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One for all: Nesting Asymmetric Stochastic Volatility models

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Keywords: EGARCH, Leverage effect, MCMC estimator, Stochastic News Impact Surface, Threshold Stochastic Volatility, VaR, WinBUGS

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One for all: Nesting Asymmetric Stochastic Volatility models[☆]

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1. Introduction

When modeling the second order dynamics of univariate financial returns, it is often observed that volatility increases are larger in response to negative than to positive past returns of the same magnitude; see Bollerslev et al. (2006) for a comprehensive list of references and Hibbert et al. (2008) for a behavioral explanation. After Black (1976), this asymmetric response of volatility is popularly known as *leverage effect* in the related literature. In order to represent the dynamic evolution of conditionally heteroscedastic time series with leverage effect, this paper focuses on Stochastic Volatility (SV) models which have been shown to have interesting properties when compared with Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models; see Carnero et al. (2004). Incorporating the leverage effect into SV models can have important implications from the point of view of financial models. For example, in the context of option valuation, Hull and White (1987) emphasize the role of the leverage effect in the Black-Scholes formula and suggest that ignoring it can cause significant biases. Also, Nandi (1998) points out the important role of the correlation between volatilities and returns when pricing and hedging S&P500 index options. More recently, Lien (2005) shows that the average optimal hedge ratios are greater when the leverage is considered. A wide variety of alternative econometric specifications are already available to choose among when dealing with SV models with leverage effect. However, in this paper, we propose a further specification. Our main motivation is that the new model, called Generalized Asymmetric SV (GASV), nests some of the most popular asymmetric volatility specifications previously available in the literature. In particular, it nests the asymmetric SV model originally proposed by Taylor (1994) and Harvey and Shephard (1996) which incorporates the leverage effect through correlation between the disturbances in the level and log-volatility equations. The second specification obtained as a particular case of the new model

was proposed by Demos (2002) and Asai and McAleer (2011) who suggest adding a noise to the log-volatility equation specified as in the EGARCH model of Nelson (1991). Finally, the third nested model is a restricted Threshold SV model in which the constant parameter of the log-volatility equation changes depending on whether past returns are positive or negative; see Asai and McAleer (2006) for the restricted Threshold SV model and Breidt (1996) and So et al. (2002) for the general one.

We derive closed-form expressions of several statistical moments of the GASV model related with the main empirical properties often observed in real financial time series, namely, excess kurtosis, positive and persistent autocorrelations of power-transformed absolute returns and negative cross-correlations between returns and future power-transformed absolute returns. We show that the GASV model allows for a large range of combinations of these moments and, consequently, it is flexible to represent a wide range of dynamics of conditionally heteroscedastic time series with leverage effect. As a marginal outcome of this analysis, we also obtain the statistical properties of the models nested within the GASV, some of which were not previously available in the literature. Comparing these properties, we are able to point out the advantages and limitations of each of the restricted specifications.

A useful tool to describe the asymmetric response of volatility to positive and negative past returns represented by alternative models is the News Impact Curve (NIC) which was originally proposed by Engle and Ng (1993) in the context of GARCH models. Yu (2012) proposes an extension of the NIC to SV models based on measuring the effect of the level disturbance on the conditional volatility. However, this is a rather difficult task due to the lack of observability of the volatility in SV models. In this paper, we suggest an alternative definition of the NIC in the context of SV models. Note that, a fundamental difference between GARCH and SV models is that in the former models there is a unique disturbance while SV models have two disturbances.

The NIC in a GARCH model relates the volatility with the unique disturbance of the model. However, in SV models, it seems more sensible to relate the volatility with their two disturbances. Therefore, in this paper, we propose to represent the response of volatility by a surface called Stochastic News Impact Surface (SNIS).¹ Analyzing the SNIS, we show that the asymmetric impact of the level disturbance on the volatility can be different depending on the volatility disturbance.

Although SV models are attractive for modeling volatility, their empirical implementation is limited by the difficulty involved in the estimation of their parameters which is complicated by the lack of a closed-form expression of the likelihood. Furthermore, the volatility itself is unobserved and cannot be directly estimated. Consequently, several simulation-based procedures have been proposed for the estimation of parameters and volatilities; see Broto and Ruiz (2004) for a survey. Among them, Monte Carlo Markov Chain (MCMC) based approaches have become popular given their good properties in estimating parameters and volatilities; see, for example, Omori et al. (2007), Omori and Watanabe (2008), Nakajima and Omori (2009), Abanto-Valle et al. (2010) and Tsiotas (2012) for MCMC estimators of SV models with leverage effect. In this paper, we consider a MCMC estimator implemented in the user-friendly and freely available WinBUGS software described by Meyer and Yu (2000). This estimator is based on a single-move Gibbs sampling algorithm and has been recently implemented in the context of asymmetric SV models by, for example, Yu (2012) and Wang et al. (2013). The MCMC estimator implemented by WinBUGS is appealing because it can handle non-Gaussian level disturbances without much programming effort. We carry out extensive Monte Carlo experiments and show that, if the level error distribution is known, it has adequate finite sample properties to estimate the

¹The SNIS proposed in this paper should not be confused with the News Impact Surface (NIS) defined in the context of multivariate models; see, for example, Asai and McAleer (2009), Savva (2009) and Caporin and McAleer (2011).

parameters and volatilities of the proposed GASV model in situations similar to those encountered when analyzing time series of real financial returns. Furthermore, we show that the restricted specifications can be adequately identified when the parameters of the GASV model are estimated using the WinBUGS software. Therefore, in empirical applications, researchers will be better off by fitting the general model proposed in this paper and letting the data choose the preferred specification of the volatility instead of choosing a particular ad hoc specification. Finally, the MCMC estimator is implemented to estimate the volatilities and Value at Risk (VaR) of daily S&P500 returns after fitting the new model proposed in this paper.

The rest of the paper is organized as follows. Section 2 describes the proposed GASV model and derives its statistical properties. The properties of the restricted specifications are analyzed and compared with each other in Section 3. Section 4 conducts Monte Carlo experiments to analyze the finite sample properties of the MCMC estimator of the parameters and underlying volatilities. Section 5 presents an empirical application to daily S&P500 returns. Finally, the main conclusions and some guidelines for future research are summarized in Section 6.

2. The Generalized Asymmetric Stochastic Volatility model

In this section, we propose a new and flexible asymmetric SV model and derive its statistical properties. In particular, we obtain closed-form expressions of the marginal variance and kurtosis, the autocorrelations of power-transformed absolute returns and cross-correlations between returns and future power-transformed absolute returns.

2.1. Model description

Let y_t be the return at time t , σ_t its volatility, $h_t \equiv \log \sigma_t^2$ and ϵ_t be an independent and identically distributed (IID) sequence with mean zero and variance one. The

GASV model is defined as follows

$$y_t = \exp(h_t/2)\epsilon_t, \quad t = 1, \dots, T \quad (1)$$

$$h_t - \mu = \phi(h_{t-1} - \mu) + f(\epsilon_{t-1}) + \eta_{t-1}, \quad (2)$$

where $f(\epsilon_t) = \alpha I(\epsilon_t < 0) + \gamma_1 \epsilon_t + \gamma_2(|\epsilon_t| - E|\epsilon_t|)$ with $I(\cdot)$ being an indicator function that takes value one when the argument is true and zero otherwise. The volatility noise, η_t , is a Gaussian white noise with variance σ_η^2 . It is assumed to be independent of ϵ_t for all leads and lags. The normality of η_t has been justified by Andersen et al. (2001a) and Andersen et al. (2001b, 2003). The scale parameter, μ , is related with the marginal variance of returns, while ϕ measures the persistence of the volatility shocks and, consequently, is related with the rate of decay of the autocorrelations of power-transformed absolute returns towards zero. The parameters α and γ_1 incorporate different asymmetries related with the leverage effect. In particular, α is a threshold parameter that deals with changes in the scale parameter depending on whether past returns are positive or negative. It captures the leverage effect observed in financial returns when $\alpha > 0$. On the other hand, γ_1 generates correlation between the volatility and the lagged level noise and, if negative, also picks up leverage effect. Note that, the GASV model in equations (1) and (2) defines the return at time t as being correlated with the volatility at time $t+1$; see Yu (2005) for the adequacy of defining the leverage effect in this way rather than including contemporaneous correlation between y_t and h_t as in Melino and Turnbull (1990), Jacquier et al. (2004) or Bandi and Renò (2012). Finally, the parameter γ_2 measures the dependence of h_t on past absolute return disturbances in the same form as in the EGARCH model. As we will show later, it allows the model to generate richer

dynamics of volatility clustering.²

It is important to mention that the only assumption made about the distribution of the level disturbance, ϵ_t , is that it is an IID sequence with mean zero and variance one. As a consequence, ϵ_t is strictly stationary. We are not assuming any particular distribution of ϵ_t . In the related literature, different assumptions about this distribution have been considered. For example, Jacquier et al. (1994), Harvey and Shephard (1996) and Asai and McAleer (2011) assume that ϵ_t is a Gaussian process. The GASV model with ϵ_t being Gaussian will be denoted as GASV-N. Although the Gaussianity of ϵ_t is the most popular assumption, there has been other proposals that consider heavy-tailed distributions such as the Student-t distribution or the Generalized Error Distribution (GED)³; see, for example, Chen et al. (2008), Choy et al. (2008) and Wang et al. (2011, 2013). When ϵ_t follows a GED distribution, the model will be denoted as GASV-G. Finally, several authors include simultaneously both leptokurtosis and skewness in the distribution of ϵ_t by assuming an asymmetric GED distribution as in Cappuccio et al. (2004) and Tsiotas (2012) or a skew-Normal and a skew-Student-t distributions as in Nakajima and Omori (2012) and Tsiotas (2012). However, in the context of daily exchange rates, Cappuccio et al. (2004) conclude that skewness in the distribution of ϵ_t is not important.

2.2. Statistical properties

To analyze the ability of the GASV model in capturing the main empirical features often observed in financial returns, we now derive its statistical properties. Theorem

²In independent work, Asai et al. (2012) mention a specification of the volatility similar to the GASV model with long-memory and Gaussian errors. However, they do not develop further the statistical properties of the model.

³The GED distribution with parameter ν is described by Harvey (1990) and has the attractiveness of including distributions with different tail thickness as, for example, the Normal when $\nu = 2$, the Double Exponential when $\nu = 1$ and the Uniform when $\nu = \infty$. The GED distribution has heavy tails if $\nu < 2$.

2.1 establishes the sufficient conditions for stationarity of y_t and derives the expression of $E(|y_t|^c)$ for any positive real number c .

Theorem 2.1. *Define y_t by the GASV model in equations (1) and (2). The process $\{y_t\}$ is strictly stationary if $|\phi| < 1$. Further, if ϵ_t follows a distribution such that $E(\exp(0.5cf(\epsilon_t))) < \infty$ and $E(|\epsilon_t|^c) < \infty$ for any positive real number c , then $\{|y_t|\}$ has finite, time-invariant moments of arbitrary order which are given by*

$$E(|y_t|^c) = \exp\left(\frac{c\mu}{2}\right) E(|\epsilon_t|^c) \exp\left(\frac{c^2\sigma_\eta^2}{8(1-\phi^2)}\right) P(0.5c\phi^{i-1}), \quad (3)$$

where $P(b_i) \equiv \prod_{i=1}^{\infty} E(\exp(b_i f(\epsilon_{t-i})))$.

Proof. See Appendix A.1. □

Theorem 2.1 establishes the strict stationarity of y_t if $|\phi| < 1$ and the existence of the expectation of y_t^2 if further $E(\exp(f(\epsilon_t))) < \infty$. Consequently, under these two conditions, y_t is also weakly stationary. The expression of the expectation of $|y_t|^c$ in (3) is the same regardless of the distribution of ϵ_t . However, in order to obtain a closed-form expression, one needs to obtain $E(|\epsilon_t|^c)$ and the expectations involved in $P(\cdot)$ that can only be derived for particular distributions of ϵ_t . If ϵ_t is assumed to have a GED distribution with parameter $\nu > 1$, then the conditions in Theorem 2.1 are satisfied and a closed-form expression of $E(|y_t|^c)$ can be derived; see Appendix B.1 for the corresponding expectations. Given that the Gaussian distribution is a special case of the GED distribution when $\nu = 2$, closed-form expressions of $E(|y_t|^c)$ can also be obtained in this case; see Appendix B.2 for the corresponding expectations. When $\nu < 1$, we cannot obtain an analytical expression of $E(|y_t|^c)$. However, in Appendix B.1, we show that $E(|y_t|^c)$ in equation (3) is finite if $\gamma_2 + |\gamma_1| \leq 0$. Given that, in the GASV model, the parameter γ_2 is nonnegative, this condition is only satisfied when $\gamma_1 = \gamma_2 = 0$. Finally, if $\nu = 1$, the condition for the existence of

$E(|y_t|^c)$ in equation (3) is $\gamma_2 + |\gamma_1| < 2\sqrt{2}/c$.⁴

The derivations in Appendix B.1, to obtain closed-form expressions of the moments of $|y_t|$, rely on the symmetry of the density of ϵ_t . Therefore, it is not straightforward to derive closed-form expressions of the expectations needed to compute $E(|y_t|^c)$ when ϵ_t has, for example, an asymmetric GED distribution. We left these derivations for further research.

From expression (3), it is straightforward to obtain expressions of the marginal variance and kurtosis of y_t as the following corollaries show.

Corollary 2.1. *Under the conditions of Theorem 2.1 with $c = 2$ and taking into account that $E(y_t) = 0$, the marginal variance of y_t is directly obtained from (3) with $c = 2$ as follows*

$$\sigma_y^2 = \exp\left(\mu + \frac{\sigma_\eta^2}{2(1 - \phi^2)}\right) P(\phi^{i-1}). \quad (4)$$

If ϵ_t has a centered and standardized GED distribution with parameter $\nu > 1$, then

$$P(\phi^{i-1}) = \prod_{i=1}^{\infty} \left\{ \exp\left(-\frac{\phi^{i-1}\gamma_2\Gamma(2/\nu)}{\sqrt{\Gamma(3/\nu)\Gamma(1/\nu)}}\right) \cdot \sum_{k=0}^{\infty} \left(\left(\frac{\Gamma(1/\nu)}{\Gamma(2/\nu)}\right)^{k/2} \frac{\Gamma((k+1)/\nu)}{2\Gamma(1/\nu)k!} \phi^{(i-1)k} [(\gamma_1 + \gamma_2)^k + \exp(\alpha\phi^{i-1})(\gamma_2 - \gamma_1)^k] \right) \right\}, \quad (5)$$

⁴The same condition should be satisfied when ϵ_t has a Student-t distribution with degrees of freedom $d > 2$.

where $\Gamma(\cdot)$ is the Gamma function. If ϵ_t is Gaussian, then

$$P(\phi^{i-1}) = \prod_{i=1}^{\infty} \left\{ \exp\left(-\phi^{i-1}\gamma_2\sqrt{2/\pi}\right) \left\{ \exp\left(\alpha\phi^{i-1} + \frac{\phi^{2i-2}(\gamma_1 - \gamma_2)^2}{2}\right) \Phi(\phi^{i-1}(\gamma_2 - \gamma_1)) \right. \right. \\ \left. \left. + \exp\left(\frac{\phi^{2i-2}(\gamma_1 + \gamma_2)^2}{2}\right) \Phi(\phi^{i-1}(\gamma_2 + \gamma_1)) \right\} \right\}, \quad (6)$$

where $\Phi(\cdot)$ is the Normal distribution function.

Note that in order to compute $P(\cdot)$ as given in (5) or (6), one needs to truncate the corresponding infinite product and summation. Our experience is that truncating the product at 500 and the summation at 1000 gives very stable results.

Corollary 2.2. *Under the conditions of Theorem 2.1 with $c = 4$ and $E(|\epsilon|^4) < \infty$, the kurtosis of y_t can be obtained as $E(y_t^4)/E(y_t^2)^2$ using expression (3) with $c = 4$ and $c = 2$ as follows*

$$\kappa_y = \kappa_\epsilon \exp\left(\frac{\sigma_\eta^2}{1 - \phi^2}\right) \frac{P(2\phi^{i-1})}{(P(\phi^{i-1}))^2}, \quad (7)$$

where κ_ϵ is the kurtosis of ϵ_t . If ϵ_t has a centered and standardized GED distribution with parameter $\nu > 1$, then $P(2\phi^{i-1})$ can be obtained similarly as in expression (5) or as in (6) if $\nu = 2$.

The kurtosis of the basic symmetric ARSV(1) model considered by Harvey et al. (1994) is given by $\kappa_\epsilon \exp\left(\frac{\sigma_\eta^2}{1 - \phi^2}\right)$. Therefore, in expression (7), we can observe that, in the GASV model, this kurtosis is multiplied by the factor $r = \frac{P(2\phi^{i-1})}{(P(\phi^{i-1}))^2}$. Figure 1 plots r as a function of the leverage parameters α and γ_1 when $\gamma_2 = 0.1$ and 0 for three different persistence parameters, namely, $\phi = 0.5, 0.9$ and 0.98 assuming Gaussian errors. First of all, we can observe that the factor is always larger than 1. Therefore, the GASV model generates returns with higher kurtosis than the corresponding basic

ARSV(1) model. Second, the effects of the parameters α , γ_1 and γ_2 on the kurtosis of returns are very different depending on the persistence. The kurtosis increases with α , $|\gamma_1|$ and γ_2 . However, their effects are only appreciable when ϕ is close to 1.

When looking at the dynamic dependencies of returns defined by the GASV model, it is easy to see that their autocorrelations are trivially zero for all positive lags. Furthermore, returns are a martingale difference process. However, they are not serially independent as the conditional heteroscedasticity generates non-zero autocorrelations of power-transformed absolute returns. The following theorem derives the autocorrelation function (acf) of power transformed absolute returns.

Theorem 2.2. *Consider a stationary process y_t defined by equations (1) and (2) with $|\phi| < 1$. If ϵ_t follows a distribution such that $E(\exp(0.5cf(\epsilon_t))) < \infty$ and $E(|\epsilon_t|^c) < \infty$ for any positive real number c , then the τ -th order autocorrelation of $|y_t|^c$ is finite and given by*

$$\rho_c(\tau) = \frac{E(|\epsilon_t|^c)M_1 \exp\left(\frac{\phi^\tau c^2 \sigma_\eta^2}{4(1-\phi^2)}\right) P(0.5c(1+\phi^\tau)\phi^{i-1})T(\tau, 0.5c\phi^{i-1}) - [E(|\epsilon_t|^c)P(0.5c\phi^{i-1})]^2}{E(|\epsilon_t|^{2c}) \exp\left(\frac{c^2 \sigma_\eta^2}{4(1-\phi^2)}\right) P(c\phi^{i-1}) - [E(|\epsilon_t|^c)P(0.5c\phi^{i-1})]^2}, \quad (8)$$

where $M_1 \equiv E(|\epsilon_t|^c \exp(0.5c\phi^{\tau-1}f(\epsilon_t)))$ and $T(n, b_i) \equiv \prod_{i=1}^{n-1} E(\exp(b_i f(\epsilon_{t-i})))$ if $n > 1$ while $T(1, b_i) \equiv 1$.

Proof. See Appendix A.2. □

The expectations needed to obtain closed-form expressions of the autocorrelations in expression (8) have been derived in Appendix B.1 for the GASV-G model with parameter $\nu > 1$ and in Appendix Appendix B.2 for the particular case of the Normal distribution, i.e. $\nu = 2$. As above, when $\nu \leq 1$, we can only obtain conditions for

the existence of the autocorrelations in (8). Notice that, in practice, most authors dealing with real time series of financial returns focus on the autocorrelations of squared and absolute returns, $\rho_2(\tau)$ and $\rho_1(\tau)$, respectively, which can be obtained from (8) when $c = 2$ and $c = 1$. As these autocorrelations are highly non-linear functions of the parameters, it is not straightforward to analyze the role of each parameter on their shape. Furthermore, by comparing the autocorrelations in (8) for absolute and squared returns, it is not easy to conclude whether the GASV model is able to generate the Taylor effect according to which the autocorrelations of absolute returns are larger than those of squares; see Ruiz and Pérez (2012) for an analysis of the Taylor effect in the context of symmetric SV models. Consequently, in order to illustrate how the autocorrelations of $|y_t|$ and y_t^2 depend on each of the parameters in the GASV model, we have considered particular GASV-N models with parameters $\phi = 0.98$, $\sigma_\eta^2 = 0.05$ and γ_2 taking values 0 or 0.1. The leverage parameters, α and γ_1 , take values between 0 and 1 and -0.25 and 0, respectively. These parameter values have been chosen to be close to those often estimated when SV models are fitted to real time series of financial returns.

The first order autocorrelations of squared and absolute returns, namely, $\rho_2(1)$ and $\rho_1(1)$, are plotted in the first row of Figure 2 as functions of the leverage parameters, γ_1 and α . In the top left panel of Figure 2, which corresponds to the autocorrelations of squares, we can observe that they are larger, the larger is γ_2 . However, both surfaces are rather flat and, consequently, the leverage parameters do not have large effects on the first order autocorrelations of squares. The corresponding first order autocorrelations of absolute returns are plotted in the top right panel of Figure 2. The autocorrelations of absolute returns are also larger the larger is the parameter γ_2 . However, we can observe that the autocorrelations of absolute returns increase with the threshold parameter α . The effect of γ_1 on the autocorrelation of absolute returns is much milder. Finally, comparing $\rho_1(1)$ with $\rho_2(1)$, we can conclude that,

the Taylor effect is stronger the larger is the leverage effect, regardless of whether this is due to α or γ_1 .

Figure 2 focuses on the first order autocorrelations, but gives no information on the shape of the acf for different lags. To illustrate this shape and the role of the distribution of ϵ_t on the acf of y_t^2 and $|y_t|$, the first two panels of the first row of Figure 3 plot the acf of squared and absolute returns for four different GASV-G models with parameters $\phi = 0.98$, $\sigma_\eta^2 = 0.05$, $\alpha = 0.07$, $\gamma_2 = 0.1$, $\gamma_1 = -0.08$ and four different values of the GED parameter, $\nu = 1.5, 1.7, 2$ and 2.5 . As expected, the acfs of $|y_t|$ and y_t^2 both have an exponential decay. Furthermore, fatter tails of ϵ_t imply smaller autocorrelations of both absolute and squared returns; see Carnero et al. (2004) for similar conclusions in the context of symmetric SV models.

The leverage effect is reflected in the cross-correlations between power-transformed absolute returns and lagged returns. The following theorem gives closed-form expressions of these cross-correlations.

Theorem 2.3. *Consider a stationary process y_t defined by equations (1) and (2) with $|\phi| < 1$. If ϵ_t follows a distribution such that $E(\exp(0.5cf(\epsilon_t))) < \infty$ and $E(|\epsilon_t|^{2c}) < \infty$ for any positive real number c , then the τ -th order cross-correlation between y_t and $|y_{t+\tau}|^c$ for $\tau > 0$ is finite and given by*

$$\rho_{c1}(\tau) = \frac{E(|\epsilon_t|^c) \exp\left(\frac{2c\phi^\tau - 1}{8(1-\phi^2)}\sigma_\eta^2\right) M_2 P(0.5(1 + c\phi^\tau)\phi^{i-1}) T(\tau, 0.5c\phi^{i-1})}{\sqrt{P(\phi^{i-1})} \sqrt{E(|\epsilon|^{2c}) \exp\left(\frac{c^2\sigma_\eta^2}{4(1-\phi^2)}\right) P(c\phi^{i-1}) - [E(|\epsilon_t|^c) P(0.5c\phi^{i-1})]^2}}, \quad (9)$$

where $M_2 \equiv E(\epsilon_t \exp(0.5c\phi^{\tau-1}f(\epsilon_t)))$.

Proof. See Appendix A.3. □

As above, the expectations needed to obtain closed-form expressions of the cross-correlations

of the GASV-G model in (9) have been derived in Appendix B.1 for $\nu > 1$ and in Appendix B.2 for the particular case of $\nu = 2$. When $\nu \leq 1$, we are just able to obtain the conditions for the finiteness of the cross-correlation function (ccf). In the second row of Figure 2, we illustrate the effect of the parameters on the first order cross-correlations between y_t and y_{t+1}^2 and $|y_{t+1}|$, $\rho_{21}(1)$ and $\rho_{11}(1)$, respectively, of a GASV-N model with the same parameters considered when dealing with the autocorrelations. First of all, observe that the first order cross-correlations between returns and future absolute and squared returns are indistinguishable for the two values of γ_2 considered in Figure 2. Second, for a given value of γ_2 , it is obvious that increasing the leverage parameters α and $|\gamma_1|$ increases the absolute cross-correlations. Note that $|\gamma_1|$ drags $\rho_{21}(1)$ in an approximately linear way while the effect of α is non-linear. On the other hand, the absolute cross-correlations between returns and future absolute returns have an approximately linear relationship with γ_1 and α and are clearly larger than those between returns and future squared returns. Therefore, it seems that when identifying conditional heteroscedasticity and leverage effect in practice, it is preferable to work with absolute returns instead of squared returns.

Moreover, the shapes of the cross-correlation functions of the GASV-G model, $\rho_{21}(\tau)$ and $\rho_{11}(\tau)$, are also illustrated in the last two panels of the first row of Figure 3, which show that the parameter ν of the GED distribution has a very mild influence on the cross-correlations, especially for $\rho_{11}(\tau)$.

To put it briefly, both ν and γ_2 increase the flexibility of the model to represent the volatility clustering while have little influence on the leverage effect. On the other hand, γ_1 affects the leverage effect and this effect is reinforced by the inclusion of α , which could influence slightly the autocorrelations of absolute returns.

Besides the cross-correlations between returns and future power-transformed absolute returns, another useful tool to describe the asymmetric response of volatility, proposed

by Engle and Ng (1993) in the context of GARCH models, is the News Impact Curve (NIC). Yu (2012) proposes to extend the NIC to SV models by defining it as a function that relates the conditional variance to the lagged return innovation, ϵ_{t-1} , holding constant all other variables. Given that, in SV models, the conditional variance is not directly specified, this definition of the NIC requires solving high-dimensional integrals using numerical methods. In this paper, we propose an alternative definition. Taking into account the information provided by the two disturbances involved in the model, we define the Stochastic News Impact Surface (SNIS) as the surface that relates σ_t^2 with ϵ_{t-1} and η_{t-1} . Therefore, evaluating the lagged volatility at the marginal variance, the SNIS of the GASV model is given by

$$\text{SNIS}_t = \exp((1 - \phi)\mu)\sigma_y^{2\phi} \exp(f(\epsilon_{t-1}) + \eta_{t-1}). \quad (10)$$

As an illustration, the top left panel of Figure 4 plots the SNIS of a GASV-N model with parameters $\{\exp(\mu/2), \alpha, \phi, \gamma_1, \gamma_2, \sigma_\eta^2\}$ given by $\{0.1, 0.07, 0.98, -0.08, 0.1, 0.05\}$. Note that due to the presence of the threshold parameter, α , this surface is discontinuous with respect to ϵ_{t-1} . Figure 4 shows that, for a given value of the lagged volatility shock, η_{t-1} , the response of volatility is stronger when ϵ_{t-1} is negative than when it is positive with the same magnitude. Furthermore, this asymmetric response depends on the log-volatility noise, η_{t-1} . The leverage effect is clearly stronger when η_{t-1} is positive and large than when it is negative. In this latter case, there are no big differences between the effects on future volatilities of positive and negative returns of the same magnitude. The SNIS obtained for GED errors with $1 < \nu < 2$ are very similar to that plotted in Figure 4 for Normal errors.

3. Alternative Asymmetric SV models

As mentioned in the Introduction, one of the main motivations to propose a further specification for asymmetric volatilities is the ability of the new model to nest some of the most popular specifications previously available in the literature. In this section, we review these nested models and analyze and compare their statistical properties which can be obtained as particular cases from those of the GASV model.

3.1. A-ARSV model

Consider the following restricted volatility specification of equation (2)

$$h_t - \mu = \phi(h_{t-1} - \mu) + \gamma_1 \epsilon_{t-1} + \eta_{t-1}, \quad (11)$$

which together with (1) is denoted as A-ARSV model. Define δ and $\sigma_{\eta^*}^2$ such that $\gamma_1 = \delta \sigma_{\eta^*}$ and $\sigma_{\eta}^2 = (1 - \delta^2) \sigma_{\eta^*}^2$. Then, the A-ARSV model with Normal errors (denoted as A-ARSV-N) is equivalent to the most popular asymmetric SV model originally proposed by Taylor (1994) and Harvey and Shephard (1996) that incorporates the leverage effect through correlation between the level and volatility noises as follows

$$h_t - \mu = \phi(h_{t-1} - \mu) + \eta_{t-1}^*, \quad (12)$$

with ϵ_t and η_t^* being jointly Normal with zero means, variances 1 and $\sigma_{\eta^*}^2$, respectively, and correlation δ ; see Asai and McAleer (2011) and Yu (2012) for the equivalence of these two specifications. Model (12) is very popular in empirical applications; see Bartolucci and De Luca (2003), Yu et al. (2006) and Tsiotas (2012) among many others. This model is also extended by Tsiotas (2012) by allowing the return disturbance to follow several asymmetric and fat-tailed distributions. However, it is important to note that the equivalence between the specifications in (11) and (12) can only be

established when ϵ_t is Normal if the volatility is assumed to be Log-Normal.

The moments of the A-ARSV-G model can be obtained from those in Section 2 by imposing $\alpha = \gamma_2 = 0$. These moments have been already derived in the literature when $\nu = 2$; see Taylor (1994, 2007), Demos (2002), Ruiz and Veiga (2008) and Pérez et al. (2009). Particularly, the marginal variance and kurtosis of y_t , given in (4) and (7), reduce to $\sigma_y^2 = \exp(\mu) \exp\left(\frac{\sigma_\eta^2 + \gamma_1^2}{2(1-\phi^2)}\right)$ and $k_y = k_\epsilon \exp\left(\frac{\sigma_\eta^2 + \gamma_1^2}{1-\phi^2}\right)$, respectively. Note that $\sigma_\eta^2 + \gamma_1^2 = \sigma_{\eta*}^2$. As a consequence, several authors conclude that, in the basic A-ARSV-N model, the variance and kurtosis of y_t do not depend on whether there is leverage effect or not; see Taylor (1994), Ghysels et al. (1996) and Harvey and Shephard (1996). One can always find a symmetric model with a larger variance of the errors that has the same variance and kurtosis as a given asymmetric model.

Expressions of the autocorrelations of $|y_t|^c$ and the cross-correlations between y_t and $|y_{t+\tau}|^c$ of the A-ARSV-G model can be also derived from the corresponding expressions (8) and (9). As an illustration, Figure 3 plots the acfs and ccfs of the A-ARSV-G models for the same parameter values of the GASV-G models represented in the first row of Figure 3 except that $\alpha = \gamma_2 = 0$. We can observe that the autocorrelations of squared and absolute returns and the absolute cross-correlations are slightly smaller than those of the corresponding GASV-G models. Therefore, including γ_2 and α in the GASV model allows for stronger volatility clustering and leverage effect. Smaller autocorrelations are observed when the tails of the distribution of the return disturbance, ϵ_t , are fatter. Once more, the thickness of the tails has very mild influence on the cross-correlations and, therefore, on the leverage effect.

Finally, consider the SNIS of the A-ARSV-N model which is obtained from (10) with $\alpha = \gamma_2 = 0$ and $\nu = 2$. The top right panel of Figure 4 illustrates the SNIS

of an A-ARSV-N model with the same parameters as in the illustration of SNIS of the GASV-N model, i.e., $\{\exp(\mu/2), \phi, \gamma_1, \sigma_\eta^2\} = \{0.1, 0.98, -0.08, 0.05\}$. Given η_{t-1} , the SNIS_t is an exponential function with exponent γ_1 . Thus, bad news generates a higher impact on volatility than good news of the same size. The magnitude of this difference increases with η_{t-1} . Moreover, it is magnified (mitigated) by positive (negative) η_{t-1} . However, for the particular model considered in Figure 4, the leverage effect is very mild when compared with that of the GASV-N model.

3.2. E-SV model

Consider now the following specification of h_t based on the EGARCH model with an added noise

$$h_t - \mu = \phi(h_{t-1} - \mu) + \gamma_1 \epsilon_{t-1} + \gamma_2 \{|\epsilon_{t-1}| - E(|\epsilon_{t-1}|)\} + \eta_{t-1}, \quad (13)$$

where all the parameters and processes are defined and interpreted as in the GASV model in (2). The model in (1) and (13), denoted as E-SV, can be obtained as a particular case of the GASV when $\alpha = 0$. Note that the E-SV model with Normal ϵ_t (denoted as E-SV-N) can also be obtained as a particular case of the model proposed by Demos (2002), who derives the acf of y_t and the ccf between y_t and y_t^2 ; see also Asai and McAleer (2011).⁵ Moreover, note that it nests the A-ARSV-N model when $\gamma_2 = 0$.

The E-SV model with ϵ_t having a GED distribution is denoted as E-SV-G. Comparing the A-ARSV-G and E-SV-G models, we can study the role of γ_2 while the role of α can be established by comparing the GASV-G and E-SV-G models.

⁵It is important to point out that the E-SV-N model has also been implemented by specifying the log-volatility using y_{t-1} instead of ϵ_{t-1} in equation (13); see Danielsson (1998) and Asai and McAleer (2005). In this case, although the estimation of the parameters is usually easier, the derivation of the properties is harder.

The third row of Figure 3 plots the autocorrelations and cross-correlations for four E-SV-G models with the same parameter values of the GASV-G models considered above except that $\alpha = 0$. Comparing the plots of the A-ARSV-G and E-SV-G models in Figure 3, we can observe that adding $|\epsilon_{t-1}|$ into the A-ARSV-G model generates larger autocorrelations of squares and absolute returns but not larger Taylor effect. However, as expected, the cross-correlations are almost identical. Therefore, the E-SV-G model is more flexible than the A-ARSV-G to represent wider patterns of volatility clustering but not of volatility leverage.

Figure 3 also illustrates that the E-SV-G model is not identified when the parameter of the GED distribution of ϵ_t , ν , is not fixed. Observe that, given a particular E-SV-G model, we can also find an A-ARSV-G model with almost the same autocorrelations and cross-correlations. Compare, for example, the autocorrelations of the E-SV-G model with $\nu = 2$ and those of the A-ARSV-G model with $\nu = 2.5$. Further, the cross-correlations are indistinguishable in any case. Therefore, if the parameter ν is a free parameter, we cannot identify the parameters γ_2 and σ_η^2 . Only by fixing the distribution of ϵ_t , i.e. choosing a particular value of ν , both parameters can be properly identified.

Finally, by comparing the GASV-G and E-SV-G models, we can observe that the autocorrelations are almost identical. Only the autocorrelations of absolute returns of GASV-G are slightly larger; see also Figure 2. Including α only has a paltry effect on the volatility clustering that the model can represent. However, the cross-correlations of the GASV-G model are stronger than those of the E-SV-G model. Therefore, α allows for a more flexible pattern of the leverage effect.

The SNIS of the E-SV-N model is illustrated in the bottom left panel of Figure 4 for a model with the same parameters chosen for the GASV-N model with $\alpha = 0$. Comparing the SNIS of the E-SV model with that of the A-ARSV model, we can

observe that these two surfaces are similar. We can identify the important role of α in the response of volatility by comparing the SNIS of the E-SV and GASV models.

3.3. *RT-SV model*

The last nested model considered in this paper is the threshold SV (T-SV) model, which specifies the log-volatility with different parameters depending on the sign of past returns. In particular, assuming normality of ϵ_t , the T-SV-N model, proposed by Breidt (1996) and So et al. (2002), is given by

$$h_t = \alpha + \alpha' I(\epsilon_{t-1} < 0) + (\phi + \phi' I(\epsilon_{t-1} < 0))h_{t-1} + \tilde{\eta}_{t-1}, \quad (14)$$

where $\tilde{\eta}_t$ is a Gaussian noise with mean zero and variance $\sigma_{\tilde{\eta}}^2 + \sigma_{\tilde{\eta}}^{2'} I(\epsilon_t < 0)$. The T-SV-N model in (14) allows the constant, persistence and the variance of the volatility noise to change depending on whether one-lagged returns are positive or negative. More recently, Chen et al. (2008) considers a standardized Student-t distribution for the return errors.

Deriving analytical properties of the T-SV-N model in (14) seems to be a difficult task. Consequently, we analyze them by simulation. The model kurtoses, first order autocorrelations of squares and first order cross-correlations between squares and levels, reported in Table 1, are obtained as the averages of the corresponding sample moments computed from $R = 1000$ series of size $T = 5000$ simulated from several T-SV-N models. Table 1 also reports the corresponding Monte Carlo standard deviations. The parameter values considered to simulate the time series reported in Table 1 have been chosen to be in concordance with the estimates often obtained when fitting the T-SV-N model to real financial returns; see So et al. (2002), Muñoz et al. (2007), Chen et al. (2008), Smith (2009), Montero et al. (2010) and Elliott et al. (2011), among others. Table 1 considers three types of T-SV-N models. The first

type are models with fixed persistence and variance of $\tilde{\eta}_t$ and in which the constant is allowed to change. Second, we consider models in which the persistence changes depending on the sign of lagged returns while both the constant and the variance are fixed. Finally, the third group of models have fixed constant and autoregressive parameter with the variance changing according to the sign of lagged returns. Table 1 shows that, in the models in which the autoregressive parameter changes, the autocorrelations of squares are not significantly different from zero. Therefore, changes in the autoregressive parameter destroy the volatility clustering and the conditional heteroscedasticity disappears. Note also that when the autoregressive parameter changes, the cross-correlations are not significantly different from zero in any of the models considered. On the other hand, when looking at the results for the models in which the variance of the volatility noise changes, we can observe that they generate significant autocorrelations of squares and, consequently, conditional heteroscedasticity. However, in these models the cross-correlations between returns and future squared returns are not significantly different from zero. Therefore, changes in the variance seem to generate conditionally heteroscedastic series without leverage effect. It is also important to note that, in these models, the kurtoses are too large when compared with those usually observed in real financial returns. Finally, consider the group of models in which both the autoregressive parameter and the variance are fixed and the constant changes. In these models, we observe that the autocorrelations of squares and the cross-correlations between returns and future squared returns are significantly different from zero when the difference between these two constants is large enough. Consequently, we focus the analysis on the following specification of volatility

$$h_t - \mu = \alpha I(\epsilon_{t-1} < 0) + \phi(h_{t-1} - \mu) + \eta_{t-1}, \quad (15)$$

with the parameters and processes defined as in (2). The model defined by equations

(1) and (15) is denoted as restricted T-SV (RT-SV) model. This model has also been considered by Asai and McAleer (2006) who assume normality of ϵ_t . In this case, we denote it as RT-SV-N. Note that when the constraints $\gamma_1 = \gamma_2 = 0$ are imposed on the GASV-N model in (2), the RT-SV-N model is obtained.

The statistical properties of the RT-SV-G model with $\epsilon_t \sim GED$ can be obtained from those of GASV-G model obtained in the previous section by restricting $\gamma_1 = \gamma_2 = 0$. The last row of Figure 3 illustrates the shape of the autocorrelations of squared and absolute returns and the cross-correlations between returns and future squared and absolute returns, for a RT-SV-G model with the same values of the parameters ϕ , σ_η^2 and ν as those considered for the GASV-G model. Comparing the autocorrelations of squares and absolute returns of the GASV-G and RT-SV-G models, we can observe that the latter are slightly smaller than the former. However, the cross-correlations are clearly smaller in the RT-SV-G model. Actually, these cross-correlations are the smallest among those of all the models considered. It seems that the presence of α in the GASV model is reinforcing the role of the leverage parameter γ_1 .

Finally, the bottom right panel of Figure 4 illustrates the SNIS of this particular RT-SV-N model which can be obtained from (10). The main characteristic of the SNIS plotted is its discontinuity with respect to ϵ_{t-1} . This surface represents different responses of volatility to positive and negative returns due to the inclusion of α . By comparing the SNIS of the GASV and RT-SV models, we can clearly observe the added flexibility to explain the leverage effect incorporated by having both α and γ_1 in the model.

4. Finite sample performance of a MCMC estimator of the parameters

Stochastic volatility models are attractive because of their flexibility to represent a high range of the dynamic properties of time series of financial returns often

observed when dealing with real data. This flexibility can be attributed to the presence of a further disturbance associated with the volatility process. However, as a consequence of the volatility being unobservable, it is not possible to obtain an analytical expression of the likelihood function. Furthermore, one needs to implement filters to obtain estimates of the latent unobserved volatilities. Thus, the main limitation of SV models is the difficulty involved in the estimation of the parameters and volatilities; see Broto and Ruiz (2004) for a survey on alternative procedures to estimate SV models. In this context, simulation based MCMC procedures are becoming very popular because of their good properties and flexibility to deal with different specifications and distributions of the errors.⁶ The first Bayesian MCMC approach to estimate SV models with leverage effect was developed by Jacquier et al. (2004). After that, there have been several proposals that try to improve the properties of the MCMC estimators. For example, Omori et al. (2007), Omori and Watanabe (2008) and Nakajima and Omori (2009) implement the efficient sampler of Kim et al. (1998) to SV models with Student-t errors and leverage effect based on $\log y_t^2$. Based on the work of Shephard and Pitt (1997) and Watanabe and Omori (2004), Abanto-Valle et al. (2010) estimate an asymmetric SV model assuming scale mixtures of Normal return distributions while SV models with skew-Student-t and skew-Normal return errors are estimated by Tsiotas (2012) using MCMC. Among the alternative MCMC estimators available in the literature, in this paper, we consider the estimator described by Meyer and Yu (2000) who propose to estimate the A-ARSV model using the user-friendly and freely available WinBUGS software. The estimator uses the single-move Gibbs sampling algorithm; see Yu (2012) and Wang et al. (2013) for empirical implementations.

⁶There are several alternative procedures proposed in the literature to estimate SV models with leverage effect. For example, Bartolucci and De Luca (2003) propose a likelihood estimator based on the quadrature methods of Fridman and Harris (1998). Alternatively, Harvey and Shephard (1996) propose a Quasi Maximum Likelihood procedure while Sandmann and Koopman (1998) implement a Simulated Maximum Likelihood procedure. Finally Liesenfeld and Richard (2003) propose a Maximum Likelihood approach based upon an efficient importance sampling.

This estimator is attractive because it reduces the coding effort allowing its empirical implementation to real time series of financial returns.

Next, we describe briefly the algorithm. Let $p(\boldsymbol{\theta})$ be the joint prior distribution of the unknown parameters $\boldsymbol{\theta} = \{\mu, \phi, \alpha, \gamma_1, \gamma_2, \sigma_\eta^2, \nu\}$. Following Meyer and Yu (2000), the prior densities of ϕ and σ_η^2 are $\phi = 2\phi^* - 1$ with $\phi^* \sim \text{Beta}(20, 1.5)$ and $\sigma_\eta^2 = 1/\tau^2$ with $\tau \sim \text{IG}(2.5, 0.025)$, respectively, where $\text{IG}(\cdot, \cdot)$ is the inverse Gaussian distribution.⁷ The remaining prior densities are chosen to be uninformative, that is, $\mu \sim N(0, 10)$, $\alpha \sim N(0.05, 10)$, $\gamma_1 \sim N(-0.05, 10)$, $\gamma_2 \sim N(0.05, 10)$ and $\nu \sim U(0, 4)$. These priors are assumed to be independent. The joint prior density of $\boldsymbol{\theta}$ and \mathbf{h} is given by

$$p(\boldsymbol{\theta}, \mathbf{h}) = p(\boldsymbol{\theta})p(h_0) \prod_{t=1}^{T+1} p(h_t|h_{t-1}, \boldsymbol{\theta}). \quad (16)$$

The likelihood function is then given by

$$p(\mathbf{y}|\boldsymbol{\theta}, \mathbf{h}) = \prod_{t=1}^T p(y_t|h_t, \boldsymbol{\theta}). \quad (17)$$

Note that the conditional distribution of y_t given h_t and $\boldsymbol{\theta}$ is $y_t|h_t, \boldsymbol{\theta} \sim \text{GED}(\nu)$.

We make use of the scale mixtures of Uniform representation of the GED distribution proposed by Walker and Gutiérrez-Peña (1999) for obtaining the conditional distribution of y_t given ν and h_t , which is given by

$$y_t|u, h_t \sim U\left(-\frac{\exp(h_t/2)}{\sqrt{2\Gamma(3/\nu)/\Gamma(1/\nu)}}u^{1/\nu}, \frac{\exp(h_t/2)}{\sqrt{2\Gamma(3/\nu)/\Gamma(1/\nu)}}u^{1/\nu}\right), \quad (18)$$

where $u|\nu \sim \text{Gamma}(1 + 1/\nu, 2^{-\nu/2})$. Given the initial values $(\boldsymbol{\theta}^{(0)}, \mathbf{h}^{(0)})$, the Gibbs sampler generates a Markov Chain for each parameter and volatility in the model

⁷Although the prior of ϕ^* is very informative, when it is changed to $\text{Beta}(1, 1)$, the results are very similar.

through the following steps:

$$\begin{aligned}
\theta_1^{(1)} &\sim p(\theta_1|\theta_2^{(0)}, \dots, \theta_K^{(0)}, h^{(0)}, \mathbf{y}); \\
&\vdots \\
\theta_K^{(1)} &\sim p(\theta_1|\theta_2^{(1)}, \dots, \theta_{K-1}^{(1)}, h^{(0)}, \mathbf{y}); \\
h_1^{(1)} &\sim p(h_1|\theta^{(1)}, h_2^{(0)}, \dots, h_{T+1}^{(0)}, \mathbf{y}); \\
&\vdots \\
h_{T+1}^{(1)} &\sim p(h_{T+1}|\theta^{(1)}, h_1^{(1)}, \dots, h_T^{(1)}, \mathbf{y}).
\end{aligned}$$

The estimates of the parameters and volatilities are the means of the Markov Chain. The posterior joint distribution of the parameters and volatilities is given by

$$p(\boldsymbol{\theta}, \mathbf{h}|\mathbf{y}) \propto p(\boldsymbol{\theta})p(h_0) \prod_{t=1}^{T+1} p(h_t|h_{t-1}, \mathbf{y}, \boldsymbol{\theta}) \prod_{t=1}^T p(y_t|h_t, \boldsymbol{\theta}). \quad (19)$$

In this section, we carry out extensive Monte Carlo experiments to analyze the finite sample performance of the MCMC estimator when estimating both the parameters and the underlying volatilities. As mentioned in Section 3, one possible problem is the parameter identification when estimating the GASV-G model. Therefore, we consider two designs for the Monte Carlo experiments. First, we treat ν as known and estimate the other parameters in the model. In this case, R replicates are generated by the GASV-N model with parameters $(\mu, \phi, \alpha, \gamma_1, \gamma_2, \sigma_\eta^2) = (0, 0.98, 0.07, -0.08, 0.1, 0.05)$. Second, R replicates are generated by the GASV-G model with the same parameters and $\nu = 1.5$. All the parameters are then estimated using the MCMC estimator. The total number of iterations in the MCMC procedure is 20,000 after a burn-in of 10,000. The results are based on $R = 500$ replicates of series with sample sizes $T = 500, 1000$ and 2000.

The left panels of Table 2 report the average and standard deviation of the posterior means together with the average of the posterior standard deviations of each parameter through the Monte Carlo replicates for the first design when ν is fixed and equal to its true value, $\nu = 2$. Once we fix ν and estimate the rest of the parameters, we observe that the Monte Carlo averages of the posterior means are rather close to the true parameter values, indicating almost no finite sample biases for series of sizes $T = 1000$ and 2000 . Also, it is important to point out that the average of the posterior standard deviations is rather close to the Monte Carlo standard deviation of the posterior means. Consequently, inference based on the posterior distributions seems to be adequate when the sample size is as large as 1000 . When $T = 500$, the estimation could suffer from small parameter bias.

On the other hand, the right panels of Table 2 report the results for the second design when ν is estimated as a further parameter. We observe that, due to the lack of identifiability mentioned above, the estimates of γ_2 , σ_η^2 and ν suffer biases that do not disappear with the sample size. Both σ_η^2 and ν are underestimated while γ_1 is over estimated. The correlation between the estimates of γ_2 and σ_η^2 is almost -0.7 while the correlation between the estimates of ν and γ_2 is as high as -0.8 . So the estimator cannot identify these parameters correctly. Also note that although the average posterior standard deviations of γ_2 are similar when ν is estimated and when it is fixed, the standard deviations of the posterior means of γ_2 are clearly larger when ν is estimated. Therefore, inference of γ_2 based on the posterior distribution can be non-reliable when ν is estimated along with all other parameters in the model. Finally, comparing the standard deviation of the posterior means with the average of the posterior standard deviations of ν , we observe that the latter are clearly smaller than the former. Consequently, inference based on the posterior distribution of ν can be misleading as we could believe that the uncertainty associated with the estimated parameter of the GED distribution is smaller than the true uncertainty.

We also want to check whether by fitting the new model proposed in this paper we are able to identify the true restricted specifications when the distribution of ϵ_t is known. With this purpose, we generate $R = 500$ replicates of sizes $T = 500$ and 1000 from each of the restricted models with Normal return errors and fit the new GASV-N model. The results, reported in Table 3, provide evidence that when ν is known it is possible to identify the true data generating process (DGP) by fitting the more general GASV-N model even when the sample size is as small as $T = 500$.

Summarizing the Monte Carlo results on the MCMC estimator considered in this paper, we can conclude that: i) Some of the parameters of the GASV model are not identified when the distribution of ϵ_t is modeled as a GED distribution with unknown parameter. ii) If ν is known and the sample size is moderately large, the posterior distribution gives an adequate representation of the finite sample distribution with the posterior mean being an unbiased estimator of the true parameter value. iii) The true restricted specifications are correctly identified after fitting the proposed GASV model when ν is known.

When dealing with conditional heteroscedastic models, practitioners are interested not only in the parameter estimates but also, and more importantly, in the volatility estimates. Consequently, in the Monte Carlo experiments above, at each time period t and for each replicate i , we also compute the relative prediction error of volatility, $e_t^{(i)} = (\sigma_t^{(i)} - \hat{\sigma}_t^{(i)})/\sigma_t^{(i)}$, where $\sigma_t^{(i)}$ is the simulated true volatility at time t in the i -th replicate and $\hat{\sigma}_t^{(i)}$ is its MCMC estimate. Table 4 reports the average and standard deviation through time of $m_t = \sum_{i=1}^R e_t^{(i)}/R$ together with the average through time of the standard deviations given by $s_t = \sqrt{\sum_{i=1}^R (e_t^{(i)} - m_t)^2/(R-1)}$ when $T = 500$ and 1000. These quantities have been computed when the GASV-N model is fitted to the series generated by the general model and by each of the restricted models assuming that $\nu = 2$. We also compute the relative volatility errors when ν is estimated as a

further parameter. Consider first the results when the GASV-N is the true DGP. We observe that the estimates of the volatility are unbiased. Further, when the restricted models are the DGPs but the general GASV-N model is fitted, the errors are also insignificant and with similar standard deviations. Finally, if the GASV-G model is fitted, the estimates of the volatility have a negative bias that does not disappear when the sample size increases. Therefore, when ν is estimated, the MCMC estimated volatilities are larger than the true underlying volatilities. Also note that the standard deviation almost does not decrease with the sample size.

5. Empirical application

5.1. Data description and estimation results

In this section, the GASV model is fitted to represent the dynamic dependence of daily S&P500 returns observed from June 17, 1996 to May 4, 2012 with $T = 4000$ observations. The returns, computed as usual as $y_t = 100 \times \Delta \log P_t$, where P_t is the adjusted close price from yahoo.finance on day t , have kurtosis 9.601 and skewness 0.042 which is not statistically significant. Therefore, it seems that it is not necessary to consider a skewed distribution of the return errors. The raw prices together with their corresponding returns are plotted in Figure 5 which suggests the presence of volatility clustering with episodes of large volatilities associated with periods of negative movements in prices. Furthermore, this association between large volatilities and negative returns can also be observed in the negative cross-correlations between returns and future squared and absolute returns plotted in Figure 6. It is clear that the volatility clustering and leverage effect are present in the daily S&P500 returns. Consequently, the GASV model is fitted first estimating ν as a free parameter and second assuming that the errors are Gaussian. Our objective is to observe empirically whether the estimated volatilities and the corresponding Value at Risk (VaR) are

affected by the distribution of ϵ_t . Recall that, according to our conclusions, both on the statistical properties of the GASV model and the simulations, if $\nu = \nu_0$, we may find another model with $\nu \neq \nu_0$ and different parameter values that represent the same dynamics of $|y_t|^c$ and the same cross-correlations between y_t and $|y_{t+\tau}|^c$. For completeness, we also fit the other six restricted models. All the parameters and volatilities have been estimated implementing the MCMC estimator of WinBUGS.

Table 5 reports the posterior mean and the 95% credible interval of the MCMC estimator of each parameter. First, we can observe that when the GASV model is fitted, the credible intervals for the threshold parameter α contain the zero regardless of whether ν is estimated as a free parameter or is fixed at $\nu = 2$. Furthermore, the DIC of the RT-SV model is larger than those of the other models regardless of whether ν is estimated or fixed; see Berg et al. (2004), Wang et al. (2013) and Tsiotas (2012) for using the DIC to compare models in the context of SV models. The Monte Carlo experiments in the previous section suggest that fitting the general GASV model proposed in this paper, one could identify the true restricted specification of the log-volatilities if the distribution of ϵ_t is known. Consequently, it seems that the threshold parameter is not needed to represent the conditional heteroscedasticity of the S&P500 returns. Therefore, we focus now on the results of the E-SV model. The estimate of the parameter of the GED distribution is $\hat{\nu} = 1.7$ which, according to our Monte Carlo results, could be underestimating the true value of ν . Comparing the estimated parameters obtained when ν is estimated with those obtained when $\nu = 2$, we observe that in the first case γ_2 is larger and σ_η^2 is smaller. Recall that σ_η^2 is underestimated while γ_2 is overestimated. Therefore, the empirical estimates are in concordance with the simulation results. The DIC seems to indicate a better fit of the E-SV-G model which is, in any case, very close to the A-ARSV-G model.

Figure 6 plots the plug-in moments implied by the estimated asymmetric SV

models together with the corresponding sample moments. First, note that for the same model with different level error distributions, GED and Normal, the plug-in moments are indistinguishable. Second, the plug-in moments of all models but the RT-SV model are similar among them and rather close to the sample moments. The RT-SV plug-in moments are somehow further away. Therefore, the RT-SV model seems not able to represent the properties of daily S&P500 returns as well as the other three asymmetric SV models which are rather similar.

Given the apparent similarity between the A-ARSV and E-SV specifications, we next check whether they can generate significant differences when predicting the VaRs.

5.2. Forecasting VaR

In this subsection, we perform an out-of-sample comparison of the ability of the alternative asymmetric SV models considered in this paper, with ϵ_t following either a GED or a Normal distribution, when evaluating the one-step-ahead VaR of the daily S&P500 returns. Given the extremely heavy computations involved in the estimation of the one-step-ahead VaR based on the MCMC estimator, we compute it using data from Jan 3, 2005 to Dec 31, 2010. The parameters are estimated using a rolling-window scheme fixing $T = 990$ observations.⁸ Moreover, one-step-ahead VaRs are obtained starting on January 4, 2010 until the end of 2010 as

$$VaR_{t+1|t}(m) = q\hat{\sigma}_{t+1|t}, \quad (20)$$

with q being the 5% quantile of the distribution with parameter ν estimated in model m or the 5% quantile of the Normal distribution when $\nu = 2$ and $\hat{\sigma}_{t+1|t}$ is the estimated one-step-ahead volatility. Finally, we obtain 252 one-step-ahead VaRs.

⁸Checking the estimates obtained, we observe that all the estimates are very stable over the year considered in the rolling window estimation.

In order to evaluate the adequacy of the interval forecasts provided by the VaRs computed as in equation (20) for each of the models, we carry out the coverage tests of Christoffersen (1998), namely, the unconditional coverage (LRuc), independence (LRind) and unconditional coverage and independence (LRcc) tests. Table 6 reports the failure rates and the likelihood ratios together with the p-values of the tests statistics. Even though the failure rate is always larger than 0.05, we do not reject the adequacy of any of the estimated VaRs.

Next, we compare the VaRs pairwise applying the Conditional Predictive Ability (CPA) statistic proposed by Giacomini and White (2006) which is based on the following asymmetric linear loss function for model m

$$\hat{\mathcal{L}}_{t+1}(m) = (0.05 - \mathbf{I}(\hat{e}_{t+1}(m) < 0))\hat{e}_{t+1}(m), \quad (21)$$

where $\hat{e}_{t+1}(m) = y_{t+1} - VaR_{t+1|t}(m)$. Given the loss function in equation (21), the null hypothesis of the CPA test is that the expected loss functions resulting from any two models, f and g are equal:

$$H_0 : E(\hat{\mathcal{L}}_{t+1}(f) - \hat{\mathcal{L}}_{t+1}(g) | F_t) = 0, \quad (22)$$

where F_t denotes the information set available at time t . The CPA statistic is computed as nR^2 , where n and R^2 are the number of observations and the uncentered R^2 of the artificial regression of $D_{t+1} = \hat{\mathcal{L}}_{t+1}(f) - \hat{\mathcal{L}}_{t+1}(g)$ on the vector $\lambda_t = (1, D_t)$, respectively. It has an asymptotic χ^2 distribution with 2 degrees of freedom.

Table 7 reports the CPA statistics of pairwise tests of equal conditional predictive ability along with the corresponding p-values in parentheses. We can observe that the p-values are always rather large. Therefore, all models have indistinguishable prediction ability in predicting the VaRs of the S&P500 returns. For example, Figure

7 represents a scatter plot of the VaRs estimated using the GASV-G model against the VaRs estimated by the GASV-N model. We can observe that both models provide nearly the same VaRs.

6. Conclusions

In this paper, we propose and derive the statistical properties of a new asymmetric SV model, the GASV, which nests some of the most popular asymmetric SV models usually implemented when modeling heteroscedastic series with leverage effect. In particular, it nests the A-ARSV model which incorporates the leverage effect through the correlation between the disturbances in the level and log-volatility equations, the E-SV model which adds a noise to the log-volatility equation specified as an EGARCH model and a restricted T-SV model, in which the constant of the volatility equation is different depending on whether one-lagged returns are positive or negative. As a marginal outcome, we also obtain the properties of all these nested models, some of which were previously unknown in the literature, and analyze the role of each parameter in the model. Closed-form expressions of the variance, kurtosis, autocorrelations of power-transformed absolute returns and cross-correlations between returns and future power-transformed absolute returns are obtained when the disturbance of the log-volatility equation is Gaussian and the disturbance of the level equation follows a GED distribution with parameter strictly larger than 1. We show that some of the parameters of the model can be non-identified when the parameter of the GED distribution is allowed to change as, in this case, the moments of returns can be undistinguishable for different combinations of the parameters and distributions. The second contribution of this paper is the proposal of the SNIS to describe the asymmetric response of volatility to positive and negative past returns in the context of SV models. We show that, in the new model proposed in this paper, the asymmetric

response of volatility is different depending on the size and sign of the volatility shock. Third, we analyze the finite sample properties of a MCMC estimator of the parameters and volatilities using the WinBUGS software. We show that estimating the proposed GASV model allows to correctly identify the true data generating process when the distribution of the level disturbance is known. However, when this distribution is assumed to be a GED and its parameter is estimated as a further parameter, we show that the MCMC estimates can be biased as a consequence of the lack of identifiability mentioned above. Finally, the GASV model is fitted to estimate the volatilities of S&P500 daily returns. For this particular data set, the threshold parameter is not significant. In any case, when estimating the VaRs all models are indistinguishable regardless of the distribution of ϵ_t .

Several possible extensions of this paper could be of interest. First, our focus is on univariate models. Extending the new asymmetric SV model proposed in this paper to a multivariate framework is worth to be considered; see, for example, Harvey et al. (1994), Asai and McAleer (2006), Chan et al. (2006), Chib et al. (2006), Jungbacker and Koopman (2006) and Yu and Meyer (2006) for multivariate SV models with leverage effect. Second, Bandi and Renò (2012) and Yu (2012) argue that the leverage effect found in many real time series of financial returns can be time-varying. Extending the model and results derived in this paper to include time-varying leverage effect is also in our research agenda. Finally, Rodríguez and Ruiz (2012) compare the properties of alternative asymmetric GARCH models to see which is closer to the empirical properties often observed when dealing with financial returns. Comparing the properties of the new model proposed in this paper with those of the best candidates within the GARCH family is also left for further research.

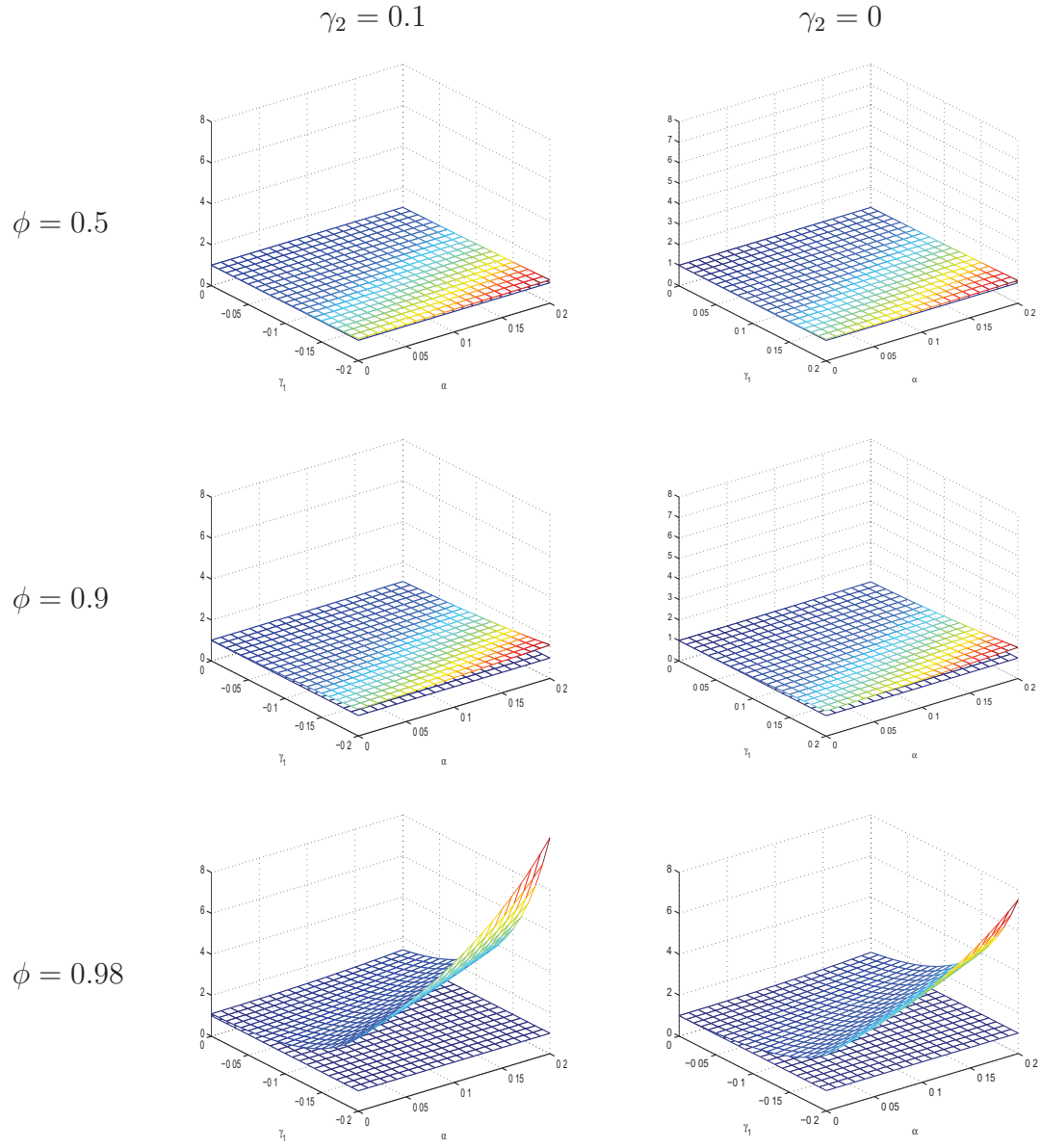


Figure 1: Ratio between the kurtoses of the GASV model and the symmetric ARSV(1) model with Gaussian errors when $\gamma_2 = 0.1$ (left column) and 0 (right column) for three different values of the persistence parameter, $\phi = 0.5$ (first row), $\phi = 0.9$ (middle row) and $\phi = 0.98$ (bottom row).

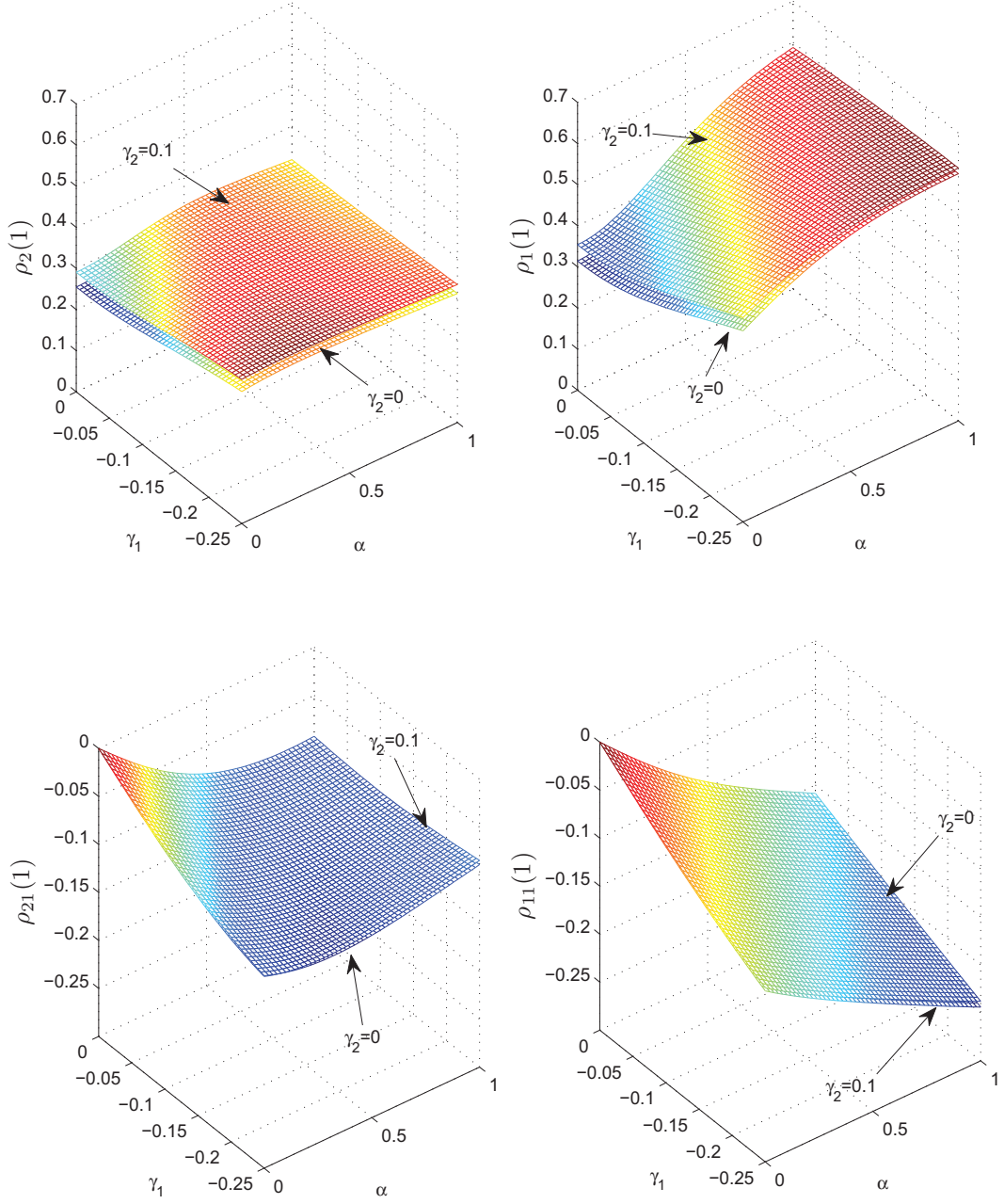


Figure 2: First order autocorrelations of squares (top left), first order autocorrelations of absolute returns (top right), first order cross-correlations between returns and future squared returns (bottom left) and first order cross-correlations between returns and future absolute returns (bottom right) of different GASV-N models with parameters $\phi = 0.98$ and $\sigma_\eta^2 = 0.05$.

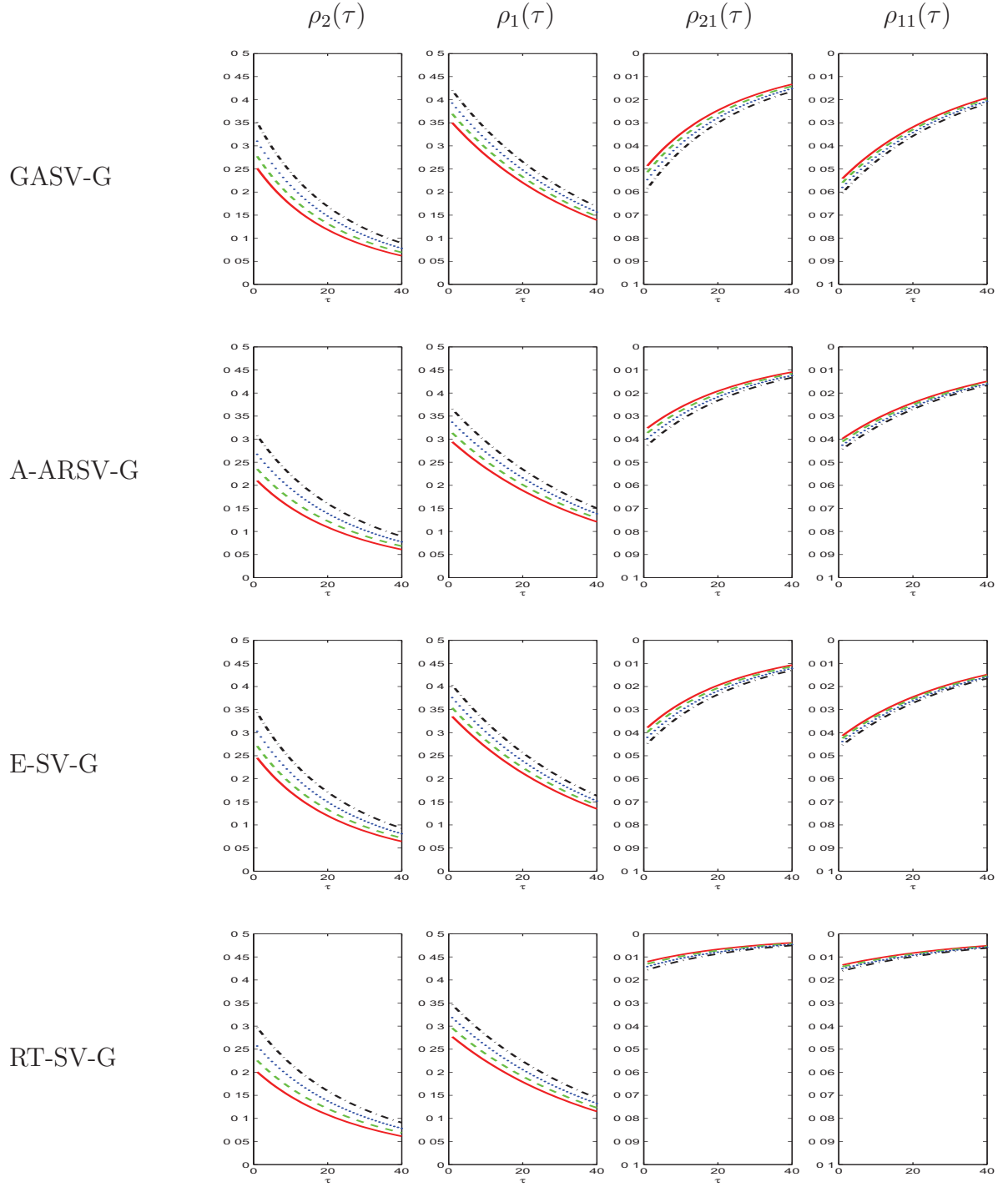


Figure 3: Autocorrelations of squares (first column), autocorrelations of absolute returns (second column), cross-correlations between returns and future squared returns (third column) and cross-correlations between returns and future absolute returns (fourth column) for different specifications of asymmetric SV models. The first row corresponds to a GASV-G model with $\alpha = 0.07, \phi = 0.98, \sigma_\eta^2 = 0.05, \gamma_1 = -0.08, \gamma_2 = 0.1$ and $\nu = 1.5$ (solid lines), $\nu = 1.7$ (dashed lines), $\nu = 2$ (dotted lines) and $\nu = 2.5$ (dashdot lines). The second row corresponds to the A-ARSV-G with $\alpha = \gamma_2 = 0$. The third row matches along with the E-SV-G model while $\alpha = 0$. Finally, the last row plots the corresponding moments of the RT-SV-G model when $\gamma_1 = \gamma_2 = 0$.

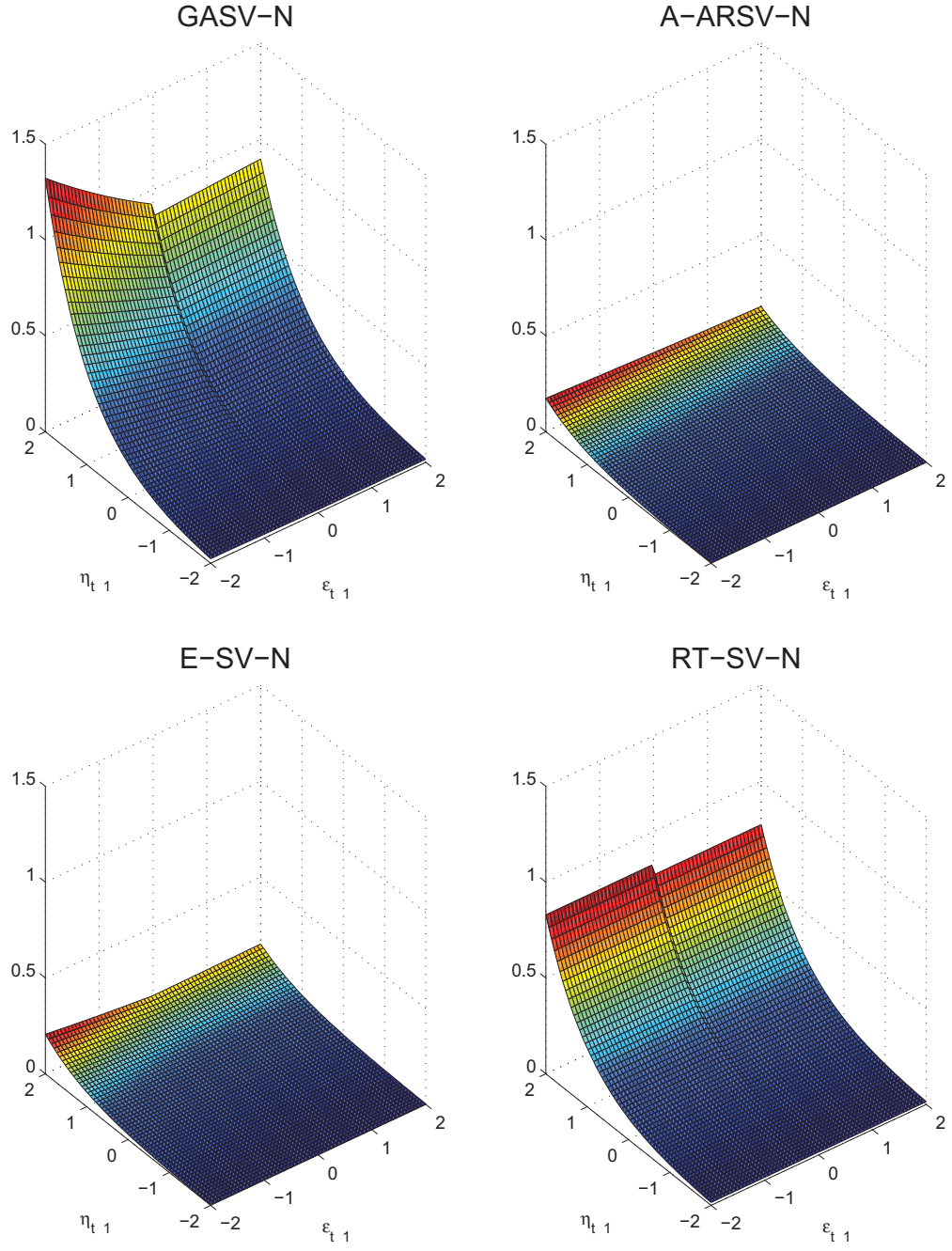


Figure 4: SNIS of different SV models with leverage effect: GASV-N (left top panel), A-ARSV-N (right top panel), E-SV-N (left bottom panel) and RT-SV-N (right bottom panel) with parameters $(\exp(\mu/2), \alpha, \phi, \gamma_1, \gamma_2, \sigma_\eta^2) = (0.1, 0.07, 0.98, -0.08, 0.1, 0.05)$.

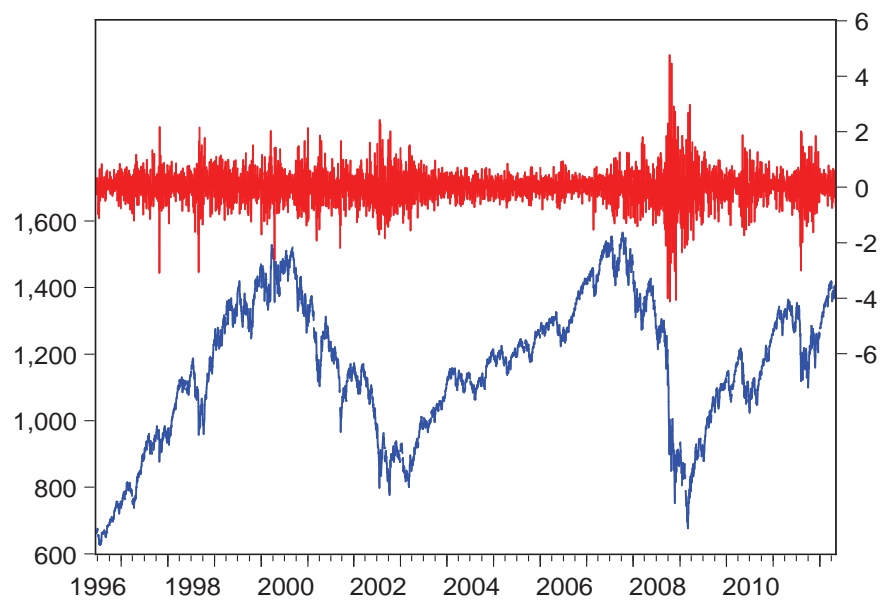


Figure 5: S&P500 daily prices (bottom line) and returns (top line) observed from Jun 17, 1996 up to May 4, 2012.

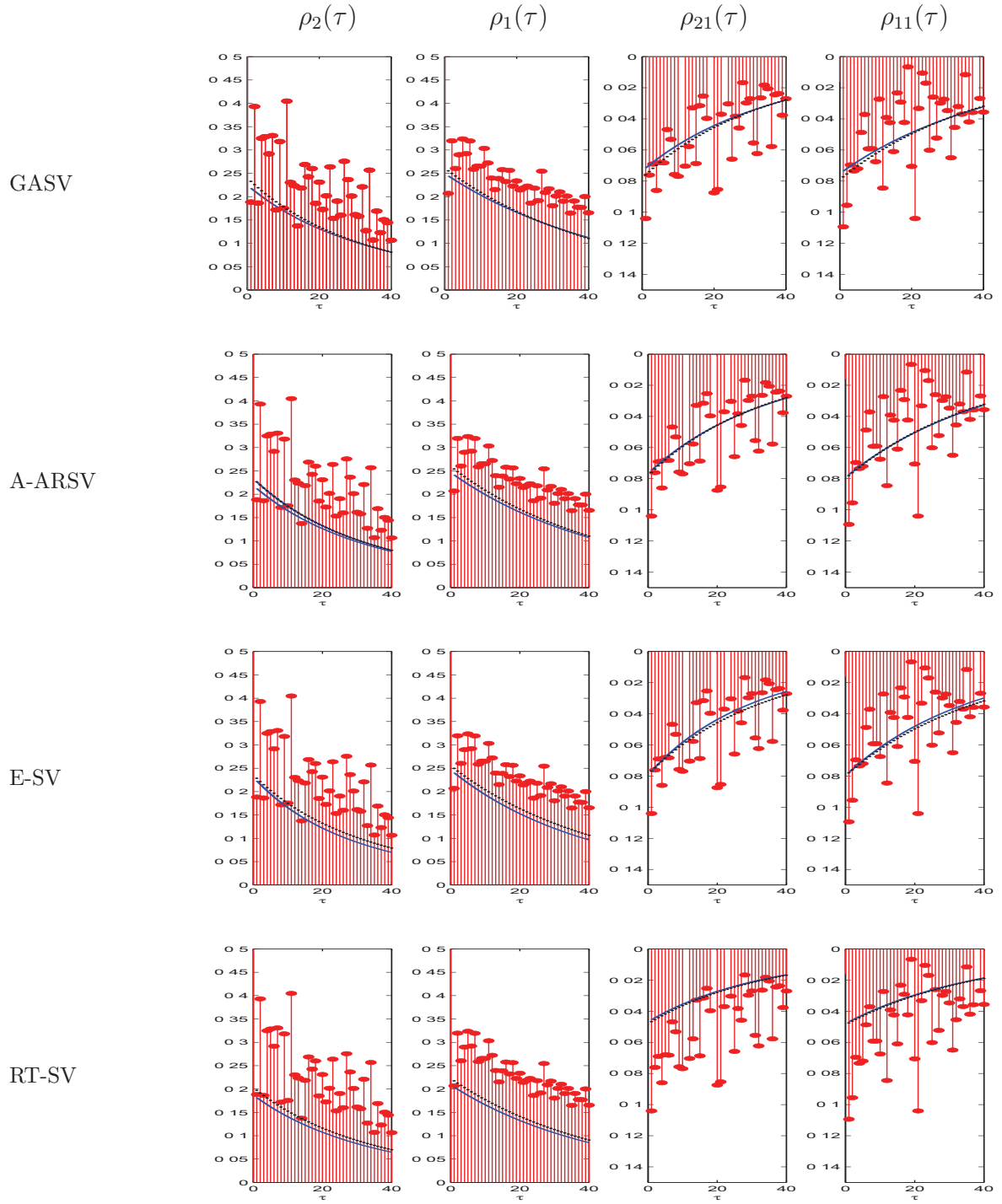


Figure 6: Sample autocorrelations of squares (first column), autocorrelations of absolute returns (second column), cross-correlations of returns and future squared returns (third column) and cross-correlations between absolute returns and lagged returns (fourth column) together with the corresponding plug-in moments obtained after fitting the GASV (first row), A-ARSV (second row), E-SV (third row) or RT-SV (fourth row) models to the daily S&P500 returns. The continuous lines correspond to the moments implied by the models estimated with a GED distribution while the dotted lines correspond to the models estimated when the distribution is Normal.

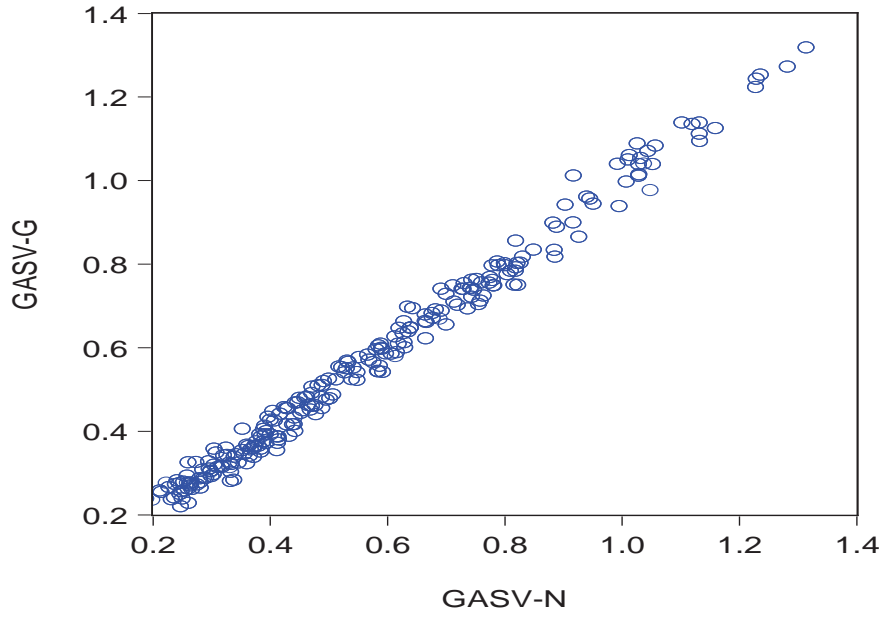


Figure 7: Scatter plot of one-step-ahead VaRs obtained after fitting the GASV-G model against those obtained by fitting GASV-N model to the daily S&P500 returns.

α	α'	ϕ	ϕ'	σ_η^2	$\sigma_{\eta'}^2$	Variance	Kurtosis	$\rho_2(1)$	$\rho_{21}(1)$
-0.2	0.2	0.98	0	0.05	0	0.014 (0.003)	11.481 (4.6000)	0.254 (0.061)	-0.038 (0.039)
-0.3	0.3	0.98	0	0.05	0	0.001 (3.730×10^{-4})	15.365 (11.042)	0.270 (0.070)	-0.057 (0.045)
-0.4	0.4	0.98	0	0.05	0	1.409×10^{-4} (4.301×10^{-5})	20.300 (21.217)	0.284 (0.078)	-0.073 (0.052)
0	0	0.9	0.08	0.05	0	1.243 (0.077)	4.654 (0.533)	0.138 (0.032)	-0.012 (0.022)
0	0	0.5	0.48	0.05	0	1.065 (0.026)	3.424 (0.139)	0.044 (0.019)	-0.021 (0.016)
0	0	0.5	0.4	0.05	0	1.053 (0.025)	3.340 (0.118)	0.033 (0.016)	-0.014 (0.015)
0	0	0.98	0	0.01	0.59	51.237 (72.848)	156.122 (173.042)	0.304 (0.123)	-0.015 (0.118)
0	0	0.98	0	0.2	0.4	144.414 (267.859)	207.445 (210.925)	0.310 (0.130)	-0.008 (0.135)
0	0	0.98	0	0.2	0.1	22.315 (21.424)	103.519 (114.620)	0.308 (0.116)	-0.005 (0.107)

Table 1: Monte Carlo means and standard deviations (in parenthesis) of the sample variance, kurtosis, first order autocorrelation of squares and first order cross-correlation between returns and future squared returns for several T-SV-N models.

	GASV-N						GASV-G						
	μ	ϕ	α	γ_1	γ_2	σ_η^2	μ	ϕ	α	γ_1	γ_2	σ_η^2	ν
True	0	0.98	0.07	-0.08	0.1	0.05	0	0.98	0.07	-0.08	0.1	0.05	1.5
T=500													
Mean	0.289 (1.572)	0.965 (0.019)	0.096 (0.113)	-0.074 (0.061)	0.145 (0.168)	0.041 (0.026)	0.186 (1.435)	0.961 (0.023)	0.095 (0.110)	-0.075 (0.064)	0.184 (0.234)	0.024 (0.018)	1.462 (0.323)
s.d.	1.922	0.014	0.109	0.059	0.115	0.021	1.861	0.016	0.111	0.060	0.117	0.017	0.178
T=1000													
Mean	0.080 (1.431)	0.974 (0.010)	0.083 (0.076)	-0.077 (0.040)	0.139 (0.106)	0.045 (0.019)	-0.041 (1.468)	0.973 (0.011)	0.082 (0.076)	-0.077 (0.042)	0.181 (0.181)	0.027 (0.017)	1.427 (0.216)
s.d.	1.737	0.008	0.078	0.042	0.081	0.015	1.710	0.009	0.077	0.041	0.084	0.014	0.112
T=2000													
Mean	-0.078 (1.281)	0.977 (0.006)	0.078 (0.055)	-0.078 (0.028)	0.119 (0.064)	0.048 (0.011)	-0.07 (1.438)	0.973 (0.011)	0.086 (0.078)	-0.074 (0.044)	0.210 (0.152)	0.023 (0.016)	1.390 (0.209)
s.d.	1.453	0.005	0.058	0.030	0.058	0.011	1.694	0.009	0.077	0.041	0.079	0.012	0.109

Table 2: Monte Carlo results of the MCMC estimator of the parameters of the GASV model. Reported are the values of the Monte Carlo average and standard deviation (in parenthesis) of the posterior means together with the Monte Carlo average of the posterior standard deviation.

T=500							T=1000					
	μ	ϕ	α	γ_1	γ_2	σ_η^2	μ	ϕ	α	γ_1	γ_2	σ_η^2
A-ARSV-N												
True	0	0.98	0	-0.08	0	0.05	0	0.98	0	-0.08	0	0.05
Mean	0.010	0.962	-0.003	-0.085	0.087	0.041	-0.009	0.973	0.002	-0.083	0.047	0.047
	(1.508)	(0.025)	(0.109)	(0.059)	(0.173)	(0.022)	(1.451)	(0.010)	(0.076)	(0.042)	(0.112)	(0.017)
s.d.	1.844	0.016	0.109	0.059	0.111	0.022	1.682	0.009	0.078	0.041	0.084	0.015
E-SV-N												
True	0	0.98	0	-0.08	0.1	0.05	0	0.98	0	-0.08	0.1	0.05
Mean	0.043	0.964	0.002	-0.082	0.190	0.036	-0.039	0.973	0.005	-0.081	0.148	0.042
	(1.497)	(0.020)	(0.106)	(0.061)	(0.149)	(0.020)	(1.382)	(0.010)	(0.072)	(0.041)	(0.100)	(0.015)
s.d.	1.885	0.015	0.107	0.059	0.109	0.020	1.697	0.009	0.077	0.041	0.079	0.015
RT-SV-N												
True	0	0.98	0.07	0	0	0.05	0	0.98	0.07	0	0	0.05
Mean	0.209	0.961	0.101	0.011	0.022	0.046	0.077	0.973	0.083	0.006	0.018	0.048
	(1.470)	(0.023)	(0.107)	(0.062)	(0.189)	(0.030)	(1.372)	(0.011)	(0.072)	(0.041)	(0.122)	(0.019)
s.d.	1.876	0.017	0.111	0.060	0.124	0.023	1.721	0.010	0.078	0.041	0.088	0.016

Table 3: Monte Carlo results of MCMC estimator of the parameters of the GASV-N model fitted to series simulated from different asymmetric SV models. Reported are the values of the Monte Carlo average and standard deviation (in parenthesis) of the posterior means together with the Monte Carlo average of the posterior standard deviation.

	GASV-N	A-ARSV-N	E-SV-N	RT-SV-N	GASV-G
T=500					
Mean	-0.035	-0.049	-0.049	-0.031	-0.065
	(0.016)	(0.061)	(0.047)	(0.018)	(0.022)
s.d.	0.235	0.252	0.238	0.242	0.283
T=1000					
Mean	-0.030	-0.036	-0.036	-0.027	-0.059
	(0.016)	(0.047)	(0.042)	(0.015)	(0.018)
s.d.	0.219	0.229	0.222	0.226	0.279

Table 4: Monte Carlo results of the relative volatility prediction errors. Reported are the values of the time average and standard deviation (in parenthesis) of $m_t = \sum_{i=1}^R e_t^{(i)} / R$ together with the time average of $s_t = \sqrt{\sum_{i=1}^R (e_t^{(i)} - m_t)^2 / (R - 1)}$, where $e_t^{(i)} = (\sigma_t^{(i)} - \hat{\sigma}_t^{(i)}) / \sigma_t^{(i)}$.

	GASV-G	A-ARSV-G	E-SV-G	RT-SV-G	GASV-N	A-ARSV-N	E-SV-N	RT-SV-N
μ	-1.059 (-2.516, 0.645)	-1.738 (-1.944,-1.540)	-1.715 (-1.891, -1.545)	-6.584 (-8.206, -5.250)	-1.254 (-2.619, 0.429)	-1.756 (-1.972,-1.548)	-1.724 (-1.932, -1.519)	-6.514 (-8.044,-5.269)
ϕ	0.982 (0.975, 0.988)	0.981 (0.975, 0.986)	0.979 (0.972, 0.986)	0.979 (0.972, 0.986)	0.980 (0.975, 0.985)	0.981 (0.974, 0.986)	0.980 (0.974, 0.986)	0.979 (0.973, 0.985)
α	-0.024 (-0.083, 0.030)			0.207 (0.152,0.263)	-0.018 (-0.082, 0.039)			0.205 (0.161,0.251)
γ_1	-0.139 (-0.169, -0.115)	-0.138 (-0.159, -0.113)	-0.133 (-0.145,-0.119)		-0.145 (-0.173, -0.119)	-0.137 (-0.164, -0.114)	-0.135 (-0.159,-0.111)	
γ_2	0.045 (-0.002,0.099)		0.109 (0.069,0.136)		0.042 (-0.013,0.081)		0.028 (-0.018,0.073)	
σ_η^2	0.011 (0.006, 0.016)	0.014 (0.007,0.018)	0.006 (0.004,0.009)	0.018 (0.012, 0.027)	0.013 (0.008, 0.019)	0.015 (0.011,0.021)	0.013 (0.009,0.019)	0.019 (0.014, 0.025)
ν	1.788 (1.658,1.905)	1.869 (1.682, 2.068)	1.706 (1.637, 1.791)	1.851 (1.721, 2.020)				
DIC	355.021	332.90	338.850	464.63	4892.570	4854.730	4888.580	4907.130

Table 5: MCMC estimates of the parameters of alternative asymmetric SV models for S&P500 daily returns. The values reported are the mean and 95% credible interval (in parenthesis) of the posterior distributions.

		Coverage Test		
		LRuc	LRind	LRcc
A-ARSV-G	0.071	2.163 (0.141)	0.082 (0.775)	2.393 (0.302)
E-SV-G	0.068	1.465 (0.226)	0.024 (0.877)	1.629 (0.443)
RT-SV-G	0.071	2.163 (0.141)	0.082 (0.775)	2.393 (0.302)
GASV-G	0.071	2.163 (0.141)	0.082 (0.775)	2.393 (0.302)
A-ARSV-N	0.068	1.465 (0.226)	0.024 (0.877)	1.629 (0.443)
E-SV-N	0.071	2.163 (0.141)	0.082 (0.775)	2.393 (0.302)
RT-SV-N	0.068	1.465 (0.226)	0.024 (0.877)	1.629 (0.443)
GASV-N	0.071	2.163 (0.141)	0.082 (0.775)	2.393 (0.302)

Table 6: Failure rates and statistics with p-values (in parenthesis) of the LRuc, LRind and LRcc tests.

Asymmetric SV models	E-SV-G	RT-SV-G	GASV-G	A-ARSV-N	E-SV-N	RT-SV-N	GASV-N
A-ARSV-G	1.453 (0.484)	2.266 (0.322)	1.359 (0.507)	0.183 (0.913)	0.010 (0.995)	3.406 (0.182)	0.430 (0.806)
E-SV-G		1.130 (0.568)	0.415 (0.813)	0.409 (0.815)	0.589 (0.745)	1.410 (0.494)	0.272 (0.873)
RT-SV-G			1.460 (0.482)	0.925 (0.630)	0.759 (0.684)	0.122 (0.941)	1.155 (0.561)
GASV-G				0.598 (0.742)	0.755 (0.686)	1.533 (0.465)	0.265 (0.876)
A-ARSV-N					0.043 (0.979)	2.526 (0.283)	1.049 (0.592)
E-SV-N						1.547 (0.462)	0.620 (0.734)
RT-SV-N							2.218 (0.330)

Table 7: CPA statistics with corresponding p-values in parenthesis.

Appendix A. Proof of Theorems

Appendix A.1. Proof of Theorem 2.1

Consider y_t , which, according to equation (1), is given by $y_t = \epsilon_t \exp(h_t/2)$. From equation (2), h_t can be written as

$$h_t - \mu = \sum_{i=1}^{\infty} \phi^{i-1} (f(\epsilon_{t-i}) + \eta_{t-i}). \quad (\text{A.1})$$

First, note that if $|\phi| < 1$ and $x = (x_1, x_2, \dots) \in \mathbb{R}_{\infty}$, then $\Psi(x) = \sum_{i=1}^{\infty} \phi^{i-1} x_i$ is a measurable function. Given that for any x_0 and $\forall \varsigma > 0$, we can find a value of $\delta = \sqrt{1 - \phi^2} \varsigma$, such that $\forall x$ satisfying $|x - x_0| = \sqrt{\sum_{i=1}^{\infty} (x_i - x_i^0)^2} < \delta$, we have $|\Psi(x) - \Psi(x_0)| = |\sum_{i=1}^{\infty} \phi^{i-1} (x_i - x_i^0)|$. Using the Cauchy-Schwarz inequality, it follows that $|\Psi(x) - \Psi(x_0)| \leq \sqrt{\sum_{i=1}^{\infty} \phi^{2i-2}} \sqrt{\sum_{i=1}^{\infty} (x_i - x_i^0)^2} < \frac{\delta}{\sqrt{1 - \phi^2}} = \varsigma$. Therefore, $\Psi(x)$ is continuous, and consequently, measurable.

Second, given that ϵ_t and η_t are both IID and mutually independent for any lag and lead, then $\{f(\epsilon_t) + \eta_t\}$ is also an IID sequence. Lemma 3.5.8 of Stout (1974) states that an IID sequence is always strictly stationary. Therefore, in (A.1), if $|\phi| < 1$, h_t is expressed as a measurable function of a strictly stationary process and, consequently, according to Theorem 3.5.8 of Stout (1974), h_t is strictly stationary. As σ_t is a continuous function of h_t , σ_t is also strictly stationary. The level noise ϵ_t is independent of σ_t and strictly stationary by definition. Therefore, it is easy to show that $y_t = \sigma_t \epsilon_t$ is strictly stationary.

When $|\phi| < 1$, y_t and σ_t^2 are strictly stationary and, consequently, any existing moments are time invariant. Next we show that σ_t and $|y_t|$ have finite moments of arbitrary positive order c when $E(|\epsilon_t|^c) < \infty$ and ϵ_t follows a distribution such that $E(\exp(0.5cf(\epsilon_t))) < \infty$. Consider y_t , which, according to equation (1), is given by $y_t = \sigma_t \epsilon_t$. Therefore, given that σ_t and ϵ_t are contemporaneously independent, the

following expression is obtained

$$E(|y_t|^c) = E(\sigma_t^c)E(|\epsilon_t|^c). \quad (\text{A.2})$$

Given that $E(|\epsilon_t|^c) < \infty$, we only need to show that if $|\phi| < 1$ and $E(\exp(0.5cf(\epsilon_t))) < \infty$, $E(\sigma_t^c)$ is finite for all c . From expression (A.1), the power-transformed volatility can be written as follows

$$\sigma_t^c = \exp(0.5c\mu) \exp \left(0.5c \sum_{i=1}^{\infty} \phi^{i-1} (f(\epsilon_{t-i}) + \eta_{t-i}) \right). \quad (\text{A.3})$$

Given that ϵ_t and η_t are mutually independent for all lags and leads, the following expression is obtained after taking expectations on both sides of equation (A.3)

$$E(\sigma_t^c) = \exp(0.5c\mu) E \left[\exp \left(0.5c \sum_{i=1}^{\infty} \phi^{i-1} f(\epsilon_{t-i}) \right) \right] E \left[\exp \left(0.5c \sum_{i=1}^{\infty} \phi^{i-1} \eta_{t-i} \right) \right]. \quad (\text{A.4})$$

As η_t is Gaussian, the last expectation in (A.4) can be evaluated using the expression of the moments of the Log-Normal. Furthermore, given that η_t and ϵ_t are both IID sequences, it is easy to show that (A.4) becomes

$$E(\sigma_t^c) = \exp(0.5c\mu) \exp \left(\frac{c^2 \sigma_\eta^2}{8(1-\phi^2)} \right) \prod_{i=1}^{\infty} E \left[\exp (0.5c\phi^{i-1} f(\epsilon_{t-i})) \right]. \quad (\text{A.5})$$

Replacing formula (A.5) into (A.2) yields the following required expression

$$E(|y_t|^c) = \exp(0.5c\mu) E(|\epsilon_t|^c) \exp \left(\frac{c^2 \sigma_\eta^2}{8(1-\phi^2)} \right) P(0.5c\phi^{i-1}), \quad (\text{A.6})$$

where $P(b_i) \equiv \prod_{i=1}^{\infty} E(\exp(b_i f(\epsilon_{t-i})))$.

Finally, we need to show that $P(0.5c\phi^{i-1})$ is finite. In general, we are going to prove that when $\sum_{i=1}^{\infty} |b_i| < \infty$ and $E(\exp(b_i f(\epsilon_{t-i}))) < \infty$, then $P(b_i)$ is always finite.

Define $a_i = E(\exp(b_i f(\epsilon_{t-i})))$. As $0 < a_i < \infty$, according to Section 0.25 of Ryzhik et al. (2007), the sufficient and necessary condition for the infinite product $\prod_{i=1}^{\infty} a_i$ to converge to a finite, nonzero number is that the series $\sum_{i=1}^{\infty} (a_i - 1)$ converge. Expanding a_i in Taylor series around $b_i = 0$, we have

$$a_i - 1 = O(b_i) \text{ as } b_i \rightarrow 0.$$

Consequently, for some $\varsigma > 0$, there exist a finite M independent of i such that

$$\sup_{|b_i| < \varsigma, b_i \neq 0} |O(b_i)| < M|b_i|.$$

$\sum_{i=1}^{\infty} |b_i| < \infty$ implies $\sum_{i=1}^{\infty} |a_i - 1| < \infty$, therefore $\sum_{i=1}^{\infty} (a_i - 1) < \infty$. Thus $P(b_i) = \prod_{i=1}^{\infty} a_i < \infty$.

Here $b_i = 0.5c\phi^{i-1}$. Therefore, if $|\phi| < 1$, $\sum_{i=1}^{\infty} |b_i| = \frac{0.5c}{1-\phi} < \infty$. Thus, the product $\prod_{i=1}^{\infty} E(\exp(0.5c\phi^{i-1}f(\epsilon_{t-i})))$ and, consequently, $E(|y_t|^c)$ are finite when $E(\exp(b_i f(\epsilon_{t-i}))) < \infty$.

Note that when $|\phi| < 1$, $E(\exp(0.5cf(\epsilon_t))) < \infty$ insures that $E(\exp(0.5c\phi^{i-1}f(\epsilon_{t-i}))) < \infty$ for any positive integer i . Then we complete the proof.

Appendix A.2. Proof of Theorem 2.2

Consider y_t as given in equations (1) and (2). We first compute the τ -th order auto-covariance of $|y_t|^c$ which is given by

$$E(|\epsilon_t|^c \sigma_t^c |\epsilon_{t-\tau}|^c \sigma_{t-\tau}^c) - [E(|y_t|^c)]^2. \quad (\text{A.7})$$

Note that from equation (2), $\sigma_t^c = \exp\{0.5ch_t\}$ can be written as follows

$$\sigma_t^c = \exp\{0.5c\mu(1 - \phi^\tau)\} \exp\left\{0.5c \sum_{i=1}^{\tau} \phi^{i-1} (f(\epsilon_{t-i}) + \eta_{t-i})\right\} \sigma_{t-\tau}^{c\phi^\tau}. \quad (\text{A.8})$$

The following expression of the auto-covariance is obtained using (A.6) and after substituting (A.8) into (A.7)

$$\begin{aligned} \text{cov}(|y_t|^c, |y_{t-\tau}|^c) = \\ E \left(|\epsilon_t|^c |\epsilon_{t-\tau}|^c \exp(0.5c\mu(1 - \phi^\tau)) \exp \left(\sum_{i=1}^{\tau} 0.5c\phi^{i-1} (f(\epsilon_{t-i}) + \eta_{t-i}) \right) \sigma_{t-\tau}^{c(\phi^\tau+1)} \right) \\ - \left\{ \exp(0.5c\mu) E(|\epsilon_t|^c) \exp \left(\frac{c^2 \sigma_\eta^2}{8(1 - \phi^2)} \right) P(0.5c\phi^{i-1}) \right\}^2. \end{aligned} \quad (\text{A.9})$$

Given that ϵ_t and η_t are IID sequences mutually independent for any lag and lead and that $\sigma_{t-\tau}$ only depends on lagged disturbances, equation (A.9) can be written as follows

$$\begin{aligned} \text{cov}(|y_t|^c, |y_{t-\tau}|^c) = \\ \exp(c\mu) E(|\epsilon_t|^c) \exp \left(\frac{1 + \phi^\tau}{4(1 - \phi^2)} c^2 \sigma_\eta^2 \right) E(|\epsilon_t|^c \exp(0.5c\phi^{\tau-1} f(\epsilon_t))) \prod_{i=1}^{\tau-1} E(\exp(0.5c\phi^{i-1} f(\epsilon_{t-i}))) \\ \cdot \prod_{i=1}^{\infty} E(\exp(0.5c(1 + \phi^\tau) \phi^{i-1} f(\epsilon_{t-i}))) - \exp(c\mu) (E(|\epsilon_t|^c))^2 \exp \left(\frac{c^2 \sigma_\eta^2}{4(1 - \phi^2)} \right) [P(0.5c\phi^{i-1})]^2. \end{aligned}$$

The required expression of $\rho_c(\tau)$ follows directly from $\rho_c(\tau) = \frac{\text{cov}(|y_t|^c, |y_{t-\tau}|^c)}{E(|y_t|^{2c}) - [E(|y_t|^c)]^2}$, where the denominator can be obtained from (A.6).

Appendix A.3. Proof of Theorem 2.3

The calculation of the cross-covariance between $|y_t|^c$ and $y_{t-\tau}$ is obtained following the same steps as in Appendix A.2. That is

$$\begin{aligned} \text{cov}(|y_t|^c, y_{t-\tau}) = \exp(0.5(c+1)\mu) E(|\epsilon_t|^c) \exp \left(\frac{1 + c^2 + 2c\phi^\tau}{8(1 - \phi^2)} \sigma_\eta^2 \right) E(\epsilon_t \exp(0.5c\phi^{\tau-1} f(\epsilon_t))) \\ \cdot \prod_{i=1}^{\infty} E(\exp(0.5(1 + c\phi^\tau) \phi^{i-1} f(\epsilon_{t-i}))) \prod_{i=1}^{\tau-1} E(\exp(0.5c\phi^{i-1} f(\epsilon_{t-i}))). \end{aligned} \quad (\text{A.10})$$

Finally, $\rho_{c1}(\tau) = \frac{\text{cov}(|y_t|^c, y_{t-\tau})}{\sqrt{E(|y_t|^{2c}) - E^2(|y_t|^c)} \sqrt{E(y_t^2)}}$ together with (A.6) and (A.10) yields the required equation (9).

Appendix B. Expectations

Appendix B.1. Expectations needed to compute $E(|y_t|^c)$, $\text{corr}(|y_t|^c, |y_{t+\tau}|^c)$ and $\text{corr}(y_t, |y_{t+\tau}|^c)$ when $\epsilon \sim \text{GED}(\nu)$

Assume that all parameters are defined as in equations (1) and (2). If ϵ has a centered and standardized GED distribution, with parameter $0 < \nu \leq \infty$, then, the density function of ϵ is given by $\psi(\epsilon) = C_0 \exp\left(-\frac{|\epsilon|^\nu}{2\lambda^\nu}\right)$, where $C_0 \equiv \frac{\nu}{\lambda^{2^{1+1/\nu}\Gamma(1/\nu)}}$ and $\lambda \equiv \left(2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)\right)^{1/2}$, with $\Gamma(\cdot)$ being the Gamma function. Thus, given that the distribution of ϵ is symmetric with support $(-\infty, \infty)$, if p is a nonnegative finite integer, then

$$\begin{aligned} E(|\epsilon|^p) &= C_0 \int_{-\infty}^{+\infty} |\epsilon|^p \exp\left(-\frac{|\epsilon|^\nu}{2\lambda^\nu}\right) d\epsilon \\ &= 2C_0 \int_0^{+\infty} \epsilon^p \exp\left(-\frac{\epsilon^\nu}{2\lambda^\nu}\right) d\epsilon. \end{aligned}$$

Substituting $s = \frac{\epsilon^\nu}{2\lambda^\nu}$ and solving the integral yields

$$E(|\epsilon|^p) = 2^{\frac{p}{\nu}} \lambda^p \Gamma((p+1)/\nu) / \Gamma(1/\nu). \quad (\text{B.1})$$

On the other hand,

$$\begin{aligned} E(|\epsilon|^p \exp(bf(\epsilon))) &= \int_{-\infty}^{+\infty} |\epsilon|^p \exp(b\alpha I(\epsilon < 0) + b\gamma_1\epsilon + b\gamma_2(|\epsilon| - E|\epsilon|)) C_0 \exp\left(-\frac{|\epsilon|^\nu}{2\lambda^\nu}\right) d\epsilon \\ &= C_0 \exp(-b\gamma_2 E|\epsilon|) \left[\int_{-\infty}^0 (-\epsilon)^p \exp(b\alpha) \exp(b(\gamma_1 - \gamma_2)\epsilon) \exp\left(-\frac{(-\epsilon)^\nu}{2\lambda^\nu}\right) d\epsilon \right. \\ &\quad \left. + \int_0^{+\infty} \epsilon^p \exp(b(\gamma_1 + \gamma_2)\epsilon) \exp\left(-\frac{\epsilon^\nu}{2\lambda^\nu}\right) d\epsilon \right]. \end{aligned}$$

Integrating by substitution with $s = -\epsilon$ in the first integral, we obtain

$$\begin{aligned}
E(|\epsilon|^p \exp(bf(\epsilon))) &= C_0 \exp(-b\gamma_2 E|\epsilon|) \left[\int_0^{+\infty} s^p \exp(b\alpha) \exp(b(\gamma_2 - \gamma_1)s) \exp\left(-\frac{s^\nu}{2\lambda^\nu}\right) ds \right. \\
&\quad \left. + \int_0^{+\infty} \epsilon^p \exp(b(\gamma_1 + \gamma_2)\epsilon) \exp\left(-\frac{\epsilon^\nu}{2\lambda^\nu}\right) d\epsilon \right] \\
&= C_0 \exp(-b\gamma_2 E|\epsilon|) \int_0^{+\infty} \epsilon^p \exp\left(-\frac{\epsilon^\nu}{2\lambda^\nu}\right) [\exp(b\alpha) \exp(b(\gamma_2 - \gamma_1)\epsilon) + \exp(b(\gamma_1 + \gamma_2)\epsilon)] d\epsilon.
\end{aligned} \tag{B.2}$$

We can rewrite equation (B.2) by replacing ϵ with $\lambda(2y)^{1/\nu}$ as follows

$$\begin{aligned}
E(|\epsilon|^p \exp(bf(\epsilon))) &= C_0 \exp(-b\gamma_2 E|\epsilon|) \frac{\lambda^{p+1} 2^{\frac{1+p}{\nu}}}{\nu} \\
&\quad \cdot \int_0^{+\infty} y^{-1+\frac{1+p}{\nu}} \exp(-y) \left[\exp(b\alpha) \exp(b(\gamma_2 - \gamma_1)\lambda 2^{\frac{1}{\nu}} y^{\frac{1}{\nu}}) + \exp(b(\gamma_1 + \gamma_2)\lambda 2^{\frac{1}{\nu}} y^{\frac{1}{\nu}}) \right] dy.
\end{aligned}$$

Expanding the expression within the square brackets in a Taylor series and substituting C_0 and $E(|\epsilon|)$ as given in (B.1), the following expression is obtained

$$\begin{aligned}
E(|\epsilon|^p \exp(bf(\epsilon))) &= \exp\left(-b\gamma_2 2^{\frac{1}{\nu}} \lambda \Gamma(2/\nu) / \Gamma(1/\nu)\right) \frac{\lambda^p 2^{\frac{p}{\nu}-1}}{\Gamma(\frac{1}{\nu})} \\
&\quad \int_0^{+\infty} \sum_{k=0}^{+\infty} \left[\exp(b\alpha) \left(b\lambda 2^{\frac{1}{\nu}} (\gamma_2 - \gamma_1)\right)^k + \left(b\lambda 2^{\frac{1}{\nu}} (\gamma_1 + \gamma_2)\right)^k \right] \frac{y^{-1+\frac{1+p+k}{\nu}} \exp(-y)}{k!} dy.
\end{aligned} \tag{B.3}$$

Define $\Delta = \max\{|b\lambda 2^{1/\nu}(\gamma_1 + \gamma_2)|, \max(\exp(b\alpha), 1)|b\lambda 2^{1/\nu}(\gamma_2 - \gamma_1)|\}$. Then, we can use the results in Nelson (1991) to show that if $\nu > 1$ then the summation and integration in (B.3) can be interchanged. Further, applying Formula 3.381 #4 of Ryzhik et al. (2007) yields the following required expression⁹

$$\begin{aligned}
E(|\epsilon|^p \exp(bf(\epsilon))) &= \exp\left[-b\gamma_2 2^{1/\nu} \lambda \Gamma(2/\nu) / \Gamma(1/\nu)\right] 2^{p/\nu} \lambda^p \\
&\quad \cdot \sum_{k=0}^{\infty} (2^{1/\nu} \lambda b)^k [(\gamma_1 + \gamma_2)^k + \exp(b\alpha)(\gamma_2 - \gamma_1)^k] \frac{\Gamma((p+k+1)/\nu)}{2\Gamma(1/\nu)k!} < \infty.
\end{aligned} \tag{B.4}$$

⁹See Nelson (1991) for the proof of finiteness of the formula.

Following the same steps, the following required expression is obtained when $\nu > 1$,

$$E(\epsilon^p \exp(bf(\epsilon))) = \exp \left[-b\gamma_2 2^{1/\nu} \lambda \Gamma(2/\nu) / \Gamma(1/\nu) \right] 2^{p/\nu} \lambda^p \cdot \sum_{k=0}^{\infty} (2^{1/\nu} \lambda b)^k \left[(\gamma_1 + \gamma_2)^k + (-1)^p \exp(b\alpha) (\gamma_2 - \gamma_1)^k \right] \frac{\Gamma((p+k+1)/\nu)}{2\Gamma(1/\nu)k!} < \infty. \quad (\text{B.5})$$

Note that the expectations (B.4) and (B.5) are only valid when $\nu > 1$. When $0 < \nu \leq 1$, it is not possible to obtain closed-form expression of the required expectations. In this case, we can only obtain the condition for the expectations to be finite. Note that

$$E(|\epsilon|^p \exp(bf(\epsilon))) = (-1)^p \int_{-\infty}^0 \epsilon^p \exp(bf(\epsilon)) \psi(\epsilon) d\epsilon + \int_0^{\infty} \epsilon^p \exp(bf(\epsilon)) \psi(\epsilon) d\epsilon$$

and

$$E(\epsilon^p \exp(bf(\epsilon))) = \int_{-\infty}^0 \epsilon^p \exp(bf(\epsilon)) \psi(\epsilon) d\epsilon + \int_0^{\infty} \epsilon^p \exp(bf(\epsilon)) \psi(\epsilon) d\epsilon.$$

Therefore, $E(|\epsilon|^p \exp(bf(\epsilon)))$ is finite if and only $E(\epsilon^p \exp(bf(\epsilon)))$ is finite. On the other hand, $E(\epsilon^p \exp(bf(\epsilon))) = \exp(b\alpha) E(\epsilon^p \exp(bg(\epsilon)) | \epsilon < 0) P(\epsilon < 0) + E(\epsilon^p \exp(bg(\epsilon)) | \epsilon \geq 0) P(\epsilon \geq 0)$, where $g(\epsilon) = \gamma_1 \epsilon + \gamma_2(|\epsilon| - E(|\epsilon|))$ and $P(\epsilon < 0)$ is the probability of $\epsilon < 0$. Thus, the conditions for the existence of $E(\epsilon^p \exp(bf(\epsilon)))$ and $E(|\epsilon|^p \exp(bf(\epsilon)))$ are the same as those of $E(\epsilon^p \exp(bg(\epsilon)))$, which are given by the Theorem A1.2 in Nelson (1991).

Therefore, if $\nu < 1$, $E(\epsilon^p \exp(bf(\epsilon)))$ and $E(|\epsilon|^p \exp(bf(\epsilon)))$ are finite if and only if

$$b\gamma_2 + |b\gamma_1| \leq 0.$$

Finally if $\epsilon \sim \text{GED}$ with $\nu = 1$ or $\epsilon \sim \text{Student-t}$ with degrees of freedom $d > 2$ and ϵ is centered and standardized to satisfy $E(\epsilon) = 0$ and $\text{var}(\epsilon) = 1$, according to

Nelson (1991), $E(\epsilon^p \exp(bf(\epsilon)))$ and $E(|\epsilon|^p \exp(bf(\epsilon)))$ are finite if and only if

$$b\gamma_2 + |b\gamma_1| < \sqrt{2}.$$

Appendix B.2. Expectations needed to compute $E(|y_t|^c)$, $\text{corr}(|y_t|^c, |y_{t+\tau}|^c)$ and $\text{corr}(y_t, |y_{t+\tau}|^c)$ when $\epsilon \sim N(0, 1)$

Assume that all the parameters are defined as in equations (1) and (2). When $\epsilon \sim N(0, 1)$, using the expression (B.2) and the formula 3.462-1 of Ryzhik et al. (2007), the following expressions for any positive integer p and any real number b are derived

$$\begin{aligned} E(|\epsilon|^p \exp(bf(\epsilon))) &= \frac{1}{\sqrt{2\pi}} \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \\ &\quad \left\{ \exp(b\alpha)\Gamma(p+1) \exp\left(\frac{b^2(\gamma_1 - \gamma_2)^2}{4}\right) D_{-p-1}(b(\gamma_1 - \gamma_2)) \right. \\ &\quad \left. + \Gamma(p+1) \exp\left(\frac{b^2(\gamma_1 + \gamma_2)^2}{4}\right) D_{-p-1}(-b(\gamma_1 + \gamma_2)) \right\} \end{aligned} \quad (\text{B.6})$$

and

$$\begin{aligned} E(\epsilon^p \exp(bf(\epsilon))) &= \frac{1}{\sqrt{2\pi}} \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \\ &\quad \cdot \left\{ (-1)^p \exp(b\alpha)\Gamma(p+1) \exp\left(\frac{b^2(\gamma_1 - \gamma_2)^2}{4}\right) D_{-p-1}(b(\gamma_1 - \gamma_2)) \right. \\ &\quad \left. + \Gamma(p+1) \exp\left(\frac{b^2(\gamma_1 + \gamma_2)^2}{4}\right) D_{-p-1}(-b(\gamma_1 + \gamma_2)) \right\}, \end{aligned} \quad (\text{B.7})$$

where $D_{-a}(\cdot)$ is the parabolic cylinder function. Particularly, when $p = 0, 1$ or 2 , the expressions are reduced to

$$\begin{aligned}
E(\exp(bf(\epsilon))) &= \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \left\{ \exp(b\alpha) \exp(\bar{A}) \Phi(\bar{C}) + \exp(\bar{B}) \Phi(\bar{D}) \right\}, \\
E(\epsilon \exp(bf(\epsilon))) &= \frac{1}{\sqrt{2\pi}} \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \\
&\quad \cdot \left\{ -\exp(b\alpha) \left[1 + \sqrt{2\pi}\bar{C} \exp(\bar{A})\Phi(\bar{C}) \right] + \left[1 + \sqrt{2\pi}\bar{D} \exp(\bar{B})\Phi(\bar{D}) \right] \right\}, \\
E(|\epsilon| \exp(bf(\epsilon))) &= \frac{1}{\sqrt{2\pi}} \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \\
&\quad \cdot \left\{ \exp(b\alpha) \left[1 + \sqrt{2\pi}\bar{C} \exp(\bar{A})\Phi(\bar{C}) \right] + \left[1 + \sqrt{2\pi}\bar{D} \exp(\bar{B})\Phi(\bar{D}) \right] \right\}
\end{aligned}$$

and

$$\begin{aligned}
E(|\epsilon|^2 \exp(bf(\epsilon))) &= \frac{1}{\sqrt{2\pi}} \exp\left(-b\gamma_2\sqrt{\frac{2}{\pi}}\right) \\
&\quad \cdot \left\{ \exp(b\alpha) \left[\bar{C} + \sqrt{2\pi}(\bar{C}^2 + 1) \exp(\bar{A})\Phi(\bar{C}) \right] + \left[\bar{D} + \sqrt{2\pi}(\bar{D}^2 + 1) \exp(\bar{B})\Phi(\bar{D}) \right] \right\},
\end{aligned}$$

where $\Phi(\cdot)$ is the Normal distribution function, $\bar{A} = \frac{b^2(\gamma_1 - \gamma_2)^2}{2}$, $\bar{B} = \frac{b^2(\gamma_1 + \gamma_2)^2}{2}$, $\bar{C} = -b(\gamma_1 - \gamma_2)$ and $\bar{D} = b(\gamma_1 + \gamma_2)$.

References

- Abanto-Valle, C., Bandyopadhyay, D., Lachos, V., Enriquez, I., 2010. Robust Bayesian analysis of heavy-tailed stochastic volatility models using scale mixtures of normal distributions. *Computational Statistics & Data Analysis* 54, 2883–2898.
- Andersen, T., Bollerslev, T., Diebold, F., Ebens, H., 2001a. The distribution of realized stock return volatility. *Journal of Financial Economics* 61, 43–76.
- Andersen, T., Bollerslev, T., Diebold, F., Labys, P., 2001b. The distribution of realized exchange rate volatility. *Journal of the American Statistical Association* 96, 42–55.

- Andersen, T., Bollerslev, T., Diebold, F., Labys, P., 2003. Modeling and forecasting realized volatility. *Econometrica* 71, 579–625.
- Asai, M., McAleer, M., 2005. Dynamic asymmetric leverage in stochastic volatility models. *Econometric Reviews* 24, 317–332.
- Asai, M., McAleer, M., 2006. Asymmetric multivariate stochastic volatility. *Econometric Reviews* 25, 453–473.
- Asai, M., McAleer, M., 2009. Multivariate stochastic volatility, leverage and news impact surfaces. *The Econometrics Journal* 12, 292–309.
- Asai, M., McAleer, M., 2011. Alternative asymmetric stochastic volatility models. *Econometric Reviews* 30, 548–564.
- Asai, M., McAleer, M., Medeiros, M.C., 2012. Asymmetry and long memory in volatility modeling. *Journal of Financial Econometrics* 10, 495–512.
- Bandi, F., Renò, R., 2012. Time-varying leverage effects. *Journal of Econometrics* 169, 94–113.
- Bartolucci, F., De Luca, G., 2003. Likelihood-based inference for asymmetric stochastic volatility models. *Computational Statistics & Data Analysis* 42, 445–449.
- Berg, A., Meyer, R., Yu, J., 2004. Deviance information criterion for comparing stochastic volatility models. *Journal of Business & Economic Statistics* 22, 107–120.
- Black, F., 1976. Studies in stock price volatility changes, in: *Proceedings of the 1976 Business Meeting of the Business and Economics Statistics Section, American Statistical Association*, 177–181.
- Bollerslev, T., Litvinova, J., Tauchen, G., 2006. Leverage and volatility feedback effects in high-frequency data. *Journal of Financial Econometrics* 4, 353–384.

- Breidt, F.J., 1996. A threshold autoregressive stochastic volatility model, in: VI Latin American Congress of Probability and Mathematical Statistics (CLAPEM), Valparaiso, Chile, Citeseer.
- Broto, C., Ruiz, E., 2004. Estimation methods for stochastic volatility models: A survey. *Journal of Economic Surveys* 18, 613–649.
- Caporin, M., McAleer, M., 2011. Thresholds, news impact surfaces and dynamic asymmetric multivariate GARCH. *Statistica Neerlandica* 65, 125–163.
- Cappuccio, N., Lubian, D., Raggi, D., 2004. MCMC Bayesian estimation of a skew-GED stochastic volatility model. *Studies in Nonlinear Dynamics & Econometrics* 8.
- Carnero, M., Peña, D., Ruiz, E., 2004. Persistence and kurtosis in GARCH and stochastic volatility models. *Journal of Financial Econometrics* 2, 319–342.
- Chan, D., Kohn, R., Kirby, C., 2006. Multivariate stochastic volatility models with correlated errors. *Econometric Reviews* 25, 245–274.
- Chen, C., Liu, F., So, M., 2008. Heavy-tailed-distributed threshold stochastic volatility models in financial time series. *Australian & New Zealand Journal of Statistics* 50, 29–51.
- Chib, S., Nardari, F., Shephard, N., 2006. Analysis of high dimensional multivariate stochastic volatility models. *Journal of Econometrics* 134, 341–371.
- Choy, B., Wai, Y., Chan, C., 2008. Bayesian Student-t stochastic volatility models via scale mixtures. *Advances in Econometrics* 23, 595–618.
- Christoffersen, P.F., 1998. Evaluating interval forecasts. *International economic review* , 841–862.

- Danielsson, J., 1998. Multivariate stochastic volatility models: Estimation and a comparison with VGARCH models. *Journal of Empirical Finance* 5, 155–173.
- Demos, A., 2002. Moments and dynamic structure of a time-varying parameter stochastic volatility in mean model. *Econometrics Journal* 5, 345–357.
- Elliott, R., Liew, C., Siu, T., 2011. On filtering and estimation of a threshold stochastic volatility model. *Applied Mathematics and Computation* 218, 61–75.
- Engle, R., Ng, V., 1993. Measuring and testing the impact of news on volatility. *The Journal of Finance* 48, 1779–1801.
- Fridman, M., Harris, L., 1998. A maximum likelihood approach for non-Gaussian stochastic volatility models. *Journal of Business & Economic Statistics* 16, 284–291.
- Ghysels, E., Harvey, A., Renault, E., 1996. Stochastic Volatility, in: Maddala, G.S., Rao, C.R., Vinod, H.D. (Eds.), *Statistical Methods in Finance*. Amsterdam: North-Holland.
- Giacomini, R., White, H., 2006. Tests of conditional predictive ability. *Econometrica* 74, 1545–1578.
- Harvey, A., 1990. *The Econometric Analysis of Time Series*. Cambridge: MIT Press. Second edition.
- Harvey, A., Ruiz, E., Shephard, N., 1994. Multivariate stochastic variance models. *The Review of Economic Studies* 61, 247–264.
- Harvey, A., Shephard, N., 1996. Estimation of an asymmetric stochastic volatility model for asset returns. *Journal of Business & Economic Statistics* 14, 429–34.
- Hibbert, A.M., Daigler, R.T., Dupoyet, B., 2008. A behavioral explanation for the negative asymmetric return–volatility relation. *Journal of Banking & Finance* 32, 2254–2266.

- Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatilities. *The Journal of Finance* 42, 281–300.
- Jacquier, E., Polson, N., Rossi, P., 1994. Bayesian analysis of stochastic volatility models: Reply. *Journal of Business & Economic Statistics* 12, 413–417.
- Jacquier, E., Polson, N., Rossi, P., 2004. Bayesian analysis of stochastic volatility models with fat-tails and correlated errors. *Journal of Econometrics* 122, 185–212.
- Jungbacker, B., Koopman, S.J., 2006. Monte Carlo likelihood estimation for three multivariate stochastic volatility models. *Econometric Reviews* 25, 385–408.
- Kim, S., Shephard, N., Chib, S., 1998. Stochastic volatility: Likelihood inference and comparison with ARCH models. *The Review of Economic Studies* 65, 361–393.
- Lien, D., 2005. A note on asymmetric stochastic volatility and futures hedging. *Journal of Futures Markets* 25, 607–612.
- Liesenfeld, R., Richard, J., 2003. Univariate and multivariate stochastic volatility models: Estimation and diagnostics. *Journal of Empirical Finance* 10, 505–531.
- Melino, A., Turnbull, S., 1990. Pricing foreign currency options with stochastic volatility. *Journal of Econometrics* 45, 239–265.
- Meyer, R., Yu, J., 2000. BUGS for a Bayesian analysis of stochastic volatility models. *The Econometrics Journal* 3, 198–215.
- Montero, J., Fernández-Avilés, G., García, M., 2010. Estimation of asymmetric stochastic volatility models: Application to daily average prices of energy products. *International Statistical Review* 78, 330–347.
- Muñoz, M.P., Marquez, M.D., Acosta, L.M., 2007. Forecasting volatility by means of threshold models. *Journal of Forecasting* 26, 343–363.

- Nakajima, J., Omori, Y., 2009. Leverage, heavy-tails and correlated jumps in stochastic volatility models. *Computational Statistics & Data Analysis* 53, 2335–2353.
- Nakajima, J., Omori, Y., 2012. Stochastic volatility model with leverage and asymmetrically heavy-tailed error using GH skew Students t-distribution. *Computational Statistics & Data Analysis* 56, 3690–3704.
- Nandi, S., 1998. How important is the correlation between returns and volatility in a stochastic volatility model? Empirical evidence from pricing and hedging in the S&P 500 index options market. *Journal of Banking & Finance* 22, 589–610.
- Nelson, D., 1991. Conditional heteroscedasticity in asset pricing: A new approach. *Econometrica* 59, 347–370.
- Omori, Y., Chib, S., Shephard, N., Nakajima, J., 2007. Stochastic volatility with leverage: Fast and efficient likelihood inference. *Journal of Econometrics* 140, 425–449.
- Omori, Y., Watanabe, T., 2008. Block sampler and posterior mode estimation for asymmetric stochastic volatility models. *Computational Statistics & Data Analysis* 52, 2892–2910.
- Pérez, A., Ruiz, E., Veiga, H., 2009. A note on the properties of power-transformed returns in long-memory stochastic volatility models with leverage effect. *Computational Statistics & Data Analysis* 53, 3593–3600.
- Rodríguez, M., Ruiz, E., 2012. GARCH models with leverage effect: differences and similarities. *Journal of Financial Econometrics* 10(4), 637–668.
- Ruiz, E., Pérez, A., 2012. Maximally autocorrelated power transformations: A closer

- look at the properties of stochastic volatility models. *Studies in Nonlinear Dynamics & Econometrics* 16.
- Ruiz, E., Veiga, H., 2008. Modelling long-memory volatilities with leverage effect: A-LMSV versus FIEGARCH. *Computational Statistics & Data Analysis* 52, 2846–2862.
- Ryzhik, I., Jeffrey, A., Zwillinger, D., 2007. Table of integrals, series and products. London: Academic Press. Seventh edition.
- Sandmann, G., Koopman, S., 1998. Estimation of stochastic volatility models via Monte Carlo maximum likelihood. *Journal of Econometrics* 87, 271–301.
- Savva, C., 2009. International stock markets interactions and conditional correlations. *Journal of International Financial Markets, Institutions and Money* 19, 645–661.
- Shephard, N., Pitt, M., 1997. Likelihood analysis of non-Gaussian measurement time series. *Biometrika* 84, 653–667.
- Smith, D., 2009. Asymmetry in stochastic volatility models: Threshold or correlation? *Studies in Nonlinear Dynamics & Econometrics* 13.
- So, M., Li, W., Lam, K., 2002. A threshold stochastic volatility model. *Journal of Forecasting* 21, 473–500.
- Stout, W.F., 1974. Almost Sure Convergence. London: Academic Press.
- Taylor, S., 1994. Modelling stochastic volatility: A review and comparative study. *Mathematical Finance* 4, 183–204.
- Taylor, S., 2007. Asset price dynamics, volatility, and prediction. Princeton, NJ: Princeton University Press.

- Tsiotas, G., 2012. On generalised asymmetric stochastic volatility models. *Computational Statistics & Data Analysis* 56, 151–172.
- Walker, S., Gutiérrez-Peña, E., 1999. Robustifying Bayesian procedures. *Bayesian Statistics* 6, 685–710.
- Wang, J.J., Chan, J.S., Choy, S.B., 2011. Stochastic volatility models with leverage and heavy-tailed distributions: A Bayesian approach using scale mixtures. *Computational Statistics & Data Analysis* 55, 852–862.
- Wang, J.J., Chan, J.S., Choy, S.B., 2013. Modelling stochastic volatility using generalized t distribution. *Journal of Statistical Computation and Simulation* 83, 340–354.
- Watanabe, T., Omori, Y., 2004. A multi-move sampler for estimating non-Gaussian time series models: Comments on Shephard & Pitt (1997). *Biometrika* 91, 246–248.
- Yu, J., 2005. On leverage in a stochastic volatility model. *Journal of Econometrics* 127, 165–178.
- Yu, J., 2012. A semiparametric stochastic volatility model. *Journal of Econometrics* 167, 473–482.
- Yu, J., Meyer, R., 2006. Multivariate stochastic volatility models: Bayesian estimation and model comparison. *Econometric Reviews* 25, 361–384.
- Yu, J., Yang, Z., Zhang, X., 2006. A class of nonlinear stochastic volatility models and its implications for pricing currency options. *Computational Statistics & Data Analysis* 51, 2218–2231.