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New isometry of Krall-Laguerre orthogonal polynomials in martingale spaces

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Abstract

In this paper we study how an inner product derived from an Uvarov transformation of the Laguerre weight function is used in the orthogonalization procedure of a sequence of martingales related to a Levy process. The orthogonalization is done by isometry. The resulting set of pairwise strongly orthogonal martingales involved are used as integrators in the so-called chaotic representation property.

Keywords: Orthogonal polynomials; Laguerre-type polynomials; Krall-Laguerre polynomials; Inner products; Lévy processes; Stochastic processes

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1 Introduction

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the Gamma distribution. Therefore, they are orthogonal with respect to the inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle p, q\rangle_\alpha = \int_0^\infty pqx^\alpha e^{-x}dx, \quad \alpha > -1,\ p, q \in \mathbb{P}. \tag{1}$$

From now on, $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ stands for the sequence of monic Laguerre polynomials orthogonal with respect to (1). Their corresponding norm is given by

$$||\widehat{L}_n^\alpha||_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{2}$$

Dealing with Laguerre polynomials, it is customary to use the normalization such that the leading coefficient of the n-th degree classical Laguerre polynomial (denoted by $L_n^{(\alpha)}(x)$) equals $\frac{(-1)^n}{n!}$, i.e.,

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}x^n + \textit{ lower degree terms,}$$

and therefore

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^\alpha(x). \tag{3}$$

It is very well known that these polynomials satisfy the following three term recurrence relation

$$x\widehat{L}_n^\alpha(x) = \widehat{L}_{n+1}^\alpha(x) + \beta_n\widehat{L}_n^\alpha(x) + \gamma_n\widehat{L}_{n-1}^\alpha(x), \quad n \geq 1, \tag{4}$$

with initial conditions $\widehat{L}_0^\alpha(x) = 1$, $\widehat{L}_1^\alpha(x) = x-(\alpha+1)$, and recurrence coefficients $\beta_n = 2n+\alpha+1$, $\gamma_n = n(n+\alpha)$ for every $n \geq 1$ (see [16], [19], [26], among others). They constitute a family of classical orthogonal polynomials (see [17] and [19]), and they are the eigenfunctions of a second order linear differential operator with polynomial coefficients. The kernel polynomials (see [3, Ch.I, §7]) associated with Laguerre polynomials will play a key role in order to prove some of the basic results of the manuscript. Let

$$K_n(x,y) = \sum_{k=0}^n \frac{\widehat{L}_k^\alpha(x)\widehat{L}_k^\alpha(y)}{||\widehat{L}_k^\alpha||_\alpha^2} \tag{5}$$

denotes the n-th kernel polynomial associated with the Laguerre orthogonal polynomials. Thus, according to the Christoffel-Darboux formula, for every $n \in \mathbb{N}$ we get the alternative expression

$$K_n(x,y) = \frac{\widehat{L}_{n+1}^\alpha(x)\widehat{L}_n^\alpha(y) - \widehat{L}_{n+1}^\alpha(y)\widehat{L}_n^\alpha(x)}{x-y}\frac{1}{||\widehat{L}_n^\alpha||_\alpha^2}.$$

The limit when $y \to x$ is known as the *confluent form* of the n-th kernel, and it reads

$$K_n(x,x) = \sum_{k=0}^n \frac{[\widehat{L}_k^\alpha(x)]^2}{||\widehat{L}_k^\alpha||_\alpha^2} = \frac{[\widehat{L}_{n+1}^\alpha(x)]'\,\widehat{L}_n^\alpha(x) - [\widehat{L}_n^\alpha(x)]'\,\widehat{L}_{n+1}^\alpha(x)}{||\widehat{L}_n^\alpha||_\alpha^2}. \tag{6}$$

Notice that if deg $f \leq n$, then the n-th kernel polynomial satisfies the so-called *reproducing property*

$$\int_0^\infty K_n(x,y)\,f(x)\,x^\alpha e^{-x}dx = f(y)\,.$$

On the other hand, from (1), let us introduce the following inner product

$$\langle p,q\rangle=\int_0^\infty pqx^\alpha e^{-x}dx+\sigma^2p(c)q(c),\quad \alpha>-1,\ p,q\in\mathbb{P} \tag{7}$$

where $\sigma^2\in\mathbb{R}_+$, and $c\in(-\infty,0]$. Notice that

$$\langle p,q\rangle=\langle p,q\rangle_\alpha+\sigma^2p(c)q(c),$$

and that (7) can be interpreted as a modification (or perturbation) of the Laguerre measure $d\mu_\alpha(x)=x^\alpha e^{-x}dx$ with a discrete measure given by a mass point at $x=c$,

$$d\tilde{\mu}_\alpha(x)=x^\alpha e^{-x}dx+\sigma^2\delta(x-c),$$

where $\delta(x-c)$ is the Dirac delta at $x=c$. This perturbation is known in the literature as the Uvarov perturbation of the measure $d\mu_\alpha(x)$ (see [6], [7], [8] and the references given there). The case $c=0$ has been deeply studied in the literature (see [4], [5], [13] among others). These polynomials are called either Laguerre-type polynomials (see, for instance, [4] and [18]) or Krall-Laguerre polynomials ([10]). They were also obtained by T.H. Koornwinder [15] as a special limit case of the Jacobi-Koornwinder (Jacobi type) orthogonal polynomials, and they are also known as Laguerre-Koorwinder polynomials. When α is a positive integer number, they are eigenfunctions of a linear differential operator of order $2\alpha+4$ with polynomial coefficients which are independent of the degree of the polynomial (see [13]). Finally, in [21] is developed an interesting application in stochastic processes.

Recently, several authors have begun to study the case when c is a negative number, i.e., the mass point is located outside the support of the Laguerre measure. The study of their asymptotic and analytic properties can be founded in [6], [7] or [8]. In the sequel, $\{\widehat{L}_n^{\alpha,c,\sigma^2}(x)\}_{n\geq 0}$ denotes the sequence of monic Laguerre-type polynomials orthogonal with respect to (7) when $c\in(-\infty,0)$, and $\{\widehat{L}_n^{\alpha,\sigma^2}(x)\}_{n\geq 0}$ stands for the monic Laguerre-type orthogonal polynomials with $c=0$. The main goal of the manuscript is to consider a natural generalization for $c\in(-\infty,0)$, of the work done in [21] for the case $c=0$. In other words, we analyze the potential application of the Laguerre-type orthogonal polynomials $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$ in connection with Lévy processes, as Schoutens [21] did for $\widehat{L}_n^{\alpha,\sigma^2}(x)$. In our opinion, the differences between these two cases are sufficient to justify a new study of the isometry of these polynomials with certain sets of martingales.

A Lévy process is a stochastic process with independent and stationary increments which consists of three basic stochastically independent parts: a deterministic part, a pure jump part and a Brownian motion. Lévy processes play an important role in many fields of science. For example, in engineering, they are used for the study of networks; in the actuarial science, for the calculation of insurance and re-insurance risk and, in economics, for continuous time-series models. In the last decades, the study of the relation between orthogonal polynomials and Lévy processes have become increasing, see [20], [21], [24] and [25]. Consider a Lévy process and let σ^2 the constant of the Brownian motion part and ν its Lévy measure. We can construct a sequence of orthogonal polynomials with respect to the measure $x^2\nu(dx)+\sigma^2\delta(dx)$. These polynomials are the building blocks of a kind of chaotic representation of the square functionals of the Lévy process proved by Nualart and Schoutens [20] and Schoutens [22]. The chaotic representation property (CRP) says that any square integrable random variable measurable with respect to normal martingales X can be expressed as an orthogonal sum of multiple stochastic integrals with respect to X.

The structure of the manuscript is as follows. In Section 2, we summarize some properties of Laguerre-type orthogonal polynomials to be used in the sequel. We briefly review the concept of Lévy process and study the orthogonalization procedure for a sequence of martingales related to the powers of the jumps of this stochastic process in Section 3. Section 4 is devoted to discuss a chaotic representation property in which the set of pairwise strongly orthogonal martingales is used as integrators. We explicitly compute the coefficients of the orthogonalization procedure by isometry with the Laguerre-type orthogonal polynomials in Section 5. Finally, in Section 6, we show an example for a particular Lévy process.

2 Laguerre-type orthogonal polynomials

In this section, we will present some properties of Krall-Laguerre (or Laguerre-type) orthogonal polynomials $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$, showing the differences which appear when $c=0$ or $c\in(-\infty,0)$. The first remarkable fact is that the position of the first (or least) zero of the Laguerre-type polynomials, strongly depends on the value of the real and positive parameter σ^2, and the position of the mass point c (see [8] for detailed study). Obviously, if $\sigma^2=0$, the zeros of the Laguerre-type polynomials trivially reduces to the zeros of the classical Laguerre polynomials. Moreover, if $\sigma^2>0$ and $c\in(-\infty,0)$, then one can find values of σ^2 for which the least zero of $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$, $n\geq 1$, is located in the interval $(c,0)$, i.e., outside of the support of the classic Laguerre measure, whereas if $\sigma^2>0$ and $c=0$ this phenomenon does not occur at all, and all the zeros of $\widehat{L}_n^{\alpha,\sigma^2}(x)$, $n\geq 1$, are located inside the interval $(0,+\infty)$ for any value of σ^2.

For every $n=1,2,\ldots$, the polynomials $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$ satisfy as well a three term recurrence relation

$$x\widehat{L}_n^{\alpha,c,\sigma^2}(x)=\widehat{L}_{n+1}^{\alpha,c,\sigma^2}(x)+\tilde{\beta}_n\widehat{L}_n^{\alpha,c,\sigma^2}(x)+\tilde{\gamma}_n\widehat{L}_{n-1}^{\alpha,c,\sigma^2}(x),\quad n\geq 1,\tag{8}$$

with recurrence coefficients

$$\begin{aligned}\tilde{\beta}_n &= \beta_n+\frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\left(1-\frac{1+\sigma^2K_{n-1}(c,c)}{1+\sigma^2K_n(c,c)}\right)-\frac{\widehat{L}_n^{\alpha}(c)}{\widehat{L}_{n-1}^{\alpha}(c)}\left(1-\frac{1+\sigma^2K_{n-2}(c,c)}{1+\sigma^2K_{n-1}(c,c)}\right),\\ \tilde{\gamma}_n &= \frac{\left(1+\sigma^2K_n(c,c)\right)\left(1+\sigma^2K_{n-2}(c,c)\right)}{\left(1+\sigma^2K_{n-1}(c,c)\right)^2}\gamma_n.\end{aligned}$$

where β_n and γ_n are the recurrence coefficients in (4) for the classical Laguerre polynomials. Notice that $\widehat{L}_n^{\alpha}(c)\neq 0$ for every $n=0,1,2,\ldots$, because c does not belong to the support of the classical Laguerre measure. The asymptotic behavior of these "perturbed" coefficients is (see [6])

$$\begin{aligned}\frac{\tilde{\beta}_n}{\beta_n} &= 1-\frac{\sqrt{|c|}}{2}n^{-3/2}+\mathcal{O}(n^{-5/2}),\\ \frac{\tilde{\gamma}_n}{\gamma_n} &= 1+2\sqrt{|c|}n^{-1/2}+\mathcal{O}(n^{-1}).\end{aligned}$$

Let $||\widehat{L}_n^{\alpha,c,\sigma^2}||$ and $||\widehat{L}_n^{\alpha}||_\alpha$ the norm of Laguerre-type and classical Laguerre orthogonal polynomials with respect to (7) and (1), respectively. For every $n\in\mathbb{N}$, the following expression holds

$$||\widehat{L}_n^{\alpha,c,\sigma^2}||^2=\frac{1+\sigma^2K_n(c,c)}{1+\sigma^2K_{n-1}(c,c)}||\widehat{L}_n^{\alpha}||_\alpha^2.$$

Concerning the asymptotic properties, uniformly on compact subsets of $\mathbb{C}\setminus[0,+\infty)$, the outer relative asymptotics of $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$, for $c\in\mathbb{R}_-$, is given by (see [7] and [6])

$$\lim_{n\to\infty}\frac{\widehat{L}_n^{\alpha,c,\sigma^2}(x)}{\widehat{L}_n^{\alpha}(x)}=\frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}},$$

meanwhile, when $c = 0$, we have (see, for example [1])

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha,\sigma^2}(x)}{\widehat{L}_n^{\alpha}(x)} = 1.$$

Finally, a very remarkable difference appear when we express the aforementioned families in terms of Gauss hypergeometric functions. The classical Laguerre polynomials $\widehat{L}_n^{\alpha}(x)$ can be expressed as hypergeometric functions of type ${}_1F_1$ (see [11],[16],[26] among others). The addition of a mass point at $c = 0$ implies that the Laguerre-type polynomials $\widehat{L}_n^{\alpha,\sigma^2}(x)$, which are used in [21], are expressed in terms of ${}_2F_2$ hypergeometric functions (see, for example [12]), meanwhile moving the mass point to $c < 0$, as in our case, implies that the Laguerre-type polynomials $\widehat{L}_n^{\alpha,c,\sigma^2}(x)$ turn to be expressed in terms of ${}_3F_3$ hypergeometric functions (see [6]).

3 Lévy processes and Teugels martingales

Let $X = \{X_t, t \geq 0\}$ be a Lévy process (meaning that X has stationary and independent increments and is continuous in probability and that $X_0 = 0$), cadlag and centered, with moments of all orders. Let us remind that a stochastic process is cadlag if its sample paths are right continuous and have left-hand limits. Denote by σ^2 the variance of the Gaussian part of X and by ν its Lévy measure. The existence of moments of all orders of X_t implies that the Lévy measure ν has moments of all orders ≥ 2. Write

$$m_n = \int_{\mathbb{R}} x^n \nu(dx) \quad \text{for} \quad n \geq 2.$$

For background on all these notions, we refer to Sato [23] and Bertoin [2]. Following Nualart and Schoutens [20], we introduce the square-integrable martingales (and Lévy processes) called Teugels martingales, related to the powers of the jumps of the process:

$$\begin{aligned} Y_t^{(1)} &= X_t, \\ Y_t^{(n)} &= \sum_{0<s\leq t} (\Delta X_t)^n - m_n t, \quad n \geq 2, \end{aligned}$$

where $\Delta X_t = X_t - X_{t^-}$ is the jump size at time t and

$$X_{t^-} = \lim_{s<t, s\to t} X_s, \;\; t > 0$$

is the left limit process. The compensated power jump process $Y^{(n)}$ of order n is a normal martingale.

An important question is the orthogonalization of the set $\{Y^{(n)}, n = 1, 2, \ldots\}$ of martingales as stochastic integrators of a kind of chaotic representation as we briefly discuss in Section 4. In [20], it is shown that the orthogonalization of $\{Y^{(n)}, n = 1, 2, \ldots\}$ can be achieved through an isometry, $x^{n-1} \longleftrightarrow Y^{(n)}$, and consequently, an orthogonalization of $\{1, x, x^2, \ldots\}$ gives an orthogonalization of $\{Y^{(1)}, Y^{(2)}, Y^{(3)}, \ldots\}$.

Here we consider a first space S_1 as the space of all real polynomials on the positive real line endowed with the scalar product $\langle \cdot,\cdot \rangle_1$ given by

$$\langle p(x), q(x) \rangle_1 = \int_{-\infty}^{+\infty} p(x)q(x)x^2\nu(dx) + \sigma^2 p(c)q(c). \tag{9}$$

In [21], it was considered the scalar product given in (9) when $c = 0$ and the standard basis $\{x^n\}_{n\geq 0}$. In this case, the scalar product $\langle x^{i-1}, x^{j-1}\rangle_1 = m_{i+j} + \sigma^2 1_{\{i=j=1\}}$ naturally induces an orthogonalization of the Teugels Martingales $\{Y^{(n)},\ n=1,2,\dots\}$ since $x^{n-1}\longleftrightarrow Y^{(n)}$ is an isometry between S_1 and the space

$$S_2=\{a_1Y^{(1)}+a_2Y^{(2)}+\cdots+a_nY^{(n)}:n\in\{1,2,\dots\},\ a_i\in\mathbb{R},\ i=1,\dots,n\}$$

endowed with the scalar product

$$\langle X,Y\rangle_2=E([X,Y]_1). \tag{10}$$

Indeed

$$\langle Y^{(i)},Y^{(j)}\rangle_2=m_{i+j}+\sigma^2 1_{\{i=j=1\}}.$$

The elements of the space S_2 are linear combinations of Teugels martingales and the orthogonalization procedure produces a set of strongly pairwise orthogonal martingales

$$\{H^{(j)}=a_{1,j}Y^{(1)}+\cdots+a_{j,j}Y^{(j)},\quad j=1,2,\dots\} \tag{11}$$

that can be used in the *predictable representation property*, as shown in Section 4.

In this paper, we focus our attention for values of $c<0$. In this case, the standard basis $\{x^n\}_{n\geq 0}$ doesn't lead to an isometry with the space S_2 since

$$\langle Y^{(i)},Y^{(j)}\rangle_2=m_{i+j}+\sigma^2c^{i+j-2}.$$

Let

$$\widetilde{Y}^{(i)}=\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\frac{1}{c^\ell}Y^{(\ell+1)} \tag{12}$$

a family of martingales that are suitable linear combinations of Teugels martingales, then S_2 can be rewritten as

$$\{a_1\widetilde{Y}^{(1)}+a_2\widetilde{Y}^{(2)}+\cdots+a_n\widetilde{Y}^{(n)}:n\in\{1,2,\dots\},\ a_i\in\mathbb{R},\ i=1,\dots,n\}$$

the set of all linear combinations of the martingales defined in (12) endowed with the scalar product defined in (10). If we consider the basis

$$\{(\frac{x}{c}-1)^n\}_{n\geq 0}$$

then it is possible to show that $(\frac{x}{c}-1)^{n-1}\longleftrightarrow \widetilde{Y}^{(n)}$ is an isometry between S_1 and S_2. Indeed, note that

$$\langle(\frac{x}{c}-1)^{i-1},(\frac{x}{c}-1)^{j-1}\rangle_1=\frac{1}{c^{i+j-2}}\int_{-\infty}^{+\infty}(x-c)^{i+j-2}x^2\nu(dx)+\sigma^2 1_{\{i=j=1\}},$$

since $(\frac{x}{c}-1)^{i-1}=\frac{1}{c^{i-1}}(x-c)^{i-1}$. Also note that $x-c=x+|c|$ since $c=-|c|<0$, then by using the Newton's binomio, we get

$$\begin{aligned}\int_{-\infty}^{+\infty}(x-c)^{i+j-2}x^2\nu(dx) &= \int_{-\infty}^{+\infty}\sum_{k=0}^{i+j-2}\binom{i+j-2}{k}x^k\,|c|^{i+j-2-k}\,x^2\nu(dx)\\ &= \sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\int_{-\infty}^{+\infty}x^{k+2}\nu(dx)\\ &= \sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\,m_{k+2}.\end{aligned}$$

Hence,

$$\langle(\frac{x}{c}-1)^{i-1},(\frac{x}{c}-1)^{j-1}\rangle_1=\frac{1}{c^{i+j-2}}\sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\,m_{k+2}+\sigma^2 1_{\{i=j=1\}}.$$

On the other hand, note that the martingale defined above can be written as

$$\widetilde{Y}^{(i)}=\frac{1}{c^{i-1}}\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}c^{i-1-\ell}\;Y^{(\ell+1)}$$

and then, the scalar product $\langle\cdot,\cdot\rangle_2$ given by

$$\begin{aligned}
\langle\widetilde{Y}^{(i)},\widetilde{Y}^{(j)}\rangle_2 &= \frac{1}{c^{i+j-2}}\langle\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}c^{i-1-\ell}\;Y^{(\ell+1)},\sum_{s=0}^{j-1}\binom{j-1}{s}c^{j-1-s}\;Y^{(s+1)}\rangle_2\\
&= \frac{1}{c^{i+j-2}}\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}c^{i-1-\ell}\sum_{s=0}^{j-1}\binom{j-1}{s}c^{j-1-s}\langle Y^{(\ell+1)},Y^{(s+1)}\rangle_2\\
&= \frac{1}{c^{i+j-2}}\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\sum_{s=0}^{j-1}\binom{j-1}{s}c^{i+j-2-(\ell+s)}m_{\ell+s+2}+\sigma^2 1_{\{i=j=1\}}\\
&= \frac{1}{c^{i+j-2}}\sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\,m_{k+2}+\sigma^2 1_{\{i=j=1\}},
\end{aligned}$$

where $k=\ell+s$, naturally leads to the isometry $(\frac{x}{c}-1)^{n-1}\longleftrightarrow\widetilde{Y}^{(n)}$. In a similar fashion of the case with $c=0$, we are able to provide a set of strongly pairwise orthogonal martingales

$$\{H^{(j),c}=a^c_{1,j}\widetilde{Y}^{(1)}+\cdots+a^c_{j,j}\widetilde{Y}^{(j)},\quad j=1,2,\dots\}$$

and in the Section 5 we will deduce the expression of the coefficients of such martingales.

4 Motivation of the paper

Let $\mathcal{M}^2$ be the space of square-integrable martingales M such that $\sup_t E(M_t^2)<\infty$ and $M_0=0$ a.s. We recall that two martingales $M,N\in\mathcal{M}^2$ are strongly orthogonal if and only if their product MN is a uniform integrable martingale.

The orthogonalized set of martingales $\{H^{(i)},i=1,2,\dots\}$ as defined in (11) can be used in a chaotic representation property (CRP). This property says that any square integrable random variable measurable with respect to normal martingales, X, can be expressed as an orthogonal sum of multiple stochastic integrals with respect to X. Nualart and Schoutens [20] showed that every random variable F in $L^2(\Omega,\mathcal{F})$ has a representation of the form

$$F=E\left[F\right]+\sum_{j=1}^{\infty}\sum_{(i_1,\dots,i_j)\in N^j}\int_0^{\infty}\int_0^{t_1-}\dots\int_0^{t_{j-1}-}f_{(i_1,\dots,i_j)}(t_1,\dots,t_j)dH^{(i_j)}_{t_j}\dots dH^{(i_2)}_{t_2}dH^{(i_1)}_{t_1}$$

where $f_{(i_1,\dots,i_j)}$'s are real deterministic functions and $N=\{1,2,3,\dots\}$. A direct consequence is the weaker predictable representation property (PRP) with respect to the same set of orthogonalized martingales, saying that every random variable F in $L^2(\Omega,\mathcal{F})$ has a representation of the form

$$F = E[F] + \sum_{j=1}^{\infty} \int_0^{\infty} \Phi_s^{(i)} dH_s^{(i)},$$

where $\Phi_s^{(i)}$ is predictable (see [20] for more details).

5 Coefficients in the orthogonalization procedure

In order to find the coefficients $\{a_{i,j}^c, 1 \leq i \leq j\}$ of the set of strongly pairwise orthogonal martingales $\{H^{(i),c}, i = 1, 2, \ldots\}$ when $c \in (-\infty, 0)$, we need to find the coefficients of the Laguerre-type polynomials $L_n^{\alpha,c,\sigma^2}(x)$. To do this, let us introduce the notation

$$\lambda_n^{\alpha,c} = \frac{L_{n+1}^{(\alpha)}(c)}{L_n^{(\alpha)}(c)}, \text{ and } \kappa_n^{\alpha,c} = 1 + \sigma^2 K_n(c, c). \tag{13}$$

Notice that $L_n^{(\alpha)}(c) \neq 0$ for every $n = 0, 1, 2, \ldots$, because c does not belong to the support of the classical Laguerre measure. In [6, Th. 1] the authors obtained a connection formula between the monic Laguerre-type orthogonal polynomials and the classical monic Laguerre orthogonal polynomials. Following [21], we will consider the alternative normalization of Laguerre-type polynomials with leading coefficient $\frac{(-1)^n}{n!}\kappa_{n-1}^{\alpha,c}$, and we will denote them by $L_n^{\alpha,c,\sigma^2}(x)$ when $c \in (-\infty, 0)$, and by $L_n^{\alpha,\sigma^2}(x)$ when $c = 0$. Using this normalization, the connection formula [6, Th. 1] between the Laguerre-type and the classical Laguerre polynomials reads

$$(x - c) L_n^{\alpha,c,\sigma^2}(x) = A_n L_{n+1}^{(\alpha)}(x) + B_n L_n^{(\alpha)}(x) + C_n L_{n-1}^{(\alpha)}(x), \tag{14}$$

where

$$\begin{aligned} A_n &= -(n+1)\kappa_{n-1}^{\alpha,c}, \\ B_n &= (n+1)\kappa_{n-1}^{\alpha,c}\lambda_n^{\alpha,c} + (n+\alpha)\frac{\kappa_n^{\alpha,c}}{\lambda_{n-1}^{\alpha,c}}, \\ C_n &= -(n+\alpha)\kappa_n^{\alpha,c}. \end{aligned}$$

Next, we would like to obtain the coefficients $\{b_{k,n}\}_{k=0}^n$ such that

$$L_n^{\alpha,c,\sigma^2}(x) = b_{n,n}x^n + b_{n-1,n}x^{n-1} + \cdots + b_{1,n}x + b_{0,n}. \tag{15}$$

Proposition 1 *Let*

$$L_n^{\alpha,c,\sigma^2}(x) = \sum_{k=0}^{n} b_{k,n}x^k$$

be the Laguerre-type polynomials, orthogonal with respect to the inner product (7). Then, the sequence $\{b_{k,n}\}_{k=0}^n$ *is given by*

$$\begin{cases} b_{n,n} = \dfrac{(-1)^n}{n!}\kappa_{n-1}^{\alpha,c}, \\ b_{k-1,n} = t_{k,n+1} + cb_{k,n}, \quad k = n, n-1, \ldots, 1, \end{cases} \tag{16}$$

where

$$t_{k,n+1} = \begin{cases} \dfrac{(-1)^n}{n!}\kappa_{n-1}^{\alpha,c}, & \text{for } k=n+1, \\ \dfrac{u_{k,n}(\alpha,c)}{n!k!}(-n)_k(\alpha+k+1)_{n-k}, & \text{for } 0\le k\le n, \end{cases}$$

$$u_{k,n}(\alpha,c) = (n+1)\left(\lambda_n^{\alpha,c}+\frac{\alpha+(n+1)}{k-(n+1)}\right)\kappa_{n-1}^{\alpha,c}+\left((k-n)+\frac{(\alpha+n)}{\lambda_{n-1}^{\alpha,c}}\right)\kappa_n^{\alpha,c}$$

and

$$\sum_{k=0}^{n+1} t_{k,n+1}c^k = 0.$$

Proof. The proof will be divided into 2 steps. First, from (14) we will obtain the coefficients $\{t_{k,n+1}\}_{k=0}^{n+1}$ of the polynomial

$$T_{n+1}(x) = (x-c)\,L_n^{\alpha,c,\sigma^2}(x) = \sum_{k=0}^{n+1} t_{k,n+1}x^k, \tag{17}$$

which is obviously related with the Laguerre-type polynomials $L_n^{\alpha,c,\sigma^2}(x)$. Second, we obtain the desired coefficients $\{b_{k,n}\}_{k=0}^{n}$ from $\{t_{k,n+1}\}_{k=0}^{n+1}$.

From (14), and the explicit coefficients for the classical Laguerre polynomials with leading coefficient $\frac{(-1)^n}{n!}$, (see, [14], [21])

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\sum_{k=0}^{n}(-n)_k(\alpha+k+1)_{n-k}\frac{x^k}{k!}, \quad \alpha>-1,$$

we get

$$(x-c)\,L_n^{\alpha,c,\sigma^2}(x) = \frac{(-1)^n}{n!}\kappa_{n-1}^{\alpha,c}x^{n+1}+\frac{1}{n!}\sum_{k=0}^{n}u_{k,n}(\alpha,c)(-n)_k(\alpha+k+1)_{n-k}\frac{x^k}{k!},$$

where

$$u_{k,n}(\alpha,c) = (n+1)\left(\lambda_n^{\alpha,c}+\frac{\alpha+(n+1)}{k-(n+1)}\right)\kappa_{n-1}^{\alpha,c}+\left((k-n)+\frac{(\alpha+n)}{\lambda_{n-1}^{\alpha,c}}\right)\kappa_n^{\alpha,c}.$$

Thus,

$$t_{k,n+1} = \begin{cases} \dfrac{(-1)^n}{n!}\kappa_{n-1}^{\alpha,c}, & \text{for } k=n+1, \\ \dfrac{u_{k,n}(\alpha,c)}{n!k!}(-n)_k(\alpha+k+1)_{n-k}, & \text{for } 0\le k\le n. \end{cases} \tag{18}$$

Next, we deduce the sequence $\{b_{k,n}\}_{k=0}^{n}$ in terms of $\{t_{k,n+1}\}_{k=0}^{n+1}$. (17) makes it obvious that, for every $n\ge 0$

$$\begin{aligned} T_{n+1}(x) &= (x-c)\,L_n^{\alpha,c,\sigma^2}(x), \\ t_{n+1,n+1}x^{n+1}+\sum_{k=1}^{n}t_{k,n+1}x^k+t_{0,n+1} &= b_{n,n}x^{n+1}+\sum_{k=1}^{n}(b_{k-1,n}-cb_k)x^k-cb_{0,n}, \end{aligned}$$

being c a root of $T_{n+1}(x)$, i.e.

$$\sum_{k=0}^{n+1} t_{k,n+1}c^k = 0.$$

Hence, the following relations matching the coefficients of $T_{n+1}(x)$ and $L_n^{\alpha,c,\sigma^2}(x)$ hold

$$\begin{cases} t_{n+1,n+1} = b_{n,n}, \\ t_{k,n+1} = b_{k-1,n} - cb_{k,n}, \quad 1 \leq k \leq n, \\ t_{0,n+1} = -cb_{0,n}. \end{cases} \tag{19}$$

The above provide a simple recursive rule to obtain the n coefficients of $L_n^{\alpha,c,\sigma^2}(x)$, as follows

$$\begin{cases} b_{n,n} = t_{n+1,n+1}, \\ b_{k-1,n} = t_{k,n+1} + cb_{k,n}, \quad k = n, n-1, \ldots, 1. \end{cases} \tag{20}$$

From (18) the statement holds easily. ■

Next, let consider the components of $L_n^{\alpha,c,\sigma^2}(x)$ in (15), as the column vector

$$L_n^{\alpha,c,\sigma^2}(x) \leftrightarrow \begin{bmatrix} b_{0,n} & b_{1,n} & \cdots & b_{n,n} \end{bmatrix}^T. \tag{21}$$

It is clear that the entries in the above vector are the coordinates of $L_n^{\alpha,c,\sigma^2}(x)$ in the standard basis $\mathcal{S} = \{x^n\}_{n\geq 0}$. Consider the basis $\mathcal{B} = \{(\frac{x}{c} - 1)^n\}_{n\geq 0}$, $c \in (-\infty, 0)$ of $\mathbb{P}$. To obtain the desired isometry between martingales and Krall-Laguerre orthogonal polynomials, we need the $\mathcal{B}$-coordinates $\{b_{k,n}^c\}_{k=0}^n$ of $L_n^{\alpha,c,\sigma^2}(x)$, but once we have its $\mathcal{S}$-coordinates $\{b_{k,n}\}_{k=0}^n$, what remains is a trivial linear algebra exercise. Indeed, it is straightforward to see that the ν-th coefficient of $L_n^{\alpha,c,\sigma^2}(x)$ in the basis $\{(\frac{x}{c} - 1)^n\}_{n\geq 0}$ is given by

$$b_{\nu,n}^c = \sum_{k=\nu}^{n} \binom{k}{\nu} c^k b_{k,n}, \quad \nu = 0, \ldots, n. \tag{22}$$

We conclude that the coefficients $\{a_{i,j}^c, 1 \leq i \leq j\}$ such that

$$\{H^{(j),c} = a_{1,j}^c Y^{(1)} + \cdots + a_{j,j}^c Y^{(j)}, \quad j = 1, 2, \ldots\}$$

are given by

$$a_{\nu,n}^c = b_{\nu-1,n-1}^c \text{ for } \nu = 1, \ldots, n. \tag{23}$$

6 Example

In this section, we consider as Lévy process $X = \{X_t, t \geq 0\}$, the one has no deterministic part and the stochastic part consists of a Brownian motion, $\{B_t, t \geq 0\}$ with parameter σ^2 and an independent pure jump part, $\{G_t, t \geq 0\}$ which is called a Gamma process. In this case, the Lévy measure is $\nu(dx) = 1_{\{x>0\}} e^{-x} x^{-1} dx$ and therefore, the polynomials orthogonal with respect this measure are $L_n^{1,c,\sigma^2}(x)$, that is, the Laguerre-type orthogonal polynomials with parameter $\alpha = 1$. In the orthogonalization of the martingales of this process, we employ the space S_1 described in Section 3 where the scalar product defined in (9) is now given by

$$\langle p(x), q(x) \rangle_1 = \int_0^\infty p(x)q(x)xe^{-x}dx + \sigma^2 p(c)q(c).$$

Note that

$$\begin{aligned}
\langle(\frac{x}{c}-1)^{i-1},(\frac{x}{c}-1)^{j-1}\rangle_1 &= \frac{1}{c^{i+j-2}}\int_0^\infty (x-c)^{i+j-2}xe^{-x}dx+\sigma^2 1_{\{i=j=1\}}\\
&= \frac{1}{c^{i+j-2}}\int_0^\infty \sum_{k=0}^{i+j-2}\binom{i+j-2}{k}x^k\,|c|^{i+j-2-k}\,xe^{-x}dx+\sigma^2 1_{\{i=j=1\}}\\
&= \frac{1}{c^{i+j-2}}\sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\int_0^\infty x^{k+1}e^{-x}dx+\sigma^2 1_{\{i=j=1\}}\\
&= \frac{1}{c^{i+j-2}}\sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{i+j-2-k}\,(k+1)!+\sigma^2 1_{\{i=j=1\}}\\
&= \sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{-k}\,(k+1)!+\sigma^2 1_{\{i=j=1\}}.
\end{aligned}$$

As in Section 3, we consider a second space $\tilde{S}_2$ which is the space of all linear transformations of the martingales of the Lévy process, i.e.

$$\tilde{S}_2=\{a_1\widetilde{Y}^{(1)}+a_2\widetilde{Y}^{(2)}+\cdots+a_n\widetilde{Y}^{(n)}:n\in\{1,2,\ldots\},\ a_i\in\mathbb{R},\ i=1,\ldots,n\},$$

where $\widetilde{Y}^{(i)}=\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\frac{1}{c^\ell}Y^{(\ell+1)}$ which is a martingale and $\tilde{S}_2$ is endowed with the following scalar product

$$\begin{aligned}
\langle\widetilde{Y}^{(i)},\widetilde{Y}^{(j)}\rangle_2 &= \langle\sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\frac{1}{c^\ell}Y^{(\ell+1)},\sum_{s=0}^{j-1}\binom{j-1}{s}\frac{1}{c^s}Y^{(s+1)}\rangle_2\\
&= \sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\frac{1}{c^\ell}\sum_{s=0}^{j-1}\binom{j-1}{s}\frac{1}{c^s}\langle Y^{(\ell+1)},Y^{(s+1)}\rangle_2\\
&= \sum_{\ell=0}^{i-1}\binom{i-1}{\ell}\sum_{s=0}^{j-1}\binom{j-1}{s}c^{-(\ell+s)}\,(\ell+s+1)!+\sigma^2 1_{\{i=j=1\}}\\
&= \sum_{k=0}^{i+j-2}\binom{i+j-2}{k}|c|^{-k}\,(k+1)!+\sigma^2 1_{\{i=j=1\}},
\end{aligned}$$

where $k=\ell+s$. So one clearly sees that $(\frac{x}{c}-1)^{n-1}\longleftrightarrow\widetilde{Y}^{(n)}$ is an isometry between S_1 and $\tilde{S}_2$. An orthogonalization of $\{1,(\frac{x}{c}-1),(\frac{x}{c}-1)^2,\ldots\}$ in S_1 gives the Laguerre-type polynomials $L_n^{1,c,\sigma^2}(x)$**,** so by isometry we also find an orthogonalization of $\{\widetilde{Y}^{(1)},\widetilde{Y}^{(2)},\widetilde{Y}^{(3)},\ldots\}$.

Next, we provide the explicit expression of the coefficients $\{b^c_{\nu,n}\}_{\nu=0}^n$ in this particular case. From (2), (3), (6) and [26, (5.1.14)], the explicit expression for (13) is

$$\begin{aligned}
\lambda_n^{1,c} &= \frac{L_{n+1}^{(1)}(c)}{L_n^{(1)}(c)},\\
\kappa_n^{1,c} &= 1+\sigma^2\frac{(n+1)}{n}\frac{L_n^{(2)}(c)L_n^{(1)}(c)-L_{n-1}^{(2)}(c)L_{n+1}^{(1)}(c)}{\Gamma(n+1)}.
\end{aligned}$$

Therefore, the sequence $\{b_{\nu,n}\}_{\nu=0}^n$ determined in (16) is

$$\begin{cases} b_{n,n}=\dfrac{(-1)^n}{n!}\kappa_{n-1}^{1,c},\\ b_{k-1,n}=t_{k,n+1}+cb_{k,n},\quad k=n,n-1,\ldots,1,\end{cases}$$

where

$$t_{k,n+1} = \begin{cases} \dfrac{(-1)^n}{n!}\kappa_{n-1}^{1,c}, & \text{for } k=n+1, \\ \dfrac{u_{k,n}(1,c)}{n!k!}(-n)_k(k+2)_{n-k}, & \text{for } 0 \le k \le n, \end{cases}$$

$$u_{k,n}(1,c) = (n+1)\left(\frac{L_{n+1}^{(1)}(c)}{L_n^{(1)}(c)} + \frac{(n+2)}{k-(n+1)}\right)\left(1+\sigma^2\frac{n}{n-1}\frac{L_{n-1}^{(2)}(c)L_{n-1}^{(1)}(c) - L_{n-2}^{(2)}(c)L_n^{(1)}(c)}{\Gamma(n)}\right)$$
$$+\left((k-n)+(n+1)\frac{L_{n-1}^{(1)}(c)}{L_n^{(1)}(c)}\right)\left(1+\sigma^2\frac{n+1}{n}\frac{L_n^{(2)}(c)L_n^{(1)}(c) - L_{n-1}^{(2)}(c)L_{n+1}^{(1)}(c)}{\Gamma(n+1)}\right).$$

Finally, the n desired coefficients for the isometry $\{b_{\nu,n}^c\}_{\nu=0}^n$, are given under the transformation (22) of the above sequence $\{b_{\nu,n}\}_{\nu=0}^n$ in this particular case, i.e., when $\alpha = 1$, and the coefficients of the set of strongly pairwise orthogonal martingales $\left\{H^{(i),c}, i = 1, 2, \dots\right\}$ are given by (23).

References

[1] R. Alvarez Nodarse and J.J. Moreno-Balcázar, *Asymptotic properties of generalized Laguerre orthogonal polynomials*, Indag. Math. N. S. **15** (2004), 151–165.

[2] J. Bertoin, Lévy Processes. Cambridge University Press, Cambridge, 1996.

[3] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York. (1978).

[4] T.S. Chihara, *Orthogonal polynomials and measures with end point masses*, Rocky Mountain J. Math. **15** (1985), 705–719.

[5] H. Dueñas and F. Marcellán, *Laguerre-Type orthogonal polynomials. Electrostatic interpretation*, Int. J. Pure and Appl. Math. **38** (2007), 345-358.

[6] H. Dueñas, E.J. Huertas and F. Marcellán, *Analytic Properties of Laguerre-type Orthogonal Polynomials*, Integral Transforms Spec. Funct. **22** (2011), 107–122.

[7] B.Xh. Fejzullahu and R.Xh. Zejnullahu, *Orthogonal polynomials with respect to the Laguerre measure perturbed by the canonical transformations*, Integral Transforms Spec. Funct. **17** (2010), 569–580.

[8] E.J. Huertas, F. Marcellán, and F.R. Rafaeli, *Zeros of orthogonal polynomials generated by canonical perturbations of measures*, Appl. Math. Comput. **218** (2012), 7109–7127.

[9] E.J. Huertas, F. Marcellán, and H.E. Pijeira, *An Electrostatic Model for Zeros of Perturbed Laguerre Polynomials*. Accepted for publication in the Proceedings of the American Mathematical Society, July 2012. In press.

[10] F.A. Grunbaum, L. Haine and E. Horozov, *Some functions that generalize the Krall-Laguerre polynomials*, J. Comput. Appl. Math. **106** (1999), 271–297.

[11] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in one variable*, Encyclopedia of Mathematics and its Applications Vol **98**, Cambridge University Press, Cambridge, UK. (2005).

[12] R. Koekoek, *Generalizations of classical Laguerre polynomials and some q-analogues*. Doctoral Dissertation, Technical University Delft.(1990)

[13] J. Koekoek and R. Koekoek, *Differential equation for Koornwinder's generalized Laguerre polynomials*. Proc. Amer. Math. Soc. **112** (1991), 1045–1054.

[14] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98-17, Delft University of Technology, 1998.

[15] T.H. Koornwinder, *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, Canad. Math. Bull. **27** (1984), 205–214.

[16] N.N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York. (1972).

[17] F. Marcellán, A. Branquinho, and J. C. Petronilho, *Classical Orthogonal Polynomials: A Functional Approach*, Acta Appl. Math. **34** (1994), 283–303.

[18] F. Marcellán and A. Ronveaux, *Differential equations for classical type orthogonal polynomials*, Canad. Math. Bull. **32** (1989), 404-411.

[19] A.F. Nikiforov and V.B. Uvarov, *Special Functions of Mathematical Physics: An Unified Approach*, Birkhauser Verlag, Basel. (1988).

[20] D. Nualart and W. Schoutens*, Chaotic and predictable representations for Lévy processes*, Stochastic processes and their applications 90 (2000) 109–122.

[21] W. Schoutens, *An application in stochastics of the Laguerre-type polynomials*, J. Comput. Appl. Math., **133**, Issues 1—2, (2001), 593-600.

[22] W. Schoutens, *Stochastic processes and orthogonal polynomials*, in Lecture Notes in Statist., vol **146**, Springer-Verlag, New York, (2000).

[23] K. Sato, Lévy processes and infinitely divisible distributions. Cambridge University Studies in Advanced Mathematics, Vol. 68. Cambridge University Press, Cambridge (1999).

[24] J.L. Solé and F. Utzet, *Time-Space harmonic polynomials relative to a Lévy processes*, Bernoulli 14 (2008), 1–13.

[25] J.L. Solé and F. Utzet, *On the orthogonal polynomials associated with a Lévy processes*, The Annals of Probability 36 (2008), 765–795.

[26] G. Szegő, *Orthogonal Polynomials*, 4^{th} ed., Amer. Math. Soc. Colloq. Publ. Series, vol **23**, Amer. Math. Soc., Providence, RI. (1975).