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Equity, Commodity and Interest Rate Volatility Derivatives

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Abstract

A new methodology to construct synthetic volatility derivatives is presented. The underlying asset price process is very general, since equity, commodities and interest rates are included. The focus is on volatility swaps and volatility swap options, but much more derivatives may be considered. The proposed methods optimize the conditional value at risk of the non-hedged risk, and yields both bid and ask prices, as well as optimal hedging strategies for both purchases and sales. Upper bounds for the broker capital losses under very negative scenarios are given. Numerical experiments are presented so as to illustrate the performance in practice of this new approach.

KEY WORDS: Incomplete and imperfect market, Risk measure, Volatility derivative, Commodity, Interest Rate

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1 Introduction

The growing interest in volatility derivatives may be justified by several reasons. These derivatives are becoming very traded in practice because they provide effective ways to diversify investors' portfolios and protect investors against market turmoils. They can be also used to hedge against Vega and/or implied volatility exposure, as well as to speculate on future volatility.

Pricing and hedging equity volatility linked derivatives (mainly variance and volatility swaps) has been broadly studied in the literature. Carr *et al* (2009) present a complete review of the historical development of volatility derivative markets. Breeden *et al* (1978), Neuberger (1994), Carr *et al* (1998) and Demeterfi *et al* (1999) replicate the log-contract by using infinitely many European puts and calls. They need to impose that every strike is available in the market. The analysis allows the authors to price and hedge the variance swap in a model-free framework. Obviously, calls or puts with every strikes are not available in a real market, although in liquid markets there exist enough strikes so as to give accurate approximations of the log-contract. Broadie *et al* (2008) extend the previous approach and minimize the standard deviation of the non-hedged component, in order to price the variance swap with the (finitely many) available options. The volatility swap is studied in this paper too. The authors consider the Heston model and hedge the volatility swap by continuously trading the variance swap. This methodology has been later extended by many others.

More complex volatility pay-offs has been recently created. Portfolio managers who desire non-linear exposure to variance are interested in other possible pay-off functions of realized variance. Some of the most popular examples include call and put options on the realized variance or volatility. Carr *et al* (2005) provide a robust dynamic hedging strategy for quite arbitrary equity linked pay-offs of realized volatility, including volatility swaps. They impose null correlation between the stock price and the variance.

In this paper we contribute to this literature by developing and testing a new methodology which can apply to price and hedge every variance/volatility linked pay-off. Furthermore, the methodology is very general, since there are no limitations for the underlying asset evolution model. As will be seen, we can go beyond equity markets and deal with commodity and interest rate volatility derivatives. Although there exists a well documented literature covering equity linked variance and volatility swaps under general assumptions or approximations, at the best of our knowledge, this is the first paper to propose a general theoretical and numerical methodology to price and hedge every volatility derivative pay-off, including vanilla options or more complex ones, on general underlying assets. Numerical examples on equity, commodity and interest rate will be provided. Moreover, the proposed methodology also yields the hedging strategy performance in monetary terms, by minimizing a coherent risk measure (Artzner *et al*, 1999) of the non-hedged risk, such as the Conditional Value at Risk (*CVaR*, Rockafellar *et al*, 2006).

Our methodology is closely related to the incomplete markets literature. Actually, though many volatility products may be studied in a complete market framework, if one has to often rebalance the position then transaction costs may imply very negative effects in real applications. Hence, if the frequency to rebalance becomes limited, we will be in an incomplete framework (there are no perfect hedging strategies). Although perfect hedges may be often possible by standard no-arbitrage methods (Cvitanic *et al*, 2004), in practice they are difficult to find.

Hedging in incomplete markets has received considerable attention in the risk management activity. Traditional approaches deal with the variance minimization of the non-hedged risk (Schweizer, 1995, Stulz, 2003, Hull, 2008, etc.). In a volatility market this may provoke some caveats, since volatility products are every asymmetric and the variance is not consistent with the second order stochastic dominance (Ogryczak and

Ruszczynski, 1999). Generalizations dealing with more complex risk measures have been proposed, such as the entropy or coherent risk measures, among others. We focus on the approach of Balbás et al (2010). Though these authors study very general risk functions, our applications in volatility markets will only deal with the $CVaR$. In fact, the $CVaR$ is consistent with the second order stochastic dominance (Ogryczak and Ruszczynski, 2002), provides the level of risk in monetary terms (potential capital losses under very negative scenarios),¹ and is sub-additive and coherent (Rockafellar *et al*, 2006), which implies that it facilitates risk diversifications.² All of these properties are making the $CVaR$ more and more popular for researchers, regulators and practitioners. By minimizing the $CVaR$ of the non-hedged component we can provide bid and ask prices in volatility markets, hedging portfolios for both purchases and sales, and upper bounds for the broker capital losses. Previous approaches, although some of them are based in a general non-parametric analyses, which provides a high level of model independence, are limited to variance swaps on an equity diffusive underlying evolution model, and are not able to provide bid and ask prices neither upper bounds for the broker capital losses.

The remainder of the paper is as follows. Section 2 is devoted to presenting the general pricing methodology we are going to deal with. Section 3 describes the most popular volatility derivative pay-offs, including variance (volatility) swaps and vanilla volatility options. Sections 4, 5 and 6 present the main contributions of this paper, since equity, commodity and interest rate derivatives are, respectively, priced and hedged. In order to shorten the exposition, Sections 5 and 6 only study volatility swaps and do not deal with volatility swap options for commodities or interest rates, but the analysis of Section 4 for equity markets may be easily extended. The last section of the paper summarizes the most important conclusions.

¹The variance does not satisfy this property.

²The Value at Risk or VaR does not satisfy this important requirement (Artzner *et al*, 1999).

2 Methodology

First of all let us summarize the pricing method proposed in Balbás *et al* (2010), which will play a critical role in our construction of synthetic volatility derivatives. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will be composed of the set of states of the world Ω , the probability measure \mathbb{P} and the σ -algebra \mathcal{F} . Additionally, we will deal with a finite-horizon $[0, T]$ economy, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T , and a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. In general, $(S_t)_{t \in \mathcal{T}}$ will denote an adapted stochastic price process. Assume that Y is a convex cone composed of super-replicable pay-offs, *i.e.*, for every $y \in Y$ there exists at least one self-financing portfolio whose replicable final pay-off is $S_T \geq y$. Denote by $\mathcal{S}(y)$ the family of such self-financing portfolios, and suppose that there exists

$$\pi(y) = \text{Inf} \{S_0; (S_t)_{t \in \mathcal{T}} \in \mathcal{S}(y)\} \quad (1)$$

for every $y \in Y$. It will be said that $\pi(y)$ is the ask price of y .

Denote by L^2 the space of \mathcal{F}_T -measurable random variables y (pay-offs at T) with finite expectation $\mathbb{E}(y)$ and variance $\sigma^2(y)$. The market will be complete if $Y = L^2$, and incomplete whenever $Y \subset L^2$ and $Y \neq L^2$. Besides, the market will be perfect if Y is a subspace of L^2 and $\pi : Y \rightarrow \mathbb{R}$ is linear, and imperfect otherwise. In general, we will impose the natural conditions, sub-additivity

$$\pi(y_1 + y_2) \leq \pi(y_1) + \pi(y_2)$$

for every $y_1, y_2 \in Y$, and positive homogeneity

$$\pi(\alpha y) = \alpha \pi(y)$$

for every $y \in Y$ and $\alpha \geq 0$. Consequently, π is a convex function such that $-\pi(-y) \leq$

$\pi(y)$ whenever y and $-y$ are super-replicable (or belong to Y). $-\pi(-y)$ is usually called the bid price of y . Finally, we will assume the existence of a risk-free asset that does not generate any friction, and r will be the risk-free rate.

Though Balbás *et al* (2010) deal with general risk measures, our applications for volatility markets will focus on the $CVaR$, as justified in the introduction. For every random variable y in L^2 , the $CVaR$ of y with the confidence level $0 < 1 - \mu_0 < 1$ may be given by two different expressions leading to the same value, namely (Rockafellar *et al*, 2006)

$$\begin{aligned} CVaR_{\mu_0}(y) &= \\ \frac{1}{\mu_0} \int_0^{\mu_0} VaR_t(y) dt &= \\ \text{Max} \left\{ -\mathbb{E}(yz); \mathbb{E}(z) = 1, 0 \leq z \leq \frac{1}{\mu_0} \right\}, \end{aligned} \quad (2)$$

VaR_t denoting the VaR of y with the confidence level $1 - t$. In order to simplify notations, the set of random variables z satisfying the constraints in the second definition above will be called sub-gradient of $CVaR_{\mu_0}$, and denoted by Δ_{μ_0} . Thus,

$$\Delta_{\mu_0} = \left\{ z; 0 \leq z \leq \frac{1}{\mu_0}, \mathbb{E}(z) = 1 \right\} \quad (3)$$

Following Balbás *et al* (2010), we can define a new pricing rule $\Pi : L^2 \rightarrow \mathbb{R}$ by solving two dual optimization problems with the same optimal value, namely³

$$\begin{aligned} \Pi(g) &= \\ \text{Min} \left\{ e^{-rT} CVaR_{\mu_0}(y - g) + \pi(y); y \in Y \right\} &= \\ \text{Max} \left\{ \mathbb{E}(gz); \pi(y)e^{rT} - \mathbb{E}(yz) \geq 0 \forall y \in Y, z \in \Delta_{\mu_0} \right\}. \end{aligned} \quad (4)$$

Briefly speaking, if a trader sells g and buys the reachable pay-off y as a hedging strategy, then he/she will minimize the price of the hedging strategy $\pi(y)$ plus the

³In Balbás *et al* (2013) one can find further analyses so as to guarantee that the optimization problems below are bounded. Needless to say, in our empirical study for volatility derivatives we will always have finite solutions.

required capital $e^{-rT} CVaR_{\mu_0}(y - g)$ that must be invested in a riskless asset in order to make the global risk vanish. In other words, if the trader sells g for $\Pi(g)$, buys y for $\pi(y)$ and invests $\Pi(g) - \pi(y)$ in the risk-free asset, then the global risk of her/his portfolio (measured with the $CVaR_{\mu_0}$) is zero. The second equality in (4) shows that $\Pi(g)$ equals the maximum price of g given by a stochastic discount factor z of the market (Cochrane, 2001) belonging to the $CVaR_{\mu_0}$ sub-gradient Δ_{μ_0} of (3). Balbás *et al* (2010) proved that Π “improves” the bid/ask spread of π in the sense that

$$-\pi(-y) \leq -\Pi(-y) \leq \Pi(y) \leq \pi(y)$$

whenever y and $-y$ are super-replicable, and Π “extends” π to the whole space L^2 in the sense that $\Pi(y) = \pi(y)$ for $y \in Y$ such that the bid/ask spread $\pi(y) + \pi(-y)$ vanishes. In particular, Π is a genuine extension of π if the initial market model is perfect (frictionless).

The pricing methodology above is quite general and Π can be built in a wide family of frameworks. Nevertheless, our numerical experiments will show that a significant simplification of the set of states Ω and the set of trading dates \mathcal{T} still allow us to give accurate prices in volatility markets, with a small bid/ask spread $\Pi(g) + \Pi(-g)$. Moreover, the clear advantage of such a simplification is that the the hedging portfolio y solving the first optimization problem in (4) is easy to create in practice. Consequently, consider a discrete framework in which securities are traded at date 0 and their pay-offs are realized at T . The unique trading dates are $\mathcal{T} = \{0, T\}$ (static approach) and there are only a finite number S of states (Ω is finite) Security j is identified by its pay-off y_j , an element of R^S ($L^2 = R^S$ in this setting) and y_{js} denotes the pay-off at T of security j under state s . It will be assumed that there exists a finite number J of securities with pay-offs y_1, \dots, y_J , $y_j \in R^S$. Under these assumptions the optimal hedging portfolio will be composed of holdings of the J avail-

able securities. These holdings may be positive, zero or negative. A positive holding of a security means a long position in that security, while a negative holding means a short position (short sale). A portfolio is denoted by a J -dimensional vector h , where h_j denotes the holding of security j . The portfolio pay-off at maturity under state s will be $\sum_{j=1}^J h_j y_{js}$. Notice that the pay-off g , to be priced and hedged, will also be considered as a general security and will take S different values, g_s , at maturity.

In this framework (4) shows that $\Pi(g)$ is the optimal value of

$$\left\{ \begin{array}{l} \text{Max } \mathbf{E}(gz), \\ \pi(y)e^{rfT} - \mathbf{E}(yz) \geq 0, \forall y \in Y, \\ \mathbf{E}(z) = 1 \\ 0 \leq z \leq \frac{1}{\mu_0} \end{array} \right. \quad (5)$$

Problem (5) characterizes our optimization problem in the general case of dealing with imperfect markets, meanwhile by means of Proposition 1 below we impose the constraints under the perfect market hypothesis.

Proposition 1. If the market is perfect and $z \in \Delta_{\mu_0}$ then

$$\pi(y)e^{rfT} - \mathbf{E}(yz) \geq 0, \forall y \in Y \iff \mathbf{E}(yz) = \pi(y)e^{rfT}, \forall y \in Y \quad (6)$$

Proof. If the market is perfect, $-y \in Y$ and $-\pi(-y) = \pi(y)$ for every $y \in Y$.

Thus, the inequality in (6) implies

$$0 \leq -\pi(y)e^{rfT} + \mathbf{E}(yz) = -(\pi(y)e^{rfT} - \mathbf{E}(yz)),$$

and the equality of (6) trivially holds.

□

Taking into account Proposition 1, the linear optimization problem (5) becomes

$$\begin{cases} \text{Max } \mathbf{E}(gz), \\ \pi(y)e^{rfT} - \mathbf{E}(yz) = 0, \forall y \in Y, \\ \mathbf{E}(z) = 1 \\ 0 \leq z \leq \frac{1}{\mu_0} \end{cases} \quad (7)$$

In this paper we will consider only the case of perfect markets in the numerical examples. Furthermore, if $z^* \in R^S$ denotes the solution of Problem (7), then Proposition 2 below characterizes the optimal hedging portfolio.

Proposition 2. The optimal hedging portfolio $h^* = (h_j^*)_{j=1}^J$ will be composed of holdings of the J available securities which equal the Lagrange multipliers of (7). In other words, $h_j^* = \Lambda_j$, $j = 1, 2, \dots, J$, Λ_j being the Lagrange multiplier associated with the j^{th} constraint of Problem (7).

Proof. Problem (7) is equivalent to

$$\begin{cases} \text{Min } -\mathbf{E}(gz), \\ \pi(y)e^{rfT} - \mathbf{E}(yz) = 0, \forall y \in Y, \\ \mathbf{E}(z) = 1, 0 \leq z \leq \frac{1}{\mu_0} \end{cases} \quad (8)$$

The Lagrangian function is

$$\mathcal{L}(z, \Lambda, \tau) = -\mathbf{E}(gz) + \sum_{j=1}^J \Lambda_j (\mathbf{E}(y_j z) - \pi(y_j)) + \beta (1 - \mathbf{E}(z)) + \sum_{s=1}^S \tau_s (z_s - \frac{1}{\mu_0}) \quad (9)$$

Reordering expression (9)

$$\mathcal{L}(z, \Lambda, \tau) = \mathbf{E}((-g + \sum_{j=1}^J \Lambda_j y_j - \beta)z) + \sum_{s=1}^S z_s \tau_s - \frac{1}{\mu_0} \sum_{s=1}^S \tau_s - \sum_{j=1}^J \Lambda_j \pi(y_j) + \beta \quad (10)$$

The dual problem of (8) can be expressed as

$$\begin{cases} \text{Max} & -\frac{1}{\mu_0} \sum_{s=1}^{s=S} \tau_s - \sum_{j=1}^{j=J} \Lambda_j \pi(y_j) + \beta \\ & (-g_s + \sum_{j=1}^{j=J} \Lambda_j y_{js} - \beta) p_s + \tau_s \geq 0, \quad s = 1, \dots, S \\ & \tau_s \geq 0, \quad s = 1, \dots, S \end{cases} \quad (11)$$

where β and Λ_j , $j = 1, \dots, J$, will be free or unconstrained parameters, and p_s is the probability associated to state of the nature s . Optimality conditions for $(z, \Lambda, \tau, \beta)$ are

$$\begin{aligned} \mathbb{E}(y_j z) &= \pi(y_j), \quad j = 1, \dots, J \\ \tau_s \left(z_s - \frac{1}{\mu_0} \right) &= 0, \quad s = 1, \dots, S \end{aligned} \quad (12)$$

$$\left[(-g_s + \sum_{j=1}^J \Lambda_j y_{js} - \beta) p_s + \tau_s \right] z_s = 0, \quad s = 1, \dots, S$$

Obviously, first condition in (12) automatically holds for $z = z^*$. In order to study the rest of conditions in (12) we will distinguish different cases:

Case: $z_s = \frac{1}{\mu_0}$. Under this scenario the last equality in (12) implies

$$\left[(-g_s + \sum_{j=1}^J \Lambda_j y_{js} - \beta) p_s + \tau_s \right] = 0, \quad s = 1, \dots, S. \quad (13)$$

Rearranging the expression above

$$\sum_{j=1}^J \Lambda_j y_{js} + \frac{\tau_s}{p_s} - \beta = g_s, \quad s = 1, \dots, S. \quad (14)$$

Then,

$$\sum_{j=1}^J \Lambda_j y_{js} \leq g_s + \beta \quad s = 1, \dots, S, \quad (15)$$

and the result follows from Theorem 13 in Balbás *et al* (2010).

Case: $z_s = 0$. By feasibility condition in (11)

$$\sum_{j=1}^J \Lambda_j y_{js} \geq g_s + \beta, \quad s = 1, \dots, S, \quad (16)$$

and once again Theorem 13 in Balbás *et al* (2010) applies.

Case: $z_s \neq 0$ and $z_s \neq \frac{1}{\mu_0}$. The second condition in (12) implies that

$$\tau_s = 0, \quad s = 1, \dots, S. \quad (17)$$

In addition, by means of the third condition in (12) it can be shown that Theorem 13 in Balbás *et al* (2010) applies because

$$\sum_{j=1}^J \Lambda_j y_{js} = g_s + \beta, \quad s = 1, \dots, S,$$

where $\sum_{j=1}^J \Lambda_j y_{js}$ may be interpreted as the pay-off of a portfolio of Λ_j units of Security j . Furthermore, this portfolio satisfies both optimality and feasibility.

□

3 Describing some volatility derivatives

Let us remind the pay-off of the volatility derivatives we are going to deal with. There are no contributions in this section, but it is worth fixing the exact pay-offs we will price and hedge in future sections.

The variance swap pay-off at maturity is

$$(\sigma_R^2 - K_{var}) \times N. \quad (18)$$

A variance swap has zero net market value at entry. At maturity, as it is shown in (18), the long side of the swap will be equal to the difference between the realized

variance over the life of the contract $[0, T]$ and a constant called the variance swap rate, K_{var} . The equivalent volatility swap payoff is constructed by substituting σ_R^2 by σ_R in (18). The absence of arbitrage implies that the variance swap rate must equal the risk-neutral expected value of the realized variance

$$K_{var} = \mathbb{E}^{\mathbb{Q}}[\sigma_R^2], \quad (19)$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes the expectation under some risk-neutral measure \mathbb{Q} . In numerical examples, the variance (volatility) swap rate, defined as (19), will be referred as the variance (volatility) price.

More complex pay-offs are those of the variance swap European call (or put) and the volatility swap European call (or put). For the variance swap European call the holder will receive

$$C_T = \max(\sigma_R^2 - K_{vol}, 0) \times N. \quad (20)$$

Meanwhile the payoff in a variance swap European put will be

$$P_T = \max(K_{vol} - \sigma_R^2, 0) \times N. \quad (21)$$

The procedure to compute the realized volatility (or variance), σ_R , is generally specified in the derivative contract and must include details about the source and observation frequency of the underlying asset, the factor AF , and the exact method to compute the volatility.

Let $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ be a partition of the time interval $[0, T]$ into n segments of length $\Delta t_i = (t_i - t_{i-1})/T$ for $i = 1, 2, \dots, n$. Most of the traded contracts

define the realized variance as

$$\sigma_R^2 = \sum_{i=1}^n \frac{AF_i}{n-1} \left(\ln \left(\frac{S_i}{S_{i-1}} \right) \right)^2, \quad (22)$$

where $AF_i = \Delta t_i/T$. For the case in which the observations are equally spaced, Expression (22) can be written as

$$\sigma_R^2 = \frac{AF}{n-1} \sum_{i=1}^n \left(\ln \left(\frac{S_i}{S_{i-1}} \right) \right)^2. \quad (23)$$

The other common contractual definition of the realized volatility, although less traded in practice, it is

$$\sigma_R^2 = \frac{AF}{n-1} \sum_{i=1}^n \left(\frac{S_i - S_{i-1}}{S_{i-1}} \right)^2. \quad (24)$$

In this paper we will consider for the numerical examples the log return definition, as in Expression (23), which corresponds to the definition of realized variance for the most traded volatility derivatives (Demeterfi *et al*, 1999).

The methodology proposed in this work is totally compatible with previous results in the literature and extends the volatility derivative knowledge by introducing a new theory for pricing and hedging commodity and interest rate volatility derivatives in an static framework. We will give empirical results for equity, commodity and interest rate linked volatility derivatives, with special focus on bid and ask prices as well as optimal hedging portfolios. Numerical examples containing more complex volatility pay-offs will be proposed, and the development of volatility linked derivatives over more general underlying evolution models will be studied in order to expand volatility derivatives to new asset classes.

4 Equity volatility derivatives

It is well-known that the dynamic hedging of a log contract captures the realized variance under general assumptions regarding the underlying evolution model. Specifically, the underlying model evolution must be diffusive,

$$\frac{dS_t}{S_t} = \mu(t, \dots)dt + \sigma(t, \dots)dW, \quad (25)$$

where μ and σ will be general arbitrary functions of time and other parameters. Following Demeterfi *et al* (1999), the Ito's lemma for $\log(S_t)$ leads to

$$d(\log S_t) = \left(\mu(t, \dots) - \frac{1}{2}\sigma(t, \dots)^2 \right) dt + \sigma(t, \dots)dW. \quad (26)$$

Combining Equations (25) and (26),

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{1}{2}\sigma(t, \dots)^2. \quad (27)$$

Integrating Equation (27),

$$V = \frac{1}{T} \int_0^T \sigma(t, \dots)^2 dt = \frac{2}{T} \left[\int_0^T \frac{dS_t}{S_t} - \log \frac{S_T}{S_0} \right]. \quad (28)$$

Hence, a trader can replicate V by a dynamically traded share position which always equals $2/S_t$ shares, a static short position in a contract paying twice the logarithm of the total return at T , and a bond position that finances the shares. It is interesting to note that the pricing and hedging of a variance swap given by (28) is non-parametric, *i.e.*, it does not depend on the volatility function $\sigma(t, \dots)$. Nevertheless, let us remark some possible limitations of this approach. Firstly, the accuracy of this strategy will depend on whether σ_R^2 is a good estimator of the discretely sample variance defined in the contract. It should be expected this to be the case if Δt_i is small enough, so the hedging strategy in (28) might not hold for sampling intervals such as weekly samples.

Secondly, in order to hedge volatility derivatives it will be required to hedge a square root derivative on the variance swap, involving a dynamic trading strategy in these log contracts which will result in excessive transaction costs. Thirdly, the hedging strategy in (28) is useless with more complex volatility pay-offs such as vanilla options or other more exotic volatility products.

Taking into account the above limitations, we apply the new methodology of Section 2. We will deal with volatility swaps and volatility swap call options. We provide, in both cases, bid and ask prices, hedging portfolios, and $P\&L$ distributions at maturity for different $CVaR$ levels. The numerical experiments developed in this section assume the following general underlying evolution:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S, t)dW. \quad (29)$$

Notice that the methodology provided in Section 2 is completely independent of the underlying evolution model and can be applied with any other equity model. The $\sigma(S, t)$ process is unrestricted. In particular, the instantaneous volatility $\sigma(S, t)$ can have stochastic drift, stochastic volatility and a stochastic jump component, among many other alternatives. It is not the object of the present paper to study the price dependency with different models neither to study the wide set of well-known volatility and equity models, although the reader can find comparisons among, jump-diffusion, local volatility with no jumps, and a constant volatility model with no jumps, under the classical pricing approach of Windeliff *et al* (2006).

As said above, there are numerous papers studying the pricing and hedging of equity volatility derivatives. For illustrative reasons, we have based our numerical examples on a well-known data set from Demeterfi *et al* (1999). We will implement a simple model, with a constant volatility given by the implied volatility of the ATM option ($\sigma_{imp} = 0.2$), and following Demeterfi *et al* (1999) data with $r = 0.05$. Un-

der these assumptions, in a risk-neutral world with a constant risk-free rate r , the underlying will follow the *SDE*

$$\frac{dS_t}{S_t} = rdt + \sigma dW. \quad (30)$$

Next, using the above underlying evolution model, we will develop the following numerical tests: First, a volatility swap with maturity in three months will be priced and hedged. Second, a volatility swap call option will be studied, presenting comparisons between bid-ask prices as well, with different strikes and *CVaR* levels.

4.1 Numerical results

4.1.1 Volatility swaps

Assume that the initial spot level is $S_0 = 100$, and the available securities composing the optimal hedging portfolio are eight call options shown in Table 1 along with the risk-free asset ($J = 9$). The underlying evolution model is simulated by Montecarlo with an Euler discretization scheme

$$S_{t+1} = S_t + S_t r \Delta t + S_t \sigma \sqrt{\Delta t} \varepsilon_i. \quad (31)$$

The volatility swap pay-off is defined as

$$\sigma_R = \sqrt{\frac{AF}{n-1} \sum_{i=1}^n \left(\ln \left(\frac{S_i}{S_{i-1}} \right) \right)^2}. \quad (32)$$

Hence, denoting by $w = 1, \dots, S$ the simulated paths, we will have S different values of (32) at maturity

$$\sigma_R(w), \quad w = 1, \dots, S. \quad (33)$$

The pay-off vector g will be composed of the S different values of $\sigma_R(w)$. Numerical

results for the $CVaR_{95\%}$, notional $N = 100000$ and ten thousand Montecarlo simulations ($S = 10000$), are shown in Table 1. Last columns present the optimal hedging portfolio for the ask price (product sale), in units of each available call option. These hedging portfolios are computed by mean of Proposition 2. In addition, the ask price equals 23370 m.u. , and the bid price is calculated by substituting g by $-g$, and equals 17180 m.u.

Table 1: (Equity Volatility Swap) Hedging Portfolio Data and Results. Parameters: $r = 0.05$, $\sigma = 0.2$, $S_0 = 100$, $T = 0.25$, $\Delta t = \text{"Daily"}$, $N = 100000 \text{ m.u.}$ and $CVaR$ confidence level = 95%. Number of simulations = 10000

Calls Strikes	Imp Vol	Price	Optimal Hedging Portfolio
100	20	4.5790	-80
105	19	2.2581	100
110	18	0.8874	-40
115	17	0.2578	120
120	16	0.0501	-20
125	15	0.0057	50
130	14	0.0003	-100
135	13	0.000006	-20

* The hedging portfolio composition will be composed of the above options plus an investment of 22530 m.u in the risk-free asset.

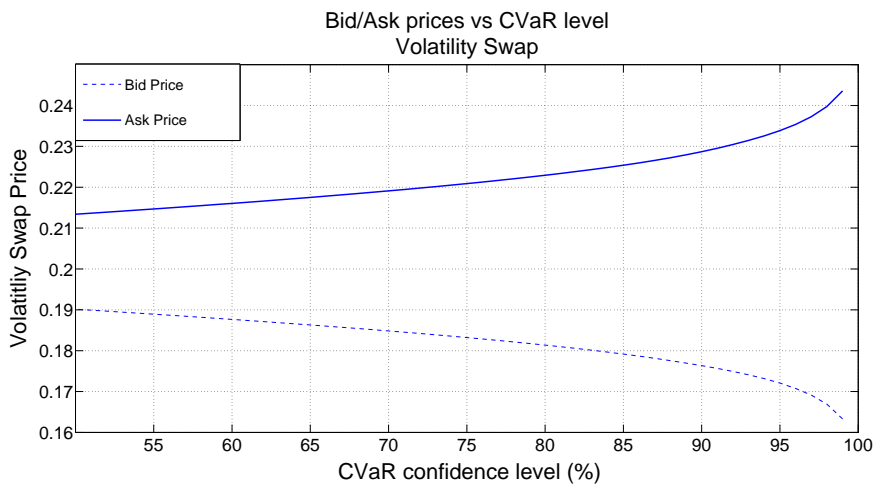


Figure 1: Pricing performance for a volatility swap. Bid and ask prices evolution with the CVaR confidence level. Parameters $r = 0.05$ and $\sigma = 0.2$. Number of simulations = 10000.

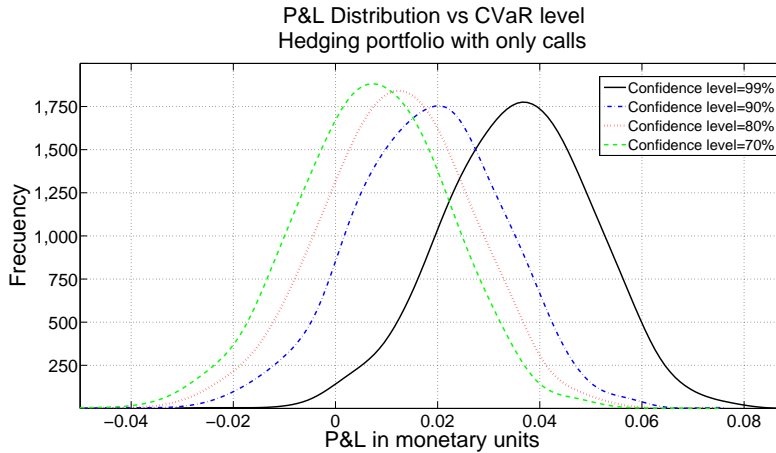


Figure 2: P&L distribution for different CVaR confidence levels. Parameters $r = 0.05$ and $\sigma = 0.2$. Number of simulations = 10000.

Figure 1 shows how the bid-ask spread evolves under different *CVaR* confidence levels, from 50% to 99%. Clearly, it can be observed how the bid-ask spread increases with more demanding *CVaR* levels, with a remarkable increment in the ask price and an important decrement in the bid price. This numerical result is coherent with the previous expectations. For a trader who is interested in selling the volatility swap, more demanding confidence levels increases the hedging accuracy requirements. Finally, the hedging portfolio performance is studied in Figure 2, computing the *P&L* distribution at maturity for confidence levels equaling 99%, 90%, 80% and 70%. The *P&L* distribution was computed as the difference between the optimal hedging portfolio pay-off and the volatility swap pay-off at maturity for each possible state of the nature.

4.1.2 Volatility swap options

The payoff vector g will be composed of the S different values of $C_T(w)$ (see Equation (20)), which represent the volatility swap call option pay-off in each state of the nature.

In this case, we will include the call option underlying asset (the volatility swap) as a new hedging instrument.⁴ The problem arises when we try to include the underlying asset (volatility swap) current price. Theorem 9 in Balbás *et al* (2010) justifies that we cannot incorporate the volatility swap with the bid/ask prices computed in the sub-section above. Indeed, if we do that then the volatility swap call bid/ask price will remain the same, *i.e.*, our methodology will lead to similar call prices with and without the volatility swap as a hedging instrument, and the volatility swap will never be in the optimal hedging strategy. In order to overcome this caveat, we deal with the available securities of Table 1, compute the bid and the ask price of the volatility swap, and use the average value $\frac{bid+ask}{2}$ as the volatility swap price.⁵ The obtained average price equals 0.1993 *m.u.* Table 2 shows numerical results for a call option and for $CVaR_{65\%}$, notional $N = 100000$, strike $K = 0.1993$ (*ATM*) and ten thousand Montecarlo simulations. Last columns give the optimal hedging portfolios of the option sale (Proposition 2). Obviously, in this example we have ten hedging instruments ($J = 10$). Under these assumptions the ask price equals 1100 *m.u.* and the bid price (option purchase) is 180 *m.u.*

Figure 3 shows the call bid-ask price evolution with respect to the strike level K . The parameters are: $CVaR_{65}$, ten thousand Montecarlo simulations ($S = 10000$) and notional of one monetary unit ($N = 1$). We replicate the classical vanilla option results and, as we should expect, the maximum bid-ask spread occurs exactly for the *ATM* strike level.

⁴Recall that the underlying asset is usually “the best” hedging instrument of every option.

⁵According to the numerical results of the sub-section above, the bid/ask average value of the volatility swap remains stable with regard to the $CVaR$ confidence level, and it also achieves quite realistic values.

Table 2: (Volatility Call Option) Hedging Portfolio Data and Results. Parameters: $r = 0.05$, $\sigma = 0.2$, $S_0 = 100$, $T = 0.25$, $\Delta t = \text{"Daily"}$ and $N = 100000$ m.u. Number of simulations = 10000

Calls Strikes	Imp Vol	Price	Optimal Hedging Portfolio
100	20	4.5790	-22
105	19	2.2581	12
110	18	0.8874	7
115	17	0.2578	48
120	16	0.0501	20
125	15	0.0057	-36
130	14	0.0003	118
135	13	0.000006	-130
Volatility Swap			50985
Risk Free Asset			-9423

* The volatility swap price is equal to 0.1993 (average between bid and ask prices).

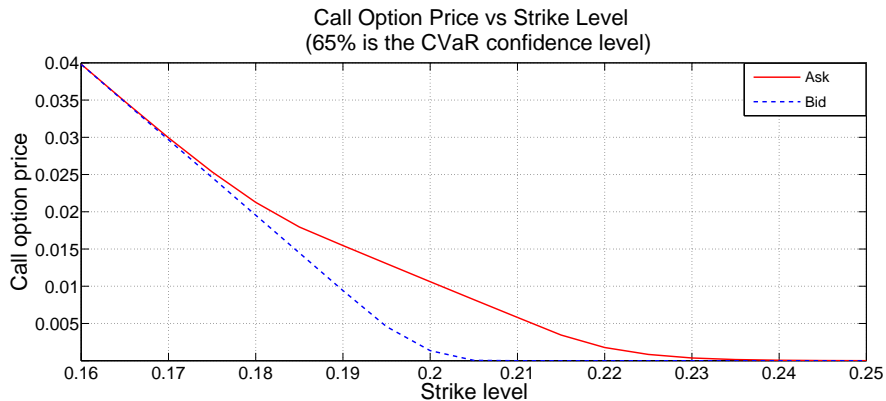


Figure 3: Call option price evolution with the strike level, K . Parameters: $CVaR$ confidence level = 65%, $N = 1$, $r = 0.05$ and $\sigma = 0.2$. Number of simulations = 10000.

5 Commodity volatility derivatives

Brennan (1991), Gibson et al (1990) and Cortazar et al (1994) show that, under a general equilibrium framework, the impact of relative supply will induce commodity prices to follow a mean reversion process. In Schwartz *et al* (1997) three different models for the stochastic behavior of commodity prices are developed, which include mean reversion and admit a simple closed expression for the related future contracts. A one factor model has been chosen to perform the empirical example. Under this

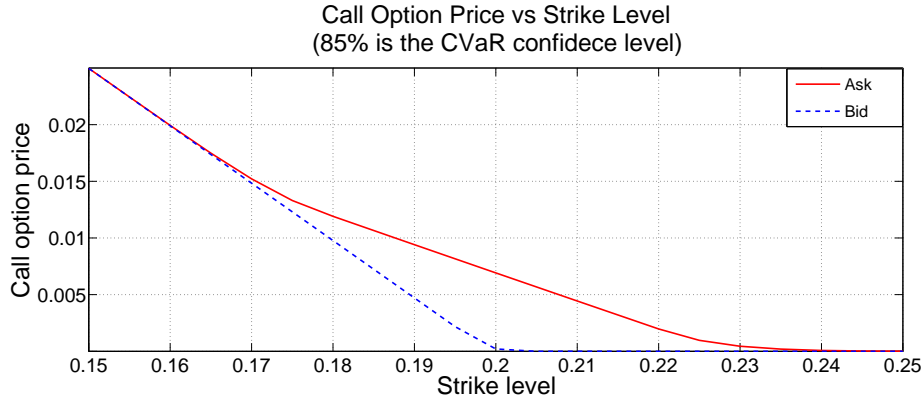


Figure 4: Call option price evolution with the strike level, K . Parameters: $CVaR$ confidence level = 85%, $N = 1$, $r = 0.05$ and $\sigma = 0.2$. Number of simulations = 10000.

assumption the commodity spot price will follow the stochastic process

$$dS = k(\mu - \ln S)Sdt + \sigma SdZ. \quad (34)$$

Applying Ito's Lemma and defining $X = \ln S$, the log price follows an Ornstein-Uhlenbeck stochastic process

$$dX = k(\alpha - X)dt + \sigma dZ, \quad (35)$$

with

$$\alpha = \mu - \frac{\sigma^2}{2k}. \quad (36)$$

In Equation (36) the parameter $k > 0$ is a measure of the level of mean reversion of the long run mean log price, α . Under common assumptions, the Ornstein-Uhlenbeck underlying evolution of (35) in a risk neutral world becomes

$$dX = k(\alpha^* - X)dt + \sigma dZ^*, \quad (37)$$

where, $\alpha^* = \alpha - \lambda$, and λ it is called market price of risk.

Assuming a constant interest rate, the commodity future price with maturity T will be computed as the expected price at T

$$F(S, T) = \mathbb{E}[S(T)] = \exp \left[e^{-kT} \ln S + (1 - e^{-kT}) \alpha^* + \frac{\sigma^2}{4k} (1 - e^{-2kT}) \right]. \quad (38)$$

We are interested in pricing and hedging a volatility swap over a forward commodity contract. We assume the above underlying evolution model from Schwartz *et al* (1997).

5.1 Numerical results

Numerical results related to the pricing and hedging of a volatility swap with maturity in one year over a future contract with delivery in one month are presented in this section. The parameter values are based on Schwartz's results for the cooper case (Model 1 in Schwartz *et al*, 1997): $k = 0.369$, $\mu = 4.854$, $\sigma = 0.233$ and $\lambda = -0.339$. On the other hand, call options prices have been computed with reverse engineering to match the model theoretical call options prices (see Table 3).

The pay-off will be

$$\sigma_R^2 = \frac{AF}{n-1} \sum_{i=1}^n \left(\ln \left(\frac{F_i}{F_{i-1}} \right) \right)^2, \quad (39)$$

where the future price is computed as

$$F_i = F(S_i, T) = \exp \left[e^{-kT} \ln S_i + (1 - e^{-kT}) \alpha^* + \frac{\sigma^2}{4k} (1 - e^{-2kT}) \right], \quad (40)$$

and the process for $X = \ln S$ is simulated by Montecarlo with an Euler discretization scheme

$$X_{t+1} = X_t + k(\alpha^* - X_t) \Delta t + \sigma \sqrt{\Delta t} \varepsilon_i. \quad (41)$$

Table 3: (Commodity Volatility Swap). Hedging Portfolio Data and Results. Parameters: $r = 0.05$, $k = 0.369$, $\mu = 4.854$, $\sigma = 0.233$, $\lambda = -0.339$, $S_0 = 100$, $T = 1$ year, $\Delta t = \text{"Daily"}$, $N = 100000$ m.u, and $CVaR$ confidence level = 95%. Number of simulations = 10000

Calls Strikes	Price	Optimal Hedging Portfolio (Units)
100	21.6406	-21
110	14.6334	42
120	9.2395	-65
125	7.1599	-49
Puts Strikes	Price	Optimal Hedging Portfolio (Units)
105	2.7704	-40
100	1.7035	51
95	0.9719	-51
90	0.5016	48

* The hedging portfolio composition will be composed of the above options plus an investment of 24081 m.u in the risk-free asset.

Numerical results are calculated with ten thousand Montecarlo simulations ($S = 10000$). As mentioned above, we are considering a forward contract with delivery in one month ($T = 1/12$). Since the volatility swap maturity is one year, we will use $\Delta t = 1/365$ for a daily computation of the realized volatility. Therefore, if we denote by $w = 1, \dots, S$ the computed paths, we will have S different values of σ_R^2 at maturity

$$\sigma_R(w), \quad w = 1, \dots, S. \quad (42)$$

The pay-off vector g will be composed of $S = 10000$ different values of $\sigma_R(w)$. The $J = 9$ hedging instruments are the four call options and the four put options of Table 3, along with the risk free asset ($r = 0.05$). Numerical results for $CVaR_{95\%}$, notional $N = 100000$ and ten thousand Montecarlo simulation ($S = 10000$) are shown in Table 3. Once again, the last column in Table 3 gives the optimal hedging portfolio (product sale). The bid price is equal to 17180 *m.u.*, and the ask price equals 23370 *m.u.* In

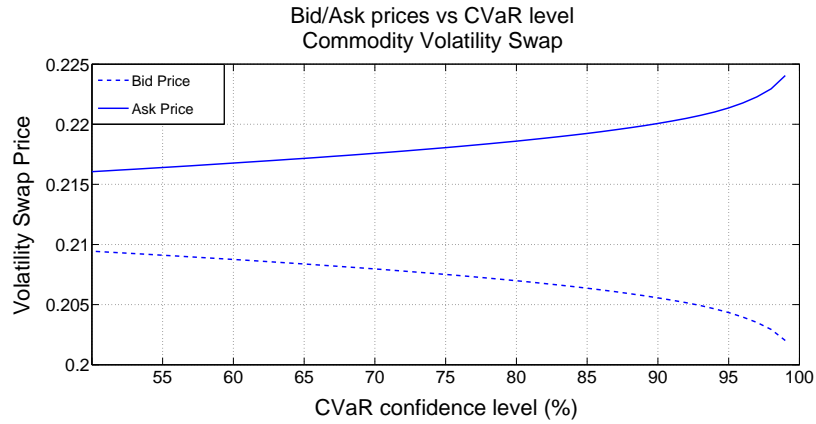


Figure 5: Pricing performance for a commodity volatility swap. Bid and ask prices evolution with the CVaR confidence level. $r = 0.05$, $k = 0.369$, $\mu = 4.854$, $\sigma = 0.233$, $\lambda = -0.339$, $S_0 = 100$, $T = 1$ year, $\Delta t = \text{"Daily"}$, $N = 1$ m.u. Number of simulations = 10000.

addition, in Figure 5 we can see how the bid-ask spread changes under different *CVaR* confidence levels. As it should be expected, the bid-ask spread increases with more demanding *CVaR* levels. To conclude this numerical example, we provide evidences about the hedging portfolio performance in Figure 6, by computing the *P&L* distribution at maturity for different *CVaR* confidence levels: 99%, 89%, and 79%.

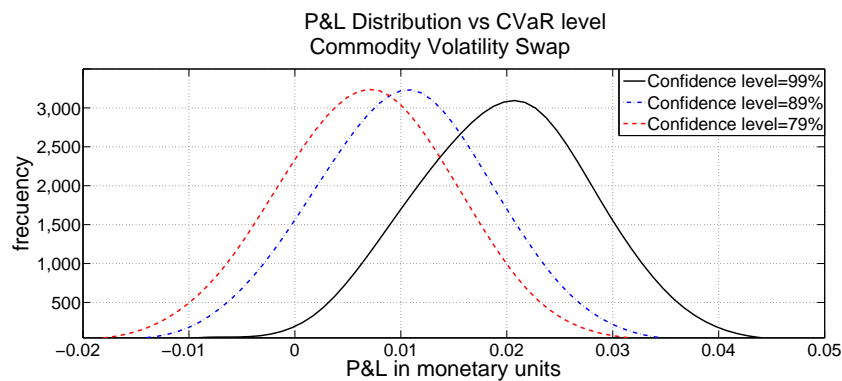


Figure 6: (Commodity Volatility Swap) *P&L* distribution for different CVaR confidence levels. Parameters: $r = 0.05$, $k = 0.369$, $\mu = 4.854$, $\sigma = 0.233$, $\lambda = -0.339$, $S_0 = 100$, $T = 1$ year, $\Delta t = \text{"Daily"}$, $N = 1$ m.u. Number of simulations = 10000.

6 Interest rate volatility derivatives

Interest rates will be the last asset class under consideration in this paper. In order to illustrate the generality of the theory proposed in this paper, we model and price volatility interest rates under a three steps approach: Firstly we construct and calibrate a recombining trinomial tree (see Hull and White, 2001). Secondly, we obtain the realized volatility for each state of the nature at maturity by using Montecarlo simulation. Thirdly, the optimization problem (7) will be solved. Therefore, the numerical experiments provided in this section assume the general interest rate model proposed by Hull and White (2001)

$$df(r) = [\theta(t) - a(t)f(r)]dt + \sigma(t)dZ, \quad (43)$$

where the function $\theta(t)$ gives term-structured parameters that will be used to fit the initial term structure. Functions $a(t)$ and $\sigma(t)$ are volatility parameters that are selected to fit the current market prices of different interest rate securities. Finally, the diffusion process, dZ , will be a standard Wiener process with zero mean and variance equal to dt . Model (43) includes some of the most popular term-structure models for interest rates, by making use of the function $f(r)$: Ho-Lee (1986), Hull-White (1990), Pelsser (1996) and Black-Karasinski (1991) are examples. For the empirical analysis Black-Karasinski (1991) is implemented, which is perhaps one of the most popular short interest rate model nowadays.

$$dln(r) = [\theta(t) - a(t)ln(r)]dt + \sigma(t)dZ. \quad (44)$$

The parameter values used in the numerical example are based on historical parameter estimations for the Black-Karasinski model: $a = 0.01$ and $\sigma = 0.25$. Meanwhile $\theta(t)$ was computed to fit the initial term structure shown in Table 4.

Table 4: Term-Structure. Date: June 30, 2011

Time to Maturity	Bond Price
0	0.9993603
0.25	0.9984632
0.5	0.9973697
0.75	0.996091
1	0.9944365
1.25	0.9921851
1.50	0.989507
1.75	0.9861916
2	0.9817949
2.25	0.9774344
2.5	0.9722901
2.75	0.9661169
3	0.9602342

6.1 Results

Assume that the initial short term interest rate is $r_0 = 0.0025$, the available J hedging instruments are caps with strikes 0.01, 0.015 and 0.02 (Table 5) plus the risk-free asset ($J = 4$). The volatility swap pay-off is defined as usually by

$$\sigma_R = \sqrt{\frac{AF}{n-1} \sum_{i=1}^n \left(\ln \left(\frac{r_i}{r_{i-1}} \right) \right)^2}. \quad (45)$$

Therefore, if we denote by $w = 1, \dots, S$ the computed paths, we will have S different values of (45) at maturity

$$\sigma_R(w), \quad w = 1, \dots, S. \quad (46)$$

The payoff vector g will be composed of the S different values of $\sigma_R(w)$. Numerical results for a $CVaR_{95\%}$, notional $N = 1$ and ten thousand Montecarlo simulation ($S = 10000$) are shown in Table 5. Last columns figures give the optimal hedging portfolio for the ask price (product sale, Proposition 2). The bid price value equals 0.1010 *m.u.*, and the ask price is 0.1266 *m.u.*

Table 5: Hedging Portfolio Data and Results. Parameters: $a = 0.01$, $\sigma = 0.25$, $r_0 = 0.0025$, $T = 3$ Years, $\Delta t = 0.25$, $N = 1$, and $CVaR$ confidence level = 95%. Number of simulations = 10000.

Caps Strikes	Price	Ask Hedging Portfolio (Units)
0.01	0.01243	5.622
0.015	0.00963	1.510
0.02	0.008624	-6.9274

* The hedging portfolio composition will be composed of the above options plus an investment of 0.0963 m.u in the risk-free asset.

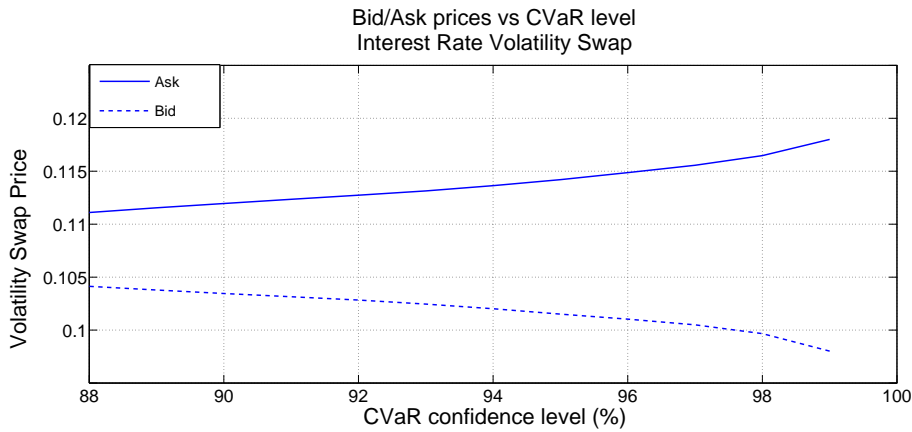


Figure 7: Pricing performance for a volatility swap. Bid and ask prices evolution with the $CVaR$ confidence level. $a = 0.01$, $\sigma = 0.25$, $r_0 = 0.0025$, $T = 3$ Years, $\Delta t = 0.25$, $N = 1$. Number of simulations = 10000.

Figure 7 shows how the bid-ask spread evolves for different $CVaR$ confidence levels, from 88% to 99%. Clearly, the bid-ask spread increases with more demanding $CVaR$ levels, with a remarkable increment in the ask price and an important decrement in the bid price. We conclude the numerical example by providing evidences about the hedging portfolio performance in Figure 8, which gives $P\&L$ distributions at maturity for different $CVaR$ confidence levels (99%, 96%, and 93%).

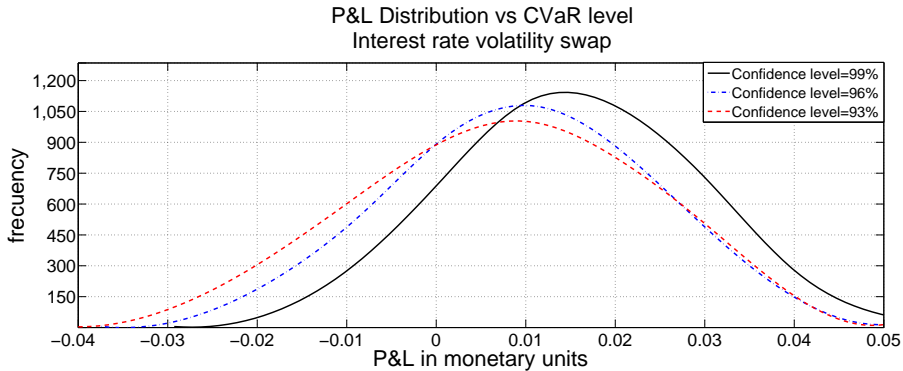


Figure 8: P&L distribution for different CVaR confidence levels. Parameters $a = 0.01$, $\sigma = 0.25$, $r_0 = 0.0025$, $T = 3$ Years, $\Delta t = 0.25$, $N = 1$. Number of simulations = 10000.

7 Conclusions

The present paper has focused on several unsolved problems about pricing and hedging volatility derivatives. The main idea of the paper is to price and hedge volatility-linked products by dealing with an incomplete market and minimizing the $CVaR$ of the non-hedged risk. The broker can buy or sell the studied volatility product for the proposed price, implement the proposed hedging strategy, and invest the received price in a riskless asset. If so, the $CVaR$ of the broker global portfolio will vanish.

The main contribution of the paper is to make it practical the idea above by creating appropriate discrete sets of scenarios with their probabilities. This allows us to give bid and ask prices for equity, commodity and interest rates volatility swaps and equity volatility options. Moreover, the analysis may be easily extended so as to deal with more sophisticated (equity, commodity or interest rate)-volatility products. The bid (ask) price is given with the associated portfolio hedging the product purchase (sale), and upper bounds for potential capital losses of the broker under very negative scenarios are given. Many numerical experiments have been presented, along with the performance in practice of the proposed pricing and hedging method.

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