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Abstract

General risk functions are becoming very important for managers, regulators and supervisors. Many risk functions are interpreted as initial capital requirements that a manager must add and invest in a risk-free security in order to protect the wealth of his clients.

This paper deals with a complete arbitrage free pricing model and a general expectation bounded risk measure, and it studies whether the investment of the capital requirements in the risk-free asset is optimal. It is shown that it is not optimal in many important cases. For instance, if the risk measure is the $CVaR$ and we consider the assumptions of the Black and Scholes model. Furthermore, in this framework and under short selling restrictions, the explicit expression of the optimal strategy is provided, and it is composed of several put options. If the confidence level of the $CVaR$ is close to 100% then the optimal strategy becomes a classical portfolio insurance. This theoretical result seems to be supported by some independent and recent empirical analyses.

If there are no limits to sale the risk-free asset, *i.e.*, if the manager can borrow as much money as desired, then the framework above leads to the existence of "good deals" (*i.e.*, sequences of strategies whose VaR and $CVaR$ tends to minus infinite and whose expected return tends to plus infinite). The explicit expression of the portfolio insurance strategy above has been used so as to construct effective good deals. Furthermore, it has been pointed out that the methodology allowing us to build portfolio insurance strategies and good deals also applies for pricing models beyond Black and Scholes, such as Heston and other stochastic volatility models.

Key words. Risk Measure, Capital Requirement, Good Deal, Portfolio Insurance.

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1 Introduction

Since Artzner *et al.* (1999) introduced the axioms and properties of the "Coherent Measures of Risk" many authors have extended the discussion. The (often legal) obligation of providing initial capital requirements has made it necessary to overcome the variance as the most used risk measure, and to introduce more general and operational risk functions with clear interpretation in monetary terms. So, among many other interesting contributions, Goovaerts *et al.* (2004) introduced the Consistent Risk Measures, Rockafellar *et al.* (2006) defined the Expectation Bounded Risk Measures, Zhiping and Wang, (2008) presented the Two-Sided Coherent Risk Measures, Brown and Sim (2009) introduced the Satisfying Measures, and Aumann and Serrano (2008) and Foster and Hart (2009) defined Indexes of Riskiness. All of these measures are more and more used by researchers, practitioners, regulators and supervisors.

Therefore, many risk measures provide regulators and supervisors with the capital reserve that a manager must add in order to protect the wealth of her/his clients. It is usually assumed that the capital requirements will be invested in a risk-free asset, though, as far as we know, nobody has proved that the investment in a risk-free asset will outperform every alternative hedging strategy. On the contrary, in particular situations, derivatives may outperform bonds when minimizing risk (Schied, 2006).

In a recent paper Artzner *et al.* (2009) consider the possibility of investing the capital requirement in an alternative "eligible asset". Filipovic (2008) also deals with similar problems, and shows that, under weak assumptions, a risky numeraire cannot reduce the capital requirements generated by the risk-free asset in a solvency test.

Balbás *et al.* (2009) have dealt with optimal reinsurance problems and have shown that for linear pricing principles the optimal contract may be a stop-loss one, though risk levels can be given by expectation bounded risk measures. Actuaries know that a stop-loss reinsurance may be understood as an "European option" whose underlying variable is the global amount paid by the insurer (claims). On the other hand, a classical viewpoint uses European puts so as to provide investors with "Portfolio Insurance". Moreover, the empirical evidence seems to reveal that classical "portfolio insurance strategies" also perform well in practice if risk levels are given by the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) (Annaert *et al.*, 2009).

The present paper considers a complete market and a general expectation bounded measure of risk and analyzes whether the investment of the capital requirements in the risk-free security outperforms the remaining feasible hedging investments. According to the ideas above, it could make sense to study the effectiveness of investing this money in adequate derivatives. There is a significant difference between our approach and those of Filipovic (2008) or Artzner *et al.* (2009). Indeed, we do not verify the existence of a fixed risky eligible asset. On the contrary, we bear in mind that the optimal investment generating a reduction of the capital requirements may be closely related to the initial portfolio we are dealing with. Thus, we follow the line of Balbás *et al.* (2009).

The notion of "Good Deal" was introduced in the seminal paper by Cochrane and Saa-Requejo (2000) . Mainly, a good deal is an investment strategy providing traders with a "very high return/risk ratio", in comparison with the value of this ratio for the Market

Portfolio. Risk is measured with the standard deviation, and the absence of good deals is imposed in an arbitrage-free model so as to price in incomplete markets.[1] This line of research has been extended for more general risk functions.[2] Moreover, some recent papers impose other conditions that are also strictly stronger than the absence of arbitrage (Dana and Le Van, 2010, Stoica and Lib, 2010, etc.). They fix a risk measure and its subgradient must contain "Equivalent Risk Neutral Probabilities".[3]

However, the fulfillment of these assumptions stronger than the arbitrage absence is not so obvious in very important Pricing Models of Financial Economics. Balbás *et al.* (2010a) have shown the existence of "pathological results" when combining some risk measures ($CVaR$, Dual Power Transform or DPT, etc.) and very popular pricing models (Black and Scholes, Heston, etc.). Indeed, for the examples above the Stochastic Discount Factor (SDF) of the pricing model and the risk measure subgradient do not satisfy some relationships, which implies the existence of sequences of portfolios whose expected returns tend to plus infinite and whose risk levels tend to minus infinite or remain bounded ($risk = -\infty$ *and* $return = +\infty$, *or bounded risk and* $return = +\infty$). This finding of Balbás *et al.* (2010a) was called by the authors "lack of compatibility between prices and risks", and is probably related to the generalizations of arbitrage and the good deals above.

The present paper seems to present several contributions, all of them related to the previous discussion. First, there are many examples where the investment of the capital requirements in a risk-free asset is outperformed by alternative hedging strategies. These examples present a SDF which does not belong to the risk measure subgradient. In these examples, if the manager must respect short selling restrictions when trading the risk-free asset then the existence of optimal alternative hedging strategies usually holds. These strategies will be called "Shadow Riskless Assets".[4] However, as a second contribution of this paper, if the manager can borrow as much money as desired and invest this money and the capital

[1] Bernardo and Ledoit (2000) also defined new concepts closely related to the notion of good deal. Besides, interesting discussions about the use of return/risk ratios may be also found in Zakamouline and Koekebbaker (2009).

[2] See Staum (2004), amongst many other interesting contributions.

[3] Thus, the existence of "Equivalent Risk Neutral Probabilities" is not sufficient. Some of them must belong the the risk measure subgradient.

[4] We have taken the expression "shadow riskless asset" from Ingersoll (1987), where the author constructs a hedging strategy in a pricing model without interest rate.

requirements in risky assets, then we will face one of the pathologies above ($risk = -\infty$, $return = +\infty$). In such a case there is no lower/upper bound for the ($risk$, $return$) couple, so every hedging strategy may be outperformed by a new one, and that leads to sequences of hedging strategies with unlimited potential gains. We have used the expression "good deal" to represent these sequences making the manager as rich as desired.

The third contribution is the explicit expression of the shadow riskless asset for the $CVaR$ and the Black and Scholes model. It is composed of a long European put, plus a short European put, plus a short binary put. If the confidence level of the $CVaR$ is close to 100% then the shadow riskless asset becomes an European put option, closely related with the notion of "portfolio insurance". This theoretical result seems to be supported by some independent and recent empirical analyses (Annaert *et al.*, 2009). This may be a surprising and important finding for researchers, practitioners, regulators and supervisors. In particular, managers can significantly reduce the capital requirements by trading options.

The fourth contribution is the effective construction of good deals, also for the $CVaR$ (or the VaR) and the Black and Scholes model. The explicit expression of the shadow riskless asset is the key to do that. Though the existence of good deals was proved in Balbás *et al.* (2010*a*), explicit computations for them had not been obtained. Furthermore, good deals had not been connected with the optimal investment of the capital requirements.[5]

Finally, it is worth pointing out that the methodology allowing us to build shadow riskless assets and good deals also applies for pricing models beyond Black and Scholes, such as the Heston model and other stochastic volatility models.

The article's outline is as follows. Section 2 will present the notations and the general framework we are going to deal with. Section 3 will present three optimization problems related to the investment of capital requirements, along with the relationships among them and the dual approach. Section 4 will show that shadow riskless assets are not risk-free for many pricing models and risk measures, along with the existence of good deals. Section 5 will extend the most important results if the risk measure is the $CVaR$. Section 6 will present the accurate construction of shadow riskless assets and good deals for the Black and

[5] As said above, we have followed the expression "good deal" of Cochrane and Saa-Requejo (2000) rather than "compatibility of prices and risks", the one used by Balbás *et al.* (2010*a*). Nevertheless, the good deals of this paper arise for the "non compatible" cases of Balbás *et al.* (2010*a*).

Scholes model, though it will be shown that the methodology also applies if we consider other pricing models. As said above, the shadow riskless asset is a combination of three European puts. Section 7 presents the most important conclusions of the paper.

2 Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. We Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual L^p (L^q) denotes the space of $\mathbb{R}-$valued random variables y on Ω such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}()$ representing the mathematical expectation ($\mathbb{E}\left(|y|^q\right) < \infty$, or y essentially bounded if $q = \infty$). According to the Riesz Representation Theorem (Horvàth, 1966), we have that L^q is the dual space of L^p. Furthermore, if B_p^+ and B_q^+ represent the intersection of the non-negative cones with the unit closed balls in L^p and L^q respectively, then it may be shown that B_q^+ is $\sigma\left(L^q, L^p\right)-$compact,

$$B_q^+ = \left\{z \in L^q; 0 \leq \mathbb{E}\left(yz\right) \leq 1, \ \forall y \in B_p^+\right\}, \tag{1}$$

and

$$\|y\|_p = Max \ \left\{\mathbb{E}\left(yz\right); \ z \in B_q^+\right\} \tag{2}$$

for every $y \in L^p$, $y \geq 0$. [6]

As usual, we will assume that prices are in L^2. Thus, consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $\left(\mathcal{F}_t\right)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We will assume that the market is complete, *i.e.*, every final pay-off $y \in L^2$ may be reached by the price process $\left(S_t\right)_{t \in \mathcal{T}}$ of a self-financing portfolio. This process is adapted to the filtration $\left(\mathcal{F}_t\right)_{t \in \mathcal{T}}$ and satisfies the equality $S_T = y$, *a.s.* Consequently, suppose also that there is a linear and continuous pricing rule

$$\Pi : L^2 \longrightarrow \mathbb{R}$$

providing us with the initial (at $t = 0$) price $\Pi(y)$ of every $y \in L^2$.

[6] As usual, $\|y\|_p = \left(\mathbb{E}\left(|y|^p\right)\right)^{1/p}$. See Horvàth (1966) for further details about the $\sigma\left(L^q, L^p\right)$ topology and $\sigma\left(L^q, L^p\right)-$compact sets.

The completeness of the model implies the existence of a risk-free asset. Thus, denote by $r_f \geq 0$ the risk-free rate, and the equality

$$\Pi(k) = ke^{-r_f T} \tag{3}$$

must hold for every $k \in \mathbb{R}$.

According to the Riesz Representation Theorem there exists a unique $z_\pi \in L^2$ such that

$$\Pi(y) = e^{-r_f T}\mathbb{E}(y z_\pi)$$

for every $y \in L^2$. Moreover, to prevent the existence of arbitrage, the strict inequality

$$z_\pi > 0 \tag{4}$$

$a.s.$ must hold (Chamberlain and Rothschild, 1983, or Duffie, 1988). z_π is usually called "Stochastic Discount Factor" (SDF), and it is closely related to the Market Portfolio of the $CAPM$ (Duffie, 1988).

Expression (3) implies that $ke^{-r_f T} = \Pi(k) = e^{-r_f T} k \mathbb{E}(z_\pi)$, which leads to

$$\mathbb{E}(z_\pi) = 1. \tag{5}$$

We will deal with risk measures that may be extended beyond L^2. Let $p \in [1, 2]$ and consider its conjugate number $q \in [2, \infty]$. Let $\rho : L^p \longrightarrow \mathbb{R}$ be the general risk function that a trader uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; -\mathbb{E}(yz) \leq \rho(y), \ \forall y \in L^p\}.$$[7]

We will assume that Δ_ρ is convex and $\sigma(L^q, L^p)$ −compact, and

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{7}$$

holds for every $y \in L^p$. Furthermore, we will also impose

$$\Delta_\rho \subset \{z \in L^q; \mathbb{E}(z) = 1\}. \tag{8}$$

Summarizing, we have:

[7] Δ_ρ is usually called the "subgradiet of ρ". (6)

Assumption 1. The set Δ_ρ given by (7) is convex and $\sigma\left(L^q, L^p\right)-$compact, $z = 1$ $a.s.$ is in Δ_ρ, (7) holds for every $y \in L^p$, and (8) holds. $\square$

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, it is easy to prove that Assumption 1 holds if and only if ρ is continuous and satisfies

$$\rho\left(y+k\right) = \rho\left(y\right) - k \tag{9}$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

$$\rho\left(\alpha y\right) = \alpha\rho\left(y\right) \tag{10}$$

for every $y \in L^p$ and $\alpha > 0$.

$$\rho\left(y_1 + y_2\right) \leq \rho\left(y_1\right) + \rho\left(y_2\right) \tag{11}$$

for every $y_1, y_2 \in L^p$.

$$\rho\left(y\right) \geq -\mathbb{E}\left(y\right) \tag{12}$$

for every $y \in L^p$.[8]

It is easy to see that if ρ is continuous and satisfies Properties (9), (10), (11), and (12) then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L^q_+ = \left\{z \in L^q; \mathbb{P}\left(z \geq 0\right) = 1\right\}. \tag{13}$$

Particular interesting examples are the Conditional Value at Risk ($CVaR$, Rockafellar *et al.,* 2006), the Weighted Conditional Value at Risk ($WCVaR$, Cherny, 2006), the Compatible Conditional Value at Risk ($CCVaR$, Balbás *et al.,* 2010a), the Dual Power Transform (DPT) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Furthermore, following the original idea of Rockafellar *et al.* (2006) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho\left(y\right) = \sigma\left(y\right) - \mathbb{E}\left(y\right) \tag{14}$$

[8] Actually, the properties above are almost similar to those used by Rockafellar *et al.* (2006) in order to introduce their Expectation Bounded Risk Measures. These authors also impose (9), (10), (11) and (12), work with $p = 2$, allow for $\rho\left(y\right) = \infty$, and impose $\rho\left(y\right) > -\mathbb{E}\left(y\right)$ if y is not zero-variance.

is continuous and satisfies (9), (10), (11), and (12) if $\sigma : L^p \longrightarrow \mathbb{R}$ is a continuous deviation, that is, if σ is continuous and satisfies (10), (11),

$$\sigma(y+k) = \sigma(y)$$

for every $y \in L^p$ and $k \in \mathbb{R}$, and

$$\sigma(y) \geq 0$$

for every $y \in L^p$. Particular examples are the classical $p-$deviation given by

$$\sigma_p(y) = \left[\mathbb{E}\left(\left|\mathbb{E}(y) - y\right|^p\right)\right]^{1/p},$$

or the downside $p-$semi-deviation given by

$$\sigma_p^-(y) = \left[\mathbb{E}\left(\left|Max\ \{\mathbb{E}(y) - y, 0\}\right|^p\right)\right]^{1/p}.$$

3 Shadow riskless assets: Primal and dual approaches

Suppose that the random variable $y_0 \in L^2$ represents a trader's final wealth. Its final risk will be given by $\rho(y_0)$, which justifies that this quantity may be an adequate final value (at T) of the capital requirement. Indeed, (9) leads to

$$\rho(y_0 + \rho(y_0)) = 0 \tag{15}$$

and the risk will vanish if the additional amount $\rho(y_0)\, e^{-r_f T}$ is invested in the risk-free security.[9] Our first purpose is to study whether the investment above in the risk-free asset is the best solution so as to make the risk vanish. Until now, in a general context it has not been proved that an alternative investment will be outperformed by the risk-free asset.

Consequently, consider the pay-off $y \in L^2$ added by the trader to his initial portfolio $y_0 \in L^2$. Suppose that

$$C > 0 \tag{16}$$

gives (the value at T of) the highest amount of money that will be invested to reduce the risk level.[10] Then the trader will choose y so as to solve

[9] See Johnston (2009) for further and recent discussions about the computation of capital requirements.

[10] If $\rho(y_0) > 0$ then (15) shows that $C = \rho(y_0)$ could be a suitable choice for C.

$$\begin{cases} Min\ \rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right) \\ \mathbb{E}\left(yz_\pi\right)\leq C \\ y\geq 0 \end{cases}. \tag{17}$$

Problem (17) considers the global risk level $\rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right)$ that the trader is facing, so it has to incorporate the value $\mathbb{E}\left(yz_\pi\right)$ of the added portfolio, that will have to be paid and will reduce the trader's wealth. Constraint $y\geq 0$ may be indicating the presence of short-selling restrictions. Since we are minimizing risk, one could consider that short sales should be allowed if they do not make the riskiness increase, so we could also deal with Problem

$$\begin{cases} Min\ \rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right) \\ \mathbb{E}\left(yz_\pi\right)\leq C \end{cases} \tag{18}$$

Some results below will show that (18) is often unbounded, *i.e.*there are sequences $(y_n)_{n=1}^{\infty}$ of (18)-feasible portfolios such that $\rho\left(y_n+y_0-\mathbb{E}\left(yz_\pi\right)\right)\rightarrow-\infty$. Furthermore, as we will prove in Proposition 1 below, if the existence of this sequence holds then it provides us with returns converging to $+\infty$. Henceforth, if there are sequences (18)-feasible (or (17)-feasible) whose riskiness converges to $-\infty$ (and therefore their expected return converges to $+\infty$) then we will say that Problem (18) (or Problem (17)) admits good deals.

Proposition 1 *If the sequence* $(y_n)_{n=1}^{\infty}$ *satisfies* $Lim_{n\rightarrow-\infty}\rho\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right)=-\infty$, *then* $Lim_{n\rightarrow-\infty}\mathbb{E}\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right)=+\infty$.

Proof. (12) shows that $\mathbb{E}\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right)\geq-\rho\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right)\rightarrow+\infty$. □

As said above, we will see that the presence of good deals for (18) often holds. Moreover, we will also show that the existence of solutions of (17) is not guaranteed, so it is worth introducing additional constraints overcoming this caveat. Let $\|.\|_2$ be the usual norm in L^2. Problem (17) may be modified according to

$$\begin{cases} Min\ \rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right) \\ \|y\|_2\leq R \\ \mathbb{E}\left(yz_\pi\right)\leq C \\ y\geq 0 \end{cases} \tag{19}$$

$R > 0$ being an arbitrary real number. Since $y = 0$ satisfies the constraints of (17), (18) and (19) (see (16)) it is obvious that these problems are feasible. Let us see that (19) is also bounded and attains its optimal value.

Proposition 2 *Problem (19) is bounded and attains its optimal value.*

Proof. From Assumption 1 it is easy to see that ρ is $\sigma(L^2, L^2)-$lower semi-continuous in L^2.[11] Moreover, the feasible set of (19) is $\sigma(L^2, L^2)-$compact due to the constraint $\|y\|_2 \leq R$ and the Alaoglu's Theorem. Therefore, the conclusion is obvious because lower semi-continuous functions always attain a global minimum in compact sets.[12] □

As said in the introduction, if the solution y^* of (17) exists then it will be called "shadow riskless asset". Similarly, the solution y^*_R of (19) will be called "$R-$shadow riskless asset".

In general, Problems (18), (17) and (19) are not differentiable because ρ is not differentiable either. Recent literature has developed several optimization methods that may solve this caveat. In this paper we will follow the procedure given in Balbás *et al.* (2010*b*). Some duality linked properties and Theorems 3 and 4 below will not be proved because they trivially follow from the results of the mentioned paper. A concrete application of the method may be also found in Balbás *et al.* (2009).

In particular, one can show that Problem

$$\begin{cases} Max \ -C\lambda - \mathbb{E}(y_0 z) \\ z \leq (1+\lambda) z_\pi \\ \lambda \in \mathbb{R},\ \lambda \geq 0,\ z \in \Delta_\rho \end{cases} \tag{20}$$

is the dual of (17), $\lambda \in \mathbb{R}$ and $z \in \Delta_\rho$ being the decision variables. Similarly,

$$\begin{cases} Max \ -\mathbb{E}(y_0 z) \\ z = z_\pi \\ z \in \Delta_\rho \end{cases} \tag{21}$$

is the dual of (18), $z \in \Delta_\rho$ being the decision variable. Finally,

$$\begin{cases} Max \ -C\lambda - \mathbb{E}(y_0 z) - R\tilde{\lambda} \\ z \leq (1+\lambda) z_\pi + \tilde{\lambda}\tilde{z} \\ \lambda, \tilde{\lambda} \in \mathbb{R}, \lambda \geq 0, \tilde{\lambda} \geq 0, z \in \Delta_\rho, \tilde{z} \in B_2^+ \end{cases} \tag{22}$$

[11] Notice that Δ_ρ is $\sigma(L^q, L^p)-$compact, so it is $\sigma(L^2, L^2)-$compact too, since $q \geq 2$.

[12] See Horvàth (1966) for further details about those mathematical properties used in this proof.

is the dual problem of (19), $\lambda, \tilde{\lambda} \in \mathbb{R}$, $z \in \Delta_\rho$ and $\tilde{z} \in B_2^+$ being the decision variables.

Following Balbás *et al.* (2010*b*) and bearing in mind Proposition 2, the following primal-dual relationships hold

Theorem 3 *Suppose that $y^* \in L^2$ and $(\lambda^*, z^*) \in \mathbb{R} \times L^2$. Then, they solve (17) and (20) if and only if the following Karush-Kuhn-Tucker conditions*

$$\left\{ \begin{array}{ll} \lambda^* \left(C - \mathbb{E}\left(y^* z_\pi \right) \right) = 0 & \\ C - \mathbb{E}\left(y^* z_\pi \right) \geq 0 & \\ \mathbb{E}\left(\left(y^* + y_0 \right) z \right) \geq \mathbb{E}\left(\left(y^* + y_0 \right) z^* \right), & \forall z \in \Delta_\rho \\ \left(\left(1 + \lambda^* \right) z_\pi - z^* \right) y^* = 0 & \\ \left(1 + \lambda^* \right) z_\pi - z^* \geq 0 & \\ y^* \in L^2,\ y^* \geq 0,\ \lambda^* \in \mathbb{R},\ \lambda^* \geq 0,\ z^* \in \Delta_\rho & \end{array} \right. \tag{23}$$

are fulfilled. Moreover, if (17) is bounded then both optimal values coincide and the dual solution is attainable. □

Theorem 4 *Suppose that $y_R^* \in L^2$ and $\left(\lambda_R^*, \tilde{\lambda}_R^*, z_R^*, \tilde{z}_R^* \right) \in \mathbb{R}^2 \times \left(L^2 \right)^2$. Then, they solve (19) and (22) if and only if the following Karush-Kuhn-Tucker conditions*

$$\left\{ \begin{array}{cr} \lambda_R^* \left(C - \mathbb{E}\left(y_R^* z_\pi \right) \right) = 0 & \\ C - \mathbb{E}\left(y_R^* z_\pi \right) \geq 0 & \\ \mathbb{E}\left(\left(y_R^* + y_0 \right) z \right) \geq \mathbb{E}\left(\left(y_R^* + y_0 \right) z_R^* \right), & \forall z \in \Delta_\rho \\ \left(\left(1 + \lambda_R^* \right) z_\pi + \tilde{\lambda}_R^* \tilde{z}_R^* - z_R^* \right) y_R^* = 0 & \\ \left(1 + \lambda_R^* \right) z_\pi + \tilde{\lambda}_R^* \tilde{z}_R^* - z_R^* \geq 0 & \\ \tilde{\lambda}_R^* \left(\mathbb{E}\left(y_R^* \tilde{z}_R^* \right) - R \right) = 0 & \\ \mathbb{E}\left(y_R^* \tilde{z} \right) \leq R, & \forall \tilde{z} \in B_2^+ \\ y_R^* \in L^2,\ y_R^* \geq 0,\ \lambda_R^*, \tilde{\lambda}_R^* \in \mathbb{R},\ \lambda_R^* \geq 0, \tilde{\lambda}_R^* \geq 0,\ z_R^* \in \Delta_\rho, \tilde{z}_R^* \in B_2^+ & \end{array} \right. \tag{24}$$

are fulfilled. Moreover, (19) and (22) are bounded and their optimal values coincide and are attainable. □

Henceforth $\rho^* \geq -\infty$ will represent the optimal value of (17) and (20), while $\rho_R^* > -\infty$ will represent optimal value of (19) and (22), for every $R > 0$.

Remark 1 *The major difference between Theorems* 3 *and* 4 *above is related to the existence of solutions, which cannot be guaranteed for (17). Actually (Balbás et al., 2010b), the three situations below may occur:*

a) Problem (17) is unbounded and Problem (20) is not feasible, i.e., $\rho^* = -\infty$ *and there are no elements satisfying the constraints of (20).*

*b*1*) Problem (17) is bounded but unsolvable, i.e., it does not attain its infimum value* $\rho^* > -\infty$*. In such a case Problem (20) is solvable, i.e., there exists* $(\lambda^*, z^*) \in \mathbb{R} \times L^2$ *that satisfies the constraints of (20) and such that* $-C\lambda^* - \mathbb{E}(y_0 z^*) = \rho^*$*.*

*b*2*) Problem (17) is bounded and solvable, i.e., there exists* y^* *satisfying the constraints of (17) such that* $\rho(y^* + y_0 - \mathbb{E}(y^* z_\pi)) = \rho^* > -\infty$*. In such a case Problem (20) is solvable, i.e., there exists* $(\lambda^*, z^*) \in \mathbb{R} \times L^2$ *that satisfies the constraints of (20) and such that* $-C\lambda^* - \mathbb{E}(y_0 z^*) = \rho^*$*.*

Next let us illustrate that Scenarios *a*) and *b*1) or *b*2) above may hold.

Proposition 5 *a) if* $\Delta_\rho = \{1\}$ *and*

$$\mathbb{P}(z_\pi < \varepsilon) > 0 \tag{25}$$

for every $\varepsilon > 0$ *then (17) is unbounded* ($\rho^* = -\infty$)*.*[13]

b) If $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ *being the level of confidence, then (17) is bounded* ($\rho^* > -\infty$)*.*

Proof. *a*) Condition (25) makes it impossible the fulfillment of $1 \leq (1+\lambda) z_\pi$ for every $\lambda \geq 0$, so (20) has no feasible solutions.

b) It is sufficient to show that (20) has feasible solutions (Balbás *et al.*, 2010*b*). Rockafellar *et al.* (2006) have shown that

$$\Delta_{CVaR_{\mu_0}} = \left\{ z \in L^\infty; \mathbb{E}(z) = 1,\ 0 \leq z \leq \frac{1}{1-\mu_0} \right\}. \tag{26}$$

[13]Notice that for $\Delta_\rho = \{1\}$ the real-valued function given by (7) satisfies Assumption 1. Furthermore, in Section 6 we will see that the Black and Scholes model satisfies (25).

On the other hand, one can consider the increasing sequence of measurable sets

$$A_n = \left\{ z_\pi \geq \frac{1}{n} \right\}$$

for every $n \in \mathbb{N}$, and

$$Lim_{n \to \infty} \mathbb{P}(A_n) = 1$$

is obvious. Take $n \in \mathbb{N}$ such that $1 - \mu_0 \leq \mathbb{P}(A_n)$. Then $1/\mathbb{P}(A_n) \leq 1/(1-\mu_0)$ and therefore, according to (26),

$$z = \begin{cases} \dfrac{1}{\mathbb{P}(A_n)}, & A_n \\ 0, & \Omega \setminus A_n \end{cases}$$

belongs to $\Delta_{CVaR_{\mu_0}}$. Take $\lambda = n/\mathbb{P}(A_n)$. Then,

$$(1+\lambda) z_\pi \geq \frac{n}{\mathbb{P}(A_n)} z_\pi \geq \frac{n}{\mathbb{P}(A_n)} \frac{1}{n} = \frac{1}{\mathbb{P}(A_n)} = z$$

on A_n, whereas $(1+\lambda) z_\pi \geq 0 = z$ on $\Omega \setminus A_n$. □

Remark 2 *In Section* 5 *we will partially improve Statement* b) *above. We will prove that Scenario* b2) *holds if* y_0 *has a finite essential infimum.* □

Remark 3 *Since (19) and (22) are always solvable we can simplify Conditions (24). Indeed, if* $\tilde{\lambda}_R^* = 0$ *then the role of* $\tilde{z}_R^*$ *in (24) may be played by every element in* B_2^+*, in the sense that* $\tilde{z}_R^*$ *may be substituted by every* $\tilde{z} \in B_2^+$ *and (24) will still hold. In particular, we can take* $\tilde{z}_R^* = y_R^*/R$*. Besides, if* $\tilde{\lambda}_R^* > 0$ *then the Cauchy-Schwartz inequality and the sixth and seventh equations in (24) imply that* $\tilde{z}_R^* = y_R^*/R$*. Thus one can always impose this equality, and the necessary and sufficient Karush-Kuhn-Tucker optimality conditions become*

$$\begin{cases} \lambda_R^* (C - \mathbb{E}(y_R^* z_\pi)) = 0 & \\ C - \mathbb{E}(y_R^* z_\pi) \geq 0 & \\ \mathbb{E}((y_R^* + y_0) z) \geq \mathbb{E}((y_R^* + y_0) z_R^*), & \forall z \in \Delta_\rho \\ \left((1+\lambda_R^*) z_\pi + \left(\tilde{\lambda}_R^*/R\right) y_R^* - z_R^* \right) y_R^* = 0 & \\ (1+\lambda_R^*) z_\pi + \left(\tilde{\lambda}_R^*/R\right) y_R^* - z_R^* \geq 0 & \\ \tilde{\lambda}_R^* (\|y_R^*\|_2 - R) = 0 & \\ \mathbb{E}(y_R^* \tilde{z}) \leq R, & \forall \tilde{z} \in B_2^+ \\ y_R^* \in L^2,\ y_R^* \geq 0,\ \lambda_R^*, \tilde{\lambda}_R^* \in \mathbb{R},\ \lambda_R^* \geq 0, \tilde{\lambda}_R^* \geq 0,\ z_R^* \in \Delta_\rho & \end{cases} \quad (27)$$

so the variable $\tilde{z}_R^* \in B_2^+$ *may be removed.* □

Let us end this section by pointing out the relationships between (17) and (19). These properties will be important so as to study whether (17) is bounded and solvable.

Theorem 6 *a)* $(\rho_R^*)_{R>0} \subset \mathbb{R}$ *is a decreasing net such that* $Lim_{R\to\infty}\rho_R^* = \rho^*$.

b) If (17) is bounded and $\left(\lambda_R^*, \tilde{\lambda}_R^*, z_R^*, \tilde{z}_R^*\right) \in \mathbb{R}^2 \times (L^2)^2$ *is a solution of (22) for every* $R > 0$, *then the nets* $(\lambda_R^*)_{R>R_0}$ *and* $\left(\tilde{\lambda}_R^*\right)_{R>R_0}$ *are bounded for some* $R_0 > 0$, *and* $Lim_{R\to\infty}\tilde{\lambda}_R^* = 0$.

c) If $\left(\lambda_R^*, \tilde{\lambda}_R^*, z_R^*, \tilde{z}_R^*\right) \in \mathbb{R}^2 \times (L^2)^2$ *is a solution of (22) for every* $R > 0$, *then (17) is bounded and solvable if and only if there exists* $R_0 > 0$ *such that* $\tilde{\lambda}_{R_0}^* = 0$. *In such a case the following assertions hold:*

c1) $\rho_R^* = \rho^*$ *for every* $R \geq R_0$.

c2) If $(\lambda^*, z^*) \in \mathbb{R} \times L^2$ *is a solution of (20), then* $\left(\lambda_R^* = \lambda^*, \tilde{\lambda}_R^* = 0, z_R^* = z^*, \tilde{z}_R^* = 0\right)$ *solves (22) for every* $R \geq R_0$.

c3) If y^* *solves (17) then it also solves (19) for* $R \geq R_0$.

Proof. *a)* (17) and (19) clearly imply that $(\rho_R^*)_{R>0} \subset \mathbb{R}$ is decreasing and $Lim_{R\to\infty}\rho_R^* \geq \rho^*$. Besides, if (a, b) with $a \in \mathbb{R}\cup\{-\infty\}$ and $b \in \mathbb{R}$ is a neighborhood of ρ^*, it is clear the existence of y (17)-feasible such that $\rho\left(y_0 + y - \mathbb{E}\left(z_\pi y\right)\right) < b$ Then, for every $R \geq \|y\|_2$ we have that $\rho_R^* < b$.

b) According to Remark 1 there exists $(\lambda^*, z^*) \in \mathbb{R} \times L^2$ solving (20). It is clear that $\left(\lambda_R = \lambda^*, \tilde{\lambda}_R^* = 0, z_R = z^*, \tilde{z}_R^* = 0\right)$ is (22)-feasible, so

$$C\lambda_R^* + \mathbb{E}\left(y_0 z_R^*\right) + R\tilde{\lambda}_R^* \leq C\lambda^* + \mathbb{E}\left(y_0 z^*\right).$$

for every $R > 0$. Since Δ_ρ is weakly$^*-$compact and $(z_R^*)_{R>0} \subset \Delta_\rho$ there exists $k \in \mathbb{R}$ such that $-\mathbb{E}\left(y_0 z_R^*\right) \leq k$ for every $R > 0$. Thus, the conclusion trivially follows from

$$\lambda_R^* \leq \lambda^* + \frac{\mathbb{E}\left(y_0 z^*\right) + k}{C}$$

and

$$\tilde{\lambda}_R^* \leq \frac{C\lambda^* + \mathbb{E}\left(y_0 z^*\right) + k}{R},$$

for every $R > 0$.

c) Suppose that (17) is bounded and solvable, and take y^* solving (17). Then y^* clearly solves (19) for every $R \geq R_1 = \|y^*\|_2$, so $c1$) and $c3$) hold if $R_0 > R_1$. Moreover, $\tilde{\lambda}_R^* = 0$ trivially follows from the sixth condition in (27). Finally, as in the proof of b), $\left(\lambda_R = \lambda^*, \tilde{\lambda}_R^* = 0, z_R = z^*, \tilde{z}_R^* = 0\right)$ is (22)-feasible, so it is sufficient to see that the optimal objective value of (22) is attained. $c1$) leads to $\rho_R^* = \rho^* = -C\lambda^* - \mathbb{E}\left(y_0 z^*\right)$.

Conversely, if $\tilde{\lambda}_{R_0}^* = 0$ then $\left(\lambda_{R_0}^*, z_{R_0}^*\right)$ is obviously (20)-feasible, and therefore

$$\rho_{R_0}^* = -C\lambda_{R_0}^* - \mathbb{E}\left(y_0 z_{R_0}^*\right) \leq \rho^*.$$

Since the opposite inequality trivially follows from a), take a solution $y_{R_0}^*$ of (19) for $R = R_0$ and $y_{R_0}^*$ is (17)-feasible and such that $\rho\left(y_0 + y_{R_0}^* - \mathbb{E}\left(z_\pi y_{R_0}^*\right)\right) = \rho^*$. □

4 Shadow riskless assets and good deals

Next let us use our previous results so as to prove two important assertions: The risk-free asset and the shadow riskless asset are often different, and the existence of good deals often holds.

Lemma 7 *The following implications hold:*

a) If (17) is bounded, (λ^*, z^*) *solves (20) and* $\lambda^* = 0$*, then* $z^* = z_\pi$*.*

b) If (17) is solvable, y^* *is its solution,* (λ^*, z^*) *solves (20) and* $\mathbb{P}\left(y^* > 0\right) = 1$*, then* $\lambda^* = 0$ *and* $z^* = z_\pi$*.*

c) If $y_R^* \in L^2$ *is a solution of (19) and* $\left(\lambda_R^*, \tilde{\lambda}_R^*, z_R^*, \tilde{z}_R^*\right) \in \mathbb{R}^2 \times \left(L^2\right)^2$ *is a solution of (22) for some* $R > 0$*, and* $\mathbb{P}\left(y_R^* > 0\right) = 1$*, then* $z_R^* = z_\pi$*.*

Proof. a) If $\lambda^* = 0$ then the constraint of (20) leads to $z^* \leq z_\pi$, and therefore $z^* = z_\pi$ because both random variables have the same expectation (see (5) and (8)).

b) If $\mathbb{P}\left(y^* > 0\right) = 1$ then the fourth equation in (23) implies that $z^* = \left(1 + \lambda^*\right) z_\pi$. Taking expectations and bearing in mind (5) and (8) we have that $1 = 1 + \lambda^*$.

c) The fourth equation in (24) leads to $\left(1+\lambda_R^*\right) z_\pi + \tilde{\lambda}_R^* \tilde{z}_R^* = z_R^*$, so $\left(1+\lambda_R^*\right) z_\pi \leq z_R^*$. Taking expectations one has $\lambda_R^* = 0$ and $z_\pi \leq z_R^*$, so once again $z_\pi = z_R^*$ because both random variables have the same expectation.. $\square$

Theorem 8 *Suppose that z_π is not essentially bounded and $\Delta_\rho \subset L^\infty$.*[14] *Then:*

a) Problem (18) is unbounded, i.e., if short-sales are allowed then there are good deals.

b) If y^* solves (17), then y^* does not satisfy $\mathbb{P}\left(y^* > 0\right) = 1$ and it is not a risk-free asset.

c) The solution y_R^* of (19) never satisfies $\mathbb{P}\left(y_R^* > 0\right) = 1$. Furthermore, y_R^* is not a risk-free asset.

Proof. *a*) Problem (21) is not feasible because z_π is not essentially bounded and the elements in Δ_ρ are essentially bounded. Then the usual primal-dual relationships (Balbás *et al.*, 2010*b*) show that (18) is unbounded.

b) If (λ^*, z^*) is a solution of (20) then Lemma 7*a* shows that $\lambda^* > 0$, since otherwise $z^* = z_\pi$, and this equality contradicts the assumptions $z_\pi \notin L^\infty$ and $\Delta_\rho \subset L^\infty$. Furthermore, $\lambda^* > 0$ and Lemma 7*b* show that $\mathbb{P}\left(y_R^* > 0\right) = 1$ cannot hold. In particular, if y^* is a risk-free asset then $\mathbb{P}\left(y^* = 0\right) = 1$. Therefore, (16) shows that the first condition in (23) cannot hold and we have a contradiction.

c) According to Lemma 7*c* the equality $\mathbb{P}\left(y_R^* > 0\right) = 1$ would lead to $z_R^* = z_\pi$, where $\left(\lambda_R^*, \tilde{\lambda}_R^*, z_R^*, \tilde{z}_R^*\right)$ solves (22). Once again we contradict $z_\pi \notin \Delta_\rho$. Thus, if y_R^* is a risk-free asset then $y_R^* = 0$. (16) and the first and sixth conditions in (24) lead to $\lambda_R^* = \tilde{\lambda}_R^* = 0$. The fifth condition implies that $z_R^* \leq z_\pi$, and the equality must hold because both random variables have the same expectation. We have the contradiction $z_\pi \in \Delta_\rho$. $\square$

Remark 4 *Theorem* 8 *implies that the shadow riskless asset y^* (if it exists) and the $R-$shadow riskless asset are frequently risky assets, as well as the existence of good deals in absence of short-selling restrictions. Indeed, suppose that ρ may be extended to the whole space L^1. Then (7) implies that $\Delta_\rho \subset L^\infty$. This property is not guaranteed (consider the*

[14] Actually, the theorem remains true if $z_\pi \notin \Delta_\rho$. In particular, every $q \in (2, \infty]$ may play the role of ∞, in the sense that $z_\pi \notin L^q$ and $\Delta_\rho \subset L^q$ are sufficient.

$CCVaR$ of Balbás et al., 2010a), but very important expectation bounded risk measures may be extended to L^1. Among others, the $CVaR$, the measure (14) if σ is the $1-$deviation (or absolute deviation) or the $1-$down-side semi-deviation (or down-side absolute semi-deviation) and the DPT of Wang (2000).[15], [16] Also the $WCVaR$ may be often extended to L^1.

Combine the previous risk measures and a pricing model with unbounded SDF. Many important examples satisfy this requirement. For instance, the Black and Scholes model, as will be seen in Section 6. Also the Heston model and other stochastic volatility models often have an unbounded SDF.[17] In these cases the shadow riskless asset (if it exists) and the $R-$shadow riskless asset are not risk-free, and there are good deals available. □

5 Dealing with the CVaR

Henceforth we will assume that $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ being the level of confidence. Bearing in mind (13) and (26), $CVaR_{\mu_0}$ is a coherent and expectation bounded measure of risk. Moreover, Ogryczak and Ruszczynski (2002) have shown that $CVaR_{\mu_0}$ is consistent with the second order stochastic dominance. These properties provoke that the $CVaR_{\mu_0}$ is becoming a very popular risk measure for both researchers and practitioners. However, Remark 4 implies that one can construct good deals for the $CVaR_{\mu_0}$ and the most important pricing models of Financial Economics, such as Black and Scholes, Heston, and other stochastic volatility models. Moreover, bearing in mind that

$$CVaR_{\mu_0}(y) \geq VaR_{\mu_0}(y), \tag{28}$$

[15] Let us remark that all of these risk measures respect the second order stochastic dominance (Ogryczak and Ruszczynski, 2002).

[16] Recall that the DPT is given by

$$DPT_a(y) = \int_0^1 VaR_{1-t}(y)\, g_a'(t)\, dt$$

for every $y \in L^1$, $a > 1$ being an arbitrary constant and $g_a : (0,1) \longrightarrow (0,1)$ given by $g_a(t) = 1 - (1-t)^a$.

[17] Though "formally" the Heston model is not complete, in practice it is assumed the existence of a volatility dependent asset. Otherwise it would be impossible to use the model so as to give a unique price of the available derivatives. Once this additional assumption is incorporated the existence of a unique risk neutral probability measure is reached, and therefore there is only one SDF. Thus, the results above apply.

one can also build good deals if the riskiness is given by the VaR_{μ_0}, *i.e.*, there are sequences of portfolios whose VaR_{μ_0} tends to minus infinite whereas their expected returns tend to plus infinite (see Proposition 1).

Proposition 5b revealed that (17) is bounded. Theorem 10 below will improve this result and will show that the solution is attainable under weak additional conditions. The existence of (non risk-free, see Remark 4) shadow riskless assets for the $CVaR$ is the major objective of this section, and it will allow us to construct particular good deals and shadow riskless assets in the next one.

Next, let us adapt (23) and (27) to the particular case we are dealing with. The third condition in (23) (or (27)) shows that the dual solution z^* must solve the mathematical programming problem

$$\left\{ \begin{array}{l} Min\ \mathbb{E}\left(\left(y^* + y_0 \right) z \right) \\ \mathbb{E}\left(z \right) = 1, z \in L^\infty, 0 \leq z \leq \dfrac{1}{1-\mu_0} \end{array} \right. \tag{29}$$

Lemma 9 *If* $y^*, y_0 \in L^2$ *and* z^* *is (29)-feasible then* z^* *solves (29) if and only if there exist* $\alpha \in \mathbb{R}$, $\alpha_1, \alpha_2 \in L^2$ *and a measurable partition* $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ *such that,*

$$\left\{ \begin{array}{ll} y^* + y_0 = \alpha - \alpha_1 + \alpha_2 & \\ \alpha_i \geq 0 & i = 1, 2 \\ \alpha_1 = \alpha_2 = 0 & on\ \Omega_0 \\ z^* = \dfrac{1}{1-\mu_0}\ and\ \alpha_2 = 0 & on\ \Omega_1 \\ z^* = 0\ and\ \alpha_1 = 0 & on\ \Omega_2 \end{array} \right. \tag{30}$$

□

The proof of the previous lemma is omitted because quite similar results may be found in Balbás *et al.* (2009).

As an obvious consequence we can use (30) and modify the third equation in (23) and (27), and we will have new necessary and sufficient optimality conditions for (17), (19), (20) and (22).

As said above, Proposition 5b guarantees that (17) is bounded ($\rho^* > -\infty$). Theorem 10 will prove that it is often solvable. Thus, Theorem 8 and its remark show the existence of an alternative investment y^* outperforming the risk-free asset if z_π is unbounded.

Theorem 10 *If y_0 has a finite essential infimum and $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ being the level of confidence, then Problem (17) is bounded and attains its optimal value.*

Proof. Without loss of generality, Expression (9) allows us to assume that $y_0 \geq 0$, *a.s.* According to Theorem 6*c*, we must prove that $\tilde{\lambda}_R^* = 0$ for some $R > 0$. Suppose that we are able to prove the existence of $R_0 > 0$ and $M > 0$ such that $0 \leq y_R^* \leq M$ for every $R \geq R_0$. Then $\|y_R^*\|_2 \leq M$, and the sixth condition in (27) implies that $\tilde{\lambda}_R^* = 0$ for every $R > Max\ \{M, R_0\}$.

If the existence of $R_0 > 0$ and $M > 0$ were false, one could find an increasing sequence $(R_n)_{n \in \mathbb{N}}$ with $Lim_{n \to \infty} R_n = \infty$ and

$$Lim_{n \to \infty} \left\| y_{R_n}^* \right\|_\infty = \infty. \tag{31}$$

Bearing in mind the third condition in (27), consider the measurable partition $\{\Omega_0^n, \Omega_1^n, \Omega_2^n\}$ and the elements $\alpha^n, \alpha_1^n, \alpha_2^n$ of lemma 9 for every $n \in \mathbb{N}$. Then, $z_{R_n}^* = 0$ on Ω_2^n implies that

$$y_{R_n}^* = 0 \tag{32}$$

on Ω_2^n, owing to (4) and the fourth condition in (27). Thus

$$y_{R_n}^* \leq y_{R_n}^* + y_0 = \alpha^n - \alpha_1^n \leq \alpha^n$$

on the whole space Ω. Hence, (31) implies that

$$Lim_{n \to \infty} \alpha^n = \infty. \tag{33}$$

We have that $y_{R_n}^* + y_0 = \alpha^n$ on Ω_0^n, which implies that

$$\mathbb{E}\left(\left(y_{R_n}^* + y_0\right) z_\pi\right) \geq \int_{\Omega_0^n} \left(y_{R_n}^* + y_0\right) z_\pi d\mathbb{P} = \alpha^n \mathbb{P}^* \left(\Omega_0^n\right),$$

where $\mathbb{P}^*$ is the $\mathbb{P}-$equivalent probability measure given by $d\mathbb{P}^* = z_\pi d\mathbb{P}$ (see (4) and (5)). Denote by $W = e^{-r_f T} \mathbb{E}\left(y_0 z_\pi\right)$ the present value of y_0. Since $\mathbb{E}\left(\left(y_{R_n}^* + y_0\right) z_\pi\right) \leq C + W e^{r_f T}$, (33) shows that

$$Lim_{n \to \infty} \mathbb{P}^* \left(\Omega_0^n\right) = 0. \tag{34}$$

Besides, Lemma 9 and (32) show that $y_0 = \alpha^n + \alpha_2^n$ on Ω_2^n. Once again,

$$We^{r_f T} = \mathbb{E}\left(y_0 z_\pi\right) \geq \int_{\Omega_2^n} \left(\alpha^n + \alpha_2^n\right) z_\pi d\mathbb{P} \geq \alpha^n \mathbb{P}^* \left(\Omega_2^n\right)$$

and (33) leads to

$$Lim_{n\to\infty} \mathbb{P}^* \left(\Omega_2^n\right) = 0. \tag{35}$$

(34) and (35) give $Lim_{n\to\infty} \mathbb{P}^* \left(\Omega_1^n\right) = 1$, and therefore

$$Lim_{n\to\infty} \mathbb{P} \left(\Omega_1^n\right) = 1, \tag{36}$$

because both probability measures are equivalent. Lemma 9 implies that

$$z_{R_n}^* = \frac{1}{1-\mu_0}$$

on Ω_1^n, so (36) leads to

$$\mathbb{E}\left(z_{R_n}^*\right) \geq \frac{1}{1-\mu_0} \mathbb{P}\left(\Omega_1^n\right) > 1$$

if $n \in \mathbb{N}$ is large enough. This inequality contradicts (8) and is provoked by (31). □

6 Dealing with the Black and Scholes model

Let us now focus on both the Black and Scholes model and the $CVaR$ (or the VaR, as well). Consequently, suppose that y_0 is the final value (at T) of a Geometric Brownian Motion (GBM). Then it is known that y_0 has a log-normal distribution. Without loss of generality we can simplify the structure of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, assume that $\Omega = (0,1)$ and $\mathbb{P}$ is the Lebesgue measure on the Borel $\sigma-$algebra of this set. Then we can take

$$y_0(\omega) = WExp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right) \tag{37}$$

for $\omega \in (0,1)$, where $W > 0$ denotes the present price of y_0 and $r > 0$ and $\sigma > 0$ denote the drift and the volatility of the GBM, respectively (Wang, 2000, or Hamada and Sherris, 2003). Obviously, $\Phi : \mathbb{R} \longmapsto (0,1)$ is the cumulative distribution function of the standard normal distribution, given by the well-known expression

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du.$$

The simplification above cannot be implemented when pricing path dependent or American style derivatives. In both situations the dynamic evolution of the GBM plays a critical role, as well as the notion of "stopping time" in the second case. Thus, when we choose the simple probability space $(\Omega, \mathcal{F}, \mathbb{P})$ above we are aware that we are missing information, and the performance of the shadow risk asset of our Theorem 11 below might be improved by those of some path dependent or American derivatives. However, our simplification is interesting because the exposition is shortened and becomes much easier, we will still obtain a shadow riskless asset that outperforms the risk-free one, and we will build concrete examples of good deals for the Black and Scholes model and both VaR and $CVaR$.

Actually, if (37) represents the trader final wealth then $\mathbb{P}(y_0 > 0) = 1$, and the risk level $\rho(y_0)$ is strictly negative. Hence, no capital requirements should be added. Nevertheless, one can consider the fund manager whose final income is given by (37) but whose liability equals a positive amount M with maturity at T. Then the manager final pay-off is given by $y_0 - M$, that can be negative, and $\rho(y_0 - M) > 0$ may hold. Thus (17) and (18) should be modified and $y_0 - M$ should play the role of y_0. However, due to (9), the solutions of both problems remain the same if y_0 replaces $y_0 - M$, so we do not miss anything if we take y_0 as in (37) and deal with (17) and (18).

Taking into account (37), it may be immediately verified that y_0 is a continuous and strictly increasing function (with respect to the ω variable) such that

$$Lim_{\omega \to 0} y_0(\omega) = 0, \tag{38}$$

and $Lim_{\omega \to 1} y_0(\omega) = \infty$. It is also easy to see (Wang, 2000) that z_π is the first derivative of the one to one increasing and convex function

$$(0,1) \ni \omega \longmapsto g(\omega) = \Phi\left(a + \Phi^{-1}(\omega)\right) \in (0,1), \tag{39}$$

$$a = \frac{r - r_f}{\sigma}\sqrt{T} \tag{40}$$

being positive because we assume, as usual, that $r > r_f$. Computing the derivative in (39) we have that

$$z_\pi(\omega) = Exp\left(-\frac{a^2}{2} - a\Phi^{-1}(\omega)\right) \tag{41}$$

$\omega \in (0,1)$, which allows us to verify that z_π is continuous and strictly decreasing,

$$Lim_{\omega \to 0} z_\pi(\omega) = \infty, \tag{42}$$

and

$$Lim_{\omega\to 1}z_\pi(\omega)=0. \tag{43}$$

Theorem 8 and its remark and Theorem 10 have shown the existence of an alternative investment y^* outperforming the risk-free asset. Theorem 11 below and its remarks will permit us to compute y^* and several good deals in practice.

Theorem 11 *Under the assumptions and notations above, if* $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ *being the level of confidence, and* y^* *solves (17) then there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $0<\beta<\alpha$, *and*

$$y^* = \begin{cases} 0 & if\ y_0 > \alpha \\ \alpha - y_0 & if\ \beta < y_0 \le \alpha \\ 0 & if\ y_0 \le \beta \end{cases} \tag{44}$$

Proof. Consider the dual solution (λ^*, z^*). Since $(1+\lambda^*)\,z_\pi$ is continuous and strictly decreasing (42) and (43) show the existence of $\gamma_1 \in (0,1)$ such that $(1+\lambda^*)\,z_\pi(\gamma_1) = \dfrac{1}{1-\mu_0}$, $(1+\lambda^*)\,z_\pi(\omega) > \dfrac{1}{1-\mu_0}$ for $\omega \in (0,\gamma_1)$ and $(1+\lambda^*)\,z_\pi(\omega) < \dfrac{1}{1-\mu_0}$ for $\omega \in (\gamma_1, 1)$. In particular, $z^*(\omega) < (1+\lambda^*)\,z_\pi(\omega)$ in $(0,\gamma_1)$, which, along with the fourth and fifth equations in (23), imply that $y^*(\omega)=0$ in $(0,\gamma_1)$. On the other hand, y_0 is continuous and strictly increasing. Take $\beta = y_0(\gamma_1)$ and we have that $y_0 \le \beta$ if and only if $(0,\gamma_1] \ni \omega$, *i.e.*, the third part of (44) has been proved.[18]

Consider the partition $(0,1)=\Omega_0 \cup \Omega_1 \cup \Omega_2$ of (30). Notice that the fourth equation in (30) and the fifth one in (23) lead to $\Omega_1 \subset (0,\gamma_1]$. Notice also that $y_0 = \alpha - \alpha_1$ in Ω_1, whereas $y_0 = \alpha + \alpha_2$ in $(0,\gamma_1] \setminus \Omega_1$, since α_1 vanishes outside Ω_1 and y^* vanishes in $(0,\gamma_1]$. Inequalities $\alpha_1, \alpha_2 \ge 0$ show that y_0 increases from Ω_1 to $(0,\gamma_1]\setminus\Omega_1$. Since y_0 is strictly increasing there will exist $\tilde{\gamma}_1 \le \gamma_1$ such that $\Omega_1 = (0,\tilde{\gamma}_1]$.

Let us see that $(\tilde{\gamma}_1,\gamma_1] \subset \Omega_2$. Indeed, otherwise in a non-null subset of $(\tilde{\gamma}_1,\gamma_1]$ we would have $y_0 = \alpha+\alpha_2 = \alpha$ (α_2 vanishes outside Ω_2), but this is a contradiction because y_0 is strictly increasing and cannot achieve any concrete value with strictly positive probability.

Assume for a few moments that Ω_0 is void. Then $\Omega_2 = (\tilde{\gamma}_1, 1)$ and $z^*=0$ in $(\tilde{\gamma}_1,1)$ (last condition in (30)). Since $(1+\lambda^*)\,z_\pi > 0$ (see (41)), the fourth equation in (23) implies

[18] y^* may be modified in $\{\gamma_1\}$ because it is a $\mathbb{P}$−null set.

$y^* = 0$ in $(0,1)$. Then $C > 0$ and $\lambda^* > 0$ provoke that the first equality in (23) does not hold, and we are facing a contradiction.

Consequently Ω_0 is not a null set. Let us see that $\tilde{\gamma}_1 = \gamma_1$. Indeed, we know that $\Omega_0 \subset (\gamma_1, 1)$. Fix λ^*. According to (20), z^* must solve

$$Min \ \{\mathbb{E}(y_0 z);\ z \leq (1+\lambda^*) z_\pi, z \in \Delta_\rho\}. \tag{45}$$

If $\tilde{\gamma}_1 < \gamma_1$ then take $v = Inf\ (\Omega_0)$, $u = Sup\ (\Omega_0)$ and

$$\tilde{z} = \begin{cases} z^*, & \omega \in \Omega_1 = (0, \tilde{\gamma}_1] \\ z^*(\omega + v - \tilde{\gamma}_1), & \tilde{\gamma}_1 < \omega < \tilde{\gamma}_1 + u - v \\ 0, & otherwise \end{cases}$$

$\tilde{z}$ trivially satisfies the constraints of (45) because so does z^*, z^* vanishes on Ω_2 and z_π is strictly decreasing. On the other hand, $\mathbb{E}(y_0\tilde{z}) < \mathbb{E}(y_0 z^*)$ trivially holds because y_0 is strictly increasing, so z^* does not solve (45). Hence, $\tilde{\gamma}_1 = \gamma_1$.

Applying a similar argument it is easy to show the existence of $\gamma_2 > \gamma_1$ such that $\Omega_0 = (\gamma_1, \gamma_2)$. Moreover, $y^* = \alpha - y_0$ in (γ_1, γ_2) implies that $y_0(\omega) \leq \alpha$ for $\omega \in (\gamma_1, \gamma_2)$, because $y^* \geq 0$. Since y_0 is continuous and strictly increasing one has that

$$\alpha \geq y_0(\gamma_2) > y_0(\gamma_1) = \beta > 0.$$

Finally, if $y_0(\omega) > \alpha$ then $\omega > \gamma_2$, so $\omega \in \Omega_2$, $z^* = 0$ (last equation in (30)), the fifth equation in (23) holds in terms of strict inequality, and the fourth equation in (23) shows that y^* vanishes. □

Remark 5 *Notice that the solution y^* above may be given by*

$$y^* = y^*_\alpha - y^*_\beta - (\alpha - \beta) y^*_{D\beta},$$

y^*_α denoting the European put option with maturity at T and strike α, y^*_β denoting the similar put with strike β, and $y^*_{D\beta}$ denoting the digital put option with maturity at T and strike β, whose pay-off is

$$y^*_{D\beta} = \begin{cases} 0 & if\ y_0 > \beta \\ 1 & if\ y_0 \leq \beta \end{cases}$$

Then the shadow riskless asset is a combination of three put options. □

Remark 6 *In order to apply our finding in practice we have to provide the values of β and α. Suppose for a few moments that we know the value of the dual solution λ^*. Then, the theorem's proof and (37) point out that β may be computed in practice by*

$$\beta = WExp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\gamma_1)\right),$$

where, according to the theorem's proof and (41),

$$\gamma_1 = z_\pi^{-1}\left(\frac{1}{(1-\mu_0)(1+\lambda^*)}\right) = \Phi\left(\frac{2L(1-\mu_0) + 2L(1+\lambda^*) - a^2}{2a}\right),$$

and a is given by (40).

Since the theorem's proof is constructive it also yields and algorithm leading to the computation of λ^*. Indeed, take in a first iteration $\gamma_1 = 1 - \mu_0$ and

$$1 + \lambda^* = \frac{1}{(1-\mu_0)z_\pi(\gamma_1)}. \tag{46}$$

In the theorem's proof this choice means that we are taking

$$z^* = \begin{cases} \frac{1}{(1-\mu_0)} & \omega \leq \gamma_1 \\ 0 & otherwise \end{cases}$$

We know that this choice does not provide the dual solution because it implies that Ω_0 is void (see the theorem's proof). Anyway, we can compute the (minus) objective of (20) in the proposed solution,

$$C\lambda^* + \mathbb{E}(z^* y_0). \tag{47}$$

Then, choose a "small enough step" $\varepsilon > 0$ and consider $\gamma_1 = 1 - \mu_0 - \varepsilon$. Take λ^* as in (46) and

$$z^* = \begin{cases} \frac{1}{(1-\mu_0)} & \omega \leq \gamma_1 \\ (1+\lambda^*) z_\pi & \gamma_1 < \omega \leq \gamma_2 \\ 0 & otherwise \end{cases},$$

where γ_2 must be selected so as to reach

$$\mathbb{E}(z^*) = \frac{\gamma_1}{1-\mu_0} + (1+\lambda^*)\int_{\gamma_1}^{\gamma_2} y_0(\omega)\, z_\pi(\omega)\, d\omega = 1.$$

Notice that the integral may be calculated by numerical methods. Then compute the (minus) objective of (20) as indicated in (47). If the value of (47) has decreased with respect to the previous one then we already reached the desired value λ^*. Otherwise take $\gamma_1 = 1 - \mu_0 - 2\varepsilon$

and repeat a new iteration of the algorithm. Once β has been computed one can calculate α because the price of y^* must equal $Ce^{-r_f T}$, i.e., $\Pi\left(y_\alpha^*\right) = Ce^{-r_f T} + \Pi\left(y_\beta^*\right) + (\alpha - \beta)\,\Pi\left(y_{D\beta}^*\right)$ must hold. □

Remark 7 *The risk measure $CVaR_{\mu_0}$ may be also given by (Rockafellar et al., 2006)*

$$CVaR_{\mu_0}(y) = \frac{1}{1-\mu_0}\int_0^{1-\mu_0} VaR_{1-t}(y)\,dt,$$

for every $y \in L^1$. Accordingly, since $VaR(y)$ only focuses on "the worst" values of y (on the left tail of y), so does $CVaR_{\mu_0}(y)$. Thus, it is not so surprising that y^* vanishes if y_0 achieves high values, since they are not affecting the global risk level.

A little bit more shocking is that y^* also vanishes if y_0 achieves its lowest values. (38) and (42) could help to interpret this finding, because it could be "very expensive" to hedge the worst values of y_0. Notice that the theorem's proof leads to $\beta = y_0(\gamma_1)$, and, according to the previous remark, $0 < \gamma_1 < 1 - \mu_0$. Therefore, $Lim_{\mu_0 \longmapsto 1}\gamma_1 = 0$, which, along with (38) and $\beta = y_0(\gamma_1)$, imply that $Lim_{\mu_0 \longmapsto 1}\beta = 0$. Thus, for a high level of confidence the lower values of y_0 become very important, and y^* almost becomes the European put option y_α^*. The limit value of α as μ_0 tends to 1 may be computed from $\Pi\left(y_\alpha^*\right) = Ce^{-r_f T}$. □

Remark 8 *There are several classical strategies providing "portfolio insurance". Maybe the most popular one is the purchase of an appropriate European put option. Theorem* 11 *highlights that for high levels of confidence the use of portfolio insurance strategies may be adequate to control the investor's risk. It is consistent with some empirical findings of recent literature. For instance, the test implemented by Annaert et al. (2009) seems to reveal that some put option-linked portfolio insurance strategies are not outperformed by other hedging methods. The authors use stochastic dominance criteria and VaR and $CVaR$ in their empirical test.* □

Remark 9 *Notice that the theorem´s proof still applies if the role of y_0 is plaid by $f(y_0)$, $f : (0,\infty) \rightarrow (a,b)$ being a continuous and strictly increasing function for some $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, $a < b$. Then, (44) still holds if y_0 is replaced by $f(y_0) - a$. Thus, there are many potential applications of Theorem* 11*. In particular, Theorem* 11 *applies for many combinations of derivatives. For example,*

$$f(y_0) = x_1 (y_0 - k)^+ - x_2 (k - y_0)^+,$$

$x_1, x_2 > 0$, which obviously represents the purchase of x_1 European calls and the sale of x_2 European puts with the same strike $k > 0$. A single European call may be easily analyzed as a limit case, since

$$(y_0 - k)^+ = Lim_{n\to\infty} (y_0 - k)^+ - \frac{1}{n}(k - y_0)^+ .$$

An alternative result may be given if $f : (0, \infty) \to (a, b)$ is a continuous and strictly decreasing function for some $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, $a < b$, though the conclusion is different. Then, the European put may be easily studied as a limit case. □

Remark 10 *Let us assume that there are no short selling restrictions, i.e., let us deal with (18) rather than (17). (26), (42) and Theorem 8a show that there are good deals. Furthermore, (28) shows that the riskiness also may become minus infinite if it is given by the VaR. In other words, one can construct sequences of portfolios such that VaR and $CVaR$ tend to minus infinite while expected returns tend to plus infinite (Proposition 1). Until now we were able to prove the existence of these sequences, but we did not give any practical example, so let us overcome this caveat. Consider $n \in \mathbb{N}$ along with an approximation of (18) given by Problem*

$$\begin{cases} Min\ CVaR_{\mu_0} (y + y_0 - \mathbb{E}(yz_\pi)) \\ \mathbb{E}(yz_\pi) \le C \\ y \ge -n \end{cases} \tag{48}$$

Then, due to (5), it is easy to see that the change of variable $x_n = y + n$ leads to

$$\begin{cases} Min\ CVaR_{\mu_0} (x_n + y_0 - \mathbb{E}(yz_\pi)) \\ \mathbb{E}(x_n z_\pi) \le C + n \\ x_n \ge 0 \end{cases}, \tag{49}$$

analogous to (17) with $\rho = CVaR_{\mu_0}$. Thus, (48) is bounded and achieves its optimal value (Theorem 10). Consider the sequence $(y_n^*)_{n=1}^\infty = (x_n^* - n)_{n=1}^\infty$ of solutions of (48), $(x_n^*)_{n=1}^\infty$ denoting the solutions of (49). It is easy to see that $(y_n^*)_{n=1}^\infty$ is "a good deal" (i.e., risk $= -\infty$, return $= +\infty$). Furthermore, every x_n^* may be computed with the algorithm of Remark 6 and takes the form of (44), so every y_n^* takes the form

$$y_n^* = x_n^* - n = \begin{cases} -n & if\ y_0 > \alpha_n \\ \alpha_n - n - y_0 & if\ \beta_n < y_0 \le \alpha \\ -n & if\ y_0 \le \beta_n \end{cases}$$

for some $0 < \beta_n < \alpha_n$*. In practice one can compute* β_n, α_n *and* y_n^* *for several values of* $n \in \mathbb{N}$ *and then stop once the objective value of (48) is "negative enough" and the expected return of* y_n^* *is "positive enough".* □

Remark 11 *The proof of Theorem* 11 *still applies if alternative pricing models substitute the Black and Scholes one. In particular, if* z_π *is strictly decreasing, (42) and (43) hold, and the manager final wealth* y_0 *is strictly increasing and bounded from below, then Theorem* 11 *still holds, and* α *and* β *may be computed by a straightforward extension of the algorithm in Remark* 6*. Moreover, Remarks* $5 - 10$ *still apply. For instance, the Heston and other stochastic volatility models may be included.* □

7 Conclusions

The paper has dealt with a complete arbitrage free market and a general expectation bounded risk measure, and has analyzed whether it is optimal to invest the capital requirements in the risk-free asset. Once the optimal strategy (shadow riskless asset) has been characterized and its existence has been studied, it has been shown that it is not the risk-free security in many important cases. For instance, if we consider the assumptions of the Black and Scholes model or the Heston model, and the risk measure is the $CVaR$, the DPT, or the expectation bounded risk measure associated with the absolute deviation or the down-side semi-deviation. Moreover, if there are no limits to sale the risk-free asset, *i.e.*, if the manager can borrow as much money as desired, then all of the examples above lead to the existence of good deals ($risk = -\infty$, $return = +\infty$). The existence of good deals also applies for the VaR.

For the $CVaR$ and the Black and Scholes model the explicit expression of the shadow riskless asset has been provided, and it is composed of a long European put, plus a short European put, plus a short binary put. If the confidence level of the $CVaR$ is close to 100% then the shadow riskless asset becomes an European put option, closely related with the notion of "portfolio insurance" This theoretical result seems to be supported by some independent and recent empirical analyses. This may be a surprising and important finding for researchers practitioners, regulators and supervisors. In particular, managers can significantly reduce the capital requirements by trading options.

The explicit expression of the shadow riskless asset has been used so as to construct good deals. Furthermore, it has been pointed out that the methodology allowing us to build the shadow riskless asset and good deals also applies for pricing models beyond Black and Scholes, such as the Heston model and other stochastic volatility models.

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The usual caveat applies.

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