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## PARAMETER UNCERTAINTY IN MULTIPERIOD PORTFOLIO OPTIMIZATION WITH TRANSACTION COSTS

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### Abstract

We study the impact of parameter uncertainty in multiperiod portfolio selection with trading costs. We analytically characterize the expected loss of a multiperiod investor, and we find that it is equal to the product of two terms. The first term corresponds with the single-period utility loss in the absence of transaction costs, as characterized by Kan and Zhou (2007), whereas the second term captures the multiperiod effects on the overall utility loss. To mitigate the impact of parameter uncertainty, we propose two multiperiod shrinkage portfolios. The first multiperiod shrinkage portfolio combines the Markowitz portfolio with a target portfolio. This method diversifies the effects of parameter uncertainty and reduces the risk of taking inefficient positions. The second multiperiod portfolio shrinks the investor's trading rate. This novel technique smooths the investor trading activity and it also may help to considerably reduce the impact of parameter uncertainty. Finally, we test the out-of-sample performance of our considered portfolio strategies with simulated and empirical datasets, and we find that ignoring transaction costs, parameter uncertainty, or both, results into large losses in the investor's performance.

**Keywords:** Estimation error, shrinkage portfolios, trading costs, out-of-sample performance.

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# Parameter Uncertainty in Multiperiod Portfolio Optimization with Transaction Costs

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## 1. Introduction

The seminal paper of Markowitz (1952) shows that an investor who cares only about the portfolio mean and variance should hold one of the portfolios on the efficient frontier. Markowitz's mean-variance framework is the main foundation of most practical investment approaches, but it relies on three restrictive assumptions. First, the investor is *myopic* and maximizes a one-period utility. Second, financial markets are frictionless. Third, the investor knows the exact parameters that capture asset price dynamics. In this manuscript, we

study the case where these three assumptions fail to hold; that is, the investor tries to maximize a *multi-period* utility in the presence of *quadratic transaction costs* and suffers from parameter uncertainty. Our contribution is threefold. First, we characterize analytically the utility loss associated with estimation error for a multiperiod mean-variance investor who faces quadratic transaction costs. Second, we use these result to propose two shrinkage portfolios designed to combat the impact of parameter uncertainty. Third, we provide evidence based on simulated and empirical datasets that the proposed shrinkage portfolios substantially outperform the portfolios of investors that ignore either parameter uncertainty or transaction costs.

There is an extensive literature on multiperiod portfolio selection in the presence of transaction costs under the assumption that there is *no parameter uncertainty*. For the case with a single-risky asset and proportional transaction costs, Constantinides (1979) and Davis and Norman (1990) show that the optimal portfolio policy of an investor with constant relative risk aversion (CRRA) utility is characterized by a no-trade region. The case with multiple-risky assets and proportional transaction costs is generally intractable analytically.<sup>1</sup> Garleanu and Pedersen (2012) show that the case with multiple-risky assets and *quadratic* transaction costs is, however, more tractable; and they provide closed-form expressions for the optimal portfolio policy of a multiperiod mean-variance investor.<sup>2</sup>

There is also an extensive literature on parameter uncertainty on portfolio selection for the case of a myopic investor who *is not subject to transaction costs*.<sup>3</sup> Kan and Zhou (2007) characterize analytically the utility loss of a mean-variance investor who suffers from parameter uncertainty. Moreover, they consider a three-fund portfolio, which is a combination of the sample mean-variance portfolio, the sample minimum-variance portfolio, and the risk-free asset. They analytically characterize those combination weights of three-fund portfolios that minimize the investor's utility loss from parameter uncertainty.<sup>4</sup>

<sup>1</sup> Liu (2004), however, characterizes analytically the case where asset returns are uncorrelated for the particular case of an investor with constant absolute risk aversion (CARA) utility.

<sup>2</sup> Quadratic transaction costs are well suited to model market impact cost; see, for instance, Engle and Ferstenberg (2007).

<sup>3</sup> This literature includes Bayesian approaches with diffuse priors (Klein and Bawa (1976), Brown (1978)), Bayesian approaches with priors based on asset pricing models (MacKinlay and Pastor (2000), Pastor (2000), Pastor and Stambaugh (2000)), shrinkage approaches Ledoit and Wolf (2004), robust optimization methods (Cornuejols and Tutuncu (2007), Goldfarb and Iyengar (2003), Garlappi et al. (2007), Rustem et al. (2000), Tutuncu and Koenig (2004)), Bayesian robust optimization (Wang (2005)), and methods based on imposing constraints (Best and Grauer (1992), Jagannathan and Ma (2003), and DeMiguel et al. (2009)).

<sup>4</sup> See also Tu and Zhou (2011), who consider a combination of the sample mean-variance portfolio with the equally-weighted portfolio.

Our work is, to the best of our knowledge, the first to consider the impact of parameter uncertainty on the performance of a multiperiod mean-variance investor facing quadratic transaction costs. As mentioned above, our contribution is threefold. Our first contribution is to give a closed-form expression for the utility loss of an investor who uses sample information to construct her optimal portfolio policy. We find that the utility loss is the product of two terms. The first term is the single-period utility loss in the absence of transaction costs, as characterized by Kan and Zhou (2007). The second term captures the effect of the multiperiod horizon on the overall utility loss. Specifically, this term can be split into the losses from the multiperiod mean-variance utility and the multiperiod transaction costs.

We also use our characterization of the utility loss to understand how the transaction costs and the investor's impatience factor affect the investor utility loss. We observe that agents that face high transaction costs are less affected by estimation risk. Although high trading costs do not diminish the investor's exposure to estimation risk, they delay its impact to future stages where the overall importance in the investor's expected utility is lower. Also, an investor with high impatience factor is less affected by estimation risk. Roughly speaking, the investor's impatience factor has a similar effect on the investor's expected utility to that of trading costs. When the investor is more impatient, the cost of making a trade takes a greater importance than the future expected payoff of the corresponding trade. Hence, larger trading costs or higher impatience factor make the investor trade less aggressively, and this offsets the uncertainty of the inputs that define the multiperiod portfolio model.

Our second contribution is to propose shrinkage portfolios designed to combat estimation risk in the multiperiod mean-variance framework with quadratic transaction costs. From Garleanu and Pedersen (2012), it is easy to show that, in the absence of estimation error, the optimal portfolio policy is to trade towards the Markowitz portfolio at a fixed trading rate every period. For this reason, we propose two approaches to combat estimation error: i) shrink the Markowitz portfolio maintaining the trading rate fixed at its nominal value; ii) shrink the trading rate. Regarding the first approach i), we propose a shrinkage portfolio that is obtained by shrinking the Markowitz portfolio towards zero. We term this portfolio as multiperiod three-fund portfolio, because it is a combination of the current portfolio, the Markowitz portfolio, and the risk-free as-

set. Then, we propose a second shrinkage portfolio obtained by shrinking the Markowitz portfolio towards a target portfolio that is less affected by estimation error, and we term the resulting shrinkage portfolio as four-fund portfolio. We show that the shrinkage intensities for the three- and four-fund portfolios are the same as for the single-period investor and provide conditions under which it is optimal to shrink. Regarding the second approach ii), the nominal trading rate given by Garleanu and Pedersen (2012) may not be optimal in the presence of parameter uncertainty. Hence, we propose versions of previous four-fund portfolio where the trading rate is also shrunk to reduce the effects of parameter uncertainty. We provide a rule to compute the optimal trading rate and we illustrate those conditions where the investor can obtain gains by shrinking the trading rate.

Our third contribution is to evaluate the out-of-sample performance of the proposed shrinkage portfolios on simulated data as well as on an empirical dataset of commodity futures similar to that used by Garleanu and Pedersen (2012). We find that the four-fund portfolios (either with fixed or optimal trading rate) substantially outperform portfolios that either ignore transaction costs, or ignore parameter uncertainty. In addition, we find that shrinking the nominal trading rate may considerably improve the investor's out-of-sample performance.

The outline of the paper is as follows. In Section 2, we introduce the setup of the economy, and we characterize the investor's expected loss when the investor uses sample information to construct the trading strategy in Section 3. In Section 4, we introduce the shrinkage portfolios that help to reduce the effects of estimation risk, and we test their out-of-sample performance in Section 5. We conclude in Section 6. Appendix A contains all the proofs and additional comments on the analytical work. Appendix B and C provide all the Tables and Figures in the paper, respectively.

## 2. General framework

We adopt the framework proposed by Garleanu and Pedersen (2012), henceforth the G&P model. In this framework, the investor maximizes her multiperiod mean-variance utility, net of quadratic transaction costs, by choosing the number of shares to hold from each of the  $N$  risky assets. The only difference between our model and the G&P model is that while G&P assume that price changes in excess of the risk-free rate are

predictable, we focus on the case where price changes are independent and identically distributed (iid) as normal with mean  $\mu$  and covariance matrix  $\Sigma$ , which is a common assumption in most of the transaction costs literature; see Constantinides (1979), Davis and Norman (1990), Liu and Loewenstein (2002), and Liu (2004).

The investor's objective is

$$\max_{\{x_i\}} U(\{x_i\}) = \sum_{i=0}^{\infty} (1-\rho)^{i+1} \left( x_i' \mu - \frac{\gamma}{2} x_i' \Sigma x_i \right) - (1-\rho)^i \left( \frac{\lambda}{2} \Delta x_i' \Sigma \Delta x_i \right), \quad (1)$$

where  $x_i \in \mathbb{R}^N$  for  $i \geq 0$  contains the number of shares held from each of the  $N$  risky assets at time  $i$ ,  $\rho$  is the investor's impatience factor, and  $\gamma$  is the risk-aversion parameter. The term  $(\lambda/2)\Delta x_i' \Sigma \Delta x_i$  is the quadratic transaction cost at the  $i$ th period, where  $\lambda$  is the transaction cost parameter, and  $\Delta x_i = x_i - x_{i-1}$  is the vector containing the number of shares traded at the  $i$ th period.

A few comments are in order. First, quadratic transaction costs are appropriate to model market impact costs, which arise when the investor makes large trades that distort market prices. A common assumption in the literature is that market *price* impact is linear on the amount traded (see Kyle (1985)), and thus market impact costs are quadratic.<sup>5</sup> Second, we adopt G&P's assumption that the quadratic transaction costs are proportional to the covariance matrix  $\Sigma$ . G&P provide micro-foundations to justify this type of trading cost.<sup>6</sup>

It is easy to adapt the results in G&P to obtain a closed-form expression for the optimal portfolio policy in our setting.

**PROPOSITION 1 (Adapted from Garleanu and Pedersen (2012)).** *The optimal portfolio at time  $i$  is:*

$$x_i = (1 - \beta) x_{i-1} + \beta x^M, \quad (2)$$

<sup>5</sup> Several authors have shown that the quadratic form matches the market impact costs observed in empirical data; see, for instance, Lillo et al. (2003) and Engle et al. (2012).

<sup>6</sup> In addition, Greenwood (2005) shows from an inventory perspective that price changes are proportional to the covariance of price changes. Engle and Ferstenberg (2007) show that under some assumptions, the cost of executing a portfolio is proportional to the covariance of price changes. Transaction costs proportional to risk can also be understood from the dealer's point of view. Generally, the dealer takes at time  $i$  the opposite position of the investor's trade and "lays it off" at time  $i + 1$ . In this sense, the dealer has to be compensated for the risk of holding the investor's trade.

where  $x^M = \frac{1}{\gamma} \Sigma^{-1} \mu$  is the static mean-variance (Markowitz) portfolio,  $\beta = \frac{\sqrt{(\gamma + \tilde{\lambda}\rho)^2 + 4\gamma\lambda} - (\gamma + \tilde{\lambda}\rho)}{2\lambda}$ ,  $\tilde{\lambda} = (1 - \rho)^{-1}\lambda$ , and  $\beta \leq 1$  is the trading rate. Moreover, the monotonicity properties of the trading rate  $\beta$  are as follows:

1.  $\beta$  is monotonically increasing with  $\gamma$ .
2.  $\beta$  is monotonically decreasing with  $\lambda$ .
3.  $\beta$  is monotonically decreasing with  $\rho$ .

Proposition 1 shows that the optimal portfolio policy is to trade every period at a trading rate  $\beta$  towards the static mean-variance (Markowitz) portfolio. The intuition is that the Markowitz portfolio is optimal in terms of the multiperiod mean-variance utility, but it is prohibitive to trade in a single period to the Markowitz portfolio due to the impact of transaction costs.

### 3. Multiperiod utility loss

In this section, we study the impact of parameter uncertainty by characterizing analytically the investor's expected loss. We consider an investor who uses a *plug-in* approach to estimate the optimal portfolio policy given by Proposition 1. Specifically, let  $r_l$  for  $l = 1, 2, \dots, T$  be the sample of excess price changes with which the investor constructs the following unbiased estimator of the Markowitz portfolio:  $\hat{x}^M = \hat{\Sigma}^{-1} \hat{\mu} / \gamma$ , where

$$\hat{\mu} = \frac{1}{T} \sum_{l=1}^T r_l, \quad \text{and} \quad \hat{\Sigma} = \frac{1}{T - N - 2} \sum_{l=1}^T (r_l - \hat{\mu})^2. \quad (3)$$

Then, the estimated optimal portfolio policy is given by replacing  $x^M$  in (2) with  $\hat{x}^M$ ,

$$\hat{x}_i = (1 - \beta) \hat{x}_{i-1} + \beta \hat{x}^M, \quad (4)$$

which results in an unbiased estimator of the optimal trading strategy.

Like Kan and Zhou (2007) we define the investor's expected utility loss as the difference between the investor's utility evaluated for the true optimal portfolio and the investor's expected utility evaluated for the estimated portfolio. For a single-period mean-variance investor in the absence of transaction costs, Kan and

Zhou (2007) characterize the expected utility loss corresponding to the sample mean-variance portfolio  $\hat{x}^M$ , which is defined as  $\delta_S(x^M, \hat{x}^M) = U_S(x^M) - E[U_S(\hat{x}^M)]$ , where  $U_S(x^M) = x^{M'}\mu - \frac{\gamma}{2}x^{M'}\Sigma x^M$ :

$$\delta_S(x^M, \hat{x}^M) = (c - 1)\frac{\theta}{2\gamma} + \frac{1}{2\gamma}c\frac{N}{T}, \quad (5)$$

where  $c = [(T - N - 2)(T - 2)]/[(T - N - 1)(T - N - 4)]$ .<sup>7</sup> We observe that the expected loss for a static investor decreases with  $\gamma$  and the sample length  $T$ , whereas it increases with  $\theta = \mu'\Sigma^{-1}\mu$  and the number of available assets  $N$ .

The following proposition provides a closed-form expression for the utility loss of a multiperiod mean-variance investor facing quadratic transaction costs that uses the plug-in approach described above.

**PROPOSITION 2.** *A multiperiod mean-variance investor who uses the plug-in approach to estimate the optimal portfolio policy has the following expected utility loss:*

$$\delta(\{x_i\}, \{\hat{x}_i\}) = \delta_S(x^M, \hat{x}^M) \times \underbrace{[AV + AC]}_{\text{Multiperiod term}}, \quad (6)$$

where  $AV$  is the multiperiod mean-variance loss factor, and  $AC$  is the multiperiod transaction cost loss factor:

$$AV = \frac{1 - \rho}{\rho} + \frac{(1 - \rho)(1 - \beta)^2}{1 - (1 - \rho)(1 - \beta)^2} - 2\frac{(1 - \rho)(1 - \beta)}{1 - (1 - \rho)(1 - \beta)}, \quad (7)$$

$$AC = \frac{\lambda}{\gamma} \frac{\beta^2}{1 - (1 - \rho)(1 - \beta)^2}. \quad (8)$$

Proposition 2 shows that the multiperiod utility loss is equal to the single-period utility loss multiplied by the summation of two terms. The first term captures the losses from the multiperiod mean-variance utility, and the second term captures the losses from the multiperiod transaction costs. Note also that the multiperiod loss factors  $AV$  and  $AC$  depend only on  $\lambda$ ,  $\gamma$ , and  $\rho$ .

Figure 1 depicts the absolute multiperiod expected losses for different values of  $\gamma$ ,  $\lambda$ , and  $\rho$ . We consider a base-case investor with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$  and  $\rho = 1 - \exp(-0.1/260)$  and  $T = 500$  observations,

<sup>7</sup> Expression (5) is not the exact expected loss that we find in Kan and Zhou (2007). This has been adapted to our estimator for the covariance matrix, that provides an unbiased estimator of the Markowitz portfolio, whereas the estimate for this element in Kan and Zhou (2007) provides a biased estimator of the Markowitz portfolio.



and we define  $\mu$  and  $\Sigma$  with the sample moments of the empirical dataset of commodity futures used in the empirical application in Section 5. We obtain three main findings from Figure 1. First, the multiperiod expected loss decreases with  $\gamma$ . Like in the static case, this is a natural result because as the investor becomes more risk averse, the investor's exposure to risky assets is lower, and then the impact of parameter uncertainty is also smaller. Second, the multiperiod expected loss decreases with  $\lambda$ . As trading costs increase, the investor delays the convergence to the Markowitz portfolio and in turn, the investor postpones the impact of parameter uncertainty to future stages where the overall importance of utility losses is smaller. This makes that the multiperiod expected loss becomes smaller with trading costs. Third, the multiperiod expected loss decreases with  $\rho$ . Roughly speaking, the investor's impatience factor has a similar effect on the investor's expected utility to that of trading costs. When the investor is more impatient, the cost of making a trade takes a greater importance than the future expected payoff of the corresponding trade.

Although the above example gives some monotonicity properties of the absolute utility loss, for interpretation it may be useful to study how the *relative* utility loss depends on the investor's risk aversion parameter  $\gamma$ , trading costs  $\lambda$ , and the investor's impatience factor  $\rho$ . Figure 2 depicts the investor's relative loss for different values of  $\gamma$ ,  $\lambda$  and  $\rho$ . From this figure, we observe that as the investor's risk aversion parameter increases, the investor's relative loss also increases but slightly. That is, the relative loss is nearly constant (but increasing) with the investor's risk aversion parameter. On the other hand, Figure 2b illustrates that larger trading costs reduce the investor's relative loss. Finally, we observe in Figure 2c that an investor with high impatience factor has a lower relative loss.

After analyzing the expected utility loss of an investor who uses sample information to construct her optimal portfolio, in section 4 we propose several shrinkage portfolios that help to reduce the effects of estimation risk on the performance of multiperiod portfolios.

#### 4. Multiperiod shrinkage portfolios

In this section we propose several shrinkage portfolios that mitigate the impact of estimation error on the multiperiod mean-variance utility of an investor who faces quadratic transaction costs. We consider two

approaches to shrink the plug-in portfolio policy defined in Equation (4): (i) shrink the estimated Markowitz portfolio  $x^M$ , and (ii) shrink the trading rate  $\beta$ .

#### 4.1. Shrinking the Markowitz portfolio

The optimal portfolio at period  $i$ , in the absence of estimation error, allocates the investor's wealth into three funds: the risk-free asset, the portfolio at period  $i - 1$ , and the Markowitz portfolio. However, this solution is not optimal when the investor suffers from parameter uncertainty. For the single period case, Kan and Zhou (2007) show that shrinking the Markowitz portfolio helps to mitigate the impact of parameter uncertainty.

We generalize their analysis to the multiperiod case. In particular, we consider two different approaches to shrink the Markowitz portfolio. First, we consider shrinking the Markowitz portfolio towards the portfolio that invests solely on the risk-free asset; that is, towards  $x = 0$ . We term the resulting shrinkage portfolio as multiperiod three-fund portfolio because the optimal portfolio at period  $i$  allocates the investor's wealth into three different funds: the portfolio at time  $i - 1$ , the Markowitz portfolio, and the risk-free asset. The resulting portfolio can be written as:

$$\widehat{x}_i^{3F} = (1 - \beta)\widehat{x}_{i-1}^{3F} + \beta\eta\widehat{x}^M, \quad (9)$$

where  $\eta$  is the shrinkage intensity.

Second, we consider a multiperiod portfolio that combines the Markowitz portfolio with a target portfolio. This combination may diversify the effects of estimation error in the sample mean-variance portfolio and reduce the risk of taking inefficient positions. We choose as a target portfolio the minimum-variance portfolio  $\widehat{x}^{Min} = (1/\gamma)\Sigma^{-1}\iota$ , which is known to be less sensitive to estimation error than the mean-variance portfolio.<sup>8</sup> We term the resulting shrinkage portfolio as four-fund portfolio:

$$\widehat{x}_i^{4F} = (1 - \beta)\widehat{x}_{i-1}^{4F} + \beta(\varsigma_1\widehat{x}^M + \varsigma_2\widehat{x}^{Min}), \quad (10)$$

where  $\varsigma_1$  and  $\varsigma_2$  are the combination parameters for the Markowitz portfolio and the minimum-variance portfolio, respectively.

<sup>8</sup> Notice that the minimum-variance portfolio does not consider  $\gamma$ . However, for expository reasons, we multiply the unscaled minimum-variance portfolio with  $(1/\gamma)$  to simplify the analysis.

Note that while Kan and Zhou (2007) consider a static mean-variance investor that is not subject to transaction costs, we consider a multiperiod mean-variance investor subject to quadratic transaction costs. Given this, one would expect that the optimal shrinkage intensities for our proposed multiperiod shrinkage portfolios would differ from those obtained by Kan and Zhou (2007) for the single-period case, but the following proposition shows that the optimal shrinkage intensities for the single-period and multiperiod cases coincide.

**PROPOSITION 3.** *The optimal shrinkage intensities for the three-fund and four-fund portfolios that minimize the utility loss of a multiperiod mean-variance investor  $\delta(\{x_i\}, \{\hat{x}_i\})$  coincide with the optimal shrinkage intensities for the single-period investor who ignores transaction costs. Specifically, the optimal shrinkage intensity for the three-fund portfolio  $\eta$  and the optimal combination parameters for the four-fund portfolio  $\varsigma_1$  and  $\varsigma_2$  are:*

$$\eta = c^{-1}, \quad (11)$$

$$\varsigma_1 = c^{-1} \frac{\Psi^2}{\Psi^2 + \frac{N}{T}}, \quad (12)$$

$$\varsigma_2 = c^{-1} \frac{\frac{N}{T}}{\Psi^2 + \frac{N}{T}} \times \frac{\mu' \Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota}, \quad (13)$$

where  $c = [(T-2)(T-N-2)]/[(T-N-1)(T-N-4)]$  and  $\Psi^2 = \mu' \Sigma^{-1} \iota - (\mu' \Sigma^{-1} \iota)^2 / (\iota' \Sigma^{-1} \iota) > 0$ .

Note that the optimal shrinkage intensities for the multiperiod three-fund and four-fund portfolios do not depend on transaction costs, given by parameter  $\lambda$ , and as a result they coincide with the optimal shrinkage intensities for the single-period case in the absence of transaction costs.

The following corollary shows that the optimal multiperiod portfolio policy that ignores estimation error is inadmissible in the sense that it is always optimal to shrink the Markowitz portfolio. Moreover, the three-fund shrinkage portfolio is also inadmissible in the sense that it is always optimal to shrink the Markowitz portfolio towards the target minimum-variance portfolio. The result demonstrates that the shrinkage approach is bound to improve performance under our main assumptions.

**COROLLARY 1.** *It is always optimal to shrink the Markowitz portfolio; that is,  $\eta < 1$ . Moreover, it is always optimal to combine the Markowitz portfolio with the target minimum-variance portfolio; that is,  $\varsigma_2 > 0$ .*

As expected from Corollary 1, the *relative improvement* in the investor's expected utility when using the proposed shrinkage portfolios in (9) and (10) is larger than that when using the plug-in portfolio in (4). In particular, Figure 3 shows that for our base-case investor, the *relative loss* when using the shrinkage three-fund portfolio in (9) is about *eight times smaller* than that when using the plug-in three fund portfolio in (4). And the *relative loss* when using the shrinkage four-fund portfolio in (10) is about *11% less* than that when using the three-fund portfolio in (9).

## 4.2. Shrinking the trading rate

In this section we study the additional utility gain associated with shrinking the trading rate in addition to the target portfolio. For the proposed shrinkage portfolios in (9) and (10), note that the *nominal* trading rate  $\beta$  as given in Proposition 1 may not be optimal in the presence of parameter uncertainty. To mitigate even more this effect, we propose to optimize the trading rate in order to minimize the investor's utility loss from estimation risk. In particular, a multiperiod mean-variance investor who uses the shrinkage four-fund portfolio in (10) may reduce the impact of parameter uncertainty by minimizing the corresponding expected utility loss,  $\delta(\{x_i\}, \{\hat{x}_i^{4F}(\beta)\})$ , respect to the trading rate  $\beta$ .

The following proposition formulates an equivalent optimization problem to obtain the optimal trading rate for the shrinkage four-fund portfolio in (10). Notice that we can apply the same proposition to the shrinkage three-fund portfolio in (9) simply by considering  $\varsigma_2 = 0$  and  $\varsigma_1 = \eta$ .

PROPOSITION 4. *For the shrinkage four-fund portfolio in (10), the optimal trading rate  $\beta$  that minimizes the expected utility loss  $\delta(\{x_i\}, \{\hat{x}_i^{4F}(\beta)\})$  can be obtained by solving the following optimization problem:*

$$\max_{\beta} \underbrace{V_1(x_{-1} - x^C)' \mu}_{\text{Excess return}} - \frac{1}{2} \underbrace{\left( E \left[ \hat{x}^{C'} \Sigma \hat{x}^C \right] V_2 + x'_{-1} \Sigma x_{-1} V_3 + x'_{-1} \Sigma x^C V_4 \right)}_{\text{Variability + Trading costs}}, \quad (14)$$

where  $x_{-1}$  is the investor's initial position,  $x^C = \varsigma_1 x^M + \varsigma_2 x^{Min}$ ,

$$E \left[ \hat{x}^{C'} \Sigma \hat{x}^C \right] = (c/\gamma^2) \left( \varsigma_1^2 (\mu' \Sigma^{-1} \mu + (N/T)) + \varsigma_2^2 \iota' \Sigma^{-1} \iota \right) + (c/\gamma^2) \left( 2\varsigma_1 \varsigma_2 \mu' \Sigma^{-1} \iota \right), \quad (15)$$

and the  $V_{i=2,3,4}$  account for the accumulated variability and trading costs:

$$V_1 = \frac{(1 - \rho)(1 - \beta)}{1 - (1 - \rho)(1 - \beta)} \quad (16)$$

$$V_2 = \gamma \left( \frac{1-\rho}{\rho} + \frac{(1-\rho)(1-\beta)^2}{1-(1-\rho)(1-\beta)^2} - 2 \frac{(1-\rho)(1-\beta)}{1-(1-\rho)(1-\beta)} \right) + \tilde{\lambda} \frac{(1-\rho)\beta^2}{1-(1-\rho)(1-\beta)^2}, \quad (17)$$

$$V_3 = \gamma \frac{(1-\rho)(1-\beta)^2}{1-(1-\rho)(1-\beta)^2} + \tilde{\lambda} \frac{(1-\rho)\beta^2}{1-(1-\rho)(1-\beta)^2}, \quad (18)$$

$$V_4 = 2\gamma \left( \frac{(1-\rho)(1-\beta)}{1-(1-\rho)(1-\beta)} - \frac{(1-\rho)(1-\beta)^2}{1-(1-\rho)(1-\beta)^2} \right) - 2\tilde{\lambda} \frac{(1-\rho)\beta^2}{1-(1-\rho)(1-\beta)^2}. \quad (19)$$

From Proposition 4 we observe that as  $\beta$  goes to zero,  $V_2$  and  $V_4$  also approximate to zero. This implies that  $V_3$  is the only element that defines the expected variability and trading costs of the multiperiod investor. Precisely, the investor's expected variability and trading costs are defined by  $((1-\rho)/\rho)x'_{-1}\Sigma x_{-1}$ , which is the accumulated variability of the investor's initial portfolio. Notice that when  $\beta$  is zero, trading costs do not affect the investor's expected utility.

In addition, we can observe that as the investor's initial position  $x_{-1}$  approximates to the static portfolio  $x^C$ , the expected return of the investor's initial portfolio in excess of the expected return of the static portfolio  $x^C$ , approximates to zero. Consequently, the optimal trading rate that we obtain from (14) must minimize the expected portfolio variability and trading costs.

To analyze the benefits of optimizing the trading rate, we study the relative loss for the multiperiod four-fund portfolio optimizing the trading rate as in (14), and the corresponding relative loss of the multiperiod four-fund portfolio with the nominal trading rate  $\beta$  as in (1). Figure 4 depicts the relative loss for our base-case investor with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$ ,  $\rho = 1 - \exp(-0.1/260)$ , and  $T = 500$ . As in the previous section, we define  $\mu$  and  $\Sigma$  with the sample moments of the empirical dataset of commodity futures that we use in Section 5.

Figures 4a, 4b and 4c depict the relative loss for an investor whose initial portfolio is  $x_{-1} = 0.1 \times x^M$  and we observe that the benefits from using the multiperiod four-fund portfolio that shrinks the trading rate are large. Moreover, we observe that the relative loss of the different multiperiod portfolios remain almost invariant to changes in  $\gamma$ ,  $\lambda$  and  $\rho$ . In addition, from Figure 4d we find that when the investor's initial portfolio is close to the static mean-variance portfolio, shrinking the trading rate  $\beta$  provides substantial

benefits. In particular, when  $x_{-1} \simeq 0.5 \times x^M$ , one can reduce the relative loss to almost zero by shrinking the trading rate. In turn, shrinking the nominal trading rate may result into a considerable reduction of the investor's expected loss, specially in those situations where the investor's initial portfolio is close to the static mean-variance portfolio. However, it does not increase the investor's expected loss when the investor's initial portfolio is not close to the true static mean-variance portfolio.

## 5. Out-of-sample performance evaluation

In this section, we compare the out-of-sample performance of the multiperiod shrinkage portfolios with that of the portfolios that ignore either transaction costs, parameter uncertainty, or both. We run the analysis with both simulated and empirical datasets.

### 5.1. Portfolio policies

We consider seven different portfolio policies. We first consider three buy-and-hold portfolios based on single-period policies that ignore transaction costs. First, the sample Markowitz portfolio, which is the portfolio of an investor who ignores transaction costs and estimation error (S-M). Second, the single period two-fund shrinkage portfolio, which is the portfolio of an investor who ignores transaction costs, but takes into account estimation error by shrinking the Markowitz portfolio (S-2F). Specifically, this portfolio can be written as

$$x^{S2F} = \eta \hat{x}^M, \quad (20)$$

where, as Kan and Zhou (2007) show, the optimal single-period shrinkage intensity  $\eta$  is as given by Proposition (3). The third portfolio is the single-period three-fund shrinkage portfolio of an investor who ignores transaction costs but takes into account estimation error by shrinking the Markowitz portfolio towards the minimum variance portfolio (S-3F-Min). Specifically, this portfolio can be written as

$$x^{S3F} = \varsigma_1 \hat{x}^M + \varsigma_2 \hat{x}^{Min}, \quad (21)$$

where the optimal single-period combination parameters are given in Proposition (3).

We then consider four multiperiod portfolios that take transaction costs into account. The first portfolio is the optimal portfolio policy of a multiperiod investor who takes into account transaction costs but

ignores estimation error (M-M), which is given by Proposition 1. The second portfolio is the multiperiod three-fund shrinkage portfolio of an investor who shrinks the Markowitz portfolio (M3F), as given by Proposition (3). The third portfolio is the multiperiod four-fund shrinkage portfolio of an investor who combines the Markowitz portfolio with the minimum-variance portfolio (M4F-Min), as given by Proposition (3). The fourth portfolio is a modified version of the multiperiod four-fund shrinkage portfolio, where in addition the investor shrinks the trading rate by solving the optimization problem given by Proposition 4 (O-M4F-Min).

## 5.2. Evaluation methodology

We evaluate the out-of-sample portfolio gains for each strategy using a *rolling-window* approach similar to DeMiguel et al. (2009). We compute the portfolio Sharpe ratio of all the considered trading strategies with the time series of the out-of-sample portfolio gains:

$$SR^i = \frac{\bar{r}^h}{\sigma^h}, \quad (22)$$

$$\text{where } (\sigma^h)^2 = \frac{1}{L-T-1} \sum_{l=T}^{L-1} \left( x_l^{h'} r_{l+1} - \bar{r}^h \right)^2, \quad (23)$$

$$\bar{r}^h = \frac{1}{L-T} \sum_{l=T}^{L-1} \left( x_l^{h'} r_{l+1} \right), \quad (24)$$

where  $x_l^h$  is the vector of asset holdings at time  $l$  for portfolio strategy  $h$ ,  $r_l$  is the vector of price changes at time  $l$ ,  $L$  is the total number of observations in the dataset, and  $T$  is the estimation window. To account for transaction costs in the empirical analysis, the definition of portfolio return is corrected by the implied cost of trading:

$$\underline{r}_{l+1}^h = x_l^{h'} r_{l+1} - \tilde{\lambda} \Delta x_l^{h'} \Sigma \Delta x_l^h, \quad (25)$$

where  $x_l^h$  denotes the estimated portfolio  $h$  at period  $l$ , and  $\Sigma$  is the covariance matrix of asset prices.<sup>9</sup> In the empirical analysis, expressions (22)-(24) are computed using portfolio returns discounted by transaction costs. We estimate the different portfolios using an estimation window of  $T=500$  observations.<sup>10</sup>

<sup>9</sup> For the simulated data, we use the population covariance matrix, whereas for the empirical dataset with commodity futures we construct  $\Sigma$  with the sample estimate of the entire dataset.

<sup>10</sup> To compute those portfolios that account for parameter uncertainty, we need to estimate the optimal combination parameters, which require the true population moments. To mitigate the impact of parameter uncertainty in these parameters, we use the shrinkage vector of means proposed in DeMiguel et al. (2013), and the shrinkage covariance matrix by Ledoit and Wolf (2004).

We measure the statistical significance of the difference between the adjusted Sharpe ratios with the stationary bootstrap of Politis and Romano (1994) with  $B=1000$  bootstrap samples and block size  $b=5$ .<sup>11</sup> Finally, we use the methodology suggested in (Ledoit and Wolf 2008, Remark 2.1) to compute the resulting bootstrap p-values for the difference of every portfolio strategy with respect to the four-fund portfolio M4F-Min.

We consider an investor with a risk aversion parameter of  $\gamma = 10^{-8}$ , which corresponds with a relative risk aversion of one for a manager who has \$100M to trade. Garleanu and Pedersen (2012) consider an investor with a lower risk aversion parameter, but because our investor suffers from parameter uncertainty, it is reasonable to establish a higher risk aversion parameter. We use a discount factor  $\rho$  equal to  $1 - \exp(-0.1/260)$ , which corresponds with an annual discount of 10%. Finally, we consider transaction costs with  $\lambda = 3 \times 10^{-7}$  as in Garleanu and Pedersen (2012). We subsequently test the robustness of our results to the values of these three parameters and observe that our main insights are robust.

Finally, we report the results for two different starting portfolios: the portfolio that is fully invested on the risk-free asset and the true Markowitz portfolio.<sup>12</sup> We have tried other starting portfolios such as the equally weighted portfolio and the portfolio that is invested in a single risky asset, but we observe that the results are similar and thus we do not report these cases to conserve space.

### 5.3. Simulated and empirical datasets

We first use simulation to generate two datasets with number of risky assets  $N = 25$  and 50. The advantage of using simulated datasets is that they satisfy the assumptions underlying our analysis. Specifically, we simulate price changes from a multivariate normal distribution. We assume that the starting prices of all  $N$  risky assets are equal to one, and the annual average price changes are randomly distributed from a uniform distribution with support  $[0.05, 0.12]$ . In addition, the covariance matrix of asset price changes is diagonal with elements randomly drawn from a uniform distribution with support  $[0.1, 0.5]$ .<sup>13</sup> Without loss

<sup>11</sup> We also compute the p-values when  $b=1$ , but we do not report these results to preserve space. These results are, however, equivalent to the block size  $b=5$ .

<sup>12</sup> For the commodity dataset, we assume the true Markowitz portfolio is constructed with the entire sample.

<sup>13</sup> Notice that for our purpose of evaluating the impact of parameter uncertainty in an out-of-sample analysis, assuming that the covariance matrix is diagonal is not a strong assumption as we know that the investor's expected loss is proportional to  $\theta = \mu' \Sigma^{-1} \mu$ .



of generality, we set the return of the risk-free asset equal to zero. Under these specifications, a level of transaction costs of  $\lambda = 3 \times 10^{-7}$  corresponds with a market that, on average, has a daily volume of \$4.66 million.<sup>14</sup>

To understand the impact of data departing from the iid normal assumption, we consider an empirical dataset similar to that used by Garleanu and Pedersen (2012). Concretely, we construct a dataset with commodity futures of Aluminum, Copper, Nickel, Zinc, Lead, and Tin from the London Metal Exchange (LME), Gas Oil from the Intercontinental Exchange (ICE), WTI Crude, RBOB Unleaded Gasoline, and Natural Gas from the New York Mercantile Exchange (NYMEX), Gold and Silver from the New York Commodities Exchange (COMEX), and Coffee, Cocoa, and Sugar from the New York Board of Trade (NYBOT). We consider daily data from July 7th, 2004 until September 19th, 2012. We collect data from those commodity futures with 3-months maturity, and for those commodity futures where we do not find data with that contract specification (i.e. 3 months maturity), we collect the data of the commodity future with the largest time series. Some descriptive statistics and the contract multiplier for each commodity is provided in Table 1.<sup>15</sup>

#### 5.4. Discussion of the out-of-sample performance

Table 2 reports the out-of-sample Sharpe ratios of the seven portfolio policies we consider on the three different datasets, together with the p-value of the difference between the Sharpe ratio of every policy and that of the multiperiod four-fund shrinkage portfolio. Panels A and B give the results for a starting portfolio that is fully invested in the risk-free asset and a starting portfolio equal to the true Markowitz portfolio, respectively.

Comparing the multiperiod portfolios that take transaction costs into account with the static portfolios that ignore transaction costs, we find that the multiperiod portfolios substantially outperform the static

<sup>14</sup> To compute the trading volume of a set of assets worth 1\$, we use the rule from Engle et al. (2012), where they assume that trading 1.59% of the daily volume implies a price change of 0.1%. Hence, for our first case we calculate the trading volume as  $1.59\% \times \text{Trading Volume} \times 3 \times 10^{-7} \times 0.3^2 \times 0.5 = 0.1\%$ .

<sup>15</sup> The contract multiplier specifies the number of units that are traded for each commodity in each contract. Also, notice that we do not report the trading volume. Unfortunately, we have not been able to obtain that type of data. However, we use the same level of transaction costs, which may be slightly high for the standard deviations of price changes that we have.

portfolios. That is, we find that taking transaction costs into account has a substantial positive impact on performance.

Comparing the shrinkage portfolios with the portfolios that ignore transaction costs, we observe that shrinking helps both for the static and multiperiod portfolios. Specifically, we find that the portfolios that shrink only the Markowitz portfolio (S2F for the static case and M3F for the multiperiod case) outperform the equivalent portfolios that ignore estimation error (S-M for the static case and M-M for the multiperiod case). Moreover, we find that shrinking the Markowitz portfolio towards the minimum-variance portfolio improves performance substantially. Specifically, we observe that the S3F-Min and M4F-Min considerably outperform the shrinkage portfolios that shrink only the Markowitz portfolios (S2F and M3F).

Finally, our out-of-sample results confirm the insight from Section 4.2 that shrinking the trading rate may help when the starting portfolio is close to the true mean-variance portfolio. Specifically, we see from Panel A that shrinking the trading rate (in addition to shrinking the Markowitz portfolio towards the minimum-variance portfolio) does not result in any gains when the starting portfolio is fully invested in the risk-free asset, but Panel B shows that it may lead to substantial gains when the starting portfolio is the true mean-variance portfolio.

Overall, the best portfolio policy is the O-M4F-Min portfolio that shrinks the Markowitz portfolio towards the minimum-variance portfolio and, in addition, shrinks the trading rate while taking transaction costs into account. This portfolio policy outperforms the M4F-Min portfolio when the starting portfolio is close to the true minimum-variance portfolio, and it performs similar to the M4F-Min for other starting points. These two policies O-M4F-Min and M4F-Min appreciably outperform all other policies, which shows the importance of taking into account both transaction costs and estimation error.

We carry out an additional analysis to test the robustness of our results for different values of the risk-aversion parameter  $\gamma$ , and number of observations  $T$ . However, we do not report robustness checks for trading costs because only modifying parameter  $\gamma$  can provide equivalent results to those when we fix  $\gamma$

and modify  $\lambda$ .<sup>16</sup> We report these results in Table 3. We consider a base-case investor with an initial portfolio equal to the true Markowitz portfolio,  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$ , and  $T = 500$ .

In general, we observe that our main insights are robust to these parameters. There are substantial losses associated with ignoring both transaction costs and estimation error, and overall the best portfolio policies are M4F-Min and O-M4F-Min. We observe that for the simulated datasets shrinking the trading rate generally helps (that is, O-M4F-Min outperforms M4F-Min), although the difference between the Sharpe ratios of these two policies are not significant.

We also observe that the static portfolio policies are very sensitive to the risk-aversion parameter, and their performance is particularly poor for the case with low risk aversion  $\gamma$ . This is because investors with low risk aversion invest more on the risky assets and thus are more vulnerable to the impact of estimation error, which is particularly large for the static investors who ignore transaction costs. The multiperiod portfolio policies are more stable because taking transaction costs into account helps to combat estimation error, even for the case with low risk aversion. In particular, the difference of performance between static portfolios and multiperiod portfolios is large when the investor's risk aversion parameter is equal to  $\gamma = 10^{-9}$ .

Finally, we observe that the performance of the static portfolio strategies is also very sensitive to the choice of estimation window  $T$ . Specifically, static portfolios perform poorly when the estimation window is small and has  $T = 250$  observations. For this estimation window, the difference between static mean-variance portfolios and multiperiod portfolios is large.

Summarizing, the out-of-sample losses associated with ignoring either transaction costs or parameter uncertainty are large. Moreover, overall the multiperiod four-fund shrinkage portfolio that combines the Markowitz portfolio with the minimum-variance portfolio achieves the best out-of-sample Sharpe ratio net of transaction costs. We also observe that shrinking the trading rate may provide considerable benefits, specially when the investor's initial portfolio is near the Markowitz portfolio.

<sup>16</sup> In particular, if we transform  $\gamma$  and  $\lambda$  by multiplying them with  $10^{-z}$  and  $10^z$ , respectively, we obtain the same multiperiod trading rate  $\beta$ , and in turn results are equivalent to those before the transformation. Then, if we want to study the impact of an increment/reduction on trading costs, we can simply reduce/increase  $\gamma$  by the same factor.

## 6. Concluding remarks

Our work is among the first to consider the impact of parameter uncertainty on multiperiod portfolio selection with transaction costs. We first provide a closed-form expression for the utility loss associated with using the plug-in approach to construct multiperiod portfolios. We observe from this closed-form expression that the investor's expected loss decreases with trading costs, the investor's impatience factor and the investor's risk aversion parameter.

Second, we propose a four-fund multiperiod shrinkage portfolio that mitigates the effects of estimation risk. We give closed-form expressions for the optimal shrinkage intensities, and we show that these intensities coincide with the shrinkage intensities for the corresponding single-period portfolio. In addition, we analytically characterize under which circumstances the four-fund shrinkage portfolio reduces the impact of parameter uncertainty, and we show that it is prohibitive to use the plug-in multiperiod portfolio and the four-fund portfolio that only shrinks the Markowitz portfolio.

Third, we propose a novel technique that reduces the investor's trading rate to the static mean-variance portfolio, and we show that this methodology can substantially improve the investor's performance. In particular, we show that this methodology improves the investor's performance when the investor's initial position is close to the Markowitz portfolio.

Finally, our out-of-sample analysis with simulated and empirical datasets shows that the losses associated with ignoring transaction costs, parameter uncertainty, or both, are large, and that the four-fund shrinkage portfolio achieves good out-of-sample performance. In addition, we observe that shrinking the trading rate helps to mitigate the impact of parameter uncertainty and helps to attain high risk-adjusted expected returns.

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## Appendix A: Proofs and additional analytical comments

### A.1. Proof Proposition 1

To solve the investor's problem, we first guess that the value function at any time  $i$ :

$$V(x_i) = -\frac{1}{2}x_i'Ax_i + x_i'B\mu + c. \quad (26)$$

Therefore, the Bellman equation becomes:

$$x_i'\mu - \frac{\gamma}{2}x_i'\Sigma x_i - \frac{\tilde{\lambda}}{2}\Delta x_i'\Sigma\Delta x_i + (1-\rho)\left(-\frac{1}{2}x_i'Ax_i + x_i'B\mu + c\right), \quad (27)$$

where  $\tilde{\lambda} = (1-\rho)^{-1}\lambda$ . The right hand side can be simplified as follows:

$$V(x_i) = -\frac{1}{2}x_i'Jx_i + x_i'h + l, \quad (28)$$

where  $J = (\gamma + \tilde{\lambda})\Sigma + (1-\rho)A$ ,  $h = \mu + \tilde{\lambda}\Sigma x_{i-1} + (1-\rho)B\mu$ , and  $l = -\frac{\tilde{\lambda}}{2}x_{i-1}'\Sigma x_{i-1} + (1-\rho)c$ . The first-order necessary condition to solve the above problem give the optimal solution:

$$x_i = J^{-1}h. \quad (29)$$

Now, plugging the solution into the value function in (28), we obtain:

$$V^*(x_i) = \frac{1}{2}h'J^{-1}h + d. \quad (30)$$

From the above expression and using (26), we obtain that  $A = -\tilde{\lambda}^2\Sigma J^{-1}\Sigma + \tilde{\lambda}\Sigma$  and  $B = \tilde{\lambda}\Sigma J^{-1}(I + (1-\rho)B)$ . Thus,  $A = \alpha\Sigma$ , which implies that

$$\alpha = -\frac{\tilde{\lambda}^2}{\gamma + \tilde{\lambda} + (1-\rho)\alpha} + \tilde{\lambda}. \quad (31)$$

Solving the above equation, we have that  $\alpha = \frac{\sqrt{(\gamma + \tilde{\lambda}\rho)^2 + 4\gamma\lambda} - (\gamma + \tilde{\lambda}\rho)}{2(1-\rho)}$ . On the other hand, the solution for  $B$  is straightforward. It takes the form

$$B = \frac{\tilde{\lambda}}{\gamma + \rho\tilde{\lambda} + (1-\rho)\alpha}I. \quad (32)$$

Thus, the optimal solution,  $x_i = J^{-1}h$ , can be expressed as follows:

$$x_i = \frac{\tilde{\lambda}}{\gamma + \tilde{\lambda} + (1-\rho)\alpha}x_{i-1} + \frac{\gamma + (1-\rho)\gamma B}{\gamma + \tilde{\lambda} + (1-\rho)\alpha} \frac{1}{\gamma}\Sigma\mu. \quad (33)$$

The above expression can be simplified as follows (see Garleanu and Pedersen (2012)):

$$x_i = (1-\beta)x_{i-1} + \beta\frac{1}{\gamma}\Sigma\mu. \quad (34)$$

where  $\beta = \alpha/\tilde{\lambda}$ .

To prove the monotonicity of the convergence rate  $\beta$ , we only need to analyze the derivative of  $\beta$  with respect to  $\gamma$ ,  $\lambda$  and  $\rho$ .

First, we show that the convergence rate  $\beta$  is a monotonic and nondecreasing function with respect to  $\gamma$ . Thus, we show that the derivative of  $\beta$  with respect to  $\gamma$  is always positive for any  $\gamma, \lambda, \rho \geq 0$ . To do that, it suffices to show that

$$2 \times (1 - \rho) \times \frac{\partial \alpha}{\partial \gamma} \geq 0.$$

Then,

$$2 \times (1 - \rho) \times \frac{\partial \alpha}{\partial \gamma} = \frac{1}{2} \frac{2(\gamma + \tilde{\lambda}\rho) + 4\lambda}{\sqrt{(\gamma + \tilde{\lambda}\rho)^2 + 4\gamma\lambda}} - 1 > 0 \Rightarrow (\gamma + \tilde{\lambda}\rho) + 2\lambda \geq \sqrt{(\gamma + \tilde{\lambda}\rho)^2 + 4\gamma\lambda}. \quad (35)$$

Now, we take the square of the above inequality, which is a monotone transformation and does not affect the results. Then:

$$(\gamma + \tilde{\lambda}\rho)^2 + 4\lambda^2 + 4\lambda(\gamma + \tilde{\lambda}\rho) \geq (\gamma + \tilde{\lambda}\rho)^2 + 4\gamma\lambda \Rightarrow \quad (36)$$

$$\Rightarrow 4\lambda^2 + 4\gamma\lambda + 4\lambda\tilde{\lambda}\rho \geq 4\gamma\lambda. \quad (37)$$

Inequality (37) is always true for any  $\gamma, \lambda, \rho \geq 0$ .

To prove that the rate of convergence  $\beta$  is a monotonic decreasing function with respect to  $\lambda$ , we show that the derivative of  $\beta$  with respect to  $\lambda$  is negative. First, let us define  $\phi = \rho/(1 - \rho)$ . Thus,

$$2 \times \frac{\left( \frac{(\gamma + \lambda\phi)\phi + 2\gamma}{\sqrt{(\gamma + \lambda\phi)^2 + 4\gamma\lambda}} - \phi \right) \lambda - \left( \sqrt{(\gamma + \lambda\phi)^2 + 4\gamma\lambda} - (\gamma + \lambda\phi) \right)}{4\lambda^2} < 0. \quad (38)$$

To prove that the above inequality holds, it suffices to prove that the numerator is negative. Thus,

$$\left( \frac{1}{2} \frac{2(\gamma + \lambda\phi)\phi + 4\gamma}{\sqrt{(\gamma + \lambda\phi)^2 + 4\gamma\lambda}} - \phi \right) \lambda - \left( \sqrt{(\gamma + \lambda\phi)^2 + 4\gamma\lambda} - (\gamma + \lambda\phi) \right) < 0. \quad (39)$$

After some straightforward manipulations, we have that

$$((\gamma + \lambda\phi)\phi + 2\gamma)\lambda < (\gamma + \lambda\phi)^2 + 4\gamma\lambda - \sqrt{(\gamma + \lambda\phi)^2 + 4\gamma\lambda} \times \gamma. \quad (40)$$

The above inequality can be expressed as:

$$\gamma\phi\lambda + \lambda^2\phi^2 + 2\gamma\lambda < \gamma^2 + \lambda^2\phi^2 + 2\gamma\phi\lambda + 4\gamma\lambda - \sqrt{(\gamma + \gamma\phi)^2 + 4\gamma\lambda} \times \gamma, \quad (41)$$

which may be simplified as

$$0 < \gamma^2 + \gamma\phi\lambda + 2\gamma\lambda - \sqrt{(\gamma + \gamma\phi)^2 + 4\gamma\lambda} \times \gamma. \quad (42)$$

Dividing by  $\gamma$ , and taking the square, we have:

$$(\gamma + \lambda\phi)^2 + 4\gamma\lambda < (\gamma + \lambda\phi)^2 + 4\lambda^2 + 4(\gamma + \lambda\phi)\lambda \Rightarrow \quad (43)$$

$$\Rightarrow 0 < 4\lambda^2 + 4\lambda^2\phi, \quad (44)$$

which shows that for any  $\gamma, \lambda, \rho > 0$ , the rate of convergence  $\beta$  is a monotonic decreasing function with respect to  $\lambda$ .

Finally, to prove that the rate of convergence  $\beta$  is a monotonic decreasing function with respect to  $\rho$ , we show that:

$$2 \times \lambda \times \frac{\partial \beta}{\partial \rho} = \frac{1}{2} \frac{2 \left( \gamma + \lambda \frac{\rho}{1-\rho} \right) \frac{\lambda}{(1-\rho)^2}}{\sqrt{\left( \gamma + \lambda \frac{\rho}{1-\rho} \right)^2 + 4\gamma\lambda}} - \frac{\lambda}{(1-\rho)^2} < 0. \quad (45)$$

After some straightforward manipulations, we have that

$$\left( \gamma + \lambda \frac{\rho}{1-\rho} \right) < \sqrt{\left( \gamma + \lambda \frac{\rho}{1-\rho} \right)^2 + 4\gamma\lambda}. \quad (46)$$

Now, taking the square of the above inequality, we have:

$$\left( \gamma + \lambda \frac{\rho}{1-\rho} \right)^2 < \left( \gamma + \lambda \frac{\rho}{1-\rho} \right)^2 + 4\gamma\lambda, \quad (47)$$

which holds for any  $\gamma, \lambda, \rho > 0$ , and thus it completes the proof that ensures that the rate of convergence  $\beta$  is a monotonic decreasing function with respect to  $\rho$ .

## A.2. Proof of Proposition 2

To prove Proposition 2, we first write the investor's expected loss:

$$\delta(\{x_i\}, \{\hat{x}_i\}) = \sum_{i=0}^{\infty} (1-\rho)^{i+1} \left\{ x'_i \mu - \frac{\gamma}{2} x'_i \Sigma x_i - \frac{\tilde{\lambda}}{2} \Delta x'_i \Sigma \Delta x_i \right. \\ \left. - E \left[ \hat{x}'_i \mu - \frac{\gamma}{2} \hat{x}'_i \Sigma \hat{x}_i - \frac{\tilde{\lambda}}{2} \Delta \hat{x}'_i \Sigma \Delta \hat{x}_i \right] \right\}. \quad (48)$$

And from the above expression, it is easy to see that the investor's expected loss is:

$$\delta(\{x_i\}, \{\hat{x}_i\}) = \sum_{i=0}^{\infty} (1-\rho)^{i+1} \left\{ E \left[ \frac{\gamma}{2} \hat{x}'_i \Sigma \hat{x}_i + \frac{\tilde{\lambda}}{2} \Delta \hat{x}'_i \Sigma \Delta \hat{x}_i \right] - \frac{\gamma}{2} x'_i \Sigma x_i - \frac{\tilde{\lambda}}{2} \Delta x'_i \Sigma \Delta x_i \right\}. \quad (49)$$

Now, we can plug the estimated investor's optimal strategy in (49) to obtain a simplified expression of the investor's expected loss. Moreover, all those elements that are linear functions with respect to the sample Markowitz portfolio disappear due to the unbiasedness of the estimator. Then, we use the following expression for the estimated multiperiod portfolio:

$$\hat{x}_i = (1-\beta)^{i+1} x_{-1} + \beta \xi_i \hat{x}^M \quad \text{and} \quad \Delta \hat{x}_i = \phi_i x_{-1} + \beta (1-\beta)^i \hat{x}^M, \quad (50)$$

where  $\xi_i = \sum_{j=0}^i (1-\beta)^j$  and  $\phi = ((1-\beta)^{i+1} - (1-\beta)^i)$ . Then, after some straightforward manipulations, we obtain that the investor's expected loss is:

$$\delta(\{x_i\}, \{\hat{x}_i\}) = \frac{1}{2\gamma} \left( E \left[ \hat{\mu}' \hat{\Sigma}^{-1} \Sigma \hat{\Sigma}^{-1} \hat{\mu} \right] - \theta \right) \times \sum_{i=0}^{\infty} (1-\rho)^{i+1} [AV_i + AC_i], \quad (51)$$

where  $\theta = \mu' \Sigma^{-1} \mu$ ,  $AV_i = \beta^2 \xi_i^2$  stands for the accumulated portfolio variability and  $AC_i = \beta^2 (\tilde{\lambda}/\gamma) (1-\beta)^{2i}$  stands for the accumulated trading costs. Then, we can substitute  $(1/2\gamma) (E [\hat{\mu}' \hat{\Sigma}^{-1} \Sigma \hat{\Sigma}^{-1} \hat{\mu}] - \theta)$  with  $\delta(x^M, \hat{x}^M)$ , and make the following simplifications for geometric series:

$$\xi_i = \sum_{j=0}^i (1-\beta)^j = \frac{1 - (1-\beta)^{i+1}}{\beta}. \quad (52)$$

In turn, we obtain that

$$\beta^2 \sum_{i=0}^{\infty} (1-\rho)^{i+1} \xi_i^2 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} + \sum_{i=0}^{\infty} (1-\rho)^{i+1} [(1-\beta)^{2i+2} - 2(1-\beta)^{i+1}]. \quad (53)$$

Because  $(1-\rho)$  and  $(1-\beta)$  are positive elements and smaller than one, we can express the above geometric series as follows:

$$AV = \beta^2 \sum_{i=0}^{\infty} (1-\rho)^{i+1} \xi_i^2 = \frac{1-\rho}{\rho} + \frac{(1-\rho)(1-\beta)^2}{1-(1-\rho)(1-\beta)^2} - 2 \frac{(1-\rho)(1-\beta)}{1-(1-\rho)(1-\beta)}. \quad (54)$$

Now, applying the same arguments, we can simplify the following expression:

$$AC = \sum_{i=0}^{\infty} (1-\rho)^{i+1} \beta^2 \frac{\tilde{\lambda}}{\gamma} (1-\beta)^{2i} = \frac{\lambda}{\gamma} \frac{\beta^2}{1-(1-\rho)(1-\beta)^2}. \quad (55)$$

In turn, we obtain that the investor's expected loss is

$$\delta(\{x_i\}, \{\hat{x}_i\}) = \delta(x^M, \hat{x}^M) \times [AV + AC]. \quad (56)$$

### A.3. Proof of Proposition 3

We now prove that the optimal combination parameter of multiperiod portfolios coincide with the optimal combination parameter in the static framework. First, let us define the investor's initial portfolio as  $x_{-1}$ . Then, we can write the investor's four-fund portfolio as:

$$\hat{x}_i = (1-\beta)^{i+1} x_{-1} + \beta \xi_i \hat{x}^C, \quad (57)$$

where  $\hat{x}^C = (\varsigma_1 \hat{x}^M + \varsigma_2 \hat{x}^{Min})$ , and

$$\Delta \hat{x}_i = \phi_i x_{-1} + \beta (1-\beta)^i \hat{x}^C, \quad (58)$$

where  $\xi_i = \sum_{j=0}^i (1-\beta)^j$  and  $\phi = ((1-\beta)^{i+1} - (1-\beta)^i)$ . Then, the investor's expected utility is defined as:

$$E \left[ \sum_{i=0}^{\infty} (1-\rho)^{i+1} \left\{ (1-\beta)^{i+1} x'_{-1} \mu + \beta \xi_i \hat{x}^{C'} \mu - \frac{\gamma}{2} \left( (1-\beta)^{2i} x'_{-1} \Sigma x_{-1} + \beta^2 \xi_i^2 \hat{x}^{C'} \Sigma \hat{x}^C + 2(1-\beta)^{i+1} \xi_i x'_{-1} \Sigma \hat{x}^C \right) - \frac{\tilde{\lambda}}{2} \left( \phi_i^2 x'_{-1} \Sigma x_{-1} + \beta^2 (1-\beta)^{2i} \hat{x}^{C'} \Sigma \hat{x}^C + 2\phi_i \beta (1-\beta)^i x'_{-1} \Sigma \hat{x}^C \right) \right\} \right] \quad (59)$$

The above expression can be simplified with the following properties of geometric series:

$$\beta \xi_i = \beta \frac{1 - (1-\beta)^{i+1}}{\beta} = 1 - (1-\beta)^{i+1} \quad (60)$$

$$\beta^2 \xi_i^2 = 1 + (1-\beta)^{2+2} - 2(1-\beta)^{i+1} \quad (61)$$

$$r_1 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} (1-\beta)^{i+1} = \frac{(1-\rho)(1-\beta)}{1-(1-\rho)(1-\beta)} \quad (62)$$

$$r_2 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} (1-\beta)^{2i+2} = \frac{(1-\rho)(1-\beta)^2}{1-(1-\rho)(1-\beta)^2} \quad (63)$$



$$r_3 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} (1-\beta)^{2i} = \frac{(1-\rho)}{1-(1-\rho)(1-\beta)^2} \quad (64)$$

$$r_4 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} (1-\beta)^{2i+1} = \frac{(1-\rho)(1-\beta)}{1-(1-\rho)(1-\beta)^2} \quad (65)$$

$$r_5 = \sum_{i=0}^{\infty} (1-\rho)^{i+1} = \frac{(1-\rho)}{\rho} \quad (66)$$

And in turn, the investor's expected utility can be simplified as follows:

$$\begin{aligned} & r_1(x_{-1} - x^C)' \mu + \frac{1-\rho}{\rho} x^{C'} \mu - \frac{\gamma}{2} \left\{ r_2 x'_{-1} \Sigma x_{-1} + (r_5 + r_2 - 2r_1) E(\widehat{x}^{C'} \Sigma \widehat{x}^C) \right. \\ & \left. + 2x_{-1} \Sigma x^C (r_1 - r_2) \right\} - \frac{\tilde{\lambda}}{2} \left\{ \beta^2 r_3 x'_{-1} \Sigma x_{-1} + E(\widehat{x}^{C'} \Sigma \widehat{x}^C) \beta^2 r_3 + 2\beta(r_4 - r_3) x'_{-1} \Sigma x^C \right\} \end{aligned} \quad (67)$$

Now, we develop the first order conditions with respect to  $\varsigma_1$ , and we obtain that the optimal value is:

$$\varsigma_1 = \frac{E[\widehat{x}^{M'} \mu]}{\gamma E[\widehat{x}^{M'} \Sigma \widehat{x}^M]} \frac{W_1}{W_2} - \frac{x'_{-1} \Sigma x^M}{\gamma E[\widehat{x}^{M'} \Sigma \widehat{x}^M]} \frac{W_3}{W_2} - \varsigma_2 \frac{E[\widehat{x}^{M'} \Sigma \widehat{x}^{Min}]}{E[\widehat{x}^{M'} \Sigma \widehat{x}^M]}, \quad (68)$$

where  $W_1 = r_5 - r_1$ ,  $W_2 = (r_5 + r_2 - 2r_1) + (\tilde{\lambda}/\gamma) \beta^2 r_3$ , and  $W_3 = \gamma(r_1 - r_2) + \tilde{\lambda} \beta(r_4 - r_3)$ . We numerically verify that  $W_1/W_2 = 1$  and  $W_3 = 0$ , so that the optimal parameter  $\varsigma_1$  takes the following expression:

$$\varsigma_1 = \frac{E[\widehat{x}^{M'} \mu]}{\gamma E[\widehat{x}^{M'} \Sigma \widehat{x}^M]} - \varsigma_2 \frac{E[\widehat{x}^{M'} \Sigma \widehat{x}^{Min}]}{E[\widehat{x}^{M'} \Sigma \widehat{x}^M]}. \quad (69)$$

Accordingly, the optimal value of  $\varsigma_2$  is

$$\varsigma_2 = \frac{E[\widehat{x}^{Min'} \mu]}{\gamma E[\widehat{x}^{Min'} \Sigma \widehat{x}^{Min}]} - \varsigma_1 \frac{E[\widehat{x}^{M'} \Sigma \widehat{x}^{Min}]}{E[\widehat{x}^{Min'} \Sigma \widehat{x}^{Min}]}. \quad (70)$$

Therefore, one can solve the system given by (69)-(70) to obtain the optimal values of  $\varsigma_1$  and  $\varsigma_2$ . This corresponds with the system of linear equations that one has to solve to obtain the optimal combination parameters in the static framework. In turn, we obtain; see Kan and Zhou (2007):

$$\varsigma_1 = c^{-1} \frac{\Psi^2}{\Psi^2 + \frac{N}{T}}, \quad (71)$$

$$\varsigma_2 = c^{-1} \frac{\frac{N}{T}}{\Psi^2 + \frac{N}{T}} \times \frac{\mu' \Sigma^{-1} \iota}{\iota' \Sigma^{-1} \iota}, \quad (72)$$

where  $c = [(T-2)(T-N-2)]/[(T-N-1)(T-N-4)]$  and  $\Psi^2 = \mu' \Sigma^{-1} \iota - (\mu' \Sigma^{-1} \iota)^2 / (\iota' \Sigma^{-1} \iota) > 0$ . Accordingly, one can obtain the optimal value of  $\eta$  by setting  $\varsigma_2 = 0$  in equation (69), and we obtain that the optimal value of  $\eta$  is:

$$\eta = \frac{E[\widehat{x}^{M'} \mu]}{\gamma E[\widehat{x}^{M'} \Sigma \widehat{x}^M]} = c^{-1} \frac{\mu' \Sigma^{-1} \mu}{\mu' \Sigma^{-1} \mu} = c^{-1}. \quad (73)$$

#### A.4. Proof of Corollary 1

We know from Proposition 3 that the optimal combination parameters coincide with the optimal combination parameters of the static case. Then, we can show that it is optimal to shrink the static mean-variance portfolio if the derivative of the investor's (static) expected utility with respect to parameter  $\eta$  is negative when  $\eta = 1$ . Deriving the investor's expected utility with respect to  $\eta$  and setting  $\eta = 1$ , we obtain that it is optimal to have  $\eta < 1$  when:

$$E\left(\widehat{x}^{M'}\mu\right) < \gamma E\left(\widehat{x}^{M'}\Sigma\widehat{x}^M\right). \quad (74)$$

If we characterize the expectations from the above expression, we obtain that  $\eta < 1$  if  $1 < c$ , where  $c = [(T - N - 2)(T - 2)]/[(T - N - 1)(T - N - 4)]$ . Because,  $c > 1$ , we observe that it is always optimal to shrink the static mean-variance portfolio.

Now, if we take derivatives of the investor's (static) expected utility with respect to parameter  $\varsigma_2$ , and then set  $\varsigma_2 = 0$ , this derivative is positive (in turn it is optimal to have  $\varsigma_2 > 0$ ) if

$$E\left(\widehat{x}^{T'}\mu\right) > \gamma_{\varsigma_1} E\left(\widehat{x}^{M'}\Sigma\widehat{x}^T\right). \quad (75)$$

Now, characterizing the above expectations, we obtain that  $\varsigma_2 > 0$  if  $1 > \varsigma_1 c$ . From the optimal expression of  $\varsigma_1$ , we obtain that  $1 > \varsigma_1 c$  if  $1 > \Psi^2/(\Psi^2 + N/T)$ , which always holds because  $\Psi^2$  can be written as  $\Psi^2 = (\mu - \mu_g)' \Sigma^{-1} (\mu - \mu_g)$ , where  $\mu_g = (\iota' \Sigma^{-1} \mu)/(\iota' \Sigma^{-1} \iota)$ , and in turn  $\Psi^2$  is nonnegative. Moreover, from the optimal expression for  $\varsigma_2$ , we observe that the optimal value is always positive because  $\Psi^2 = \mu' \Sigma^{-1} \iota - (\mu' \Sigma^{-1} \iota)^2 / (\iota' \Sigma^{-1} \iota) > 0$ , and it means that  $\mu' \Sigma^{-1} \iota$  should be positive, otherwise  $\Psi^2 > 0$  would not hold. This means that all the elements require to compute the optimal  $\varsigma_2$  are positive, and in turn the optimal  $\varsigma_2$  is positive.

#### A.5. Proof of Proposition 4

Writing the expected utility for an investor using the four-fund portfolio as in (67), it is straightforward to see that we can obtain the optimal  $\beta$  that minimizes the investor's expected loss by solving the following problem:

$$V_1(x_{-1} - x^C)' \mu - \frac{1}{2} \left( E \left[ \widehat{x}^{C'} \Sigma \widehat{x}^C \right] V_2 + x'_{-1} \Sigma x_{-1} V_3 + x'_{-1} \Sigma x^C V_4 \right), \quad (76)$$

where  $V_i$  accounts for the accumulated variability and trading costs of  $\widehat{x}^C$  and the investor's initial position  $x_{-1}$ , and they take the form:

$$V_1 = \frac{(1 - \rho)(1 - \beta)}{1 - (1 - \rho)(1 - \beta)} \quad (77)$$

$$V_2 = \gamma \left( \frac{(1 - \rho)}{\rho} + \frac{(1 - \rho)(1 - \beta)^2}{1 - (1 - \rho)(1 - \beta)^2} - 2 \frac{(1 - \rho)(1 - \beta)}{1 - (1 - \rho)(1 - \beta)^2} \right) + \tilde{\lambda} \frac{(1 - \rho)\beta^2}{1 - (1 - \rho)(1 - \beta)^2}, \quad (78)$$

$$V_3 = \gamma \frac{(1 - \rho)(1 - \beta)^2}{1 - (1 - \rho)(1 - \beta)^2} + \tilde{\lambda} \frac{(1 - \rho)\beta^2}{1 - (1 - \rho)(1 - \beta)^2}, \quad (79)$$

$$V_4 = 2\gamma \left( \frac{(1 - \rho)(1 - \beta)}{1 - (1 - \rho)(1 - \beta)} - \frac{(1 - \rho)(1 - \beta)^2}{1 - (1 - \rho)(1 - \beta)^2} \right) - 2\tilde{\lambda} \frac{(1 - \rho)\beta^2}{1 - (1 - \rho)(1 - \beta)^2}. \quad (80)$$

Now, we characterize  $E \left[ \widehat{x}^{C'} \Sigma \widehat{x}^C \right]$ , which is defined as:

$$E \left[ \widehat{x}^{C'} \Sigma \widehat{x}^C \right] = \frac{c}{\gamma^2} \left( \varsigma_1^2 \left( \mu' \Sigma^{-1} \mu + \frac{N}{T} \right) + \varsigma_2^2 \iota' \Sigma^{-1} \iota + 2\varsigma_1 \varsigma_2 \mu' \Sigma^{-1} \iota \right), \quad (81)$$

where  $c = [(T - N - 2)(T - 2)]/[(T - N - 1)(T - N - 4)]$ .

## Appendix B: Tables

**Table 1** Commodity futures:

This table provides some descriptive statistics of the data from the commodity futures, as well as the contract multiplier.

Commodity	Average Price	Volatility price changes	Contract multiplier
Aluminium	56,231.71	888.37	25
Copper	161,099.45	3,268.96	25
Nickel	127,416.45	3,461.62	6
Zinc	54,238.84	1,361.69	25
Lead	45,925.04	1,227.02	25
Tin	78,164.60	1,733.53	5
Gasoil	69,061.48	1,571.89	100
WTI Crude	75,853.55	1,798.93	1000
RBOB Crude	88,503.62	2,780.74	42,000
Natural Gas	63,553.35	3,4439.78	10,000
Coffee	58,720.11	940.55	37,500
Cocoa	23,326.21	458.50	10
Sugar	18,121.58	462.35	112,000
Gold	94,780.87	1,327.11	100
Silver	87,025.94	2,415.69	5,000

**Table 2** Sharpe ratio discounted with transaction costs

This table reports the annualized out-of-sample Sharpe ratio for the different portfolio strategies that we consider. Sharpe ratios are discounted by quadratic transaction costs with  $\lambda = 3 \times 10^{-7}$ . The number in parentheses are the corresponding p-values for the difference of each portfolio strategy with the four-fund portfolio that combines the static mean-variance portfolio with the minimum-variance portfolio. Our considered base-case investor has an absolute risk aversion parameter of  $\gamma = 10^{-8}$  and an impatience factor of  $\rho = 1 - \exp(-0.1/260)$ .

	Panel A: <i>Start from zero</i>			Panel B: <i>Start from <math>x^M</math></i>		
	N=25	N=50	Com.	N=25	N=50	Com.
<i>Static trading strategies</i>						
S-M	-0.266 (0.000)	-0.345 (0.000)	-0.459 (0.000)	-0.266 (0.000)	-0.337 (0.000)	-0.452 (0.000)
S2F	0.076 (0.000)	0.105 (0.000)	0.102 (0.050)	0.068 (0.000)	0.101 (0.000)	0.106 (0.036)
S3F-Min	0.678 (0.000)	0.633 (0.000)	0.739 (0.126)	0.678 (0.000)	0.637 (0.000)	0.769 (0.148)
<i>Multiperiod trading strategies</i>						
M-M	0.150 (0.000)	0.297 (0.008)	0.056 (0.036)	0.153 (0.004)	0.295 (0.008)	0.052 (0.042)
M3F	0.202 (0.004)	0.307 (0.004)	0.269 (0.106)	0.212 (0.000)	0.298 (0.008)	0.259 (0.094)
M4F-Min	0.765 (1.000)	0.771 (1.000)	0.874 (1.000)	0.772 (1.000)	0.764 (1.000)	0.868 (1.000)
O-M4F-Min	0.765 (0.786)	0.771 (0.774)	0.874 (0.628)	0.911 (0.144)	0.865 (0.376)	0.895 (0.742)

**Table 3** Sharpe ratio: some robustness checks (RC)

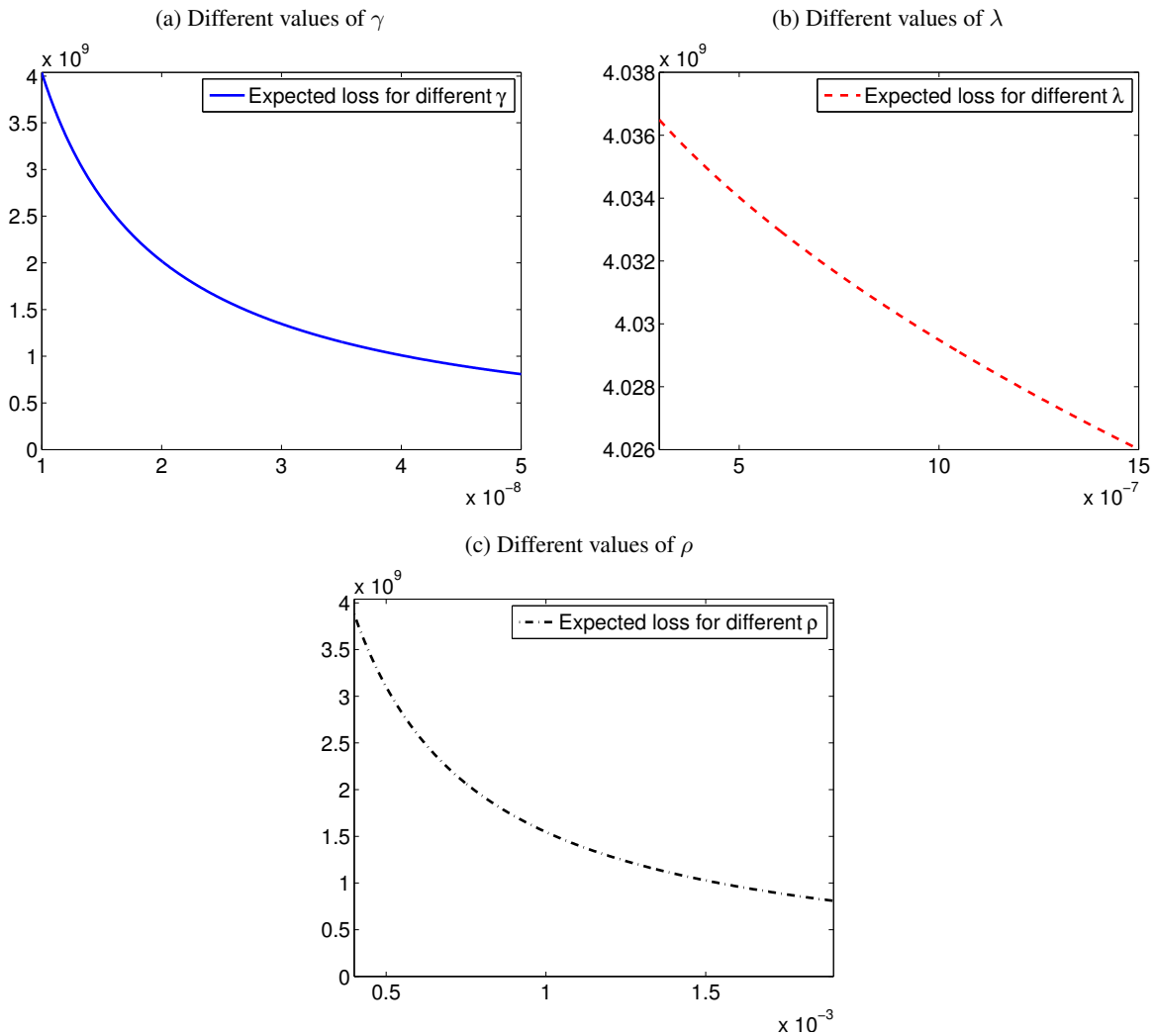
This table reports the annualized out-of-sample Sharpe ratio for the different portfolio strategies that we consider. Our considered base-case investor has an absolute risk aversion parameter of  $\gamma = 10^{-8}$  and an impatience factor of  $\rho = 1 - \exp(-0.1/260)$  and faces quadratic transaction costs with  $\lambda = 3 \times 10^{-7}$ . The number in parentheses are the corresponding p-values for the difference of each portfolio strategy with the four-fund portfolio that combines the static mean-variance portfolio with the minimum-variance portfolio.

	Panel A: RC for different $\gamma$						Panel B: RC for different $T$					
	$\gamma = 10^{-9}$			$\gamma = 10^{-7}$			T=250			T=750		
	N=25	N=50	Com.	N=25	N=50	Com.	N=25	N=50	Com.	N=25	N=50	Com.
<i>Static trading strategies</i>												
S-M	-3.623	-4.020	-2.636	0.141	0.248	-0.044	-1.126	-1.458	-0.766	-0.023	0.207	-0.156
	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)	(0.006)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.110)
S2F	-1.242	-1.383	-1.625	0.209	0.272	0.324	-0.186	-0.096	-0.073	0.087	0.435	-0.100
	(0.000)	(0.000)	(0.000)	(0.004)	(0.004)	(0.112)	(0.000)	(0.000)	(0.008)	(0.000)	(0.010)	(0.174)
S3F-Min	-0.195	-0.425	-0.740	0.765	0.748	0.984	0.349	0.416	0.089	0.718	0.822	0.417
	(0.000)	(0.000)	(0.000)	(0.066)	(0.076)	(0.030)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.900)
<i>Multiperiod trading strategies</i>												
M-M	0.086	0.219	-0.067	0.179	0.306	0.034	0.055	0.218	0.635	0.216	0.545	0.049
	(0.000)	(0.000)	(0.030)	(0.000)	(0.000)	(0.018)	(0.000)	(0.000)	(0.752)	(0.000)	(0.044)	(0.310)
M3F	0.194	0.305	0.115	0.226	0.290	0.311	0.171	0.321	0.515	0.179	0.569	-0.122
	(0.000)	(0.006)	(0.086)	(0.004)	(0.008)	(0.102)	(0.000)	(0.000)	(0.476)	(0.000)	(0.036)	(0.142)
M4F-Min	0.752	0.767	0.767	0.779	0.762	0.936	0.633	0.734	0.730	0.775	0.909	0.441
	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)
O-M4F-Min	0.895	0.887	0.843	0.918	0.845	0.921	0.742	0.905	0.711	0.831	0.949	0.536
	(0.166)	(0.318)	(0.432)	(0.108)	(0.488)	(0.846)	(0.182)	(0.156)	(0.512)	(0.552)	(0.768)	(0.104)

## Appendix C: Figures

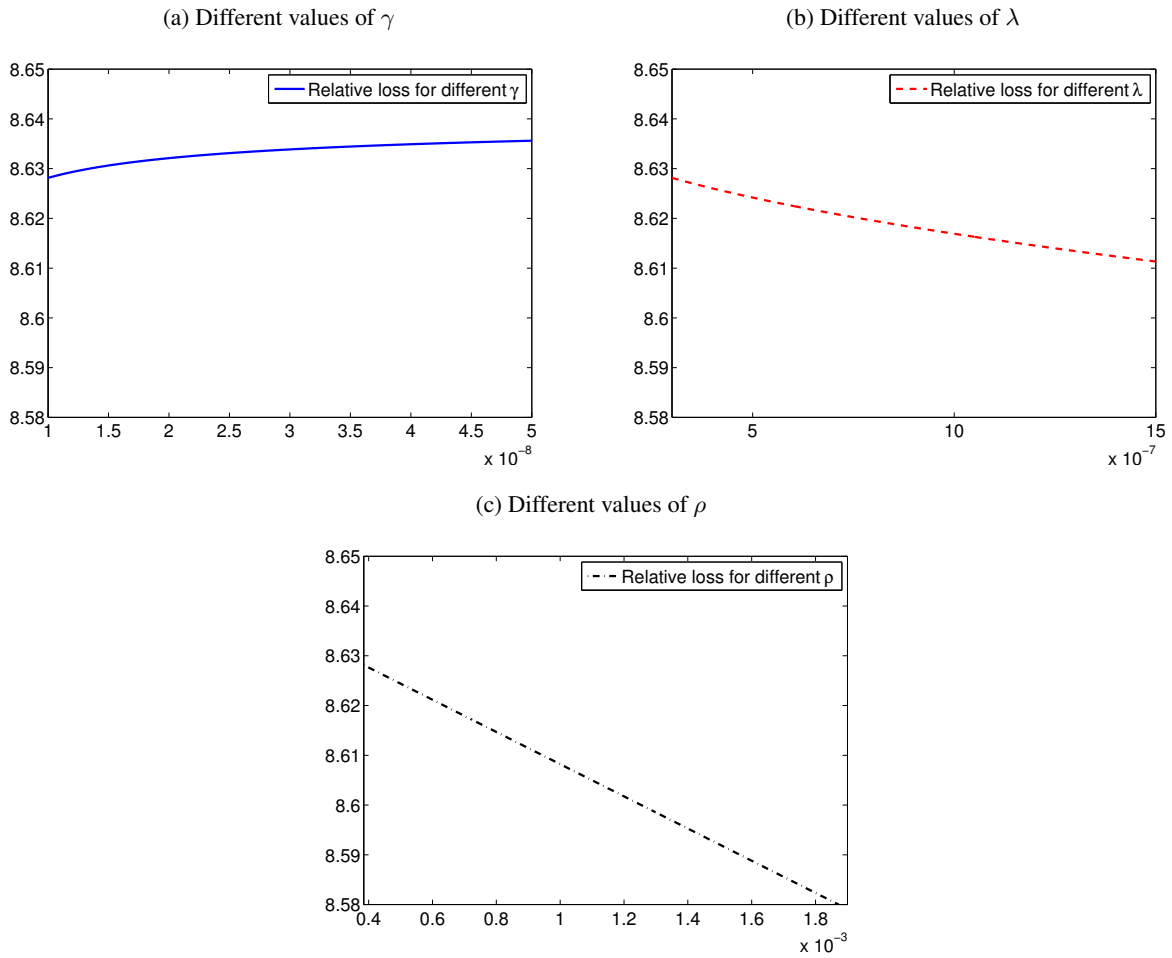
**Figure 1** Absolute loss of multiperiod investor

This plot depicts the investor's absolute expected loss for different values of  $\gamma$ ,  $\lambda$ , and  $\rho$ . Our base-case investor is defined with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$  and  $\rho = 1 - \exp(-0.1/260)$ . We consider an investor that has 500 observations to construct the optimal trading strategy whose parameters are defined with the sample moments of the empirical dataset formed with commodities that we consider in the empirical application.



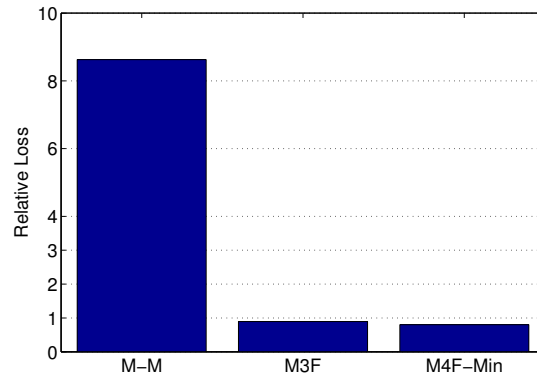
**Figure 2** Relative loss of multiperiod investor

This plot depicts the investor's relative loss for different values of  $\gamma$ ,  $\lambda$ , and  $\rho$ . Our base-case investor is defined with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$  and  $\rho = 1 - \exp(-0.1/260)$ . We consider an investor that has 500 observations to construct the optimal trading strategy whose parameters are defined with the sample moments of the empirical dataset of commodity futures that we consider in the empirical application.

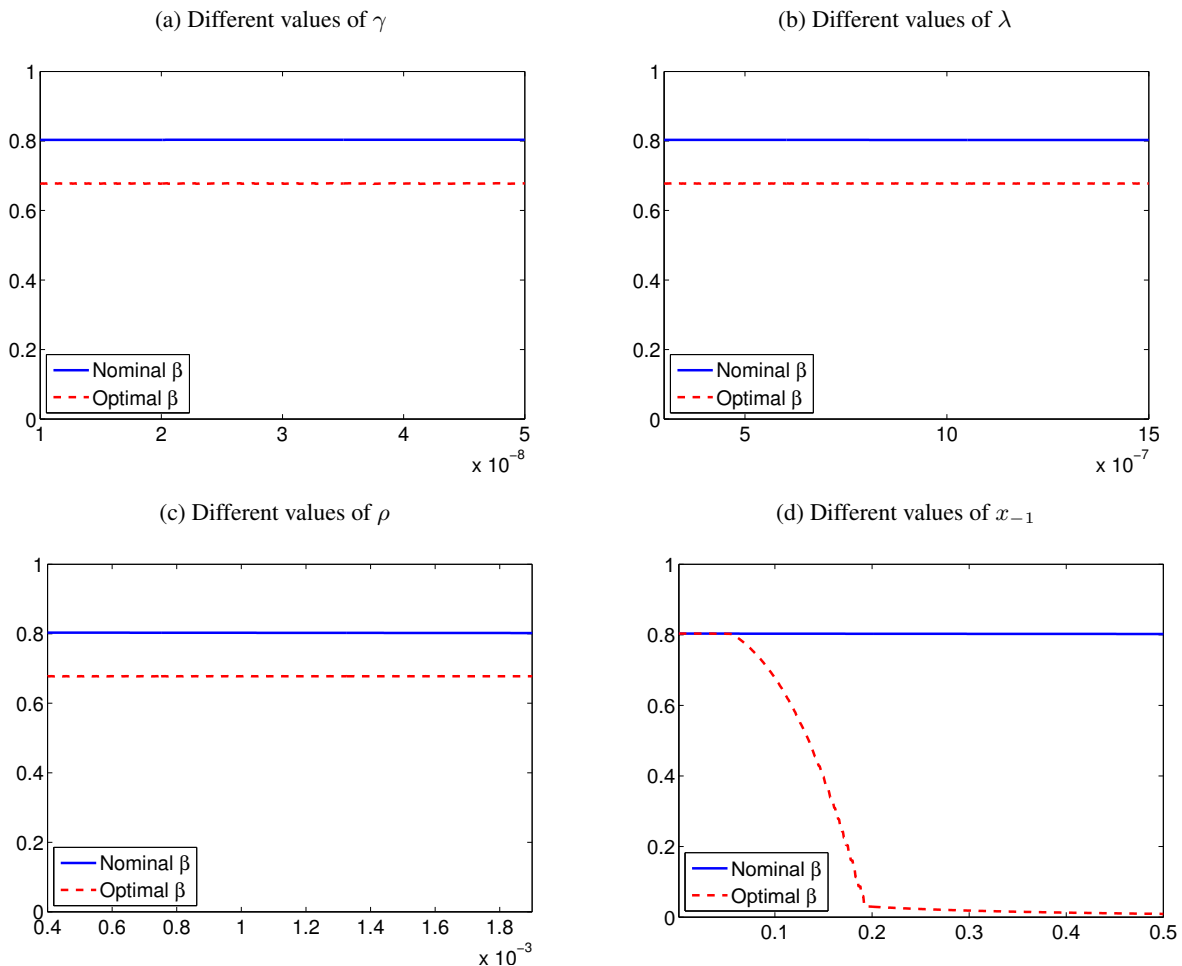


**Figure 3** Relative loss of different multiperiod investor

This plot depicts the investor's relative loss of the plug-in multiperiod investor (M-M), the multiperiod investor that shrinks the static mean-variance portfolios (M3F), and the multiperiod four-fund portfolio that combines the static mean-variance portfolio with the minimum-variance portfolio (M4F-Min). Our base-case investor is defined with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$  and  $\rho = 1 - \exp(-0.1/260)$ . The investor has 500 observations to construct the optimal trading strategy whose parameters are defined with the sample moments of the empirical dataset of commodity futures that we consider in the empirical application.

**Figure 4** Nominal Vs Optimal four-fund portfolios: Comparison of relative losses

This plot depicts the investor's relative loss for different values of  $\gamma$ ,  $\lambda$ , and  $\rho$ . Our base-case investor is defined with  $\gamma = 10^{-8}$ ,  $\lambda = 3 \times 10^{-7}$  and  $\rho = 1 - \exp(-0.1/260)$ . We consider an investor that has 500 observations to construct the optimal trading strategy whose parameters are defined with the sample moments of the empirical dataset of commodity futures that we consider in the empirical application.



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