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The definition of the selection risk measure is based on calling a set-valued portfolio acceptable if it possesses a selection with all individually acceptable marginals. The obtained risk measure is coherent (or convex), law invariant and has values being upper convex closed sets. We describe the dual representation of the selection risk measure and suggest efficient ways of approximating it from below and from above. In case of Kabanov's exchange cone model, it is shown how the selection risk measure relates to the set-valued risk measures considered by Kulikov (2008), Hamel and Heyde (2010), and Hamel et al. (2013).

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Ignacio Cascos* and Ilya Molchanov†

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Since risky positions in multivariate portfolios can be offset by various choices of capital requirements that depend on the exchange rules and related transaction costs, it is natural to assume that the risk measures of random vectors are set-valued. Furthermore, it is reasonable to include the exchange rules in the argument of the risk and so consider risk measures of set-valued portfolios. This situation includes the classical Kabanov's transaction costs model, where the set-valued portfolio is given by the sum of a random vector and an exchange cone, but also a number of further cases of additional liquidity constraints.

The definition of the selection risk measure is based on calling a set-valued portfolio acceptable if it possesses a selection with all individually acceptable marginals. The obtained risk measure is coherent (or convex), law invariant and has values being upper convex closed sets. We describe the dual representation of the selection risk measure and suggest efficient ways of approximating it from below and from above. In case of Kabanov's exchange cone model, it is shown how the selection risk measure relates to the set-valued risk measures considered by Kulikov (2008), Hamel and Heyde (2010) and Hamel et al. (2013).

1 Introduction

Most studies of risk measures and utilities deal with the univariate case, where the gains or liabilities are expressed by a random variable. We refer to [11] and [23] for a thorough treatment of univariate risk measures. The main purpose of univariate risk measures is to determine the capital that need to be added to (or can be released from) a position to make the position acceptable.

Multiasset portfolios in practice are often represented by their total monetary value in a fixed currency with the subsequent calculation of univariate risk measures that can be used

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to determine the overall capital requirements. The main emphasis is put on the dependency structure of the various components of the portfolio, see [2, 7] and [9]. Numerical risk measures for a multivariate portfolio X have been also studied in [7, 25]. The key idea is to consider the expected scalar product of $(-X)$ with a random vector Z and take supremum over all random vectors that share the same distribution with X and possibly over a family of random vectors Z.

However, in many natural applications it is necessary to assess the risk of a vector X in $\mathbb{R}^d$ whose components represent different currencies or gains from various business lines, where profits/losses from one line or currency cannot be used directly to offset the position in a different one. Even in the absence of transaction costs, the exchange rates fluctuate and so may influence the overall risk assessment. Also the regulatory requirements may be very different for different lines (e.g. in case of several states within the same currency area), and moving assets may be subject to transaction costs, taxes or other restrictions. For such cases, it is important to determine the necessary reserves that should be allocated in each line (currency or component) of X in order to make the overall position acceptable. The simplest solution would be to treat each component separately and allocate reserves accordingly, which is not in the interest of financial institutions who might want to use profits from one line to compensate for eventual losses in other ones. Thus, in addition of assessing the risk of the original vector X, one can also evaluate the risk of any other portfolio that may be obtained from X by allowed transactions. In view of this, it is natural to assume that the acceptability may be achieved by several possible choices of capital requirements that form a set of possible values for the risk measure. This suggests the idea of working with *set-valued* risk measures.

Since the first work on multivariate risk measures [17], by now it is accepted that multi-asset risk measures (or utility functions) can be naturally considered as taking values in the space of sets, see [3, 12, 21]. The risk measures for random vectors are mostly considered in relation to Kabanov's transaction costs model, whose main ingredient is a cone $\boldsymbol{K}$ of portfolios available at price zero at the chosen time horizon. If X is the terminal gain, then each random vector with values in $X + \boldsymbol{K}$ is possible to obtain by converting the gain X following the rules determined by $\boldsymbol{K}$. In other words, instead of measuring the risk of X we consider the whole family of random vectors taking values in $X + \boldsymbol{K}$. In relation to this, note that families of random vectors representing attainable gains are often considered in the financial studies of transaction costs models, see e.g. [26].

To relate this framework to the classical setting, a univariate random gain X is replaced by half-line $(-\infty, X]$ and we measure risks of all random variables dominated by X. The monotonicity property of a chosen risk measure r implies that all these risks build the set $\rho(X) = [r(X), \infty)$. If r is subadditive, then ρ becomes superadditive in the inclusion order, i.e. $\rho(X+Y) \supset \rho(X)+\rho(Y)$. Furthermore, $r(X) \leq 0$ if and only if $\rho(X)$ contains the origin. While in the univariate case this construction leads to half-lines that can be summarised by a number, in the multivariate situation it naturally gives rise to so-called upper convex sets, see [12, 21]. A portfolio is acceptable if its risk measure contains the origin. The value of the risk measure is the set of all $a \in \mathbb{R}^d$ such that the portfolio becomes acceptable if a is added to its value.

Note that in the setting of real-valued risk measures [7], the family of all $a \in \mathbb{R}^d$ that make $X + a$ acceptable is a half-space, which apparently could not be the case for cone-based transaction costs models. The setting of Riesz spaces (partially ordered linear spaces), in particular Fréchet lattices and Orlicz spaces has become already common in the theory of risk measures, see [1, 4]. However, these spaces are mostly used to describe the arguments of risk measures whose values belong to the (extended) real line. Furthermore, the space of sets is no longer a Riesz space — while the addition is well defined, the matching subtraction does not exist.

The dual representation for risk measures of random vectors in case of a deterministic exchange cone is obtained in [12] and for the random case in [13], see also [21] who considers both deterministic and random exchange cones. However in case of a random exchange cone, it does not produce law-invariant risk measures — the risk measure in [12, 13, 14, 21] is defined as a function of a random vector X representing the gain, while identically distributed gains might exhibit different properties in relation to the random exchange cone. While the dual representations from [12, 13, 21] are general, they are rather difficult to use in order to calculate risks for given portfolios, since they are given as intersections of half-spaces determined by a rather rich family of random vectors from the dual space. Recent advances in vector optimisation have led to a substantial progress in computation of set-valued risk measures, see [14]. However, the dual approximation also does not explicitly yield the relevant trading (or exchange) strategy that determines transactions suitable to compensate for risks. The construction of set-valued risk measures from [3] is based on the concept of the depth-trimmed region, and their values are easy to calculate numerically or analytically, but it only applies for deterministic exchange cones and often results in marginalised risks (so that the risk measure is a translate of the exchange cone reflected with respect to the origin).

In order to come up with a law invariant risk measure and also cover the case of random exchange cones, we assume that the argument of a risk measure is a random closed set that consists of all attainable portfolios. This set may be generated from a random vector and an exchange cone as $X + \boldsymbol{K}$ (which has been the most important example so far) or can be defined otherwise. In any such case we speak about a set-valued portfolio $\boldsymbol{X}$.

In this paper we suggest a rather simple and intuitive way to measure risks for set-valued portfolios. Our construction is based on the intuitive perception of the family of all gains that may be attained after some exchanges are performed. The crucial step is to consider all random vectors taking values in a random set $\boldsymbol{X}$ (selections of $\boldsymbol{X}$) as possible gains and regard the random set acceptable if it possesses a selection with all acceptable components. In view of this, we do not only determine the necessary capital reserves, but also the way of converting the terminal value of the portfolio into an acceptable one. In particular, our construction applies to random exchange cones and in all cases yields law invariant risk measures. In case of exchange cones, we relate our construction to the dual representation from [12, 21]. Throughout the paper we concentrate on the coherent case and one-period setting, but occasionally comment on convex and multi-period extensions.

We show how to approximate the values of the risk measure both from above (which is the aim of the market regulator) and from below (as the financial institution would aim to do). The bounds provide a feasible alternative to exact analytical calculations of risk. Increasing the family of selections used to define the risk is in the interest of the financial institution, since it makes the capital requirements less stringent. Therefore, in this case the computational burden is passed to the financial institution who aims to increase the family of possible scenarios, quite differently to the dual construction of [21], where the market regulator faces the task of making the acceptance criterion more stringent by approximating from above the exact value of the risk measure.

Section 2 introduces the main concepts of set-valued portfolios and risk measures. Section 3 defines the selection risk measure, which relies on d univariate risk measures $r_1, \dots, r_d$ applied to the components of selections for a set-valued portfolio. In particular, the coherency property of the selection risk measure is established. While throughout the paper we work with coherent risk measures defined on L^p spaces with $p \in [1, \infty]$, the construction can be also based on non-coherent and non-convex univariate risk measures, so that it yields their non-coherent set-valued analogues, such as the value-at-risk.

Section 4 derives lower and upper bounds for risk measures. In Section 5 it is shown that, for the exchange cones setting, the upper bound corresponds to the dual representation of risk measures from [12] and [21]. The special case of deterministic exchange cones is considered in Section 6. In particular, the set-valued risk measures are easy to calculate for comonotonic portfolios.

We briefly comment on scalarisation issues in Section 7, i.e. explain relationships to univariate risk measures constructed for set-valued portfolios, which in case of a deterministic exchange cone are related to those considered in [7, 25].

Section 8 establishes the dual representation of the selection risk measures. The idea is to handle set-valued portfolios through their support functions. The key difficulty consists in handling possibly unbounded values of the support functions. In case of the deterministic exchange cone model and for random exchange cones with $p \in [1, \infty)$, the selection risk measure has the same dual representation as in [12, 21].

Section 9 presents several numerical examples of set-valued risk measures. The algorithms used to approximate risk measures are very transparent and easy to implement in comparison with a considerably more sophisticated approach from [22] used in [14] in order to come up with exact values of set-valued risk measures.

2 Set-valued portfolios and risk measures

Let $\boldsymbol{X}$ be an almost surely non-empty *random closed convex set* in $\mathbb{R}^d$ (often shortly called random set) that represents all feasible terminal gains on d assets expressed in physical units and is called the *set-valued portfolio*. Assume that $\boldsymbol{X}$ is defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Any attainable terminal gain is a random vector ξ that almost surely takes a value from $\boldsymbol{X}$, i.e. $\xi \in \boldsymbol{X}$ a.s., and such ξ is called a *selection* of $\boldsymbol{X}$ or a *feasible portfolio*. We refer to [24] for the modern mathematical theory of random closed sets.

Since the free disposal of assets is allowed, with each point x, the set $\boldsymbol{X}$ also contains all points dominated by x coordinatewisely and so $\boldsymbol{X}$ is said to be a *lower* set in $\mathbb{R}^d$. The *efficient part* of $\boldsymbol{X}$ is the set $\partial^+\boldsymbol{X}$ of all points $x \in \boldsymbol{X}$ such that no other point of $\boldsymbol{X}$ dominates x in the coordinatewise order. Note that $\boldsymbol{X}$ is never bounded, and $\boldsymbol{X}$ is called *quasi-bounded* if $\partial^+\boldsymbol{X}$ is a.s. bounded.

In order to handle set-valued portfolios, we need to define several important operations with sets in $\mathbb{R}^d$. The closure of a set M is denoted by $\operatorname{cl}(M)$. Further,

$$\check{M} = \{-x : x \in M\}$$

denotes the centrally symmetric set to M. The sum $M + L$ of two (deterministic) sets M and L in a linear space is defined as the set $\{x+y : x \in M, y \in L\}$. If one of the summands is compact and the other is closed, the set of pairwise sums is also closed. In particular, the sum $x + M$ of a point and a set is given by $\{x + y : y \in M\}$. For instance, $x + \mathbb{R}^d_-$ is the set of points dominated by x.

The *norm* of a set M is defined as $\|M\| = \sup\{\|x\| : x \in M\}$. A set M is said to be *upper*, if $x \in M$ and $x \leq y$ imply that $y \in M$, where all inequalities between vectors are understood coordinatewisely. Inclusions of sets are always understood in the non-strict sense, e.g. $M \subset L$ allows for the equality $M = L$.

The ε-envelope M^ε of a closed set M is defined as the set of all points x such that the distance between x and the nearest point of M is at most ε. The *Hausdorff distance* $\mathfrak{d}_{\mathrm{H}}(M_1, M_2)$ between two closed sets M_1 and M_2 in $\mathbb{R}^d$ is the smallest $\varepsilon \geq 0$ such that $M_1 \subset M_2^\varepsilon$ and $M_2 \subset M_1^\varepsilon$. The Hausdorff distance metrises the family of compact sets, while it can be infinite for unbounded sets.

The *support function* (see [27, Sec. 1.7]) of a set M in $\mathbb{R}^d$ is defined as

$$h_M(u) = \sup\{\langle u, x\rangle : x \in M\}, \qquad u \in \mathbb{R}^d,$$

where $\langle u, x\rangle$ denotes the scalar product. The support function may take infinite values if M is not bounded. Denote by

$$M' = \{u : |h_M(u)| \neq \infty\}$$

the efficient domain of the support function of M. The set M' is always a convex cone in $\mathbb{R}^d$. If K is a cone in $\mathbb{R}^d$, then K' equals the dual cone to K defined as

$$K^* = \{u \in \mathbb{R}^d : \langle u, x\rangle \leq 0 \text{ for all } x \in K\}. \tag{1}$$

It is well known that an almost surely non-empty random closed set admits at least one measurable selection. In view of subsequent use of risk measures we need selections satisfying some integrability properties. Fix $p \in [1, \infty]$ and consider the space $L^p(\mathbb{R}^d)$ of p-integrable random vectors in $\mathbb{R}^d$ defined on $(\Omega, \mathfrak{F}, \mathbf{P})$. The L^p-norm of ξ is denoted by $\|\xi\|_p$. Furthermore, the family of p-integrable selections of $\boldsymbol{X}$ is denoted by $L^p(\boldsymbol{X})$, $L^\infty(\boldsymbol{X})$ is the family of all essentially bounded selections, and $L^0(\boldsymbol{X})$ is the family of all selections. In the following we assume that $\boldsymbol{X}$ contains at least one p-integrable selection, so that $L^p(\boldsymbol{X})$ is non-empty. Then $\boldsymbol{X}$ is called an *p-integrable* set-valued portfolio.

By [24, Th. 2.1.6], all subsets $\mathcal{X} \subset L^p(\mathbb{R}^d)$ that can be represented as $L^p(\boldsymbol{X})$ for a set-valued portfolio $\boldsymbol{X}$ can be characterised as closed convex decomposable lower subsets of $L^p(\mathbb{R}^d)$. Recall that $\mathcal{X}$ is said to be *decomposable* if with each $\xi_1, \xi_2 \in \mathcal{X}$, the family $\mathcal{X}$ also contains the random vector $\xi_1 \mathbf{1}_A + \xi_2 \mathbf{1}_{A^c}$ for each $A \in \mathfrak{F}$. Furthermore, $\mathcal{X}$ is a lower set in $L^p(\mathbb{R}^d)$ if with each $\xi \in \mathcal{X}$, the family $\mathcal{X}$ also contains all random vectors $\eta \in L^p(\mathbb{R}^d)$ such that $\eta \leq \xi$ a.s. coordinatewisely.

In order to handle diversification effects and the multiperiod setting we need to define the sum of random set-valued portfolios $\boldsymbol{X}$ and $\boldsymbol{Y}$. For this, we start with the sum of $L^p(\boldsymbol{X})$ and $L^p(\boldsymbol{Y})$ being the set of pairwise sums of p-integrable selections of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. The closure of $L^p(\boldsymbol{X}) + L^p(\boldsymbol{Y})$ in $L^p(\mathbb{R}^d)$ is a convex closed decomposable lower subset of $L^p(\mathbb{R}^d)$ and so can be represented as $L^p(\boldsymbol{X} \boxplus \boldsymbol{Y})$ for a random set $\boldsymbol{X} \boxplus \boldsymbol{Y}$. The latter set may be a strict subset of the random set defined pointwisely as $\boldsymbol{X}(\omega) + \boldsymbol{Y}(\omega)$ for $\omega \in \Omega$ using the sum of sets in $\mathbb{R}^d$. However, the both approaches yield the same result for the sum of a set-valued portfolio $\boldsymbol{X}$ and another bounded random closed set with finite p-integrable norm, most importantly a random singleton. The two definitions also coincide in the univariate case, where portfolios are given by half-lines.

Consider now several important examples of set-valued portfolios.

Example 2.1 (Univariate portfolios). If $d = 1$, then $\boldsymbol{X} = (-\infty, X]$ is a half-line and the monotonicity of risks implies that it suffices to consider only its upper bound X as in the classical theory of risk measures.

Example 2.2 (Exchange cones). Let $X \in L^p(\mathbb{R}^d)$ represent gains from d assets. Furthermore, let $\boldsymbol{K} \supset \mathbb{R}^d_-$ be a (possibly random) *exchange cone* representing the family of portfolios available at price zero. Formally, $\boldsymbol{K}$ is a random closed set with values being cones, see [24]. Since we exclude the trivial case of $\boldsymbol{K}$ being the whole space, the origin lies on the boundary of K. Define $\boldsymbol{X} = X + \boldsymbol{K}$, so that selections of $\boldsymbol{X}$ correspond to portfolios that are possible to obtain from X following the exchange rules determined by $\boldsymbol{K}$. Note that $\boldsymbol{X}' = \boldsymbol{K}^*$. If $\boldsymbol{K}$ does not contain any line, then the market has an *efficient friction*. Otherwise, some exchanges are free from transaction costs. The cone $\boldsymbol{K}$ is a half-space if and only if all exchanges do not involve transaction costs. If $\boldsymbol{K}$ is deterministic, then we denote it by K.

Example 2.3 (Cones generated by bid-ask matrix). In case of d currencies, the cone $\boldsymbol{K}$ is usually generated by a *bid-ask matrix*, as in Kabanov's transaction costs model, see [18, 21, 26]. Let $\Pi = (\pi^{(ij)})$ be a (possibly random) matrix of exchange rates, so that $\pi^{(ij)}$ is the number of units of currency i needed to buy one unit of currency j. It is assumed that the elements of Π are positive, the diagonal elements are all one and $\pi^{(ij)} \leq \pi^{(ik)}\pi^{(kj)}$ meaning that a direct exchange is always cheaper than a chain of exchanges. The cone $\boldsymbol{K}$ describes the family of portfolios available at price zero, so that $\boldsymbol{K}$ is spanned by vectors $-e_i$ and $e_j - \pi^{(ij)} e_i$ for $i, j = 1, \dots, d$, where $e_1, \dots, e_d$ are standard basis vectors in $\mathbb{R}^d$.

If the gain X contains derivatives drawn on the exchange rates, then we arrive at the situation when X and the exchange cone $\boldsymbol{K}$ are dependent.

The following examples describe several quasi-bounded set-valued portfolios. Despite the fact that some of them are generated by random vectors, it is essential to treat these portfolios as sets, e.g. for possible diversification effects. The latter means that a sum of such set-valued portfolios is not necessarily equal to the set-valued portfolio generated by the sum of the generating random vectors.

Example 2.4 (No exchanges). The random set $\boldsymbol{X} = X + \mathbb{R}^d_-$ for a random vector X describes the case when no exchanges are allowed.

Example 2.5 (Restricted liquidity). Let $\boldsymbol{X} = X + K$, where $K = \{x : \sum x_i \leq 0, x_i \leq 1, i = 1, \dots, d\}$. Then the exchanges up to the unit volume are at the unit rate free from transaction costs while other exchanges are not allowed. A similar example with transaction costs and a random exchange cone $\boldsymbol{K}$ can be constructed as $\boldsymbol{X} = X + (\boldsymbol{K} \cap (\mathbb{R}^d_- + a))$ for some $a \in \mathbb{R}^d_+$. Further variants can be constructed by combining several cones in order to model the situation with liquidity problems for larger transactions.

Example 2.6. Let $X^{(1)}, \dots, X^{(n)}$ be random vectors in $\mathbb{R}^d$ that represent terminal gains in d lines (e.g. currencies) of n investments. The random set $\boldsymbol{X}$ is defined as the set of all points in $\mathbb{R}^d$ dominated by at least one convex combination of the gains. In other words, $\boldsymbol{X}$ is the sum of $\mathbb{R}^d_-$ and the convex hull of $X^{(1)}, \dots, X^{(n)}$.

Example 2.7. Assume that $\boldsymbol{X} = X + B_\varepsilon + \mathbb{R}^d_-$, where B_ε is the ball of fixed radius ε centred at the origin. This model corresponds to the case, when infinitesimally small transactions are free to exchange at the rate that depends on the balance between the portfolio components.

Further examples also deal with sets of portfolios generated by random vectors. However, in these cases the set generated by $X + a$ is not equal to the set generated by X and then translated by a.

Example 2.8 (Transactions maintaining solvency). Let $\boldsymbol{K}$ be an exchange cone from Example 2.2 and let X be the value of a portfolio. Define $\boldsymbol{X}$ to be the set of points coordinatewisely dominated by a point from $(X + \boldsymbol{K}) \cap \check{\boldsymbol{K}}$ if X belongs to the solvency cone $\check{\boldsymbol{K}} = \{-x : x \in \boldsymbol{K}\}$, and $\boldsymbol{X} = X + \mathbb{R}^d_-$ if $X \notin \check{\boldsymbol{K}}$. In this case no transactions are allowed in the non-solvent case and otherwise all transactions should maintain the solvency of portfolio. Only a solvent portfolio $\boldsymbol{X}$ can be split into the sum of two non-trivial components.

Example 2.9 (Restricted liquidity with transaction costs). Let X be a random vector in $\mathbb{R}^d$ and let $\boldsymbol{X} = (X + \boldsymbol{K}) \cap (c|X| + \mathbb{R}^d_-)$, where $\boldsymbol{K}$ is the exchange cone from Example 2.2, $|X|$ is the random vector composed of the absolute values of the coordinates of X and $c > 1$. In this case, exchanges are allowed up to the absolute amount of the asset in each coordinate times a certain constant factor.

Example 2.10. Let $\boldsymbol{X}$ be the set of points dominated by the segment with end-points (X_1, X_2) and (X_2, X_1) for a bivariate random vector $X = (X_1, X_2)$. This situation corresponds to an arbitrary profit allocation between two different lines without transaction costs up to the amount $|X_1 - X_2|$.

Definition 2.11. A function $\rho(\boldsymbol{X})$ defined on p-integrable set-valued portfolios is called a *set-valued coherent risk measure* if it takes values being upper convex sets and satisfies the following conditions.

1. $\rho(\boldsymbol{X} + a) = \rho(\boldsymbol{X}) - a$ for all $a \in \mathbb{R}^d$ (cash invariance).

2. If $\boldsymbol{X} \subset \boldsymbol{Y}$ a.s., then $\rho(\boldsymbol{X}) \subset \rho(\boldsymbol{Y})$ (monotonicity).

3. $\rho(c\boldsymbol{X}) = c\rho(\boldsymbol{X})$ for all $c > 0$ (homogeneity).

4. $\rho(\boldsymbol{X} \boxplus \boldsymbol{Y}) \supset \rho(\boldsymbol{X}) + \rho(\boldsymbol{Y})$ (superadditivity for inclusion).

The risk measure ρ is said to be *closed-valued* coherent risk measure if its values are closed sets. Furthermore, ρ is said to be a *convex* set-valued risk measure if the homogeneity and superadditivity conditions are replaced by

$$\rho(t\boldsymbol{X} \boxplus (1-t)\boldsymbol{Y}) \supset t\rho(\boldsymbol{X}) + (1-t)\rho(\boldsymbol{Y}) . \tag{2}$$

While (2) actually reads that ρ is concave for the inclusion order, its values can be also ordered by the reverse inclusion relationship, which justifies keeping the name "convex" for it. Similarly, the superadditivity condition becomes subadditivity if the sets are ordered by the reverse inclusion.

Definition 2.11 for closed-valued ρ appears in [21] and [12], with the argument of ρ being a random vector X and for a fixed exchange cone $\boldsymbol{K}$, which in our formulation means that the argument of ρ is the random set $X + \boldsymbol{K}$.

The set-valued portfolio $\boldsymbol{X}$ is *acceptable* if $0 \in \rho(\boldsymbol{X})$. The superadditivity of ρ means that the acceptability of $\boldsymbol{X}$ and $\boldsymbol{Y}$ entails the acceptability of $\boldsymbol{X} \boxplus \boldsymbol{Y}$, exactly as the classical case of coherent risk measures. A risk measure ρ is said to be *proper* if its values are distinct from the whole space on a considered family of set-valued portfolios.

Example 2.12. In the univariate case $d = 1$, $\boldsymbol{X} = (-\infty, X]$ and $\rho(\boldsymbol{X}) = [r(X), \infty)$ for a coherent risk measure r. The portfolio $\boldsymbol{X}$ is acceptable if and only if $r(X) \le 0$.

Example 2.13. If $0 \in \boldsymbol{X}$ a.s., then $\boldsymbol{X}$ is acceptable under any closed-valued coherent risk measure. Indeed, then $\rho(\boldsymbol{X}) \supset \rho(\mathbb{R}^d_-)$, while $\rho(\mathbb{R}^d_-)$ contains the origin by the homogeneity and superadditivity properties and the closedness of the values for ρ. In case of a non-coherent ρ, it is sensible to impose the normalisation condition $\rho(\mathbb{R}^d_-) = \mathbb{R}^d_+$.

Remark 2.14 (Capital requirements in the exchange cones setting). The value of the risk measure $\rho(\boldsymbol{X})$ determines possible capital amounts $a \in \mathbb{R}^d$ that make $\boldsymbol{X} + a$ acceptable. Consider $\boldsymbol{X} = X + \boldsymbol{K}$ for a possibly random exchange cone $\boldsymbol{K}$. The necessary capital should be allocated at time zero, when the exchange rules are determined by a non-random exchange cone K_0. Thus, the initial capital x should be chosen so that $x + K_0$ intersects $\rho(X + \boldsymbol{K})$. In other words, the family of all possible initial capital requirements is given by

$$A_0 = \rho(X + \boldsymbol{K}) + \check{K}_0 .$$

Optimal capital requirements are given by the extremal points from A_0 in the order generated by the cone K_0. If $\boldsymbol{K} = K_0$ is not random, $A_0 = \rho(X + K_0)$. If K_0 is a half-space, meaning that the initial exchanges are free from transaction costs, then A_0 is a half-space too. In this case, the sensible initial capital is given by the tangent point to $\rho(X + \boldsymbol{K})$ in direction of the normal to K_0.

If A_0 is the whole space, which might be the case, for instance, if $\boldsymbol{K}$ and K_0 are two different deterministic half-spaces, then it is possible to release an infinite capital from the position, and this situation should be excluded for the modelling purposes.

Remark 2.15 (Eligible portfolios). In order to simplify the presentation, we assume throughout that all portfolios $a \in \mathbb{R}^d$ can be used to offset the risk. It is straightforward to extend this setting by assuming that the set of eligible portfolios is a proper linear subspace M of $\mathbb{R}^d$, cf. [12, 13, 14].

Remark 2.16 (Multiperiod setting). If $\boldsymbol{X}_0, \dots, \boldsymbol{X}_T$ is a sequence of set-valued portfolios, then the terminal risk is the value of the risk measure on the set valued portfolio obtained as $\boldsymbol{X}_0 \boxplus \cdots \boxplus \boldsymbol{X}_T$.

3 Selection risk measure for set-valued portfolios

Below we explicitly construct set-valued risk measures based on selections of $\boldsymbol{X}$. For this, let $r_1, \dots, r_d$ be law invariant coherent risk measures defined on the space $L^p(\mathbb{R})$ with values in $\mathbb{R} \cup \{\infty\}$. For a random vector $\xi = (\xi_1, \dots, \xi_d) \in L^p(\mathbb{R}^d)$ write

$$\mathbf{r}(\xi) = (r_1(\xi_1), \dots, r_d(\xi_d)) .$$

Random vector ξ is said to be *acceptable* if $\mathbf{r}(\xi) \leq 0$, i.e. $r_i(\xi_i) \leq 0$ for all $i = 1, \dots, d$.

Example 3.1. For $X = (X_1, \dots, X_d) \in L^p(\mathbb{R}^d)$,

$$\mathbf{r}(X) + \mathbb{R}^d_+ = \times_{i=1}^d [r_i(X_i), \infty) ,$$

is the upper orthant generated by $\mathbf{r}(X)$. If $\boldsymbol{X} = X + \mathbb{R}^d_-$ is the random set of all points smaller than or equal to X in the coordinatewise order (as in Example 2.4), then $\rho(\boldsymbol{X}) = \mathbf{r}(X) + \mathbb{R}^d_+$ is a simple set-valued coherent risk measure. It is called the *regulator risk measure* in [14].

Definition 3.2. A p-integrable set-valued portfolio $\boldsymbol{X}$ is said to be *acceptable* if $\mathbf{r}(\xi) \leq 0$ for at least one selection $\xi \in L^p(\boldsymbol{X})$.

The monotonicity property of univariate risk measures $r_1, \dots, r_d$ implies that $\boldsymbol{X}$ is acceptable if and only if its efficient part $\partial^+ \boldsymbol{X}$ admits an acceptable selection.

In the setting of Example 2.2, the acceptability of $\boldsymbol{X} = X + \boldsymbol{K}$ means that it is possible to transfer the assets given by the components of X according to the exchange rules determined by $\boldsymbol{K}$, so that the resulting random vector $X + \eta$ with $\eta \in L^p(\boldsymbol{K})$ has all acceptable components. In Example 2.6, the acceptability of $\boldsymbol{X}$ means that a convex combination of $X^{(1)}, \dots, X^{(n)}$ (possibly with random weights) has all acceptable components.

Definition 3.3. The *selection risk measure* of $\boldsymbol{X}$ is defined as the set of deterministic portfolios x that make $\boldsymbol{X} + x$ acceptable, i.e.

$$\rho_{\mathrm{s},0}(\boldsymbol{X}) = \{x \in \mathbb{R}^d : \boldsymbol{X} + x \text{ is acceptable}\} . \tag{3}$$

Its closed-valued variant is $\rho_{\mathrm{s}}(\boldsymbol{X}) = \operatorname{cl} \rho_{\mathrm{s},0}(\boldsymbol{X})$.

Theorem 3.4. *The selection risk measure $\rho_{\mathrm{s},0}$ defined by* (3) *and its closed-valued variant ρ_{s} are law invariant set-valued coherent risk measures, and*

$$\rho_{\mathrm{s},0}(\boldsymbol{X}) = \bigcup_{\xi \in L^p(\boldsymbol{X})} (\mathbf{r}(\xi) + \mathbb{R}^d_+) . \tag{4}$$

Proof. We show first that $\rho_{s,0}(\boldsymbol{X})$ is an upper set. Let $x \in \rho_{s,0}(\boldsymbol{X})$ and $y \geq x$. If $\xi + x$ is acceptable for some $\xi \in L^p(\boldsymbol{X})$, then also $\xi + y$ is acceptable because of the monotonicity of the components of $\mathbf{r}$. Hence $y \in \rho_s(\boldsymbol{X})$. If $x \in \rho_s(\boldsymbol{X})$ and $y \geq x$, then $x_n \to x$ for a sequence $x_n \in \rho_{s,0}(\boldsymbol{X})$. Consider any $y' > y$, then $x_n \leq y'$ for sufficiently large n, so that $y' \in \rho_{s,0}(\boldsymbol{X})$. Letting y' decrease to y yields that $y \in \rho_s(\boldsymbol{X})$, so that $\rho_s(\boldsymbol{X})$ is an upper set.

In order to confirm the convexity of $\rho_{s,0}(\boldsymbol{X})$, assume that $x, y \in \rho_{s,0}(\boldsymbol{X})$ with $\mathbf{r}(\xi+x) \leq 0$ and $\mathbf{r}(\eta+y) \leq 0$ and take any $\lambda \in (0,1)$. The subadditivity of components of $\mathbf{r}$ implies that

$$\begin{aligned}\mathbf{r}(\lambda\xi + (1-\lambda)\eta + \lambda x + (1-\lambda)y) &= \mathbf{r}(\lambda(\xi + x) + (1-\lambda)(\eta + y)) \\ &\leq \lambda\mathbf{r}(\xi + x) + (1-\lambda)\mathbf{r}(\eta + y) \leq 0\,.\end{aligned}$$

It remains to note that $\lambda\xi + (1-\lambda)\eta$ is also a selection of $\boldsymbol{X}$ in view of the imposed convexity assumption on $\boldsymbol{X}$. Then $\rho_s(\boldsymbol{X})$ is also convex as the closure of a convex set.

The law invariance property is not immediate, since identically distributed random closed sets might have rather different families of selections, see [24, p. 32]. Denote by $\mathfrak{F}_{\boldsymbol{X}}$ the σ-algebra generated by the random closed set $\boldsymbol{X}$, see [24, Def. 1.2.4]. If $\boldsymbol{X}$ is acceptable, then $\mathbf{r}(\xi) \leq 0$ for some $\xi \in L^p(\boldsymbol{X})$. The dilatation monotonicity of law invariant numerical coherent risk measures (see [5]) implies that

$$\mathbf{r}(\mathbf{E}(\xi|\mathfrak{F}_{\boldsymbol{X}})) \leq \mathbf{r}(\xi) \leq 0\,.$$

Therefore, the conditional expectation $\eta = \mathbf{E}(\xi|\mathfrak{F}_{\boldsymbol{X}})$ is also acceptable. The convexity of $\boldsymbol{X}$ implies that η is a p-integrable $\mathfrak{F}_{\boldsymbol{X}}$-measurable selection of $\boldsymbol{X}$. Therefore, $\boldsymbol{X}$ is acceptable if and only if it has an acceptable $\mathfrak{F}_{\boldsymbol{X}}$-measurable selection. It remains to note that two identically distributed random sets have the same families of selections which are measurable with respect to the minimal σ-algebras generated by these sets, see [24, Prop. 1.2.18]. In particular, the intersections of these families with $L^p(\mathbb{R}^d)$ are identical. Thus, $\rho_{s,0}$ (and also ρ_s) are law invariant.

Representation (4) follows from

$$\rho_{s,0}(\boldsymbol{X}) = \bigcup_{\xi \in L^p(\boldsymbol{X})} \{x : \ \mathbf{r}(\xi + x) \leq 0\} = \bigcup_{\xi \in L^p(\boldsymbol{X})} (\mathbf{r}(\xi) + \mathbb{R}^d_+)\,. \tag{5}$$

The first two properties of coherent risk measures follow directly from the definition of $\rho_{s,0}$. The homogeneity and superadditivity follow from the fact that all acceptable random sets build a cone. Indeed, if $\xi \in L^p(\boldsymbol{X})$ is acceptable, then $c\xi$ is an acceptable selection of $c\boldsymbol{X}$ and so $c\boldsymbol{X}$ is acceptable. If $\xi \in L^p(\boldsymbol{X})$ and $\eta \in L^p(\boldsymbol{Y})$ are acceptable, then $\xi + \eta$ is acceptable because the components of $\mathbf{r}$ are coherent risk measures and so $L^p(\boldsymbol{X} \boxplus \boldsymbol{Y})$ contains an acceptable random vector. Thus, $\rho_{s,0}(\boldsymbol{X} \boxplus \boldsymbol{Y}) \supset \rho_{s,0}(\boldsymbol{X}) + \rho_{s,0}(\boldsymbol{Y})$ and by passing to the closure we arrive at the superadditivity property of ρ_s. $\square$

Representation (4) for $p = 1$ was used in [14] to define the so-called *market extension* of a regulator risk measure. It is easy to see that the selection risk measure provides the smallest coherent extension of the regulator risk measure for portfolios defines by means of exchange cones.

Conditions for closedness of $\rho_{s,0}(X + \boldsymbol{K})$ for $p = 1$ in the exchange cone setting was obtained in [10]. The following result establishes the closedness of $\rho_{s,0}(\boldsymbol{X})$ for portfolios with p-integrably bounded essential part. Recall that a random compact set $\boldsymbol{Y}$ is called *p-integrably bounded* if $\mathbf{E}\|\boldsymbol{Y}\|^p < \infty$. If $p = \infty$, this amounts to $\boldsymbol{Y} \subset M$ a.s. for a deterministic compact set M. Further results concerning closedness of $\rho_{s,0}$ are presented in Corollary 8.6 and Corollary 8.8.

Theorem 3.5. *If $\partial^+\boldsymbol{X}$ is p-integrably bounded, then the selection risk measure $\rho_{s,0}(\boldsymbol{X})$ is closed.*

Proof. Let $\xi_n + x_n$ be acceptable and $x_n \to x$. Note that $L^p(\partial^+\boldsymbol{X})$ is weak compact in case $p = 1$ by [24, Th. 2.1.19] and for $p \in (1, \infty)$ by its boundedness in view of the reflexivity of $L^p(\mathbb{R}^d)$. By passing to subsequences we can assume that ξ_n weakly converges to ξ in $L^p(\mathbb{R}^d)$. The dual representation of L^p-risk measures yields that for a random variable η_n, $r_i(\eta_n)$ equals the supremum of $\mathbf{E}(\eta\zeta)$ over a family of random variables ζ in $L^q(\mathbb{R}^d)$. If η_n weakly converges to η in $L^p(\mathbb{R}^d)$, then $\mathbf{E}(\eta_n\zeta) \to \mathbf{E}(\eta\zeta)$, so that $r_i(\eta) \le \liminf r_i(\eta_n)$. Applying this argument to the components of ξ_n we obtain that

$$\mathbf{r}(\xi + x) \le \liminf \mathbf{r}(\xi_n + x_n) \le 0\,, \tag{6}$$

so that $x \in \rho_{s,0}(\partial^+\boldsymbol{X})$. If $p = \infty$, then $\{\xi_n\}$ are uniformly bounded and so have a subsequence that converges in distribution and so can be realised on the same probability space as an almost surely convergent sequence. The Fatou property of the components of $\mathbf{r}$ yields (6). □

Theorem 3.6 (Lipschitz property)**.** *Assume that all components of $\mathbf{r}$ take finite values on $L^p(\mathbb{R})$. Then there exists a constant $C > 0$ such that $\mathfrak{d}_{\mathrm{H}}(\rho_s(\boldsymbol{X}), \rho_s(\boldsymbol{Y})) \le C\|\mathfrak{d}_{\mathrm{H}}(\boldsymbol{X}, \boldsymbol{Y})\|_p$ for all p-integrable set-valued portfolios $\boldsymbol{X}$ and $\boldsymbol{Y}$.*

Proof. For each $\xi \in L^p(\boldsymbol{X})$ there exists $\eta \in L^p(\boldsymbol{Y})$ such that $\|\xi - \eta\|_p \le \varepsilon$. The Lipschitz property of L^p-risk measures (see [11, Lemma 4.3] for $p = \infty$ and [19] for $p \in [1, \infty)$) implies that $\|\mathbf{r}(\xi) - \mathbf{r}(\eta)\| \le C\varepsilon$ for a constant C. By (4), the Hausdorff distance between $\rho_s(\boldsymbol{X})$ and $\rho_s(\boldsymbol{Y})$ is bounded by $C\varepsilon$. □

Corollary 3.7. *Assume that all components of $\mathbf{r}$ take finite values on $L^p(\mathbb{R})$. Let $\mathcal{X}$ be a convex decomposable lower subset of $L^p(\mathbb{R}^d)$, and let $\rho_{s,0}(\mathcal{X})$ be the set of all $x \in \mathbb{R}^d$ such that $\mathcal{X} + x$ contains an acceptable element. Then $\operatorname{cl}\rho_{s,0}(\mathcal{X}) = \rho_{s,0}(\operatorname{cl}\mathcal{X})$, where the closure of $\mathcal{X}$ is taken in $L^p(\mathbb{R}^d)$.*

Proof. Consider $x \in \mathbb{R}^d$ such that $x + \operatorname{cl}\mathcal{X}$ contains an acceptable element $\xi \in L^p(\mathbb{R}^d)$. Then $\|\xi_n - \xi\|_p \to 0$ for $\xi_n \in \mathcal{X}$ and so $\mathbf{r}(x + \xi_n) \le (\varepsilon_n, \dots, \varepsilon_n)$ for $\varepsilon_n \downarrow 0$ in view of the Lipschitz property of $\mathbf{r}$. Thus, $x + (\varepsilon_n, \dots, \varepsilon_n) \in \rho_{s,0}(\mathcal{X})$ and so $x \in \operatorname{cl}\rho_{s,0}(\mathcal{X})$. □

Example 3.8 (Selection expectation). If $\mathbf{r}(\xi) = \mathbf{E}(-\xi)$ is the expectation of $-\xi$, then

$$\rho_s(\boldsymbol{X}) = \mathbf{E}\check{\boldsymbol{X}} + \mathbb{R}^d_+\,,$$

where $\mathbf{E}\check{\boldsymbol{X}}$ is the *selection expectation* of $\check{\boldsymbol{X}}$, i.e. the closure of the set of expectations $\mathbf{E}\xi$ for all integrable selections $\xi \in L^1(\boldsymbol{X})$, see [24, Sec. 2.1]. Thus, a general selection risk measure yields a superadditive generalisation of the selection expectation.

Example 3.9. Assume that $p = \infty$ and all components of $\mathbf{r}$ are given by $r_i(\xi_i) = -\operatorname{essinf} \xi_i$. Then $\boldsymbol{X}$ is acceptable if and only if $\boldsymbol{X} \cap \mathbb{R}_+^d$ is almost surely non-empty, and $\rho_{\mathrm{s}}(\boldsymbol{X})$ is the set of points that belong to the set $\check{\boldsymbol{X}} + \mathbb{R}_+^d$ with probability one.

Example 3.10. If $\boldsymbol{X} = \boldsymbol{K}$ is an exchange cone, then $\rho_{\mathrm{s}}(\boldsymbol{K})$ is a deterministic convex cone that contains $\mathbb{R}_+^d$. If $\boldsymbol{K} = K$ is deterministic, then $\rho_{\mathrm{s}}(K) = \check{K}$.

Remark 3.11. Selection risk measures have a number of further properties.
I. If $\boldsymbol{X}$ is quasi-bounded and, moreover, $\boldsymbol{X} \subset \xi + \mathbb{R}_-^d$ for p-integrable ξ, then

$$\rho_{\mathrm{s}}(\boldsymbol{X}) \subset \mathbf{r}(\xi) + \mathbb{R}_+^d .$$

In particular, this is the case if $\partial^+ X$ is p-integrably bounded.
II. Assume that $\mathbf{r} = (r, \dots, r)$ has all identical components. If $\boldsymbol{X}$ is acceptable, then its orthogonal projection on the linear subspace $\mathbb{H}$ of $\mathbb{R}^d$ generated by any $u_1, \dots, u_k \in \mathbb{R}_+^d$ is also acceptable, and so the risk of the projected $\boldsymbol{X}$ contains the projection of $\rho_{\mathrm{s}}(\boldsymbol{X})$. Indeed, if $\mathbf{r}(\xi) \leq 0$ for $\xi \in L^p(\boldsymbol{X})$, then the projection ξ on $\mathbb{H}$ has coordinates given by linear combinations of coordinates of ξ with non-negative coefficients, and so are acceptable by the subadditivity of r.
III. The conditional expectation of random set $\boldsymbol{X}$ with respect to a σ-algebra $\mathfrak{B}$ is defined as the closure of the set of conditional expectations for all its integrable selections, see [24, Sec. 2.1.6]. The dilatation monotonicity property of components of $\mathbf{r}$ implies that if ξ is acceptable, then $\mathbf{E}(\xi|\mathfrak{B})$ is acceptable. Therefore, ρ_{s} is also *dilatation monotone* meaning that

$$\rho_{\mathrm{s}}(\mathbf{E}(\boldsymbol{X}|\mathfrak{B})) \supset \rho_{\mathrm{s}}(\boldsymbol{X}) .$$

In particular, $\rho_{\mathrm{s}}(\boldsymbol{X}) \subset \rho_{\mathrm{s}}(\mathbf{E}\boldsymbol{X}) = \mathbf{E}\check{\boldsymbol{X}}$. Therefore, the integrability of the support function $h_{\boldsymbol{X}}(u)$ for at least one u provides an easy sufficient condition that guarantees that $\rho_{\mathrm{s}}(\boldsymbol{X})$ is proper, i.e. not equal to the whole space.

Remark 3.12. In the setting of Example 2.2, $\rho_{\mathrm{s}}(X + \boldsymbol{K})$ written as a function of X only becomes a centrally symmetric variant of the coherent utility function considered in [21, Def. 2.1]. It should be noted that the utility function from [21] depends on both X and $\boldsymbol{K}$ and on the dependency structure between them, which might influence the risk if $\boldsymbol{K}$ is random and two identically distributed versions of X are considered. Thus, the utility function from [21] is *not law invariant* as function of X only, if $\boldsymbol{K}$ is random.

If the components of $\mathbf{r}$ are convex risk measures, so that the homogeneity assumption is dropped, then $\rho_{\mathrm{s}}(\boldsymbol{X})$ is a convex set-valued risk measure, which is not necessarily homogeneous. If the components of $\mathbf{r}$ are not law invariant, then ρ_{s} is a possibly not law invariant set-valued risk measure. It is also possible to define the acceptability of selections using a numerical multivariate risk measure from [2, 7]. The following two examples mention *non-convex* risk measures, which are also defined using selections.

Example 3.13. Assume that the components of $\mathbf{r}$ are general monetary risk measures without imposing any convexity properties, e.g. are values-at-risk at the level α, bearing in mind that the resulting set-valued risk measure is no longer coherent and not necessarily law invariant.

Then $\boldsymbol{X}$ is acceptable if and only if there exists $\xi \in L^0(\boldsymbol{X})$ such that $\mathbf{P}\{\xi_i \geq 0\} \geq \alpha$ for all i.

Example 3.14. Let K_0 be a deterministic exchange cone and fix some acceptance level α. Call random vector ξ acceptable if $\mathbf{P}\{\xi \in \check{K}_0\} \geq \alpha$ and note that this condition differs from requiring that $\mathbf{P}\{\xi_i \geq 0\} \geq \alpha$ for all i. Then a set-valued portfolio $\boldsymbol{X}$ is acceptable if and only if $\mathbf{P}\{\boldsymbol{X} \cap \check{K}_0 \neq \emptyset\} \geq \alpha$. If $\boldsymbol{X} = X + \mathbb{R}^d_-$ and $K_0 = \mathbb{R}^d_-$, then $\{x : \mathbf{P}\{X \geq -x\} \geq \alpha\}$ is sometimes termed a multivariate quantile or the value-at-risk of X, see [9] and [12].

4 Bounds for selection risk measures

The family of selections for a random set is typically very rich. A *lower bound* for $\rho_{\mathrm{s}}(\boldsymbol{X})$ can be obtained by restricting the choice of possible selections. The convexity property of the selection risk measure implies that it is bounded from below by the convex hull of the union of $\mathbf{r}(\xi) + \mathbb{R}^d_+$ for the chosen selections ξ.

For instance, it is possible to consider deterministic selections, also called the fixed points of $\boldsymbol{X}$, i.e. the points which belong to $\boldsymbol{X}$ with probability one. For instance, if $a \in \boldsymbol{X}$ a.s., then $\rho_{\mathrm{s}}(\boldsymbol{X}) \supset -a + \mathbb{R}^d_+$. However, this set of fixed points is typically rather poor to reflect essential features related to the variability of $\boldsymbol{X}$.

Another possibility would be to consider selections of $\boldsymbol{X}$ of the form $\xi + a$ for a fixed random vector ξ and a deterministic a. If $\boldsymbol{X} \supset \xi + M$ for a deterministic set M (which always can be chosen to be convex in view of the convexity of $\boldsymbol{X}$), then

$$\rho_{\mathrm{s}}(\boldsymbol{X}) \supset \mathbf{r}(\xi) + \mathbb{R}^d_+ + \check{M}\,. \tag{7}$$

It is possible to tighten the bound by taking the convex hull for the union of the right-hand side for several ξ. The inclusion in (7) can be strict even if $\boldsymbol{X} = \xi + M$, since taking *random* selections of M makes it possible to offset the risks as the following example shows.

Example 4.1. Let $\boldsymbol{X} = \xi + M$, where M is the unit ball and $X = (X_1, X_2)$ is the standard bivariate normal vector. Consider the risk measure $\mathbf{r}$ with two identical components being expected shortfalls at level 0.05. Then $\mathbf{r}(X) + M$ is the upper set generated by the ball of radius one centred at $\mathbf{r}(X) = (2.063, 2.063)$. Consider the selection of M given by $\xi = (\mathbf{1}_{X_1<X_2}, \mathbf{1}_{X_1>X_2})$. By numerical calculation of the risks, it is easily seen that $\mathbf{r}(X + \xi) = (1.22, 1.22)$, which does not belong to $\mathbf{r}(X) + M$.

Below we describe an *upper bound* for $\rho_{\mathrm{s}}(\boldsymbol{X})$, which is also a set-valued coherent risk measure itself. For $x, y \in \mathbb{R}^d$, xy (resp. x/y) denote the vectors composed of pairwise products (resp. ratios) of the coordinates of x and y. If M is a set in $\mathbb{R}^d$, then $My = \{xy : x \in M\}$. By agreement, let $\{0\}/0 = \mathbb{R}$ and $0/0 = -\infty$.

Let $\boldsymbol{Z} \subset L^q(\mathbb{R}^d)$ be a non-empty family of non-negative q-integrable random vectors $Z = (\zeta_1, \dots, \zeta_d)$ in $\mathbb{R}^d$, where $p^{-1} + q^{-1} = 1$.

Recall that $\mathbf{E}(\check{\boldsymbol{X}}Z)$ denotes the selection expectation of $\check{\boldsymbol{X}}$ with the coordinates scaled according to the components of Z, see Example 3.8. It exists, since $\boldsymbol{X}$ is assumed to possess at least one p-integrable selection and so $L^1(\check{\boldsymbol{X}}Z) \neq \emptyset$. Define the set-valued risk measure

$$\rho_{\boldsymbol{Z}}(\boldsymbol{X}) = \bigcap_{Z \in \boldsymbol{Z}} \frac{\mathbf{E}(\check{\boldsymbol{X}}Z)}{\mathbf{E}Z}\,, \tag{8}$$

which is similar to the classical dual representation of coherent risk measures, see [6]. The univariate risk measures (components of $\mathbf{r}$) can be represented as

$$r_i(\xi_i) = \sup_{\zeta_i \in \mathcal{Z}_i} \frac{\mathbf{E}(-\xi_i\zeta_i)}{\mathbf{E}\zeta_i}\,, \quad i = 1, \dots, d\,, \tag{9}$$

where $\mathcal{Z}_1, \dots, \mathcal{Z}_d$ are families of non-negative q-integrable random variables that appear as

$$\mathcal{Z}_i = \{\zeta \in L^q(\mathbb{R}_+) :\ \mathbf{E}(\zeta\xi) \geq 0 \text{ for all } \xi \text{ with } r_i(\xi) \leq 0\}\,. \tag{10}$$

Despite $\mathcal{Z}_i$ contains a random variable a.s. equal zero, letting $0/0 = -\infty$ ensures the validity of (9).

Theorem 4.2. *Assume that the components of $\mathbf{r} = (r_1, \dots, r_d)$ admit the dual representations* (9)*. Then $\rho_{\mathrm{s}}(\boldsymbol{X}) \subset \rho_{\boldsymbol{Z}}(\boldsymbol{X})$ for any family $\boldsymbol{Z}$ of q-integrable random vectors $Z = (\zeta_1, \dots, \zeta_d)$ such that $\zeta_i \in \mathcal{Z}_i$, $i = 1, \dots, d$.*

Proof. In view of the dual representation (9),

$$[r_i(\xi_i), \infty) = \bigcap_{\zeta_i \in \mathcal{Z}_i} [\frac{\mathbf{E}(-\xi_i\zeta_i)}{\mathbf{E}\zeta_i}, \infty)\,,$$

so that

$$\mathbf{r}(\xi) + \mathbb{R}^d_+ \subset \bigcap_{Z \in \boldsymbol{Z}} (\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}^d_+)\,.$$

By (4),

$$\begin{aligned}\rho_{\mathrm{s}}(\boldsymbol{X}) &\subset \operatorname{cl} \bigcup_{\xi \in L^p(\boldsymbol{X})} \bigcap_{Z \in \boldsymbol{Z}} (\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}^d_+) \\ &\subset \operatorname{cl} \bigcap_{Z \in \boldsymbol{Z}} \bigcup_{\xi \in L^p(\boldsymbol{X})} (\frac{\mathbf{E}(-\xi Z)}{\mathbf{E}Z} + \mathbb{R}^d_+) \subset \bigcap_{Z \in \boldsymbol{Z}} \frac{\mathbf{E}(\check{\boldsymbol{X}}Z)}{\mathbf{E}Z}\,.\end{aligned}$$

Note that for each $\xi \in L^p(\boldsymbol{X})$ and $a \in \mathbb{R}^d_+$, the random vector $(-\xi + a)Z$ is an integrable selection of $\check{\boldsymbol{X}}Z$. The closure is omitted, since the selection expectation is already closed by definition. $\square$

Corollary 4.3. *The selection risk measure $\rho_{\mathrm{s}}(\boldsymbol{X})$ is proper if $h_{\boldsymbol{X}}(Zu)$ is integrable for some $u \in \mathbb{R}^d_+$ and $Z = (\zeta_1, \dots, \zeta_d)$ with $\zeta_i \in \mathcal{Z}_i$, $i = 1, \dots, d$.*

Proof. It suffices to note that the imposed condition ensures that $\mathbf{E}(\check{\boldsymbol{X}}Z)$ is a strict subset of $\mathbb{R}^d$. $\square$

Theorem 4.4. *Assume that $\boldsymbol{Z}$ is a non-empty family of non-negative q-integrable random vectors. The functional $\rho_{\boldsymbol{Z}}(X)$ is a closed-valued coherent risk measure, and*

$$\rho_{\boldsymbol{Z}}(\boldsymbol{X}) = \bigcap_{Z\in\boldsymbol{Z}, u\in\mathbb{R}^d_+} \{x:\ \mathbf{E}\langle x, uZ\rangle \geq -\mathbf{E}h_{\boldsymbol{X}}(uZ)\}\,. \tag{11}$$

Proof. The closedness and convexity of $\rho_{\boldsymbol{Z}}(X)$ follow from the fact that it is intersection of half-spaces and it is an upper set since the normals to these half-spaces belong to $\mathbb{R}^d_+$. It is evident that $\rho_{\boldsymbol{Z}}(X)$ is monotonic, monetary and homogeneous. In order to check the superadditivity, note that

$$\begin{aligned}\rho_{\boldsymbol{Z}}(\boldsymbol{X}\boxplus\boldsymbol{Y}) &= \bigcap_{Z\in\boldsymbol{Z}} \frac{\mathbf{E}[(\check{\boldsymbol{X}}\boxplus\check{\boldsymbol{Y}})Z]}{\mathbf{E}Z} \supset \bigcap_{Z\in\boldsymbol{Z}} \frac{\mathbf{E}(\check{\boldsymbol{X}}Z)}{\mathbf{E}Z} + \frac{\mathbf{E}(\check{\boldsymbol{Y}}Z)}{\mathbf{E}Z}\\ &\supset \bigcap_{Z\in\boldsymbol{Z}} \frac{\mathbf{E}(\check{\boldsymbol{X}}Z)}{\mathbf{E}Z} + \bigcap_{Z\in\boldsymbol{Z}} \frac{\mathbf{E}(\check{\boldsymbol{Y}}Z)}{\mathbf{E}Z}\,.\end{aligned}$$

Recall that the support function of the expectation of a set equals the expected value of the support function. Since $\mathbf{E}(\check{\boldsymbol{X}}Z)$ is an upper set,

$$\begin{aligned}\rho_{\boldsymbol{Z}}(\boldsymbol{X}) &= \bigcap_{Z\in\boldsymbol{Z}^q} \frac{\mathbf{E}(\check{\boldsymbol{X}}Z)}{\mathbf{E}Z}\\ &= \bigcap_{Z\in\boldsymbol{Z}^q}\bigcap_{u\in\mathbb{R}^d_-} \{x:\ \mathbf{E}h_{\check{\boldsymbol{X}}Z}(u) \geq \langle x, u\mathbf{E}Z\rangle\}\\ &= \bigcap_{Z\in\boldsymbol{Z}^q}\bigcap_{u\in\mathbb{R}^d_-} \{x:\ \mathbf{E}h_{\boldsymbol{X}}(-Zu) \geq \mathbf{E}\langle x, uZ\rangle\}\\ &= \bigcap_{Z\in\boldsymbol{Z}^q}\bigcap_{u\in\mathbb{R}^d_+} \{x:\ -\mathbf{E}h_{\boldsymbol{X}}(Zu) \leq \mathbf{E}\langle x, uZ\rangle\}\,.\end{aligned}$$

$\square$

Remark 4.5. The risk measure $\rho_{\boldsymbol{Z}}$ is not law invariant in general. It is possible to construct a law invariant (and also tighter) upper bound for the selection risk measure by extending $\boldsymbol{Z}$ to $\tilde{\boldsymbol{Z}}$, so that, with each Z, the family $\tilde{\boldsymbol{Z}}$ contains all random vectors $\tilde{Z}$ that share the distribution with Z.

Proposition 4.6. *Let all components of $\mathbf{r} = (r, \dots, r)$ be identical univariate risk measures whose dual representation* (9) *involves the same family $\mathcal{Z}$ of a.s. non-negative random variables. Consider the family $\boldsymbol{Z}_0$ that consists of all $Z = (\zeta, \dots, \zeta)/\mathbf{E}\zeta$ for $\zeta \in \mathcal{Z}$. Then*

$$\rho_{\boldsymbol{Z}_0}(\boldsymbol{X}) = \bigcap_{u\in\mathbb{R}^d_+} \{x:\ \langle x, u\rangle \geq r(h_{\boldsymbol{X}}(u))\}\,, \tag{12}$$

where $r(h_{\boldsymbol{X}}(u)) = -\infty$ if $h_{\boldsymbol{X}}(u) = \infty$ with positive probability.

Proof. By (11),

$$\rho_{\boldsymbol{Z}_0} = \bigcap_{\zeta\in\mathcal{Z},u\in\mathbb{R}^d_+} \{x:\ \langle x,u\rangle\mathbf{E}\zeta \geq -\mathbf{E}(h_{\boldsymbol{X}}(u)\zeta)\} = \bigcap_{u\in\mathbb{R}^d_+}\{x:\ \langle x,u\rangle \geq \sup_{\zeta\in\mathcal{Z}} \frac{\mathbf{E}(-h_{\boldsymbol{X}}(u)\zeta)}{\mathbf{E}\zeta}\},$$

so it remains to identify the supremum as the dual representation for $r(h_{\boldsymbol{X}}(u))$. □

The bounds for selection measures for set-valued portfolios determined by exchange cones are considered in the subsequent sections. Below we mention two examples of quasi-bounded portfolios.

Example 4.7. Consider portfolio $\boldsymbol{X}$ with $\partial^+\boldsymbol{X}$ being the segment in the plane with end-points X' and X''. Then

$$\rho_{\boldsymbol{Z}_0}(\boldsymbol{X}) = \bigcap_{u\in\mathbb{R}^2_+}\{x:\ \langle x,u\rangle \geq r(\max(\langle X',u\rangle,\langle X'',u\rangle))\}.$$

Example 4.8. Consider portfolio $\boldsymbol{X}$ from Example 2.7. Then $\rho_{\boldsymbol{Z}}(\boldsymbol{X})$ is the intersection of sets

$$\{x:\ \mathbf{E}\langle x,Zu\rangle \geq -\mathbf{E}\langle X,Zu\rangle - \varepsilon\mathbf{E}\|Zu\|\}$$

over $u\in\mathbb{R}^d_+$ and $Z\in\boldsymbol{Z}$. If all components of $\mathbf{r}$ are identical, then

$$\rho_{\boldsymbol{Z}_0}(\boldsymbol{X}) = \bigcap_{u\in\mathbb{R}^d_+}\{x:\ \langle x,u\rangle \geq r(\langle X,u\rangle) - \varepsilon\|u\|\}.$$

5 Random exchange cones

Let $\boldsymbol{X} = X + \boldsymbol{K}$ for an essentially bounded random vector X and a (possibly random) exchange cone $\boldsymbol{K}$. Then

$$\mathbf{r}(X) + \rho_{\mathrm{s}}(\boldsymbol{K}) \subset \rho_{\mathrm{s}}(X+\boldsymbol{K}) \subset \rho_{\mathrm{s}}(\mathbf{E}(X|\boldsymbol{K}) + \boldsymbol{K}), \tag{13}$$

where the first inclusion relation is due to the superadditivity of $\rho_{\mathrm{s}}((X+\mathbb{R}^d_-)+\boldsymbol{K})$ and the second one follows from the dilatation monotonicity of law invariant risk measures, by conditioning on $\boldsymbol{K}$. If X and $\boldsymbol{K}$ are independent, the upper bound becomes $-\mathbf{E}X + \rho_{\mathrm{s}}(\boldsymbol{K})$.

Theorem 4.2 provides a tighter upper bound. Since $\boldsymbol{X} = X+\boldsymbol{K}$ is unbounded, $\mathbf{E}h_{\boldsymbol{X}}(uZ)$ is infinite unless uZ almost surely belongs to the dual cone $\boldsymbol{K}^*$, see (1). Then $h_{\boldsymbol{X}}(uZ) = \langle X,uZ\rangle$ and so the upper bound (11) turns into

$$\rho_{\boldsymbol{Z}}(X+\boldsymbol{K}) = \bigcap_{u\in\mathbb{R}^d_+,\,Z\in\boldsymbol{Z},\,uZ\in\boldsymbol{K}^*\text{ a.s.}} \{x:\ \mathbf{E}\langle x,uZ\rangle \geq -\mathbf{E}\langle X,uZ\rangle\}. \tag{14}$$

If $\boldsymbol{X} = X + \boldsymbol{K}$, then the support function $h_{\boldsymbol{X}}(\cdot)$ is additive on $\boldsymbol{K}^*$ and so the intersection in (14) can be taken only over Z being extreme points of $\boldsymbol{Z}$.

The right-hand side of (14) corresponds to the dual representation for set-valued risk measures (converted from the utilities by the central symmetry) from [21], where it is written as function of X only. It should be noted that the upper bound $\rho_{\boldsymbol{Z}_0}$ from Proposition 4.6 is too conservative for a random exchange cone $\boldsymbol{K}$, since the intersection in (12) is reduced to u that belong to $\boldsymbol{K}^*$ with probability one.

Consider now the frictionless case, where $\boldsymbol{K}$ is a random half-space, so that $\boldsymbol{K}^* = \{t\upsilon : t \geq 0\}$ for a random direction $\upsilon \in \mathbb{R}^d_+$. Then

$$\rho_{\boldsymbol{Z}}(X + \boldsymbol{K}) = \bigcap_{\zeta \in \mathcal{Z}} \{x : \mathbf{E}(\langle x, \upsilon \rangle \zeta) \geq -\mathbf{E}(\langle X, \upsilon \rangle \zeta)\},$$

where $\mathcal{Z}$ is a family of non-negative q-integrable random variables such that the components of $Z = \upsilon\zeta$ belong to the families $\mathcal{Z}_1, \dots, \mathcal{Z}_d$.

Example 5.1 (Bivariate frictionless random exchanges). Consider two currencies exchangeable at random rate $\pi = \pi^{(21)}$ without transaction costs, see Example 2.3. The selection risk measure of $\boldsymbol{X} = X + \boldsymbol{K}$ is the closure of the set of $(r(X_1 + \eta), r(X_2 - \eta\pi))$ for essentially bounded random variables η, see Example 9.3 for a numerical illustration. The value of $\rho_{\mathrm{s}}(\boldsymbol{K})$ is useful to bound the selection risk measure of $\boldsymbol{X} = X + \boldsymbol{K}$, see (13).

Let $\boldsymbol{K}$ be the corresponding exchange cone, being the half-plane with normal $(\pi, 1)$. Assume that $\mathbf{r} = (r, r)$ for two identical L^∞-risk measures, and that π has support bounded away from 0 and ∞. For the purpose of computation of the risk measure it suffices to consider selections of the form $(\eta, -\eta\pi)$, where η is any essentially bounded random variable. Furthermore, it suffices to consider separately almost surely positive and almost surely negative η. Then $\rho_{\mathrm{s}}(\boldsymbol{K})$ is a cone in $\mathbb{R}^2$, and the slopes of the two half-lines that form its boundary can be calculated as

$$\gamma_1 = \sup_{\eta \in L^\infty(\mathbb{R}_+)} \frac{r(-\eta\pi)}{r(\eta)}, \qquad \gamma_2 = \inf_{\eta \in L^\infty(\mathbb{R}_+)} \frac{r(\eta\pi)}{r(-\eta)}.$$

The canonical choices $\eta = 1$ and $\eta = 1/\pi$ yield that

$$\frac{1}{r(1/\pi)} = \max(-r(-\pi), \frac{1}{r(1/\pi)}) \leq \gamma_1 \leq \gamma_2 \leq \min(r(\pi), \frac{-1}{r(-1/\pi)}) = r(\pi),$$

where the maximal and minimal elements above are obtained after applying the Jensen inequality to the dual representation of a univariate risk measure (observe that $f(x) = 1/x$ is convex on the positive half-line). Therefore, $\rho_{\mathrm{s}}(\boldsymbol{K})$ contains the cone with slopes given by $(r(1/\pi))^{-1}$ and $r(\pi)$.

An upper bound for $\rho_{\mathrm{s}}(\boldsymbol{K})$ relies on a lower bound for γ_2 and an upper bound for γ_1. Using the dual representation of r as $r(\xi) = \sup_{\zeta \in \mathcal{Z}} \mathbf{E}(-\zeta\xi)/\mathbf{E}\zeta$ for a family $\mathcal{Z} \subset L^q(\mathbb{R}_+)$ of random variables with positive expectation, for any (positive) η, π we have

$$\frac{r(\eta\pi)}{r(-\eta)} = \frac{\sup_{\zeta_1 \in \mathcal{Z}} \mathbf{E}(-\eta\pi\zeta_1/\mathbf{E}\zeta_1)}{\sup_{\zeta_2 \in \mathcal{Z}} \mathbf{E}(\eta\zeta_2/\mathbf{E}\zeta_2)} = -\inf_{\zeta_1, \zeta_2 \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta_1/\mathbf{E}\zeta_1)}{\mathbf{E}(\eta\zeta_2/\mathbf{E}\zeta_2)} \geq -\inf_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)},$$

where the inequality follows from setting $\zeta_1 = \zeta_2$. Then

$$\begin{aligned}\gamma_2 &\geq \inf_{\eta \in L^\infty(\mathbb{R}_+)} \left\{ -\inf_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)} \right\} \\ &= -\sup_{\eta \in L^\infty(\mathbb{R}_+)} \inf_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)} \\ &\geq -\inf_{\zeta \in \mathcal{Z}} \sup_{\eta \in L^\infty(\mathbb{R}_+)} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)}.\end{aligned}$$

With a similar argument,

$$\gamma_1 \leq -\sup_{\zeta \in \mathcal{Z}} \inf_{\eta \in L^\infty(\mathbb{R}_+)} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)}.$$

If r is the expected shortfall ES_α, so that $\mathcal{Z}$ consists of all random variables taking value 1 with probability α and 0 otherwise, and if π is continuous with a convex support, then

$$\begin{aligned}-\inf_{\zeta \in \mathcal{Z}} \sup_{\eta \in L^\infty(\mathbb{R}_+)} \frac{\mathbf{E}(\eta\pi\zeta)}{\mathbf{E}(\eta\zeta)} &= -\inf_{\zeta \in \mathcal{Z}} \sup_{\eta \in L^\infty(\mathbb{R}_+)} \frac{\mathbf{E}(\pi\zeta)(\eta\zeta)}{\mathbf{E}(\eta\zeta)} \\ &\geq -\inf_{\zeta \in \mathcal{Z}} \sup_{\eta \in L^\infty(\mathbb{R}_+)} \frac{\mathbf{E}(\pi\zeta\eta)}{\mathbf{E}(\eta)} = -\inf_{\zeta \in \mathcal{Z}} \operatorname{esssup}(\pi\zeta) = \mathrm{VaR}_\alpha(\pi),\end{aligned}$$

where VaR_α denotes the value-at-risk, while the upper bound for γ_1 is $\mathrm{VaR}_{1-\alpha}(\pi)$, see Figure 3 the corresponding lower and upper bounds. Thus, for any $\alpha \leq 1/2$ the upper bound for γ_1 is not greater than the lower bound for γ_2 meaning that $\rho_{\mathrm{s}}(\boldsymbol{K})$ is a proper risk measure.

The arguments presented in Example 5.1 yield the following result.

Proposition 5.2. *Let $p = \infty$ and let $d = 2$. Consider the selection measure ρ_{s} generated by $\mathbf{r}$ with all components being expected shortfalls at level $\alpha \leq 1/2$. If $\boldsymbol{X} \subset \xi + \boldsymbol{K}$, where $\boldsymbol{K}$ is a half-plane with normal $(\pi, 1)$ such that $\pi > 0$ a.s. and has the support bounded away from 0 and ∞, and ξ and π are independent, then $\rho_{\mathrm{s}}(\boldsymbol{X})$ is distinct from the whole space.*

6 Deterministic exchange cones

Assume that $\boldsymbol{X} = X + K$ for a deterministic exchange cone K and a p-integrable random vector X. Then $\rho_{\mathrm{s}}(K) = \check{K}$ and (13) yield

$$\mathbf{r}(X) + \check{K} \subset \rho_{\mathrm{s}}(X + K) \subset -\mathbf{E}X + \check{K}.$$

In particular, if $X = a$ is deterministic, then both bounds coincide, so that $\rho_{\mathrm{s}}(a + K) = -a + \check{K}$.

A tighter upper bound can be obtained by Theorem 4.2 as

$$\mathbf{r}(X) + \check{K} \subset \rho_{\mathrm{s}}(X + K) \subset \rho_{\boldsymbol{Z}}(X + K).$$

While the right-hand side is a set-valued coherent risk measure, $\mathbf{r}(X) + \check{K}$ fails to be a risk measure, because it is not always monotone. This is explained by the fact that $\mathbf{r}(\cdot)$ is assumed to be monotone only for the coordinatewise ordering of the argument, while $X \leq Y$ is not necessarily the case if $X + K \subset Y + K$.

Example 6.1. Let K be the cone with the points $(-2, 1)$ and $(1, -2)$ on its boundary, the random vector X takes values $(2, -1)$ and $(-1, 2)$ with equal probabilities, and $Y = (0, 0)$ with probability 1. Clearly $Y + K \subset X + K$, but if we take the negative of the essential infimum as risk measure, then $\mathbf{r}(X)+\check{K} = (1, 1)+\check{K}$ is not a subset of $(0, 0)+\check{K} = \mathbf{r}(Y)+\check{K}$.

The upper bound from Theorem 4.2 turns into

$$\rho_{\boldsymbol{Z}}(X + K) = \bigcap_{Z\in\boldsymbol{Z}} \frac{\mathbf{E}(-XZ) + \mathbf{E}(\check{K}Z)}{\mathbf{E}Z}\,. \tag{15}$$

Note that $\mathbf{E}(\check{K}Z)/\mathbf{E}Z \supset \check{K}$ with a possibly strict inclusion, since the selection expectation of $\check{K}Z$ is defined as the family of expectations of ηZ for all (and possibly random) selections η of $\check{K}$.

If all components of $\mathbf{r}$ are identical risk measures r whose dual representations involve the same family of q-integrable non-negative random variables $\mathcal{Z}$ with positive expectation, it is useful to consider the upper bound $\rho_{\boldsymbol{Z}_0}(X + K)$ for $\rho_{\mathrm{s}}(X + K)$ described in Proposition 4.6. Then

$$\rho_{\boldsymbol{Z}_0}(X + K) = \bigcap_{\zeta\in\mathcal{Z}} \Big(\frac{\mathbf{E}(-X\zeta)}{\mathbf{E}\zeta} + \check{K}\Big)\,.$$

Proposition 6.2. *Assume that* $\mathbf{r}$ *has all identical components being* r*. Then*

$$\rho_{\boldsymbol{Z}_0}(X + K) = \bigcap_{u\in K^*} \{x : r(\langle X, u\rangle) \leq \langle x, u\rangle\}\,, \tag{16}$$

where K^* *is the dual cone to* K*, see* (1).

Proof. The result follows from Proposition 4.6 and the fact that $h_{X+K}(u) = \langle X, u\rangle$ if $u \in K^*$ and otherwise the support function is infinite. □

The coherency of r implies that the intersection in (16) can be taken over all u being extreme elements of K^*, which in dimension 2, implies that $\rho_{\boldsymbol{Z}_0}(X+K)$ is a translate of cone $\check{K}$. In general dimension, if K is a *Riesz cone* (i.e. $\mathbb{R}^d$ with the order generated by K is a Riesz space), then $\rho_{\boldsymbol{Z}_0}(X + K) = a + \check{K}$ with $a \in \mathbb{R}^d$ being the supremum of $\mathbf{E}(-X\zeta)/\mathbf{E}\zeta$ in the order generated by K. Similar risk measures were proposed in [3, Ex. 6.6], where instead of $\{\mathbf{E}(-X\zeta)/\mathbf{E}\zeta : \zeta \in \mathcal{Z}\}$ a depth-trimmed region was considered. In general, $\rho_{\boldsymbol{Z}}(X + K)$ is not necessarily a translate of $\check{K}$.

Example 6.3. Let $\mathcal{Z}$ be the family of indicator random variables $\zeta = \mathbf{1}_A$ for all measurable $A \subset \Omega$ with $\mathbf{P}(A) > \alpha$ for some fixed $\alpha \in (0, 1)$. Then $\rho_{\boldsymbol{Z}_0}(X+K)$ becomes the vector-valued worst conditional expectation (WCE) of X introduced in [17, Ex. 2.5].

If all components r of $\mathbf{r}$ are univariate expected shortfalls ES_α at level α, then for the corresponding family $\mathcal{Z}$ of random variables, (16) yields the set-valued expected shortfall (ES) defined as

$$\mathrm{ES}_\alpha(X+K) = \bigcap_{u\in K^*} \{x:\ \mathrm{ES}_\alpha(\langle X,u\rangle) \le \langle x,u\rangle\}\,. \tag{17}$$

The explicit expression of the set-valued ES if K is a Riesz cone is given in [3, eq. (7.1)] — in that case $\mathrm{ES}_\alpha(X+K)$ is a translate of $\check{K}$. Since the univariate ES and WCE coincide for non-atomic random variables, their multivariate versions coincide by Proposition 4.6 if X has a non-atomic distribution. Since (17) is the risk measure of type $\rho_{\boldsymbol{Z}_0}$, it provides only an upper bound for the selection risk measure defined by applying ES to the individual components of selections of $X+K$. In particular, the acceptability of $X+K$ under $\mathrm{ES}_\alpha(X+K)$ does not necessarily imply the existence of an acceptable selection of $X+K$.

The following result deals with the *comonotonic* case, see e.g. [11, p. 91] for the definition of comonotonicity.

Theorem 6.4. *Assume that* $\mathbf{r}$ *has all identical comonotonic additive components* r*. If the components of* X *are comonotonic and* K *is a deterministic exchange cone, then* $\rho_{\mathrm{s}}(X+K) = \mathbf{r}(X)+\check{K}$*.*

Proof. Since r is comonotonic additive and X is comonotonic, $r(\langle X,a\rangle) = \langle \mathbf{r}(X),a\rangle$ for any $a\in K^*\subset\mathbb{R}^d_+$. Then

$$\mathbf{r}(X)+\check{K} = \bigcap_{a\in K^*} \{x:\ \langle \mathbf{r}(X),a\rangle \le \langle x,a\rangle\}\,,$$

and consequently $\rho_{\mathrm{s}}(X+K) = \mathbf{r}(X)+\check{K} = \rho_{\boldsymbol{Z}_0}(X+K)$. □

Notice that $\mathbf{r}(X)$ is solely determined by the marginal distributions of X. Consequently, if $\tilde{X}$ is a comonotonic rearrangement of X (i.e. a random vector with the same marginal distributions and comonotonic coordinates), then

$$\rho_{\mathrm{s}}(X+K) \supset \mathbf{r}(X)+\check{K} = \mathbf{r}(\tilde{X})+\check{K} = \rho_{\mathrm{s}}(\tilde{X}+K)\,.$$

Recall that the definition of $\rho_{\mathrm{s}}(X+K)$ relies on the family of all selections of $X+K$. The following result shows that $\rho_{\mathrm{s}}(X+K)$ does not change if instead of all selections one uses only those being deterministic functions of X.

Theorem 6.5. *Set-valued portfolio* $\boldsymbol{X} = X+K$ *is acceptable if and only if* $\mathbf{r}(X+\eta(X))\le 0$ *for a selection* $\eta(X)\in K$*, which is a deterministic function of* X*.*

Proof. If $\mathbf{r}(X+\eta(X))\le 0$, then X is acceptable. For the reverse implication, the dilatation monotonicity of the components of $\mathbf{r}$ yields that

$$\mathbf{r}(X+\mathbf{E}(\eta|X)) \le \mathbf{r}(X+\eta)\le 0\,.$$

It remains to note that the conditional expectation $\mathbf{E}(\eta|X)$ is a function of X and also belongs to K, since K is a deterministic convex cone. □

Example 6.6 (No exchanges). If $K = \mathbb{R}^d_-$, then no exchanges are allowed and $\mathbf{r}(X+\eta) \geq \mathbf{r}(X)$ for all $\eta \in K$, so adding η to X could not turn a not acceptable position into an acceptable one. Therefore,

$$\rho_{\mathrm{s}}(X+K) = \mathbf{r}(X) + \mathbb{R}^d_+ ,$$

that is, each component of X should allocate own reserves based on the particular marginal risk measure.

Example 6.7 (Frictionless market with deterministic exchange rates). Assume that the market is frictionless, so that K is a half-space. In terms of the corresponding bid-ask matrix Π (see Example 2.3), the market is frictionless if $\pi^{(ij)} = 1/\pi^{(ji)}$ and $\pi^{(ij)} = \pi^{(ik)}\pi^{(kj)}$ for all i, j, k, so that the matrix Π is determined, by one (say first) column (or row). Assume that

$$K = \{x : \langle x, u\rangle \leq 0\}$$

for a unit vector u, which is the normalised version of $(1, \pi^{(21)}, \dots, \pi^{(d1)})$.

Further assume that all components of $\mathbf{r}$ are identical. Observe that $K^* = \{\lambda u : \lambda \geq 0\}$. If $0 \in \rho_{\boldsymbol{Z}}(X+K)$, then $r(\langle X, u\rangle) \leq 0$. Define $\eta = \langle X, u\rangle u - X \in K$. Then

$$\mathbf{r}(X+\eta) = \mathbf{r}(\langle X, u\rangle u) = r(\langle X, u\rangle)u \leq 0 ,$$

so that X is acceptable. Thus,

$$\rho_{\mathrm{s}}(X+K) = \{x : \; r(\langle X, u\rangle) \leq \langle x, u\rangle\}$$

is a half-space. In the trivial case, when all components of X represent the same currency and so are freely exchangeable at rate one, we obtain that X is acceptable if and only if $r(X_1 + \cdots + X_d) \leq 0$. Otherwise the amount $a = r(X_1 + \cdots + X_d) \leq 0$ should be allocated arbitrarily to the components of X in order to make X acceptable.

Example 6.8. Assume that the components of X represent the same currency and so are freely exchangeable at rate one, so that K is the half-space with normal $(1, \dots, 1)$, while the acceptance criteria for each component differ and so $\mathbf{r} = (r_1, \dots, r_d)$ with possibly different components whose dual representations involve families of q-integrable random variables $\mathcal{Z}_1, \dots, \mathcal{Z}_d$, see (10). It will be shown later on in Corollary 8.8 that $\rho_{\boldsymbol{Z}}(X+K) = \rho_{\mathrm{s}}(X+K)$ for some family $\boldsymbol{Z}$. Then $Z = (\zeta_1, \dots, \zeta_d) \in K^* = \{(t, \dots, t) : \; t \geq 0\}$ if and only if all components of Z are identical random variables, which then belong to $\mathcal{Z}_* = \cap_i \mathcal{Z}_i$. The family $\mathcal{Z}_*$ determines the coherent risk measure r_* called the convex convolution of the components of $r_1, \dots, r_d$. Then X is aceptable under ρ_{s} if and only if $r_*(X_1 + \cdots + X_d) \leq 0$ and

$$\rho_{\mathrm{s}}(X+K) = \rho_{\boldsymbol{Z}}(X+K) = (r_*(X_1 + \cdots + X_d), 0, \dots, 0) + \check{K} .$$

A risk measure of $X + K$ for a deterministic cone K is said to *marginalise* if its values are translates of $\check{K}$ for all $X \in L^p(\mathbb{R}^d)$. Note that $\rho_{\boldsymbol{Z}_0}(X+K)$ marginalises if K is a Riesz cone. While the above examples and Theorem 6.4 provide $\rho_{\mathrm{s}}(X+K)$ of the marginalised form $\mathbf{r}(X) + \check{K}$, this is not always the case, as numerical examples in Section 9 confirm.

The calculation of ρ_{s} for $d = 2$ and $p = \infty$ can be facilitated by using the following result. It shows that $\rho_{\mathrm{s}}(X+K)$ and $\rho_{\boldsymbol{Z}_0}(X+K)$ coincide sufficiently far away from the origin.

Proposition 6.9. *Assume that $p = \infty$ and $\mathbf{r}$ has all identical comonotonically additive components r. If $\boldsymbol{X} = X + K$ for a deterministic cone K and an essentially bounded random vector X in dimension $d = 2$, then for all x with sufficiently large norm, $x \in \rho_{\boldsymbol{Z}_0}(X + K)$ implies that $x \in \rho_{\mathrm{s}}(X + K)$.*

Proof. In dimension 2, we can always assume that the cone K is generated by a bid-ask matrix (see Example 2.3), so that K and its polar are given by

$$K = \{\lambda b_1 + \delta b_2 : \lambda, \delta \geq 0\}, \qquad K^* = \{\lambda a_1 + \delta a_2 : \lambda, \delta \geq 0\},$$

where

$$\begin{aligned} b_1 &= (1, -\pi^{(21)}), \qquad & b_2 &= (-\pi^{(12)}, 1) \\ a_1 &= (\pi^{(21)}, 1), \qquad & a_2 &= (1, \pi^{(12)}). \end{aligned}$$

By Proposition 6.2 and by the coherence of r,

$$\rho_{\boldsymbol{Z}_0}(X + K) = \bigcap_{i=1,2} \{x : r(\langle X, a_i \rangle) \leq \langle x, a_i \rangle\}. \tag{18}$$

In order to obtain a point lying on the boundaries of both $\rho_{\mathrm{s}}(X + K)$ and $\rho_{\boldsymbol{Z}_0}(X + K)$, consider the positive random variable

$$\zeta = (X_1 - \operatorname{essinf} X_1)/\pi^{(12)}.$$

Then

$$X + \zeta b_2 = \big(\operatorname{essinf} X_1, X_2 + (X_1 - \operatorname{essinf} X_1)/\pi^{(12)}\big)$$

has the first a.s. deterministic coordinate and so its components are comonotonic. Since ζ is a.s. non-negative, $X + \zeta b_2$ is a selection of $X + K$. Define

$$x_1 = \mathbf{r}(X + \zeta b_2) = \left(-\operatorname{essinf} X_1, \frac{r(\langle X, a_2 \rangle) + \operatorname{essinf} X_1}{\pi^{(12)}}\right).$$

The fact that $X + \zeta b_2 \in X + K$ guarantees that $x_1 \in \rho_{\mathrm{s}}(X + K) \subset \rho_{\boldsymbol{Z}_0}(X + K)$. Since $a_2 \in \mathbb{R}^2_+$ and r is comonotonic additive, and the components of $X + \zeta b_2$ are comonotonic,

$$\langle x_1, a_2 \rangle = r(\langle X + \zeta b_2, a_2 \rangle) = r(\langle X, a_2, \rangle), \tag{19}$$

where the last equality holds because a_2 and b_2 are orthogonal. Because of (18) and (19), x_1 lies on the supporting line of $\rho_{\boldsymbol{Z}_0}(X + K)$ which is normal to a_2, whence it lies on the boundaries of $\rho_{\mathrm{s}}(X + K)$ and $\rho_{\boldsymbol{Z}_0}(X + K)$.

By a similar argument,

$$x_2 = \left(\frac{r(\langle X, a_1 \rangle) + \operatorname{essinf} X_2}{\pi^{(21)}}, -\operatorname{essinf} X_2\right)$$

lies on the supporting line of $\rho_{\boldsymbol{Z}_0}(X + K)$ which is normal to a_1, so on the boundaries of $\rho_{\mathrm{s}}(X + K)$ and $\rho_{\boldsymbol{Z}_0}(X + K)$.

Let b be the radius of a closed ball centred at the origin and containing the triangle with vertices at x_1, x_2, and the vertex of cone $\rho_{\boldsymbol{Z}_0}(X + K)$. If $x \in \rho_{\boldsymbol{Z}_0}(X + K)$ with $\|x\| \geq b$, then clearly $x \in \lambda x_1 + (1 - \lambda) x_2 + \check{K}$ for some $0 \leq \lambda \leq 1$, which by the convexity of $\rho_{\mathrm{s}}(X + K)$ and the fact that $\rho_{\mathrm{s}}(X + K) = \rho_{\mathrm{s}}(X + K) + \check{K}$ guarantees that $x \in \rho_{\mathrm{s}}(X + K)$. $\square$

7 Numerical risk measures for set-valued portfolios

The set-valued risk measure ρ_{s} gives rise to several numerical coherent risk measures, i.e. functionals $\mathsf{r}(\boldsymbol{X})$ with values in $\mathbb{R} \cup \{\infty\}$ that satisfy the following properties.

1. There exists $u \in \mathbb{R}^d$ auch that $\mathsf{r}(\boldsymbol{X} + a) = \mathsf{r}(\boldsymbol{X}) - \langle a, u\rangle$ for all $a \in \mathbb{R}^d$.
2. If $\boldsymbol{X} \subset \boldsymbol{Y}$, then $\mathsf{r}(\boldsymbol{X}) \geq \mathsf{r}(\boldsymbol{Y})$.
3. $\mathsf{r}(c\boldsymbol{X}) = c\mathsf{r}(\boldsymbol{X})$ for all $c > 0$.
4. $\mathsf{r}(\boldsymbol{X} + \boldsymbol{Y}) \leq \mathsf{r}(\boldsymbol{X}) + \mathsf{r}(\boldsymbol{Y})$.

The canonical scalarisation construction relies on the support function of $\rho_{\mathrm{s}}(\boldsymbol{X})$. Denote

$$\mathsf{r}_u(\boldsymbol{X}) = -h_{\rho_{\mathrm{s}}(\boldsymbol{X})}(-u) = \inf\{\langle u, x\rangle :\ x \in \rho_{\mathrm{s}}(\boldsymbol{X})\}\,, \quad u \in \mathbb{R}^d_+\,.$$

Then $\mathsf{r}_u(\boldsymbol{X})$ is a law invariant coherent risk measure. Note that $\mathsf{r}_u(\boldsymbol{X})$ is infinite for $u \notin \mathbb{R}^d_+$. Furthermore, $\boldsymbol{X}$ is acceptable under ρ_{s}, i.e. $0 \in \rho_{\mathrm{s}}(\boldsymbol{X})$, if and only if $\boldsymbol{X}$ is acceptable under r_u for all $u \in \mathbb{R}^d_+$.

If the exchange cone is trivial, i.e. $\boldsymbol{X} = X + \mathbb{R}^d_-$, then $\rho_{\mathrm{s}}(\boldsymbol{X}) = \mathbf{r}(X) + \mathbb{R}^d_+$ and so

$$\mathsf{r}_u(\boldsymbol{X}) = \langle \mathbf{r}(X), u\rangle\,, \quad u \in \mathbb{R}^d_+\,,$$

is given by a linear combination of the marginal risks.

The constructed real-valued risk measure depends on set-valued portfolio $\boldsymbol{X}$ and so cannot be directly related to the framework of [2, 7, 25], since the risk measures considered there depend on random vectors without using the corresponding exchange cones.

Let $Z \in L^q(\mathbb{R}^d)$. The *max-correlation* risk measure of p-integrable random vector X is defined as

$$\Phi_Z(X) = \sup_{\tilde{Z} \sim Z} \frac{\mathbf{E}\langle -X, \tilde{Z}\rangle}{\mathbf{E}Z}\,,$$

where the supremum is taken over all random vectors $\tilde{Z}$ distributed as Z, see [2, 25]. A general coherent numerical risk measure of X can be represented as the supremum of $\Phi_Z(X)$ over a family $Z \in \boldsymbol{Z}$. Then the set of $x \in \mathbb{R}^d$ that make X acceptable is given by

$$\{x \in \mathbb{R}^d :\ \mathbf{E}\langle x, \tilde{Z}\rangle \geq -\mathbf{E}\langle X, \tilde{Z}\rangle,\ \tilde{Z} \sim Z,\ Z \in \boldsymbol{Z}\}\,.$$

Thus, the set-valued risk measure corresponding to the max-correlation risk is a half-space, while general coherent numerical risk measures are of the type $\rho_{\boldsymbol{Z}}$ given by (14).

8 Dual representation

The representation of the selection risk measure by (5) can be regarded as its primal representation. In this section we arrive at its dual representation.

In order to handle functionals on unbounded random sets, we consider here some special families of random convex sets representing set-valued portfolios. Fix a (possibly random) closed convex cone $\mathbb{G} \subset \mathbb{R}^d_+$, define $\mathbb{G}_1 = \{x \in \mathbb{G} : \|x\| = 1\}$ and consider the space $\mathrm{Lip}^p(\mathbb{G})$ of all random convex closed sets $\boldsymbol{X}$ such that $\boldsymbol{X} = \boldsymbol{X} + \mathbb{R}^d_-$, the effective domain of $h_{\boldsymbol{X}}(\cdot)$ is $\mathbb{G}$, and $h_{\boldsymbol{X}}(u)$ is p-integrable and a.s. Lipschitz on $\mathbb{G}_1$ with the Lipschitz constant $\|\boldsymbol{X}\|_{\mathrm{Lip}}$ having finite pth moment. Since $\mathbb{G}_1$ is a compact subset of the unit sphere, the values $|h_{\boldsymbol{X}}(u)|$, $u \in \mathbb{G}_1$, are bounded by a p-integrable random variable. The norm $\|\boldsymbol{X}\|_L$ of $\boldsymbol{X}$ from $\mathrm{Lip}^p(\mathbb{G})$ is defined as the maximum of the L^p-norms of the maximum of $|h_{\boldsymbol{X}}(u)|$ over $u \in \mathbb{G}_1$ and $\|\boldsymbol{X}\|_{\mathrm{Lip}}$.

Example 8.1. The random set $\boldsymbol{X} = X + \boldsymbol{K}$ for an exchange cone $\boldsymbol{K}$ belongs to $\mathrm{Lip}^p(\boldsymbol{K}^*)$ if X is p-integrable. Considering portfolios of the form $X + \boldsymbol{K}$ from the space $\mathrm{Lip}^p(\boldsymbol{K}^*)$ means that all of them share the same exchange cone $\boldsymbol{K}$. This is reasonable, since the diversification effects affect only the portfolio components, while the (possibly random) exchange cone remains the same for all portfolios.

A quasi-bounded portfolio $\boldsymbol{X}$ belongs to $\mathrm{Lip}^p(\mathbb{R}^d_+)$ if $\partial^+\boldsymbol{X}$ is p-integrably bounded random compact set. If $M = \{(x_1, x_2) \in \mathbb{R}^2_- : x_1x_2 \geq 1\}$, then $\boldsymbol{X} = X + M$ does not belong to $\mathrm{Lip}^p(\mathbb{R}^2_+)$, no matter what X is.

If the effective domain of the support function of $\boldsymbol{X}$ is random, the value for the support function $h_{\boldsymbol{X}}(u)$ at a deterministic u may be infinite. In order to handle this situation, we identify a random set $\boldsymbol{X}$ from $\mathrm{Lip}^p(\mathbb{G})$ with its support function $h_{\boldsymbol{X}}(Y)$ evaluated at Y from the family $L^0(\mathbb{G}_1)$ of all selections of $\mathbb{G}_1$. Since $\mathbb{G}_1$ is a subset of the unit sphere, all its selections are a.s. bounded. It is immediately seen that

$$\mathbf{E}|h_{\boldsymbol{X}}(Y) - h_{\boldsymbol{X}}(Y')|^p \leq \mathbf{E}\|\boldsymbol{X}\|^p_{\mathrm{Lip}}\|Y - Y'\|_\infty\,,$$

so that $h_{\boldsymbol{X}}(\cdot)$ is Lipschitz as a map from $L^0(\mathbb{G}_1)$ with L^∞-norm to the space $L^p(\mathbb{R})$, and the Lipschitz constant of this map equals the L^p-norm of $\|\boldsymbol{X}\|_{\mathrm{Lip}}$. The family of such Lipschitz functions is called the *Lipschitz space* and is also denoted by $\mathrm{Lip}^p(\mathbb{G})$.

Consider linear functionals acting on $\mathrm{Lip}^p(\mathbb{G})$ as

$$\langle \mu, \boldsymbol{X} \rangle = \mathbf{E}\int h_{\boldsymbol{X}}(u)\mu(du) = \mathbf{E}\sum_{i=1}^n \eta_i h_{\boldsymbol{X}}(Y_i)\,, \tag{20}$$

where μ is a signed measure with q-integrable weights $\eta_1, \dots, \eta_n$ assigned to its atoms $Y_1, \dots, Y_n \in L^0(\mathbb{G}_1)$ and any $n \geq 1$. These functionals form a linear space and build a complete family, so their values identify the distribution of $\boldsymbol{X}$. Indeed, by the Cramér–Wold device, it suffices to take μ with atoms located at selections of $\mathbb{G}$ scaled by q-integrable random variables in order to determine the joint distribution of the values of $h_{\boldsymbol{X}}$ at these selections. Note that (20) does not change if μ is a signed measure attaching weights 1 or -1 to $Z_i = \eta_i Y_i \in L^q(\mathbb{G})$, $i = 1, \dots, n$.

It is shown in [16, Th. 4.3] that a sequence from a Lipschitz space of functions defined on any metric space with values in a Banach space E weak-star converges if and only if the norms of the functions are uniformly bounded and their values at each given point weak-star converge in E. In case of set-valued risk measures, E is the L^p space. Thus, the sequence $\boldsymbol{X}_n$ weak-star converges to $\boldsymbol{X}$ in case $p \in [1, \infty)$ if and only if $\|\boldsymbol{X}_n\|_L$ are uniformly bounded and $h_{\boldsymbol{X}_n}(Y)$ weakly converges in L^p to $h_{\boldsymbol{X}}(Y)$ for each $Y \in L^0(\mathbb{G}_1)$. If $p = \infty$ one assumes that $\|\boldsymbol{X}_n\|$, $n \geq 1$, are uniformly bounded and $h_{\boldsymbol{X}_n}(Y)$ converges in probability to $h_{\boldsymbol{X}}(Y)$ for each $Y \in L^0(\mathbb{G}_1)$. Note that the weak convergence and topological duals to Lipschitz spaces have not yet been characterised, see [15].

Consider a general set-valued coherent risk measure ρ from Definition 2.11. If its acceptance set $\mathcal{A}_\rho = \{\boldsymbol{X} \in \mathrm{Lip}^p(\mathbb{G}) : \ \rho(\boldsymbol{X}) \ni 0\}$ is weak-star closed, the risk measure ρ is said to satisfy the *Fatou property*. In view of the cash invariance property, this formulation of the Fatou property is equivalent to its conventional variant saying that $\rho(\boldsymbol{X}) \supset \limsup \rho(\boldsymbol{X}_n)$ if $\boldsymbol{X}_n$ weak-star converges to $\boldsymbol{X}$ in $\mathrm{Lip}^p(\mathbb{G})$, cf. [21]. Recall that the upper limit, $\limsup M_n$, of a sequence of sets $\{M_n, n \geq 1\}$ is the set of limits for all convergent subsequences of $\{x_n\}$, where $x_n \in M_n$ for all n, see [24, Def. B.4].

Theorem 8.2. *A function ρ on random sets from $\mathrm{Lip}^p(\mathbb{G})$ with values being convex closed upper sets is a set-valued coherent risk measure with the Fatou property if and only if $\rho(\boldsymbol{X}) = \rho_{\boldsymbol{Z}}(\boldsymbol{X})$ given by (11) for a certain family $\boldsymbol{Z} \subset L^q(\mathbb{G})$.*

Proof. Necessity. Note first that (11) can be equivalently written as

$$\rho_{\boldsymbol{Z}}(\boldsymbol{X}) = \{x : \ \langle x, \mathbf{E} \int u\mu(du)\rangle \geq -\mathbf{E} \int h_{\boldsymbol{X}}(u)\mu(du)\,,\ \mu \in \mathcal{M}\} \tag{21}$$

for $\mathcal{M}$ being the family of counting measures with atoms from $\boldsymbol{Z}$.

The dual cone $\mathcal{M}$ to the family $\mathcal{A}_\rho$ of acceptance sets for ρ is the family of signed measures μ on $\mathbb{G}$ with q-integrable total variation such that $\langle \mu, \boldsymbol{X}\rangle \geq 0$ for all $\boldsymbol{X} \in \mathcal{A}_\rho$. Since, $\boldsymbol{X} = \boldsymbol{Y} + \mathbb{G}^*$ is acceptable for each random compact convex set $\boldsymbol{Y}$ containing the origin, each $\mu \in \mathcal{M}$ is non-negative. By the bipolar theorem, $\mathcal{A}_\rho$ is the dual cone to $\mathcal{M}$.

Sufficiency. By Theorem 4.4, $\rho_{\boldsymbol{Z}}$ is a set-valued coherent risk measure. By an application of the dominated convergence theorem and noticing that the weak-star convergence implies the uniform boundedness of the norms, its acceptance set is weak-star closed. □

Theorem 8.2 holds also for a set-valued coherent risk measure defined on a convex subfamily of random convex sets from $\mathrm{Lip}^p(\mathbb{G})$, if the intersection of this subfamily with the acceptance set of ρ is weak-star closed.

Example 8.3 (Random cones). If $\boldsymbol{X} = X + \boldsymbol{K}$ for a random exchange cone $\boldsymbol{K}$ and p-integrable random vector X, then $\boldsymbol{X} \in \mathrm{Lip}^p(\boldsymbol{K}^*)$. Therefore, set-valued risk measures defined on sets $\boldsymbol{X} = X + \boldsymbol{K}$ and satisfying the Fatou property can be represented by (11), which is exactly the dual representation from [21] obtained for $p = \infty$. The Fatou property is formulated as $\rho(X + \boldsymbol{K}) \supset \limsup \rho(X_n + \boldsymbol{K})$ if X_n converges to X in probability and has norms uniformly a.s. bounded by one.

Since the selection risk measure ρ_{s} is a special case of a general set-valued coherent risk measure, $\rho_{\mathrm{s}} = \rho_{\boldsymbol{Z}}$ for a suitable (and possibly non-unique) family $\boldsymbol{Z}$ provided ρ_{s} satisfies the Fatou property. Note that the Fatou property of ρ_{s} is weaker than the Fatou property of $\rho_{\mathrm{s},0}$, which is established in the following theorem.

Theorem 8.4. *For $p \in [1, \infty)$, the selection risk measure $\rho_{\mathrm{s},0}$ satisfies the Fatou property on random sets from* $\mathrm{Lip}^p(\mathbb{G})$.

Proof. Assume that $\boldsymbol{X}_n$ weak-star converges to $\boldsymbol{X}$, so that $\|\boldsymbol{X}_n\|_{\mathrm{Lip}} \le c$ a.s. for a constant c and $h_{\boldsymbol{X}_n}(Y)$ weakly converges in L^p for each $Y \in L^0(\mathbb{G}_1)$.

Fix any $\varepsilon > 0$ and choose a finite set of selections $\boldsymbol{Y} = \{Y_1, \dots, Y_N\} \subset L^0(\mathbb{G}_1)$ such that the (ε/c)-neighbourhood of $\boldsymbol{Y}$ covers $\mathbb{G}_1$ with probability one. Since the uniform norm of $h_{\boldsymbol{X}_n}(Y_i)$ is bounded by c, it is possible to find a subsequence $n(k)$ such that the joint distribution of $h_{\boldsymbol{X}_{n(k)}}(Y_1), \dots, h_{\boldsymbol{X}_{n(k)}}(Y_N)$ converges to the distribution of $h_{\boldsymbol{X}}(Y_1), \dots, h_{\boldsymbol{X}}(Y_N)$. By the extended Skorohod coupling (see [20, Cor. 6.12]) it is possible to realise the random sets $\boldsymbol{X}_{n(k)}$ on the same probability space in order to ensure the convergence almost surely.

The dominated convergence theorem implies that $\|h_{\boldsymbol{X}_{n(k)}}(Y_i) - h_{\boldsymbol{X}}(Y_i)\|_p \to 0$ as $k \to \infty$. Since the Lipschitz constants of $h_{\boldsymbol{X}_{n(k)}}$ are bounded by c,

$$\sup_{u \in \mathbb{G}_1} \|h_{\boldsymbol{X}_{n(k)}}(u) - h_{\boldsymbol{X}}(u)\|_p \le \frac{\varepsilon}{2} + \max_{i=1,\dots,N} \|h_{\boldsymbol{X}_{n(k)}}(Y_i) - h_{\boldsymbol{X}}(Y_i)\|_p \,.$$

Let ξ_n be an acceptable selection of $\boldsymbol{X}_n$, $n \ge 1$. The uniform convergence of the support functions of $\boldsymbol{X}_{n(k)}$ to the support function of $\boldsymbol{X}$ implies that for each $\varepsilon > 0$ there exists a selection $\xi \in L^p(\boldsymbol{X})$ such that $\|\xi_n - \xi\|_p \le \varepsilon$. The monotonicity property of univariate risk measures implies that $\mathbf{r}(\xi) \le (\varepsilon, \dots, \varepsilon)$, whence ξ is acceptable and so $\boldsymbol{X}$ is acceptable under $\rho_{\mathrm{s},0}$. □

Theorem 8.5. *For $p = \infty$, the selection risk measure $\rho_{\mathrm{s},0}$ satisfies the* Fatou property *on quasi-bounded set-valued portfolios.*

Proof. If necessary by passing to a subsequence, assume that $x_n \in \rho_{\mathrm{s},0}(\boldsymbol{X}_n)$ and $x_n \to 0$. Then, for all n there exists $\xi_n \in L^p(\partial^+ \boldsymbol{X}_n)$ such that $\mathbf{r}(\xi_n + x_n) \le 0$.

The quasi-bounded set-valued portfolios belong to the space $\mathrm{Lip}^p(\mathbb{R}^d_+)$. The weak-star convergence of $\boldsymbol{X}_n$ implies that the norms of support functions on $\mathbb{R}^d_+$ are uniformly bounded, so that $\partial^+ \boldsymbol{X}_n$, $n \ge 1$, are all subsets of a fixed compact set M. Replace M by its ε-envelope, so that $\partial^+ \boldsymbol{X}_n + x_n$ is also a subset of M. By [24, Th. 1.6.21], the convergence in probability of the values $h_{\boldsymbol{X}_n}(u)$ for all $u \in \mathbb{R}^d_+$ and the uniform boundedness of the sets imply that $\boldsymbol{X}_n$ converges to $\boldsymbol{X}$ in probability as random closed sets. Since the family of convex subsets of a compact set is compact in the Hausdorff metric and $\xi_n \in M$, it is possible to find a subsequence of $(\partial^+ \boldsymbol{X}_n, \xi_n)$ that converges to $(\partial^+ \boldsymbol{X}, \xi)$. Recall also that the convergence in probability implies the weak convergence of random closed sets, see [24, Cor. 1.6.22]. Pass to the chosen subsequence and realise the pairs $(\boldsymbol{X}_n, \xi_n)$ on the same probability space, so that $\boldsymbol{X}_n \to \boldsymbol{X}$ and $\xi_n \to \xi$ a.s.

Since the components of $\mathbf{r}$ are law invariant, $\{\xi_n\}$ are uniformly bounded, and the components of $\mathbf{r}$ have the Fatou property (ensured by imposing the law invariance),

$$\mathbf{r}(\xi) \le \liminf \mathbf{r}(\xi_n + x_n) \le 0\,.$$

Thus, $0 \in \rho_{\mathrm{s},0}(\boldsymbol{X})$. $\square$

Corollary 8.6. *The selection risk measure $\rho_{\mathrm{s}}(\boldsymbol{X})$ on set-valued portfolios $\boldsymbol{X}$ from* $\mathrm{Lip}^p(\mathbb{G})$ *for $p \in [1, \infty)$ and on quasi-bounded portfolios for $p = \infty$ equals $\rho_{\mathrm{s},0}(\boldsymbol{X})$ and admits representation as $\rho_{\boldsymbol{Z}}(\boldsymbol{X})$ for a family $\boldsymbol{Z} \subset L^q(\mathbb{R}^d)$.*

Proof. Consider the sequence $\boldsymbol{X}_n = \boldsymbol{X}$, $n \ge 1$. Then the Fatou property of $\rho_{\mathrm{s},0}$ established in Theorems 8.4 and 8.5 implies that the upper limit of $\rho_{\mathrm{s},0}(\boldsymbol{X})$ (being the closure of $\rho_{\mathrm{s},0}(\boldsymbol{X})$) is a subset of $\rho_{\mathrm{s},0}(\boldsymbol{X})$, so that $\rho_{\mathrm{s},0}(\boldsymbol{X})$ is closed. $\square$

The following result establishes the Fatou property for the selection risk measure in case $p = \infty$ for portfolios obtained as $\boldsymbol{X} = X + K$ for a deterministic exchange cone K.

Theorem 8.7. *For $p = \infty$, the selection risk measure $\rho_{\mathrm{s},0}$ satisfies the Fatou property on the family of portfolios obtained as $X + K$ for an essentially bounded random vector X and a fixed deterministic exchange cone K.*

Proof. In case of a fixed deterministic exchange cone K, the weak-star convergence of $\boldsymbol{X}_n = X_n + K$ means that X_n, $n \ge 1$, converge in probability and are uniformly bounded. For each $u \in K^*$, the numerical risk measure $-h_{\rho_{\mathrm{s},0}(X+K)}(-u)$ defined as a function of a random vector X is law invariant and coherent risk measure, so that it satisfies the Fatou property by [8, Th. 2.6] which also applies if the cash invariance property is relaxed as in [2]. Thus, if X_n converges to X in probability and $\|X_n\| \le 1$ a.s. for all n, then

$$h_{\rho_{\mathrm{s},0}(X+K)}(u) \ge \limsup h_{\rho_{\mathrm{s},0}(X_n+K)}(u)\,.$$

If $x \in \limsup \rho_{\mathrm{s},0}(X_n + K)$, then $x = \lim x_{n_k}$ for a certain sequence $x_{n_k} \in \rho_{\mathrm{s},0}(X_{n_k} + K)$. Therefore,

$$\langle x, u\rangle = \lim \langle x_{n_k}, u\rangle \le \lim h_{\rho_{\mathrm{s},0}(X_{n_k}+K)}(u) \le \limsup h_{\rho_{\mathrm{s},0}(X_n+K)}(u)\,.$$

Thus,

$$h_{\rho_{\mathrm{s},0}(X+K)}(u) \ge h_{\limsup \rho_{\mathrm{s},0}(X_n+K)}(u)\,, \quad u \in K^*\,,$$

so that

$$\rho_{\mathrm{s},0}(X+K) \supset \limsup \rho_{\mathrm{s},0}(X_n + K)\,.$$

$\square$

Corollary 8.8. *In case $p = \infty$, the selection risk measure $\rho_{\mathrm{s},0}(\boldsymbol{X})$ for set-valued portfolios $\boldsymbol{X} = X + K$ with a fixed deterministic exchange cone K has closed values and is equal to $\rho_{\boldsymbol{Z}}$ for a certain family of integrable random vectors $\boldsymbol{Z}$.*

Proof. It is shown in [21] that a set-valued risk measure with $p = \infty$ satisfying the Fatou property has the dual representation as $\rho_{\boldsymbol{Z}}$, and so ρ_{s} admits exactly the same dual representation. $\square$

It should be noted that not all risk measures $\rho_{\boldsymbol{Z}}$ are selection risk measures, and so the acceptability of $\boldsymbol{X}$ under $\rho_{\boldsymbol{Z}}$ does not immediately imply the existence of an acceptable selection and so does not guarantee the existence of a trading strategy that eliminates the risk. While the calculation of the risk measure $\rho_{\boldsymbol{Z}}$ in [21] is rather complicated, its representation as the selection risk measure opens a possibility for an approximation of its values from below by exploring selections of $X + K$.

9 Computation and approximation of risk measures

The evaluation of $\rho_{\mathrm{s}}(\boldsymbol{X})$ involves calculation of $\mathbf{r}(\xi)$ for all selections $\xi \in L^p(\boldsymbol{X})$. The family of such selections is immense, and in application only several possible selections can be considered. A wider choice of selections is in the interest of the financial institution, since it produces a larger set approximating $\rho_{\mathrm{s}}(\boldsymbol{X})$ from below and so reduces the required capital reserves. For upper bounds, one can use the risk measure $\rho_{\boldsymbol{Z}}(\boldsymbol{X})$ or its superset obtained by restricting the family $\boldsymbol{Z}$, e.g. as $\boldsymbol{Z}_0$.

In view of (4) and the convexity of its values, $\rho_{\mathrm{s}}(\boldsymbol{X})$ contains the convex hull of the union of $\mathbf{r}(\xi) + \mathbb{R}^d_+$ for any collection of selections $\xi \in L^p(\boldsymbol{X})$. It should be noted that this convex hull is not subadditive in general (unless the family of selections builds a cone) and so itself cannot be used as a risk measure, while providing a reasonable approximation for it. It is possible to start with some "natural" selections ξ' and ξ'' and consider all their convex combinations or combine them as $\xi' 1_A + \xi'' 1_{A^c}$ for events A.

For the deterministic exchange cone model $\boldsymbol{X} = X + K$, it is sensible to consider selections $X + t\eta$ for all $t \geq 0$ and a selection η taking values from the boundary of K. Due to the dilatation monotonicity, see Theorem 6.5, it suffices to work with selections of K which are functions of X, and write them as $\eta(X)$. Since the aim is to minimise the risk, it is natural to choose η which is a sort of "countermonotonic" with respect to X, while choosing comonotonic X and η does not yield any gain in risk for their sum.

Assume that the components of $\mathbf{r}$ are expected shortfalls at level α. Then in order to approximate ρ_{s}, it is possible to use a "favourable" selection $\eta_* \in K$ constructed by projecting X onto K following the two-step procedure. First, X is translated by subtracting the vector of univariate α-quantiles in order to obtain random vector Y whose univariate α-quantiles are zero. Then Y is projected onto the boundary of the solvency cone $\check{K}$ and $\eta_*(X)$ if defined as the opposite of such projection. Consider all selections of the form $X + t\eta_*(X)$ for $t > 0$. With this choice, the components of X assuming small values are partially compensated by the remaining components. The points from the centrally symmetric cone to K^* are mapped to the origin and no compensation will be applied to them.

An alternative procedure is to modify the projection rule, so that only some part of the boundary of the solvency cone is used for compensation. In dimension $d = 2$, η_1 and η_2 are defined by projecting Y onto either one of the two half-lines that form the boundary of $\check{K}$.

Example 9.1. Consider the random vector X taking values $(-2,4)$ and $(4,-2)$ with equal probabilities. Let K be the cone with the points $(-5,1)$ and $(1,-5)$ on its boundary, so the corresponding bid-ask matrix has entries $\pi^{(12)} = \pi^{(21)} = 5$. Assume that $\mathbf{r}$ consist of two identical components being the expected shortfall ES_α at level $\alpha = 3/4$. Observe that $\mathrm{ES}_\alpha(X) = (0,0)$ and for

$$\eta_1 = \begin{cases} (1.2,-6) & \text{if } X = (-2,4)\,, \\ (0,0) & \text{if } X = (4,-2)\,, \end{cases} \qquad \eta_2 = \begin{cases} (0,0) & \text{if } X = (-2,4)\,, \\ (-6,1.2) & \text{if } X = (4,-2)\,, \end{cases}$$

we obtain $\mathrm{ES}_\alpha(X+\eta_1) = (-0.8,2)$ and $\mathrm{ES}_\alpha(X+\eta_2) = (2,-0.8)$.

By Theorem 6.5, the boundary of $\rho_{\mathrm{s}}(X+K)$ is given by $\mathbf{r}(X+\eta(X))$ with $\eta(X)$ belonging to the boundary of K. In order to compensate the risks of X it is natural to choose η as function of X such that $\eta(-2,4) = t(-5,1)$ or $\eta(-2,4) = t(1,-5)$ and $\eta(4,-2) = s(-5,1)$ or $\eta(-2,4) = s(1,-5)$ for $t,s > 0$. The minimisation problem over t and s can be easily solved analytically (or numerically) and yields the boundary of $\rho_{\mathrm{s}}(X+K)$. In Figure 1, the boundary of $\rho_{\boldsymbol{Z}_0}(X+K)$ is shown as dashed line, the boundary of $\mathbf{r}(X)+\check{K}$ is dotted line, while $\rho_{\mathrm{s}}(X+K)$ is the shaded region.

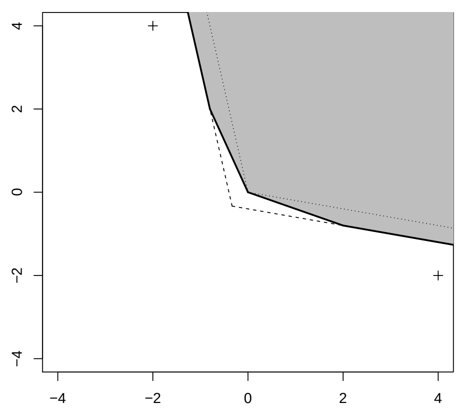
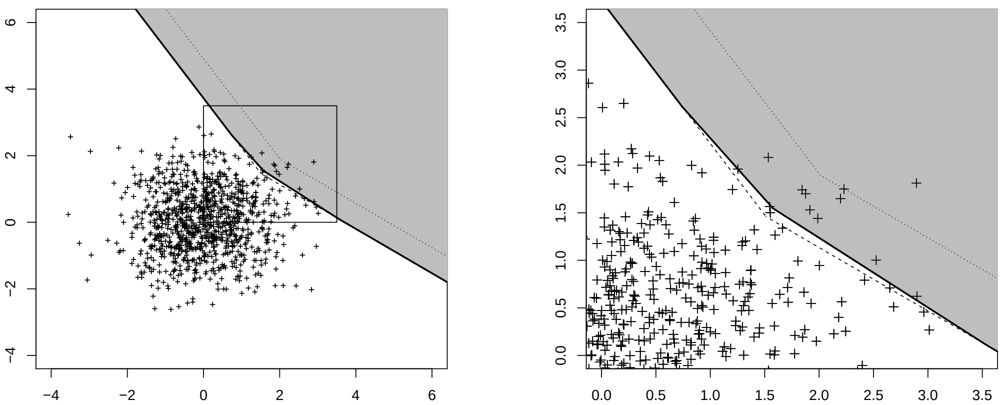


Figure 1: Two values for X, selection risk measure (shaded region) and its bounds from below and from above for Example 9.1.

Example 9.2. Consider the cone K with the points $(-1.5,1)$ and $(1,-1.5)$ on its boundary, so the corresponding bid-ask matrix has entries $\pi^{(12)} = \pi^{(21)} = 1.5$. Let $\mathbf{r} = (\mathrm{ES}_{0.05}, \mathrm{ES}_{0.05})$ consist of two identical components. At this time, we approximate the risk of $X_n + K$, where X_n is the empirical distribution for a sample of n observations from the bivariate standard normal distribution.

In order to approximate $\rho_{\mathrm{s}}(X_n + K)$ from below, we first determine the set A of $\mathbf{r}(\xi)$ for the selections of $X_n + K$ obtained as $\xi = X + t\eta_*(X_n)$, $X_n + t\eta_1(X_n)$, and $X_n + t\eta_2(X)$, with $t > 0$, where η_*, η_1, and η_2 are described above using projection on $\check{K}$ and the two half-lines from its boundary. Then we add to A the points x_1 and x_2 described in the proof to Proposition 6.9, find the convex hull of A, and finally add $\check{K}$ to the convex hull.

Figure 2(a) shows a sample of $n = 1000$ observations of a standard bivariate normal distribution and the approximation to the true value of $\rho_{\mathrm{s}}(X_n + K)$ described above, on the right panel a detail of the same plot is presented. The constructed approximation to $\rho_{\mathrm{s}}(X_n + K)$ is the shaded region, the boundary of $\rho_{\boldsymbol{Z}_0}(X_n + K)$ obtained as described in Proposition 6.2 is plotted as dashed line, and the boundary of $\mathbf{r}(X_n) + \check{K}$ is plotted as dotted line.



(a) (b)

Figure 2: (a) An approximation to the selection risk measure and its bounds for a sample of normally distributed gains and a deterministic exchange cone; (b) enlarged part of the plot.

Example 9.3. Consider the random frictionless exchange of two currencies described in Example 5.1. Transactions that diminish the risk of $X = (X_1, X_2)$ can be constructed by projecting X onto the boundary line to half-plane $\boldsymbol{K}$ with normal $(\pi, 1)$ and then subtracting the scaled projection from X. This leads to the family of selections $X + t\eta_*$ for $t \geq 0$ and $\eta_* \in \boldsymbol{K}$ given by

$$\eta_* = -\Big(\frac{X_1 - \pi X_2}{1 + \pi^2}, \frac{\pi^2 X_2 - \pi X_1}{1 + \pi^2}\Big)\,. \tag{22}$$

Assume that the exchange rate $\pi = \pi^{(21)}$ is log-normally distributed with mean one and volatility σ. We approximate the risk of $X + \boldsymbol{K}$ with X having a bivariate standard normal distribution and independent of π (equivalently independent of $\boldsymbol{K}$) for $\mathbf{r}$ with two identical components being ES_α. Observe that $\mathrm{ES}_\alpha(\pi) = -\alpha^{-1}\Phi(\Phi^{-1}(\alpha) - \sigma)$ and $\mathrm{ES}_\alpha(1/\pi) =$

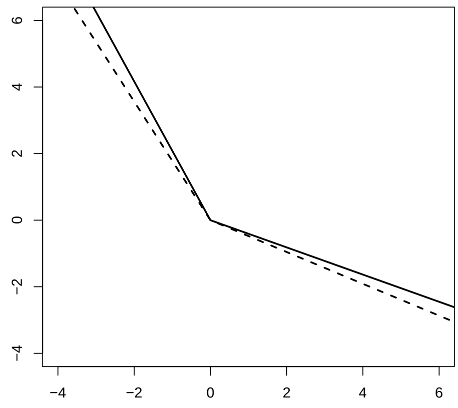


Figure 3: The boundaries of two cones C_1 and C_2 such that $C_1 \subset \rho_{\mathrm{s}}(\boldsymbol{K}) \subset C_2$.

$-\alpha^{-1}e^{\sigma^2}(1-\Phi(\Phi^{-1}(1-\alpha)+\sigma))$, where Φ is the cumulative distribution function of a standard normal random variable and Φ^{-1} its quantile function.

Fix $\alpha = 0.05$ and the volatility $\sigma = 0.4$ and denote by C_1 the cone in $\mathbb{R}^2$ with vertex at the origin of coordinates and half-lines with slopes $\mathrm{ES}_{0.05}(\pi) = -0.4086929$ and $1/\mathrm{ES}_{0.05}(1/\pi) = -2.085047$ as boundaries, while C_2 is the cone with vertex at the origin of coordinates and half-lines with slopes $\mathrm{VaR}_{0.05}(\pi) = -0.4780971$ and $\mathrm{VaR}_{0.95}(\pi) = -1.782366$ so that $C_1 \subset \rho_{\mathrm{s}}(\boldsymbol{K}) \subset C_2$, see Example 5.1. Figure 3 shows the boundaries of C_1 and C_2.

In order to approximate the risk of $X + \boldsymbol{K}$, we take a sample of n observations of (X, π). Denote by (X_n, π_n) a random vector whose distribution is the empirical distribution of the sample of (X, π) and by $\boldsymbol{K}_n$ a random half-space with normal $(\pi_n, 1)$. Note that the empirical distribution of the exchange rate is bounded away from the origin and infinity.

Figure 4(a) shows a sample of $n = 1000$ observations of a standard bivariate normal distribution and an approximation to the true value of $\rho_{\mathrm{s}}(X_n + \boldsymbol{K}_n)$ obtained by means of the convex hull of the risks of the selections of $X_n + \boldsymbol{K}_n$ described in (22) plus C_1 (solid line). The boundary of $\mathbf{r}(X_n) + C_1$ is shown as a dotted line.

Example 9.4. Consider the restricted liquidity situation presented in Example 2.5. Let $\boldsymbol{X} = X + (\boldsymbol{K} \cap ((1,1) + \mathbb{R}^2_-))$, where X follows a bivariate standard normal distribution and $\boldsymbol{K}$ is the half-plane from Example 9.3. Assume that $\mathbf{r} = (\mathrm{ES}_{0.05}, \mathrm{ES}_{0.05})$. Transactions that diminish the risk of $\boldsymbol{X}$ can be constructed by projecting $X = (X_1, X_2)$ onto the boundary line to half-plane $\boldsymbol{K}$ with normal $(\pi, 1)$ and then subtracting the projection from X. If the obtained point lies out of the line segment with end-points $X + (1, -\pi)$ and $X + (-1/\pi, 1)$, we take the nearest of the two end-points. This leads to the selection ξ_* of $\boldsymbol{X}$ given by

$$\xi_* = \begin{cases} X + (1, -\pi)\,, & \text{if } \frac{X_1 - \pi X_2}{1+\pi^2} \le -1\,, \\ X + (-1/\pi, 1)\,, & \text{if } \frac{\pi^2 X_2 - \pi X_1}{1+\pi^2} \le -1\,, \\ X - \left(\frac{X_1 - \pi X_2}{1+\pi^2}, \frac{\pi^2 X_2 - \pi X_1}{1+\pi^2}\right), & \text{otherwise}\,. \end{cases} \tag{23}$$

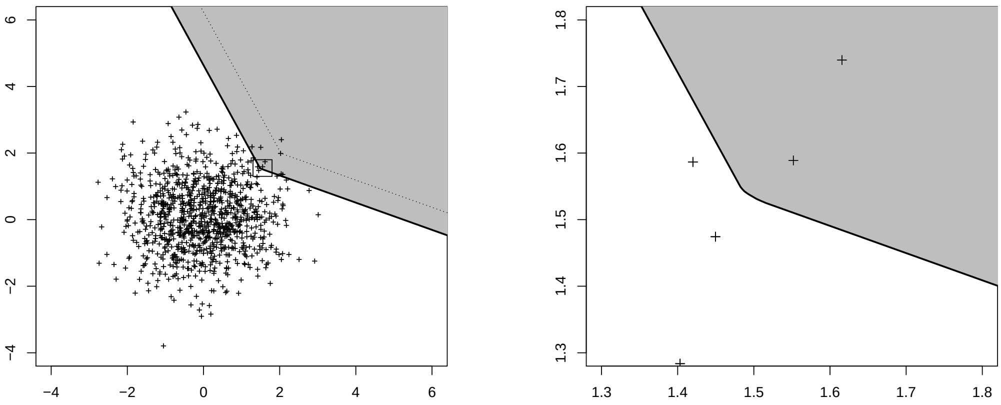


Figure 4: (a) An approximation to the selection risk measure and it lower bound for a normal sample in a random frictionless exchange case; (b) enlarged part of the plot.

Other relevant selections of $\boldsymbol{X}$ are $X + (1, -\pi)$ and $X + (-1/\pi, 1)$.

In order to approximate the risk of $\boldsymbol{X}$, we take a sample of $n = 1000$ observations of (X, π). Denote by (X_n, π_n) a random vector whose distribution is the empirical distribution of the sample, by $\boldsymbol{K}_n$ a random half-space with normal $(\pi_n, 1)$, and let $\boldsymbol{X}_n = X_n + (\boldsymbol{K}_n \cap ((1,1) + \mathbb{R}^2_-))$.

Figure 5(a) shows a sample of $n = 1000$ observations of X and an approximation to the true value of $\rho_{\mathrm{s}}(\boldsymbol{X}_n)$ obtained by means of all convex combinations of the selection ξ_* of $\boldsymbol{X}_n$ described in (23) with $X_n + (1, -\pi_n)$ and $X_n + (-1/\pi_n, 1)$ respectively. The shaded region with boundary plotted as a dotted line is $\mathbf{E}\check{\boldsymbol{X}}_n$, the reflected selection expectation of $\boldsymbol{X}_n$. The boundary of $\rho_{\boldsymbol{Z}_0}(\boldsymbol{X}_n)$ is plotted as a dashed line, and the boundary of $\mathbf{r}(X_n) + \rho_{\mathrm{s}}(\boldsymbol{K}_n \cap (\mathbb{R}^2_- + (1,1)))$ is plotted as a dash-dot line. On the right panel a detail of the same plot is presented.

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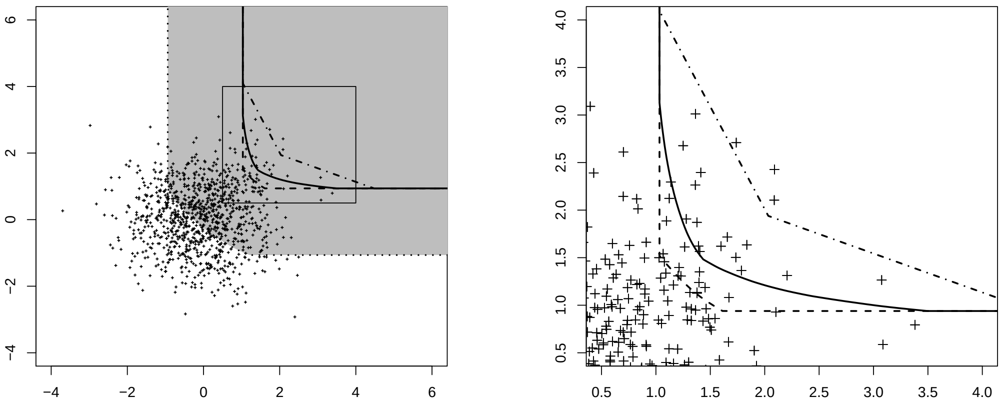

(a) (b)

Figure 5: (a) An approximation to the selection risk measure and bounds for a normal sample in a random frictionless exchange case with restricted liquidity; (b) enlarged part of the plot.

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