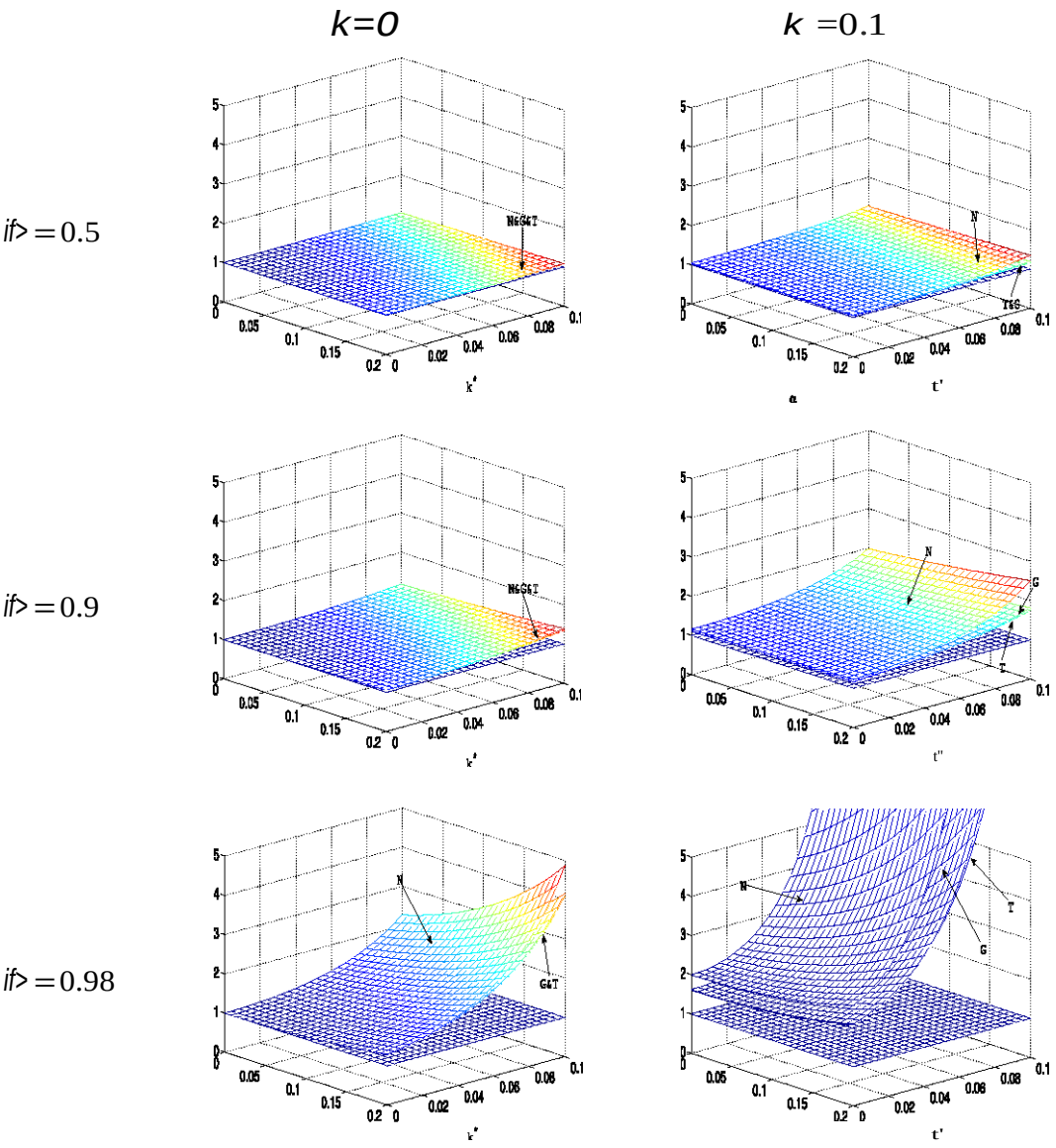

Figure 2: Ratio between the kurtoses of the GAS^{2}V model and the symmetric ARSV(l) model with Gaussian (N), GED (G) and Student-t (T) errors when $k = 0$ (left column) and 0.1 (right column) for three different values ofthe persistence parameter, $\langle/J = 0.5$ (first row), $rjJ = 0.9$ (middle row) and $rP = 0.98$ (bottom row).

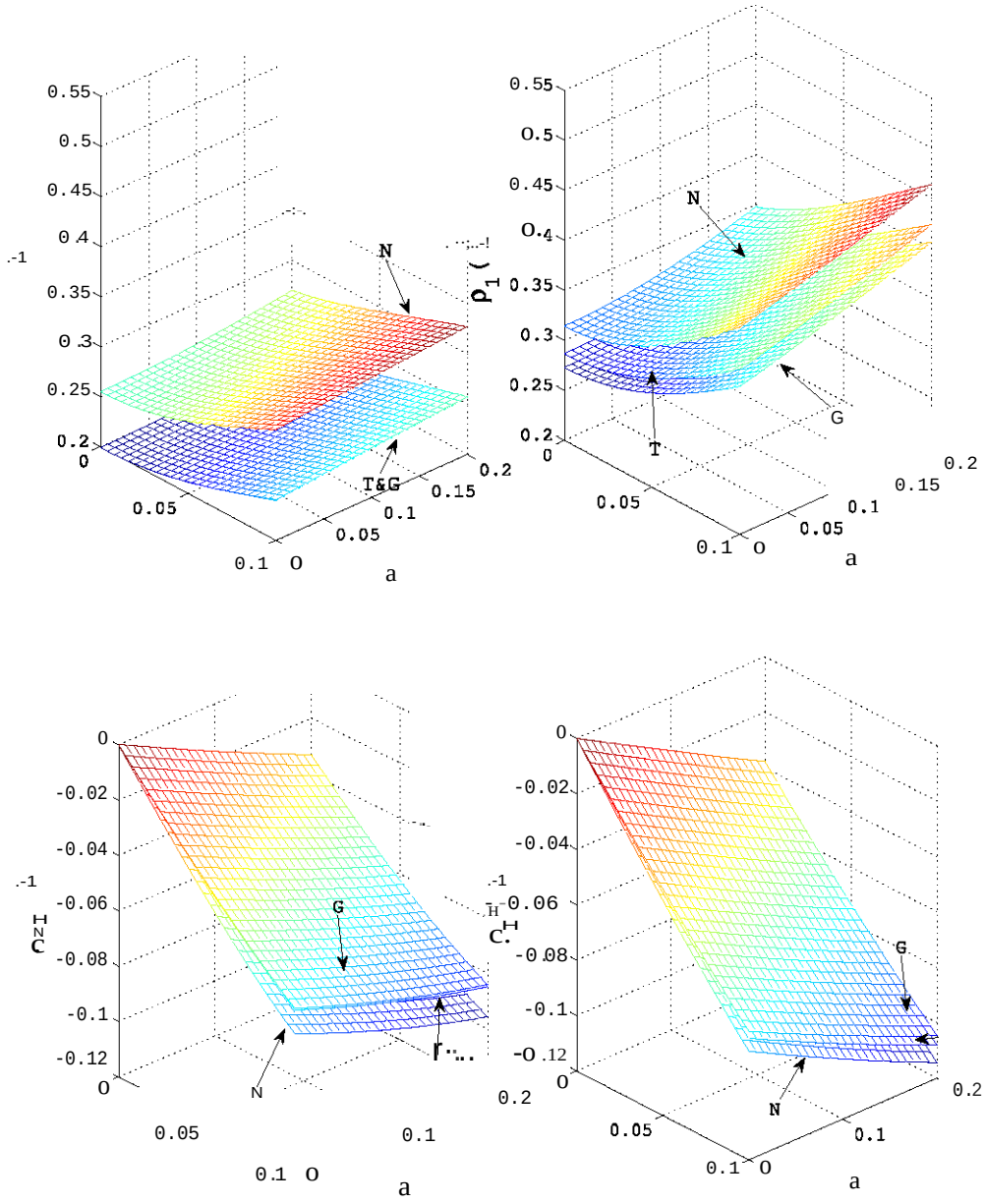

Figure 3: First order autocorrelations of squares (top left), first order autocorrelations of absolute returns (top right) , first order cross-correlations between returns and future squared returns (bottom left) and first order cross-correlations between returns and future absolute returns (bottom right) of different GAS^{2}V models when $\lambda = 0$, $rP = 0.98$, $\alpha = 0.05$, $v = 1.5$, $v_0 = 11.8745$ and $k = 0$. The surface N represents the moments of the GAS^{2}V-N model, T represents the moments of the GAS^{2}V-T model and G represents the moments of the GAS^{2}V-G model.

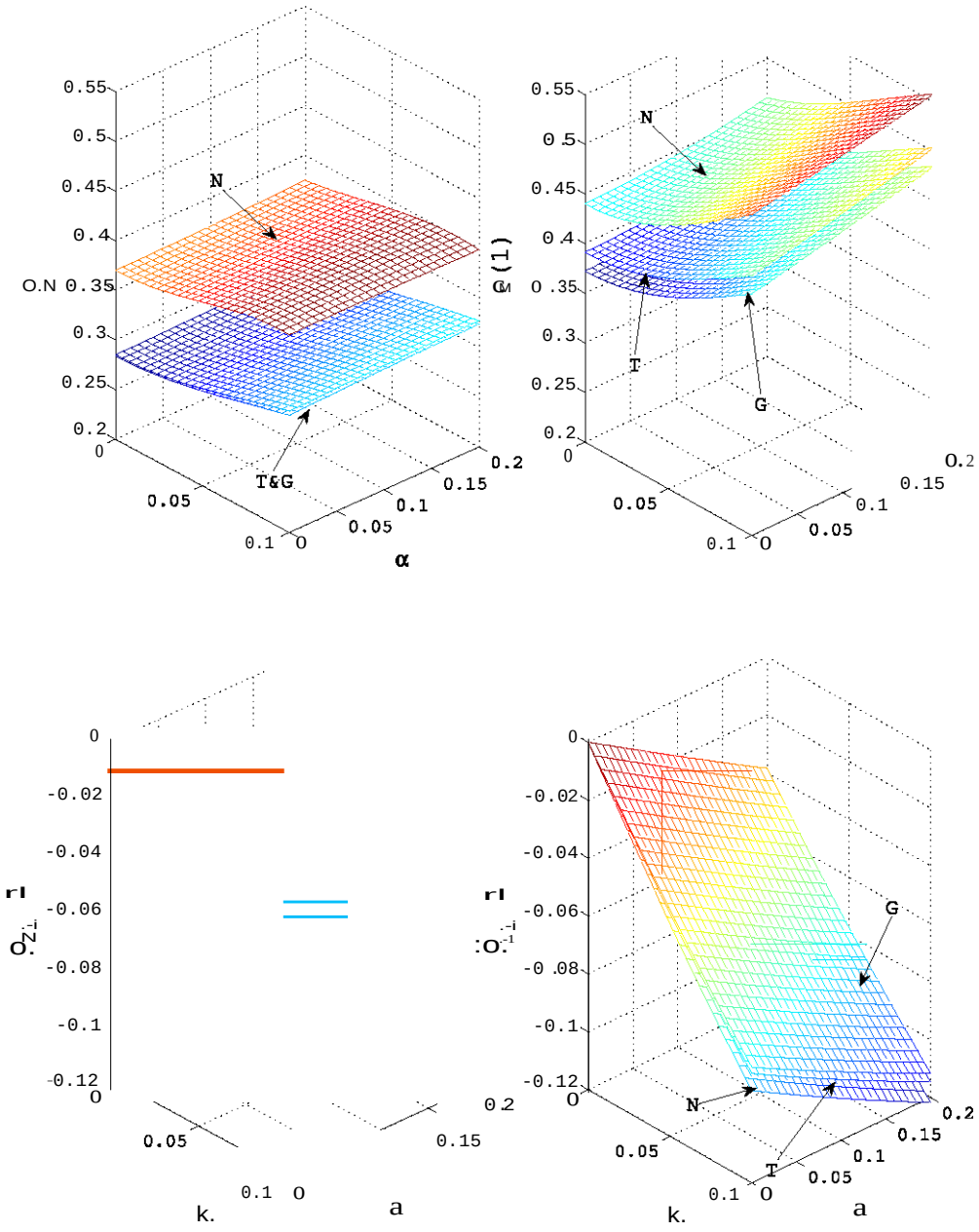

Figure 4: First order autocorrelations of squares (top left), first order autocorrelations of absolute returns (top right) , first order cross-correlations between returns and future squared returns (bottom left) and first order cross-correlations between returns and future absolute returns (bottom right) of different GAS^{2}V models when $Ji. = 0$, $rjJ = 0.98$, $\alpha = 0.05$, $v = 1.5$, $Vo = 11.8745$ and $k = 0.1$. The surface N represents the moments of the GAS^{2}V-N model, T represents the moments of the GAS^{2}V-T model and G represents the moments of the GAS^{2}V-G model.

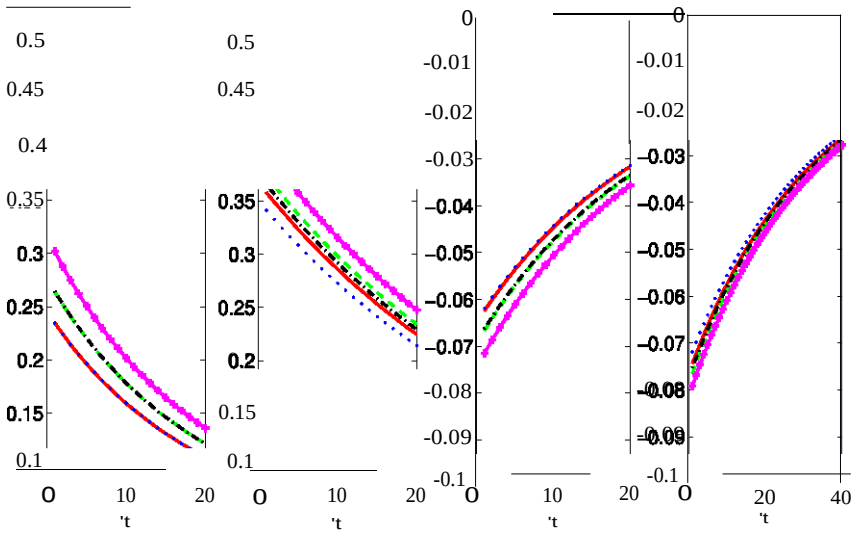

Figure 5: Autocorrelations of squares (first column), autocorrelations of absolute returns (second column), cross-correlations between returns and future squared returns (third column) and cross-correlations between returns and future absolute returns (fourth column) for different specifications of GAS^{2}V models when $4J = 0.98$, $\boldsymbol{u} = 0.05$, $a = 0.07$, $k^* = 0.08$ amd $k = O$. The solid line corresponds to the moments ofthe GAS^{2}V-T model with $v_0 = 11.8745$ while $v_0 = 19.8387$ for dashed lines. The dotted and dashdot lines corresponds to the moments of the GAS^{2}V-G model when $v = 1.5$ and 1.7, respectively. Finally, the '+-' line represents the moments of the GAS^{2}V-N model.

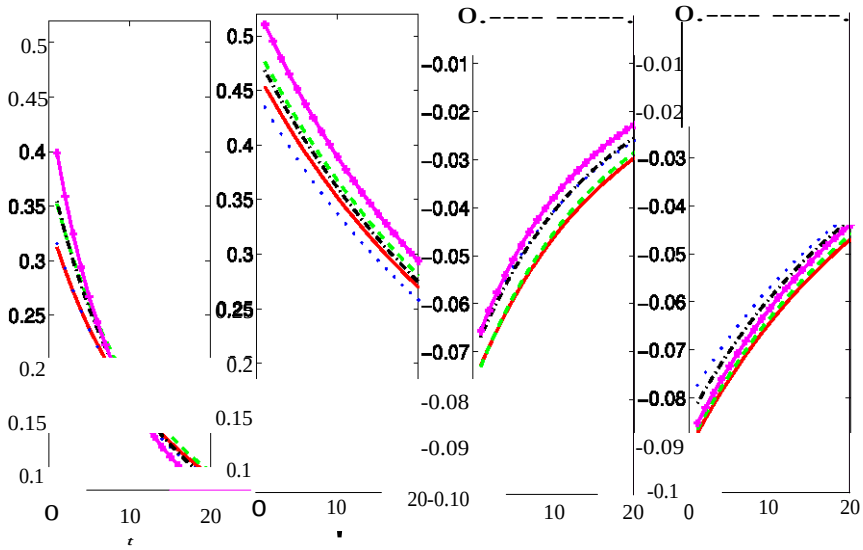

Figure 6: Autocorrelations of squares (first column), autocorrelations of absolute returns (second column), cross-correlations between returns and future squared returns (third column) and cross-correlations between returns and future absolute returns (fourth column) for different specifications of GAS^{2}V models when $\phi = 0.98$, $\alpha = 0.05$, $a = 0.07$, $k^* = 0.08$ and $k = 0.1$. The solid line corresponds to the moments ofthe GAS^{2}V-T model with $\nu_0 = 11.8745$ while $\nu_0 = 19.8387$ for dashed lines. The dotted and dashdot lines corresponds to the moments of the GAS^{2}V-G model when $\nu = 1.5$ and 1.7, respectively. Finally, the '+-' line represents the moments of the GAS^{2}V-N model.

	$1-\phi$	ϕ	α	k^*	σ^2	ν
GAS^{2}V-N						
True	0	0.98	0.07	0.08	0.06	
Mean	0.131	0.976	0.067	0.083	0.054	
	(1.259)	(0.010)	(0.056)	(0.021)	(0.018)	
s.d.	1.548	0.007	0.060	0.020	0.014	
GAS^{2}V-T						
True	0	0.98	0.07	0.08	0.05	11.8745
Mean	0.108	0.974	0.076	0.084	0.059	10.602
	(1.274)	(0.010)	(0.056)	(0.026)	(0.027)	(2.007)
s.d.	1.362	0.008	0.058	0.022	0.016	2.845
GAS^{2}V-G						
True	0	0.98	0.07	0.08	0.05	1.5
Mean	0.257	0.973	0.071	0.081	0.055	1.522
	(1.438)	(0.011)	(0.073)	(0.029)	(0.025)	(0.147)
s.d.	1.529	0.008	0.067	0.024	0.016	0.104

Table 1: Monte Cario results of the MCMC estimator of the parameters of the GAS^{2}V model. The value reported are the Monte Carla average and standard deviation (in parenthesis) of the posterior means together with the Monte Carla average of the posterior standard deviation.

	GAS^{2}V-N	GAS^{2}V-T	GAS^{2}V-G
1'	-2.401	-1.435	-1.892
	(-3.530, -0.215)	**(-1.782, -0.669)**	**(-2.518, -0.169)**
III	0.978	0.980	0.982
	(0.966,0.989)	**(0.968, 0.989)**	**(0.973, 0.992)**
▪▪	0.104	0.080	0.067
	(0.003, 0.145)	**(0.048, 0.108)**	**(0.050, 0.093)**
k'	0.056	0.087	0.073
	(0.041, 0.071)	**(0.069, 0.104)**	**(0.060, 0.080)**
▪▪	0.020	0.011	0.008
	(0.009, 0.030)	**(0.007, 0.021)**	**(0.001, 0.002)**
v		3.929	1.395
		(2.733, 3.176)	**(1.267, 1.422)**
Log-Lik.elihood	-6.070	-5.853	-5.900

Table 2: Estimation results from daily S&P500. The values reported are the mean and 95% credible interval (parenthesis) of the posterior distributions.

	Median	**Maximum**	**Mínimum**	**Std. Dev.**	**Sk.ewness**	**Kurtosis**	**P2(1)**	P1(1)	P21(1)	*Pn* (1)
S&P500	0.125	11.245	-20.195	2.404	**-0.813"'**	**10.354...**	**0.297***	**0.332***	**-0.254***	**-0.229***
NIKKEI225	0.136	11.529	-27.805	3.113	**-0.741***	**9.945"'**	**0.120"'**	**0.171"'**	**-0.125"'**	**-0.139"'**

"' Significant at 1% level.

Table 3: Sample moments of mean adjnsted weekly S&P500 and NIKKEI225 retnrns observed from Jan 13, 1992 to Dec 27, 2010.

Data	Model	Log MargLik	1"	4>	U'	III	"y¡Jk•	v
S&P500	GAS^{2}V-N	-1.579	-2.534	0.964	0.030	0.281	0.012	
				(-3.914, -1.171)	**(0.947, 0.982)**	**(0.013, 0.047)**	**(0.204, 0.380)**	**(-0.015, 0.037)**
	GAS^{2}V-T	-1.629	-2.032	0.971	0.018	0.205	0.022	15.490
			(-3.651, -1.097)	**{0.9542, 0.9839)**	**(0.004856, 0.02706)**	**{0.1375, 0.3335)**	**(-0.005294, 0.04658)**	**(9.894, 20.05)**
	GAS^{2}V-G	-1.577	-2.419	0.966	0.029	0.245	0.014	2.010
			(-5.262, -0.492)	**(0.937' 0.982)**	**(0.015, 0.054)**	**(0.106, 0.339)**	**(-0.017, 0.051)**	**{1.802, 2.215)**
	T-GASV-N	-1.579	-1.548	0.962	0.032	0.230	-0.047	
			(-3.576, 0.09023)	**(0.9436, 0.974)**	**(0.01539, 0.05349)**	**(0.09363, 0.3669)**	**(-0.1132, 0.007123)**	
	T-GASV-G	-1.527	-1.394	0.957	0.040	0.221	-0.058	2.257
			(-4.066, 0.516)	**{0.9283, 0.9788)**	**{0.01514, 0.07397)**	**{0.07043, 0.353)**	**(-0.1335, 0.02006)**	**{1.969, 2.526)**
NIKKEI225	GAS^{2}V-N	-7.975	1.288	0.882	0.060	0.164	0.057	
			(0.599, 1.873)	**(0.832, 0.932)**	**(0.029, 0.105)**	**(0.024, 0.306)**	**(0.006, 0.107)**	
	GAS^{2}V-T	-7.559	1.505	0.917	0.033	0.090	0.074	13.050
			(0.821, 2.125)	**(0.848, 0.957)**	**(0.015, 0.065)**	**(-0.024, 0.222)**	**(0.029, 0.112)**	**(8.252, 21.04)**
	GAS^{2}V-G	-7.966	1.348	0.877	0.064	0.152	0.064	2.002
			(0.793, 2.017)	**(0.797' 0.929)**	**(0.031, 0.119)**	**(0.016, 0.276)**	**(0.025, 0.114)**	**(1.824,2.252)**
	T-GASV-N	-7.523	1.800	0.893	0.059	0.041	-0.150	
			(0.700, 2.630)	**(0.833, 0.932)**	**(0.033, 0.100)**	**(-0.175, 0.258)**	**(-0.276, -0.041)**	
	T-GASV-G	-7.208	1.897	0.873	0.074	0.030	-0.158	2.164
			(1.068, 2.666)	**(0.801, 0.924)**	**(0.039, 0.126)**	**(-0.170, 0.229)**	**(-0.273, -0.055)**	**(1.964, 2.518)**

Table 4: MOMO estimates of paxameters of GAS^{2}V and GASV models fitted to weekly S&P500 and NIKKEI225. The values reported axe the mean and, in paxenthesis, 95% credible intervals of the posterior distributions.

MAE*lOOO					
	GAS^2V-N	GAS^2V-T	GAS^2V-G	T-GASV-N	T-GASV-G
S&P500	6.220	6.378	6.248	6.206	6.220
NIKKEI 225	9.171	9.536	9.157	9.235	8.992
LPS					
S&P500	-2.047	-2.047	-2.064	-2.049	-2.062
NIKKEI 225	-2.572	-2.524	-2.764	-2.579	-2.672

Table 5: Forecasting results from weekly data. MAE refers to the mean absolute forecasting error and LPS refers to the log-predictive likelihood.

Appendix A. Closed-form of E(e exp(bf(e,))) and E(letlcexp(bf(e,))}

Appendix A.1. Et *Normal*

Proposition I. *Let* e *be a non-negative integer and b* E lR *and* Et *and f(* Et) *are defined as in GAf!l V-N model. If bk* + lbk*l < !, *then*

$$E(IEtl^{0}\exp(bj(Et))) = \exp(-bk)\,)r\left(\frac{e+1}{2}\right)\left[\exp(ba)\left(\frac{1}{2}\right)^{-(e+1)}\left(1-b(k+k^*)\right)^{-(e+1)} + \left(\frac{1}{2}\right)^{-(e+1)}\left(1-b(k-k^*)\right)^{-(e+1)}\right]$$

and (A.1)

$$E(E\exp(bj(Et))) = ex:\,k)r\,(e;^{1})\,[e^{-1)^{0}}\exp(ba)\,G-\,b(k+k^*)r\bullet!' + G-\,b(k-k^*)r\cdot!'l$$

(A.2)

Proof. $= 1:($

$$)c\exp(ba + bkE - bk + bk^*E)\qquad \exp(-)\,dEt$$

E(IEtlc *exp(bf(* Et)))

-Et

$$+ lXJ_{(E,)^{0}}\exp(bkE - bk - bk^*E)\qquad \exp(-)dE, \qquad (A.3)$$

Integrating by substitution with $s, = -E,$ in the finite integral, we obtain

E(IEtlc *exp(bf(* Et)))

$$\exp(b(a-k))\,roo\,\frac{1}{\qquad}\qquad \exp(-bk)\,roo\,\frac{1}{\qquad}$$

$$= \quad ,j2ir \quad lo \quad (s,)^{0} \exp((b(k + k^{*}) - 2)s)ds, + \quad ,j2ir \quad lo \quad (Et)^{0} \exp((b(k - k^{*}) - 2)E)dE,. \tag{A.4}$$

According to the formula 3.326-2 of Ryzhik et al. (2007), when e 2:: 0 and $bk + |bk^*| <$ 'the former equation reduces to

$$E\left(\epsilon_t^c \exp(bf(\epsilon_t))\right) = \int_{-\infty}^{0} \epsilon_t^c \exp(b\alpha + bk\epsilon_t^2 - bk + bk^*\epsilon_t^2)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\epsilon_t^2}{2}\right)d\epsilon_t + \int_{0}^{\infty} (\epsilon_t)^c \exp(bk\epsilon_t^2 - bk - bk^*\epsilon_t^2)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\epsilon_t^2}{2}\right)d\epsilon_t$$

$$= \frac{\exp(b(\alpha-k))}{\sqrt{2\pi}}\int_0^{\infty}(-s_t)^c \exp((b(k+k^*)-\frac{1}{2})s_t^2)ds_t + \frac{\exp(-bk)}{\sqrt{2\pi}}\int_0^{\infty}(\epsilon_t)^c \exp((b(k-k^*)-\frac{1}{2})\epsilon_t^2)d\epsilon_t$$

$$= \frac{\exp(b(\alpha-k))}{\sqrt{2\pi}}\frac{(-1)^c\Gamma\left(\frac{c+1}{2}\right)}{2\left(\frac{1}{2}-b(k+k^*)\right)^{\frac{c+1}{2}}} + \frac{\exp(-bk)}{\sqrt{2\pi}}\frac{\Gamma\left(\frac{c+1}{2}\right)}{2\left(\frac{1}{2}-b(k-k^*)\right)^{\frac{c+1}{2}}}$$

$$= \frac{\exp(-bk)}{2\sqrt{2\pi}}\Gamma\left(\frac{c+1}{2}\right)\left[(-1)^c\exp(b\alpha)\left(\frac{1}{2}-b(k+k^*)\right)^{-\frac{c+1}{2}} + \left(\frac{1}{2}-b(k-k^*)\right)^{-\frac{c+1}{2}}\right]. \quad \text{(A.6)}$$

D

Appendix A.2. *t '''t11*

Proposition 2. *Let* e *be a nonnegative integer and* bE *R and* <:t *and J(t) defined as in GASJ V-T model, then when v* >e

$$E(|\epsilon_t|^e \exp(bf(f_t))) = (v-2)e/^2 \exp(-bk) \frac{B(!, T)}{B(\frac{2}{2}, 2)}$$

$$\cdot \{ \exp (ba:) [1 + L.,., (rri-l \; c + 1 + 2j) \; \frac{(b(v+1)(k+k^*))il}{v + 1 + 2 \cdot J} \; '!$$

$$+ [1 + E (fi \; \frac{e + 1 + 2j.)}{v + 1 + 2J} \; \frac{(b(v+1)\_(k-k'''))íl\}}{z!}$$

(A.7)

and

$$E(^e_{ft} \exp (bf(_{ft}))) = (v-2YI^2 \exp(-bk) \frac{B(, T)}{B(\frac{!}{2'}, 2)}$$

$$\cdot \{(-1)eexp(ba) [1 + E (fi \; \frac{+ \; + j\cdot)}{+ \; + J} \; (b(v+1). k+k'''))il$$

$$+ [1 + E (fi \; \frac{e + 1 + 2j.)}{// + 1 + 2J} \; \frac{(b(v+l)\_(k-k'''))il\}.}{t!}$$

(A.8)

Proof. The probability density function of ft is '1/Jo(t:t) = J:&.l)(1+ 2)-"t 1 where 'Po – y----¡;--, then u_t – (v + 1)b$_t$ – 1 and ι_t – E/(lli¡¡.>) ... $Beta(\frac{1}{2}, \frac{11}{2})$, see Harvey (2013).

$$E(IEtle \exp(bj(E,))) = I:(-E,)e\exp(b(a-k)) \exp(b(v+1)(k+k^*)b,) 1/>o(E,)dE,$$

$$+ \int^{+00} E \exp(-bk) \exp(b(v+1)(k-k^*)b,) 1/>o(Et)dEt$$

$$= \exp(b(a-k)) \int E \exp(b(v+1)(k+k^*)b,) 1/>o(Et)dEt$$

$$+ \exp(-bk) la'''' E \exp(b(v+1)(k-k^*)b,) 1/>o(Et)dEt$$

$$= \exp(b(a-k)) E(IEtle\exp(b(v+1)(k+k^*)b,)) \;\; 2$$

$$+ \exp(bk)) E (IEtle\exp(b(v+1)(k-k^*)b,)), \qquad (A.9)$$

We proceed to work out the expectation E(IEtle exp(mb,)) with respect toE,. Note that E(IEtle exp(mb,)) $= cp()E(vef^2 b/^2/(1-b,)''/^2$ exp(mb,)) with respect to b, BetaG,). It follows that

$$E(IEtle \exp(m)b,)$$

$$= 'Pe_0 \,{}_i 1vef2be/2/(1-b_t)e/2 \exp(mb_t) \frac{bf-\backslash 1-b,)-1db}{B(l\ '')_{2'\,2}}{}_t$$

$$- e\ e/2B(,''?') 11 \quad)b;'-(1-b,)'T-1db \quad {}_{,}\ {}^{1+c\ \ 1}$$

$$- 'Pov \frac{B(l\ '')}{2'2} \;0\; \exp m., \quad B(\frac{He}{2'2} ''----'')\ {}_t$$

$$= <pevef^2_0 \frac{B(He\ \ v-e)_{2\ ,\ 2}}{B(l\ '')_{2'\,2}} E(\exp(mb_t)) \qquad (A.1O)$$

with the expectation taken with respect toa $Betae!e, v e)$ when $v > e$, which is the moment generating function of b, $Betae!e, v e)$. It yields that

$$E(IEtle \exp(mb,) = 'Pove/2 \frac{B\ t\ (He\ \ v-e)}{Beta(2,2)} e\ a\ ,-:; \left\{ 1 + \sum_{k=l}^{oo} \left(\prod_{r=O}^{k-1} \frac{e+\frac{1}{2}+r}{v+1+2r} \right) \frac{m_{,}^{k}}{k.} \right\} . \qquad (A.ll)$$

Combing equation (A.9) and (A.ll) gives the expression. On the other hand,

E(E exp(bj(Et))) = ¡ :Efexp *(b(a- k))* exp *(b(v* + 1)(k + *k*)bt)* 1/lo(Et)dEt

+ 1+00 Ef exp(-*bk)* exp *(b(v* + 1)(k- *k*)bt) 1/lo(Et)dEt*

= (-1r*exp(b(a-k))* ¡ : Ef exp *(b(v* + 1)(k + *k*)bt) 1/Jo(Et)dEt*

+ exp(-*bk)* 1 oo Ef exp *(b(v* + 1)(k- *k*)bt)* 1/lo(Et)dEt

= (-l)cexp(b(a- *k)) E* (lftlcexp *(b(v* + 1)(k + *k*)bt))*
2

+ exp(-*bk))* *E(IEticexp(b(v+1)(k-k*)bt))* (A.12)
2

The proof is completed. D

Appendix A.9. ft"' *GED(v)*

Proposition 3. *Let* e *be a nonnegative integer and b* E lR. ft *and f(* Et) *defined as in GASJ V-G model. Then, when bk* + *lbk*i* < 1/v,

2 1

E(lftlcexp(bf(Et))) = *exp(b(a-k))* (r(1/v)t/- ; ( ) (1-*vb(k* + *k*))*- ,
2 (r( )r1

+ exp(-*bk)* (r(1/v)t/2-1 r( ) (1- *vb(k- k*))*- ' (A.13)
2 (r( )r/2

and

E(fe e xp *(bj(E*))) = (-1Yexp(b(a- *k))* (r(l/v)t/2-1 r( ) (1- *vb(k* + *k*))*-
t *t* 2 (r(!))c/2

+ exp(-*bk)* (r(l/v)t/2-1 r( ) (1- *vb(k*- k*))_ctl. (A.14)
2 (r(!)t/2

Proof.

$$E(|E_t|^c \exp(bj(E_t)))$$
$$= \int_{-\infty}^{0} (-E_t)^c \exp(ba + bkU_t + bk^*(u_t + 1))'1/l(E_t)dE_t + \int_{0}^{+\infty} E^c \exp(bku_t - bk^*(u_t + 1))'1/l(E_t)dE_t$$
$$= \exp(b(a-k)) \int_{0}^{+\infty} E^c \exp(b(k+k^*)'igt)'I/J(E_t)dE_t + \exp(-bk) \int_{0}^{+\infty} E^c \exp(b(k-k^*)'igt)'I/J(E_t)dE_t$$
$$\exp(b(a-k))$$
$$= \underline{cpc\exp(b(a-k))} E(\quad (b(k$$
$$= \exp(b(a-k)) E(|E_t|^c \exp(b(k+k^*)\,gt)) + \exp(-bk) E(|E_t|^c \exp(b(k-k^*)\,gt))$$
$$= E(cpcg[\exp(b(k+k^*)'igt)) + E(cpcg[\exp(b(k-k^*)'igt))$$
$$2 \quad 9t \exp \quad + \quad 2 \quad 9t \quad + \quad 2 \quad 9t \exp \quad 2 \quad 9t$$

According to the Appendix B.2 of Harvey (2013), when $E(\exp(b(k+k^*Hgt)) < \infty$ and $E(\exp(b(k-k^*)\,gt)) < \infty$, the previous equation can be written

$$cpc\exp(b(a-k)) \frac{2\,\Gamma(\quad)}{\Gamma(\quad)} E(\quad (vb(k+k^*)\_)) \quad \underline{cpc\exp(-bk)} \frac{2\,\Gamma(\quad)}{\Gamma(\quad)} E(\quad (\underline{vb(k-k^*)}\_))$$
$$2 \quad \exp \quad 2 \quad 9t \quad + \quad 2 \quad \exp \quad 2 \quad 9t$$

where $Yt \sim Gamma(2, c^{-1})$. When $bk + |bk^*| < '$ both $E(\exp(b(k+k^*Hgt))$ and $E(\exp(b(k-k'')\,gt))$ are finite and given by the generating moments function of the Gamma distribution, then

$$E(|\epsilon_t|^c \exp(bf(\epsilon_t))) = \frac{\exp(b(\alpha-k))}{2} \frac{(\Gamma(1/\nu))^{c/2-1}\Gamma(\frac{c+1}{\nu})}{(\Gamma(\frac{3}{\nu}))^{c/2}} (1-\nu b(k+k^*))^{-\frac{c+1}{\nu}}$$
$$+ \frac{\exp(-bk)}{2} \frac{(\Gamma(1/\nu))^{c/2-1}\Gamma(\frac{c+1}{\nu})}{(\Gamma(\frac{3}{\nu}))^{c/2}} (1-\nu b(k-k^*))^{-\frac{c+1}{\nu}}$$

The expression for $E(|E_t|^c \exp(bj(E_t)))$ can be obtained following the similar steps.

D

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