Almost rejectionless sampling from Nakagami-m distributions ($m \geq 1$)

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The Nakagami-m distribution is widely used for the simulation of fading channels in wireless communications. A novel, simple and extremely efficient acceptance-rejection algorithm is introduced for the generation of independent Nakagami-m random variables. The proposed method uses another Nakagami density with a half-integer value of the fading parameter, $m_p = n/2 \leq m$, as proposal function, from which samples can be drawn exactly and easily. This novel rejection technique is able to work with arbitrary values of $m \geq 1$, average path energy, Ω, and provides a higher acceptance rate than all currently available methods.

Introduction: The Nakagami m distribution is widely used to model the wireless fading channel because of its good agreement with empirical channel measurements for some urban multipath environments [1]. The Nakagami probability density function (PDF) is $p_o(x) = C_p p(x)$, with $C_p = 2m^m/[\Omega\Gamma(m)]$ and

$$p(x) = x^{2m-1}\exp\left(-\frac{m}{\Omega}x^2\right), \qquad x \geq 0 \tag{1}$$

where $m \geq 0.5$ is the fading parameter, which indicates the fading depth, and $\Omega > 0$ is the average received power.

Several schemes for drawing samples from a Nakagami m PDF have been proposed. On the one hand, when m is an integer or half integer (i.e. $m = n/2$ with $n \in \mathbb{N}$), independent samples can be generated through the square root of a sum of squares of n zero mean independent identically distributed (IID) Gaussian random variables (RVs). On the other hand, for $m \neq n/2$ several techniques have been proposed for drawing correlated samples from (1) [2–4], but all of them present limit ations in terms of complexity, applicability or poor performance for some values of m. Alternatively, several simple and efficient accep tance rejection methods, using different proposals and with increasing accuracy, have been recently introduced [5–7]. Currently, the best results are provided by [7] using a truncated Gaussian PDF as the proposal.

In this Letter we provide an extremely efficient acceptance rejection method for drawing independent samples from non truncated (i.e. without any restriction in the domain) Nakagami PDFs with $m \geq 1$. As a proposal, we consider another Nakagami PDF with an integer or half integer fading parameter, $m_p = n/2 \leq m$, from which samples can be easily and efficiently drawn [8]. Our approach is valid for arbi trary values of the fading parameters $m \geq 1$ (for many practical channels $1 \leq m \leq 15$, as discussed in [9]) and $\Omega > 0$. Furthermore, since our proposal is another Nakagami PDF, the novel rejection scheme provides a very good fit of the target, thus achieving very high acceptance rates that tend to 100% (i.e. exact or rejectionless sampling) when $m \rightarrow +\infty$ and outperforming all the alternative techniques reported in the literature.

Acceptance rejection algorithm: Rejection sampling (RS) is a classical technique for generating samples from an arbitrary target PDF, $p_o(x) = C_p p(x)$ with $x \in \mathcal{D}$ and $C_p = [\int_{\mathcal{D}} p(x)dx]^{-1}$, using an alterna tive simpler proposal PDF, $\pi_o(x) = C_\pi \pi(x)$ with $x \in \mathcal{D}$ and $C_\pi = [\int_{\mathcal{D}} \pi(x)dx]^{-1}$, such that $\pi(x) \geq p(x)$, i.e. $\pi(x)$ is a hat function w.r.t. $p(x)$. RS works by generating samples from the proposal density, $x' \sim \pi_o(x)$, accepting them when $u' \leq p(x')/\pi(x')$, with u' uni formly distributed in [0,1], and rejecting them otherwise. The key per formance measure for RS is the average acceptance rate, $a_R = \int_{\mathcal{D}} p(x)/\pi(x)\pi_o(x)dx = C_\pi/C_p \leq 1$. The value of a_R depends on how close the proposal is to the target, and determines the efficiency of the approach. Hence, the main difficulty when designing an RS algor ithm is finding a good hat function, $\pi(x) \geq p(x)$, such that $\pi(x)$ and $p(x)$ are as close as possible and drawing samples from $\pi_o(x) = C_\pi\pi(x)$ can be done easily and efficiently.

In this work, we consider as target density the PDF given by (1) with $m \geq 1$. As proposal PDF, we suggest using another Nakagami function with different parameters, namely

$$\pi_o(x) \propto \pi(x) = \alpha_p x^{2m_p-1}\exp\left(-\frac{m_p}{\Omega_p}x^2\right), \qquad x \geq 0 \tag{2}$$

with $m_p = n/2, n = \lfloor 2m \rfloor$ (with $\lfloor x \rfloor$ denoting the integer part of $x \in \mathbb{R}$), and the remaining parameters (α_p and Ω_p) adjusted to obtain the same location and value of the maximum in the proposal as in the target:

$$\Omega_p = \frac{2m_p}{2m_p-1}x_{\max}^2 = \Omega\frac{m_p(2m-1)}{m(2m_p-1)} \tag{3}$$

$$\alpha_p = \frac{p(x_{\max})}{x_{\max}^{2m_p-1}\exp\left(-m_p x_{\max}^2/\Omega_p\right)}$$
$$= \exp(m_p - m)\left(\frac{\Omega(2m-1)}{2m}\right)^{m-m_p} \tag{4}$$

where $x_{\max}$ is the location of the maximum of the Nakagami PDF, obtained solving $dp(x)/dx = 0$, which results in

$$x_{\max} = \sqrt{\frac{(2m-1)\Omega}{2m}} \tag{5}$$

Note that we always have $m_p \leq m$, with m_p being an integer or half integer value. Thanks to this choice of m_p and the parameters derived in (3) and (4), we can ensure that: (a) we can draw samples exactly from $\pi_o(x) \propto \pi(x)$ [8]; (b) $\pi(x) \geq p(x)$ for all $x \geq 0$, as proved in the sequel. Fig. 1a shows an example of the target, $p(x)$, our proposal, $\pi(x)$, and the proposal used in [7] for an unbounded domain, which fits the true PDF in a much looser way than ours, thus leading to worse acceptance rates.
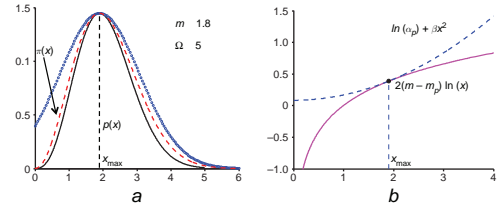
Fig. 1 *Example (Fig. 1a) of Nakagami PDF (solid line) with $m = 1.8$ and $\Omega = 5$, our proposal (dashed line), and Gaussian proposal, $\pi(x) = p(x_{max})\exp(-m/\Omega(x - x_{max})^2)$ used in [7] for an unbounded domain (circles). The two functions (Fig. 1b), $\ln(\alpha_p) + \beta x^2$ (dashed line) and $2(m - m_p)\ln(x)$ (solid line) in (9), when $m = 1.8$ and $\Omega = 5$*

Therefore, our algorithm follows these three simple steps: (a) calculate the parameters of the proposal PDF, $\pi_o(x) \propto \pi(x)$; (b) draw a sample x' from $\pi_o(x)$ using the direct approach described in [8]: generate $2m_p$ IID Gaussian RVs, $z_k \sim \mathcal{N}(0,1)$ for $1 \leq k \leq 2m_p$, and set

$$x' = \sqrt{\frac{\Omega_p}{2m_p}\sum_{k=1}^{2m_p}z_k^2} \tag{6}$$

(c) accept x' with probability $p(x')/\pi(x')$ and discard otherwise. Steps (b) and (c) are repeated until the desired number of samples has been obtained.

Proof of RS inequality: To apply the RS technique we need to ensure that $\pi(x) \geq p(x)$, i.e.

$$\alpha_p x^{2m_p-1}\exp\left(-\frac{m_p x^2}{\Omega_p}\right) \geq x^{2m-1}\exp\left(-\frac{mx^2}{\Omega}\right) \quad \forall x \geq 0 \tag{7}$$

Alternatively, (7) can be easily rewritten as

$$\alpha_p\exp(\beta x^2) \geq x^{2(m-m_p)}, \qquad \forall x \geq 0 \tag{8}$$

where $\beta \triangleq m/\Omega - m_p/\Omega_p$ and $x^{2(m-m_p)}$ presents a sub linear growth, since $0 \leq 2(m - m_p) < 1$. Finally, taking the logarithm on both sides of (8),

$$\ln\alpha_p + \beta x^2 \geq 2(m - m_p)\ln x, \qquad \forall x \geq 0 \tag{9}$$

Now, since $m \geq m_p$ and Ω_p is given by (3), we note that

$$\beta = \frac{m}{\Omega} - \frac{m_p}{\Omega_p} = \frac{m}{\Omega}\left(1 - \frac{2m_p-1}{2m-1}\right) \geq 0 \tag{10}$$

Hence, since we have $\alpha_p > 0$ from (4), the parabola on the left hand side of (9) is an increasing function with an increasing first derivative (i.e. a convex function). Moreover, since $m \geq m_p$, the logarithmic function on the right hand side of (9) is also an increasing function, but with a

decreasing first derivative (i.e. a concave function). Consequently, since both functions are increasing for $x \geq 0$, but $\ln \alpha_p + \beta x^2$ is convex and $2(m - m_p) \ln x$ is concave, they can have at most two intersection points. However, as shown in Fig. 1b, the two functions are tangent at $x = x_{\max}$, which is the only contact point between both curves. To prove this, we need to show that both functions are equal at $x = x_{\max}$, i.e.

$$\ln \alpha_p + \beta x_{\max}^2 = 2(m - m_p) \ln x_{\max}$$
$$= (m - m_p) \ln\left(\frac{\Omega(2m - 1)}{2m}\right) \quad (11)$$

and also that their first derivatives are equal, i.e.

$$\left.\frac{d(\ln \alpha_p + \beta x^2)}{dx}\right|_{x = x_{\max}} = \left.\frac{d(2(m - m_p) \ln x)}{dx}\right|_{x = x_{\max}}$$
$$= \frac{2(m - m_p)}{x_{\max}} = \sqrt{\frac{8m(m - m_p)^2}{(2m - 1)\Omega}} \quad (12)$$

Therefore, since x^2 grows faster than $\ln x$, we can guarantee that $\ln \alpha_p + \beta x^2 \geq 2(m - m_p) \ln x \ \forall x \geq 0$, with equality only at $x = x_{\max}$, and the RS inequality in (7) is satisfied.

Results: To analyse the performance of the algorithm, we have compared the acceptance rate (AR), a_R, of our approach and the Gaussian proposal used in [7] to draw samples from a Nakagami PDF without truncation. The AR of our technique can be obtained analytically:

$$a_{R1} = (2e)^{m-m_p} \frac{\Gamma(m)(2m_p - 1)^{m_p}}{\Gamma(m_p)(2m - 1)^m} \quad (13)$$

with $\Gamma(m)$ denoting the gamma function, whereas the AR for the proposal used in [7] can be approximated for $m \geq 4$ as

$$a_{R2} \simeq \frac{e^{m-1/2}\Gamma(m)(2m - 1)^{1/2-m}}{\sqrt{\pi}2^{m+1/2}} \quad (14)$$

Note that in both cases the AR is independent of the average received power, Ω. Fig. 2 shows this AR, obtained empirically after drawing $N = 6 \times 10^5$ independent samples, for both approaches and several values of the fading depth, m. It can be seen that our technique is extremely efficient, outperforming the proposal used in [7] and providing the best results ever reported in the literature for $m \geq 1$. Furthermore, our technique provides exact sampling (i.e. $a_{R1} = 1$) when m is an integer or half integer, since our proposal is equal to the target in these cases.
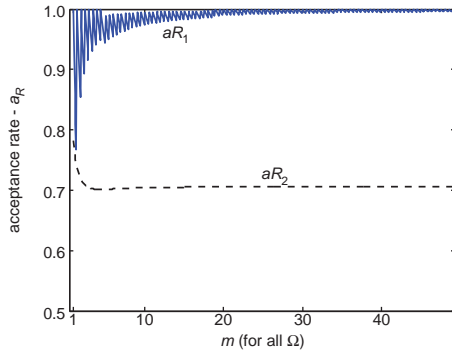
Fig. 2 *Acceptance rate (AR) using our proposal (continuous line) and the one from [7] for an unbounded domain (dashed line) for $1 \leq m \leq 50$*

Conclusion: We have proposed a rejection sampling (RS) scheme for generating Nakagami random variables, with arbitrary values of $m \geq 1$ and Ω, where the proposal PDF is itself another Nakagami m density. The proposed algorithm is simple and extremely efficient, providing the best acceptance rates ever reported in the literature for $m \geq 1$.

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