Spectral problems and orthogonal polynomials on the unit circle

Kenier Castillo Rodríguez
Department of Mathematics
Carlos III University of Madrid

A thesis submitted for the degree of
Doctor of Philosophy in Mathematics
September 2012

Spectral problems and orthogonal polynomials on the unit circle [1]

Advisor

Francisco Marcellán Español
Full Professor
Department of Mathematics
Carlos III University of Madrid

[1]This work was supported by FPU Research Fellowships, Re. AP2008-00471 and Dirección General de Investigación, Ministerio de Ciencia e Inovación of Spain, grant MTM2009-12740-C03-01.

Acknowledgements

I would like to express my deepest gratitude to my advisor F. (Paco) Marcellán for his patience and guidance during my doctoral studies. Additionally, my sincere thanks to L. E. Garza and F. (Fe) R. Rafaeli who helped me with ideas and suggestions during the last years. My appreciation to my friends and colleagues from State University of São Paulo, E. X. L. de Andrade, C. F. Bracciali, D. K. Dimitrov, R. L. Lamblém, M. V. Mello, and A. Sri Ranga, as well as from Catholic University of Leuven, M. Humet and M. Van Barel. I am grateful to my friends from Carlos III University of Madrid, J. Grujic, G. Miritello, and J. M. Pérez-Pardo. I am indebted to F. M. Dopico, A. G. García, E. Moro, and J. M. Rodríguez by their teachings. I am thankful to Ministerio de Educación, Gobierno de España for supporting part of my Ph.D. research. Finally, I wish to thank my entire extended family, without their support this work would not have been possible.

— K. Castillo
Toledo, Spain. May 6, 2012

Abstract

The main purpose of the work presented here is to study transformations of sequences of orthogonal polynomials associated with a hermitian linear functional $\mathcal{L}$, using spectral transformations of the corresponding C-function F. We show that a rational spectral transformation of F is given by a finite composition of four canonical spectral transformations. In addition to the canonical spectral transformations, we deal with two new examples of linear spectral transformations. First, we analyze a spectral transformation of $\mathcal{L}$ such that the corresponding moment matrix is the result of the addition of a constant on the main diagonal or on two symmetric sub-diagonals of the initial moment matrix. Next, we introduce a spectral transformation of $\mathcal{L}$ by the addition of the first derivative of a complex Dirac linear functional when its support is a point on the unit circle or two points symmetric with respect to the unit circle. In this case, outer relative asymptotics for the new sequences of orthogonal polynomials in terms of the original ones are obtained. Necessary and sufficient conditions for the quasi-definiteness of the new linear functionals are given. The relation between the corresponding sequence of orthogonal polynomials in terms of the original one is presented. We also consider polynomials which satisfy the same recurrence relation as the polynomials orthogonal with respect to the linear functional $\mathcal{L}$, with the restriction that the Verblunsky coefficients are in modulus greater than one. With positive or alternating positive-negative values for Verblunsky coefficients, zeros, quadrature rules, integral representation, and associated moment problem are analyzed. We also investigate the location, monotonicity, and asymptotics of the zeros of polynomials orthogonal with respect to a discrete Sobolev inner product for measures supported on the real line and on the unit circle.

Keywords: Orthogonal polynomials on the real line; orthogonal polynomials on the unit circle; Szegő polynomials on the real line; Hankel matrices; Toeplitz matrices; discrete Sobolev orthogonal polynomials on the real line; discrete Sobolev orthogonal polynomials on the unit circle; outer relative asymtotics; zeros; $\mathcal{S}$-functions; C-functions; rational spectral transformations; canonical spectral transformations.
2010 MSC: 42C05, 33C45

Resumen

El objetivo principal de este trabajo es el estudio de las sucesiones de polinomios ortogonales con respecto a transformaciones de un funcional lineal hermitiano $\mathcal{L}$, usando para ello las transformaciones de la correspondiente C-función F. Un primer resultado es que las transformaciones espectrales racionales de F están dadas por una composición finita de cuatro transformaciones espectrales canónicas. Además de estas transformaciones canónicas se estudian dos ejemplos de transformaciones espectrales lineales que son novedosos en la literatura. El primero de estos ejemplos está dado por una modificación del funcional lineal $\mathcal{L}$, de modo que la correspondiente matriz de momentos es el resultado de la adición de una constante en la diagonal principal o en dos subdiagonales simétricas de la matriz de momentos original. El segundo ejemplo es una transformación de $\mathcal{L}$ mediante la adición de la primera derivada de una delta de Dirac compleja cuando su soporte es un punto sobre la circunferencia unidad o dos puntos simétricos respecto a la circunferencia unidad. En este caso se obtiene la asintótica relativa exterior de la nueva sucesión de polinomios ortogonales en términos de la original. Se dan condiciones necesarias y suficientes para que los funcionales derivados de las perturbaciones estudiadas sean cuasi-definidos, y se obtiene la relación entre las correspondientes sucesiones de polinomios ortogonales. Se consideran además polinomios que satisfacen las mismas ecuaciones de recurrencia que los polinomios ortogonales con respecto al funcional lineal $\mathcal{L}$, agregando la restricción de que sus coeficientes de Verblunsky son en valor absoluto mayores que 1. Cuando estos coeficientes son positivos o alternan signo, se estudian los ceros, las fórmulas de cuadratura, la representación integral y el problema de momentos asociado. Asimismo, se estudia la localización, monotonicidad y comportamiento asintótico de los ceros asociados a polinomios discretos ortogonales de Sobolev para medidas soportadas tanto en la recta real como en la circunferencia unidad.

Palabras claves: Polinomios ortogonales en la recta real; polinomios ortogonales en la circunferencia unidad; polinomios de Szegő en la recta real; matrices de Hankel; matrices de Toeplitz; polinomios discretos ortogonales de Sobolev en la recta real; polinomios ortogonales de Sobolev discreto en la circunferencia unidad; asintótica relativa exterior; ceros; $\mathcal{S}$-funciones; C-funciones; transformaciones espectrales racionales; transformaciones espectrales canónicas.
2010 MSC: 42C05, 33C45

Contents

List of Figures

List of Tables

Nomenclature

Roman Symbols

a_n Jacobi parameter

b_n Jacobi parameter

$\mathbb{C}$ complex numbers

$\mathbb{C}_+$ $\{z \in \mathbb{C}; \Im z > 0\}$

c_n moments associated with $\mathcal{L}$ or $d\sigma$

$\mathbb{D}$ unit disc, $\{z \in \mathbb{C}; |z| < 1\}$

det determinant

deg degree of a polynomial

$F(z)$ C-function

$\mathbf{H}$ Hankel matrix

H_n Hermite polynomial

$\Im$ imaginary part

$\mathbf{I}$ identity matrix

I support of the measure μ, $\mathrm{supp}(\mu) = \{x \in \mathbb{R};\ \mu(x-\epsilon, x+\epsilon) > 0 \text{ for every } \epsilon > 0\}$

$\mathbf{J}$ Jacobi matrix

K_n kernel polynomial

$\mathbf{k}_n$ $\langle \Phi_n, \Phi_n \rangle_{\mathcal{L}} = ||\Phi_n||_{\sigma}^2$

$\mathcal{L}$ hermitian linear functional

$L_n^{(\alpha)}$ Laguerre polynomial

$\mathcal{M}$	linear functional
$\mathcal{N}$	Nevai class
P_n	monic orthogonal polynomial on the real line
$P_n^{(\alpha,\beta)}$	Jacobi polynomial
p_n	orthonormal polynomial on the real line
$\mathbb{P}$	linear space of polynomials with complex coefficients, $\mathrm{span}\{z^k\}_{k\in\mathbb{Z}_+}$
$\mathbb{P}_n$	linear space of polynomials with complex coefficients and degree at most n
$\Re$	real part
$\mathbb{R}$	real line
$\mathbb{R}_+$	positive half of the real line, $\{x \in \mathbb{R}; x > 0\}$
$\mathrm{sgn}(x)$	$\lvert x\rvert^{-1}x$, $x \neq 0$
S	$\mathcal{S}$-function
$\mathcal{S}$	Szegő class
$\mathbf{T}$	Toeplitz matrix
$\mathbb{T}$	unit circle $\{z \in \mathbb{C}; \lvert z\rvert = 1\}$
T_n	Chebyshev polynomials of the first kind
$\mathbf{v}^H$	transpose conjugate of $\mathbf{v}$
$\mathbf{v}^T$	transpose of $\mathbf{v}$
$\mathbb{Z}$	integers $\{0, \pm1, \pm2, \dots\}$
$\mathbb{Z}_+$	non-negative integers $\{0, 1, 2, \dots\}$
$\mathbb{Z}_-$	negative integers $\{0, -1, -2, \dots\}$

Greek Symbols

$\delta_{n,m}$	Kronecker delta
μ	measure supported on the real line
σ	measure supported on the unit circle
κ_n	leading coefficient of $\phi_n(z) = \kappa_n z^n + \cdots$, $\kappa_n = \lVert\Phi_n\rVert_\sigma^{-1}$

Λ	linear space of Laurent polynomials with complex coefficients, $\mathrm{span}\{z^k\}_{k\in\mathbb{Z}}$
μ_n	moments associated with $\mathcal{M}$ or $d\mu$
Ω_n	second kind orthogonal polynomial on the unit circle
Φ_n	monic orthogonal polynomial on the unit circle
ϕ_n	orthonormal polynomial on the unit circle
Ψ_n	perturbed monic orthogonal polynomial on the unit circle
ψ_n	perturbed orthonormal polynomial on the unit circle
$\Phi_n^*(z)$	reverse polynomial $z^n\overline{\Phi}_n(z)$
$(\Phi_n(z))_*$	$\overline{\Phi}_n(z^{-1})$

Other Symbols

$(x)_n^+$	Pochhammer's symbol for the rising factorial, $x(x+1)\cdots(x+n-1)$, $n\geq 1$

Chapter 1

Introduction

I tend to write about what interests me, in the hope that others will also be interested.

— J. Milnor. *Interview with John Milnor* Raussen and Skau [2012]

Orthogonal polynomials have very useful properties in the solution of mathematical and physical problems. Their relations with moment problems Jones et al. [1989]; Simon [1998], rational approximation Bultheel and Barel [1997]; Nikishin and Sorokin [1991], operator theory Kailath et al. [1978]; Krall [2002], analytic functions (de Branges's proof de Branges [1985] of the Bieberbach conjecture), interpolation, quadrature Chihara [1978]; Gautschi [2004]; Meurant and Golub [2010]; Szegő [1975], electrostatics Ismail [2009], statistical quantum mechanics Simon [2011], special functions Askey [1975], number theory Berg [2011] (irrationality Beukers [1980] and transcendence Day and Romero [2005]), graph theory Cámara et al. [2009], combinatorics, random matrices Deift [1999], stochastic process Schoutens [2000], data sorting and compression Elhay et al. [1991], computer tomography Louis and Natterer [1983], and their role in the spectral theory of linear differential operators and Sturm-Liouville problems Nikiforov and Uvarov [1988], as well as their applications in the theory of integrable systems Flaschka [1975]; Golinskii [2007]; Morse [1975a,b] constitute some illustrative samples of their impact.

1.1 Motivation and main objectives

Let consider the classical mechanical problem of a 1-dimensional chain of particles with neighbor interactions. Assume that the system is homogeneous (contains no impurities) and that the mass of each particle is m. We denote by y_n the displacement of the n-th particle, and by $\varphi(y_{n+1} - y_n)$ the interaction potential between neighboring particles. We can consider this system as a chain of infinitely many particles joined together with non-linear springs; see Figure 1.1.

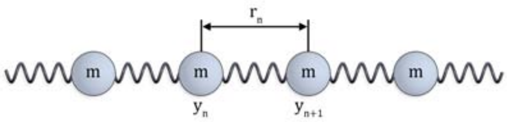


Figure 1.1: A model for 1-dimensional lattice

Therefore, if

$$F(r) = -\frac{d}{dr}\varphi(r) = -\varphi'(r)$$

is the force of the spring when it is stretched by the amount r, and $r_n = y_{n+1} - y_n$ is the mutual displacement, then by Newton's law, the equation that governs the evolution is

$$my_n'' = \varphi'(y_{n+1} - y_n) - \varphi'(y_n - y_{n-1}).$$

If $F(r)$ is proportional to r, that is, when $F(r)$ obeys Hooke's law, the spring is linear and the potential can be written as $\varphi(r) = \frac{\kappa}{2}r^2$. Thus, the equation of motion is

$$my_n'' = \kappa(y_{n-1} - 2y_n + y_{n+1}),$$

and the solutions $y_n^{(l)}$ are given by a linear superposition of the normal modes. In particular, when the particles $n = 0$ and $n = N+1$ are fixed,

$$y_n^{(l)} = C_n \sin\left(\frac{\pi l}{N+1}\right)\cos(\omega_l t + \delta_l), \quad l = 1, 2, \ldots, N,$$

where $\omega_l = 2\sqrt{\kappa/m}\sin(\pi l/(2N+2))$, the amplitude C_n of each mode is a constant determined by the initial conditions. In this case there is no transfer of energy between the models. Therefore, the linear lattice is non-ergodic, and cannot be an object of statistical mechanics unless some modification is made. In the early 1950s, the general belief was that if a non-linearity is introduced in the model, then the energy flows between the different modes, eventually leading to a stable state of statistical equilibrium Fermi et al. [1965]. This phenomenon was explained by the connection to solitons [1].

There are non-linear lattices which admit periodic behavior at least when the energy is not too high. Lattices with exponential interaction have the desired properties. The Toda lattice Toda [1989] is given by setting

$$\varphi(r) = e^{-r} + r - 1.$$

Flaschka Flaschka [1975] (see also Morse [1975a,b]) proved complete integrability for the Toda lattice by recasting it as a Lax equation for Jacobi matrices. Later, Van Moerbeke Moerbeke [1976], following a similar work McKean and Moerbeke [1975] on Hill's equation Magnus and Winkler [1966], used the Jacobi matrices to define the Toda hierarchy for the periodic Toda lattices, and to find the corresponding

[1] In mathematics and physics, a soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of non-linear and dispersive effects in the medium.

Lax pairs.

Flaschka's change of variable is given by

$$a_n = \frac{1}{2}e^{-(y_{n+1}-y_n)/2}, \quad b_n = \frac{1}{2}y_n'.$$

Hence the new variables obey the evolution equations

$$a_n' = a_n(b_{n+1}-b_n), \tag{1.1}$$

$$b_n' = 2(a_n^2 - a_{n-1}^2), \quad a_{-1}=0, \quad n \geqslant 0, \tag{1.2}$$

with initial data $b_n = b_n(0) = \overline{b_n(0)}$, $a_n = a_n(0) > 0$, which we suppose uniformly bounded [1]. Set $\mathbf{J}_t$ to be the semi-infinite Jacobi matrix associated with the system (1.1)-(1.2), that is

$$\mathbf{J}_t = \begin{bmatrix} b_0(t) & a_0(t) & 0 & 0 & \cdots \\ a_0(t) & b_1(t) & a_1(t) & 0 & \cdots \\ 0 & a_1(t) & b_2(t) & a_2(t) & \ddots \\ 0 & 0 & a_2(t) & b_3(t) & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

We use the notation $\mathbf{J}_0 = \mathbf{J}_\mu$, which is the matrix of the operator of multiplication by x in the basis of orthonormal polynomials on the real line. Favard's theorem says that, given any Jacobi matrix $\widetilde{\mathbf{J}}$, there exists a measure μ on the real line for which $\widetilde{\mathbf{J}} = \mathbf{J}_\mu$. In general, μ is not unique.

Flaschka's main observation is that the equations (1.1)-(1.2) can be reformulated in terms of the Jacobi matrix $\mathbf{J}_t$ as Lax pairs

$$\mathbf{J}_t' = [\mathbf{A}, \mathbf{J}_t] = \mathbf{A}\mathbf{J}_t - \mathbf{J}_t\mathbf{A},$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & a_0(t) & 0 & 0 & \cdots \\ -a_0(t) & 0 & a_1(t) & 0 & \cdots \\ 0 & -a_1(t) & 0 & a_2(t) & \ddots \\ 0 & 0 & -a_2(t) & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix} = (\mathbf{J}_t)_+ - (\mathbf{J}_t)_-,$$

where we use the standard notation $(\mathbf{J}_t)_+$ for the upper-triangular, and $(\mathbf{J}_t)_-$ for the lower-triangular projection of the matrix $\mathbf{J}_t$. At the same time, the corresponding orthogonality measure $d\mu(\cdot,t)$ goes through a simple spectral transformation,

$$d\mu(x,t) = e^{-tx}d\mu(x,0), \quad t > 0. \tag{1.3}$$

[1] $\sup_n (|a_n| + |b_n|) < \infty$.

Notice that spectral transformations of orthogonal polynomials on the real line play a central role in the solution of the problem. Indeed, the solution of Toda lattice is a combination of the inverse spectral problem from $\{a_n\}_{n\geqslant 0}$, $\{b_n\}_{n\geqslant 0}$ associated with the measure $d\mu = d\mu(\cdot, 0)$, the spectral transformation (1.3), and the direct spectral problem from $\{a_n(t)\}_{n\geqslant 0}$, $\{b_n(t)\}_{n\geqslant 0}$ associated with the measure $d\mu(\cdot, t)$.

Given the infinity matrix $\mathbf{J}_\mu$, which is a bounded self-adjoint operator in $\ell^2(\mathbb{Z}_+)$, we can define Gesztesy and Simon [1997]; Simon [2004] the so-called $\mathcal{S}$-function by

$$S(x) = \left\langle \mathbf{e}_1, (\mathbf{J}_\mu - x)^{-1}\mathbf{e}_1 \right\rangle,$$

where $\{\mathbf{e}_i\}_{i\geq 0} = \{(\delta_{i,j})_{j\geq 0}\}_{i\geq 0}$ is a vector basis in $\ell^2(\mathbb{Z}_+)$. In terms of the spectral measure μ associated with $\mathbf{J}_\mu$,

$$S(x) = \int_I \frac{d\mu(y)}{x-y}. \tag{1.4}$$

In many problems, (1.4) has more simple analytical and transformation properties than the measure μ, and, hence, S is often much more convenient for analysis. Recently, Peherstorfer, Spiridonov, and Zhedanov Peherstorfer et al. [2007] established a correspondence between the Toda lattice and differential equations for (1.4), using an alternative approach proposed in Peherstorfer [2001]. If the coefficients $\{a_n(t)\}_{n\geqslant 0}$, $\{b_n(t)\}_{n\geqslant 0}$ satisfy the system of equations (1.1)-(1.2) with $a_0(t)$ and $b_0(t)$ taken as arbitrary initial functions of time, then the corresponding $\mathcal{S}$-function $S(\cdot, t)$ satisfies the Riccati equation

$$\frac{\partial}{\partial t} S(x,t) = -1 + (x - b_0(t))S(x,t) - a_0^2(t)S^2(x,t).$$

Usually, the Toda lattice is studied using matrix spectral functions Novikov et al. [1984].

The problem of classifying all possible spectral transformations of orthogonal polynomials corresponding to a rational spectral transformation of the $\mathcal{S}$-function S, i.e.,

$$\widetilde{S}(x) = \frac{a(x)S(x) + b(x)}{c(x)S(x) + d(x)}, \quad a(x)d(x) - b(x)c(x) \neq 0, \tag{1.5}$$

where a, b, c, and d are coprime polynomials, in other words, the description of a generator system of the set of rational spectral transformations, was raised by Marcellán, Dehesa, and Ronveaux Marcellán et al. [1990] in 1990. Two years later, Peherstorfer Peherstorfer [1992] analyzed a particular class of rational spectral transformations. Indeed, he deduced the relation between the corresponding linear functionals. In 1997, Zhedanov Zhedanov [1997] proved that a generic linear spectral transformation ($c = 0$) of the $\mathcal{S}$-function (1.4) can be represented as a finite composition of Christoffel Szegő [1975] and Geronimus spectral transformations Geronimus [1940a,b], and also that any rational spectral transformation can be obtained as a finite composition of linear and associated elementary transformations Zhedanov [1997]. Here a natural question arises. What we can say about the generator system for rational spectral transformations of C-functions in the theory of orthogonal polynomials on the unit circle? This work is organized around this question.

Surprisingly, the theory of orthogonal polynomials with respect to non-trivial probability measures supported on the unit circle had not been so popular until the mid-1980's. The monographs by Szegő Grenander and Szegő [1984]; Szegő [1975], Freud Freud [1971], and Geronimus Geronimus [1954] were the main (and the few) major contributions to the subject, despite the fact that people working in linear prediction theory and digital signal processing used as a basic background orthogonal polynomials on the unit circle; see Delsarte and Genin [1990] and references therein. The recent monograph by Simon Simon [2005] constitutes an updated overview of the most remarkable directions of research in the theory, both from a theoretical approach (extensions of the Szegő theory from an analytic point of view), as well as from their applications in the spectral analysis of unitary operators, GGT Geronimus [1944]; Gragg [1993]; Teplyaev [1992] and CMV Cantero et al. [2003] matrix representations of the multiplication operator, quadrature formulas, and integrable systems (Ablowitz-Ladik systems Nenciu [2005], which include Schur flows as particular case), among others. Many concepts developed on orthogonal polynomials on the real line have an analogous in this theory.

The Schur flow equation – which can be naturally called Toda lattice for the unit circle – is given by

$$\alpha_n' = (1-|\alpha_n|^2)(\alpha_{n+1}-\alpha_{n-1}), \quad \alpha_{-1}=0, \quad n \geqslant 0, \tag{1.6}$$

where $\{\alpha_n\}_{n\geqslant 0}$ is a complex function sequence with $|\alpha_n|<1$, initially occurred in Ablowitz and Ladik [1975, 1976], under the name of discrete modified KdV equation, as a spatial discretization of the modified Korteweg-de Vries equation Korteweg and de Vries [1895]

$$\frac{\partial}{\partial t}f(x,t) = 6f^2(x,t)\frac{\partial}{\partial x}f(x,t) - \frac{\partial^3}{\partial x^3}f(x,t).$$

In a very recent work Golinskii [2007], Golinskii proved that the solution of the system (1.6) reduces to the combination of the direct and the inverse spectral problems related by means of the Bessel transformation

$$d\sigma(z,t) = C(t)e^{t(z+z^{-1})}d\sigma(z,0), \quad t>0, \tag{1.7}$$

where σ is a non-trivial probability measure supported on the unit circle and $C(t)$ is a normalization factor. Additionally, using CMV matrices the Lax pair for this system is found, and the dynamics of the corresponding spectral measures are described.

Given (1.4), the natural '$\mathcal{S}$-function' in the theory of orthogonal polynomials on the unit circle is the $\mathcal{C}$-function F Simon [2004] given by

$$F(z) = \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\sigma(\theta), \tag{1.8}$$

where σ is a non-trivial probability measure supported on $[-1,1]$. The Cauchy kernel has the Poisson kernel as its real part, and this is positive, so

$$\Re F(z) > 0, \quad |z|<1, \quad F(0)=1.$$

Hence, (1.8) is the function introduced by Carathéodory in Carathéodory [1907].

In this work – in the more general framework of hermitian linear functionals which are not necessarily positive definite – we consider, sequences of orthogonal polynomials deduced from spectral transformations of the corresponding C-function F. Our aim is to obtain and analyze the generator system of rational spectral transformations for non-trivial C-functions given by

$$\widetilde{F}(z) = \frac{A(z)F(z)+B(z)}{C(z)F(z)+D(z)}, \quad A(z)D(z) - B(z)C(z) \neq 0, \tag{1.9}$$

where A, B, C, and D are coprime polynomials. This result can be considered as a 'unit circle analogue' of the well known result by Zhedanov for orthogonal polynomials on the real line Zhedanov [1997]. Furthermore, we introduce and study relevant examples of linear spectral transformations ($C = 0$) associated with the addition of Lebesgue measure and derivatives of complex Dirac's deltas. However, in this, as well as in all research directions, more problems related to our original problem have arisen; some have been solved, and the rest are in Chapter 7 as a part of the open problems formulated therein.

1.2 Overview of the text

The original contributions of this work appear in twelve articles whose content is distributed as follows.

Chapter 3 develops the results of Castillo et al. [2012d,e]. Chapter 4 corresponds to Castillo et al. [2010a, 2011a, 2012h]. The results of Chapter 5 are contained in Castillo [2012f]; Castillo et al. [2011b, 2012a,b]. In Chapter 6 are included the results of Castillo and Marcellán [2012c]; Castillo et al. [2010b]. Finally, the Appendixes A and B contain results discussed in Castillo et al. [2011b, 2012g].

Chapter 2 is meant for non-experts and therefore it contains some introductory and background material. We give a brief outline of orthogonal polynomials on the real line and on the unit circle, respectively. However, proofs of statements are not given. The emphasis is focussed on spectral transformation of the corresponding $\mathcal{S}$-functions and C-functions. This chapter could be omitted without destroying the unity or completeness of the work. The original content of this work appears in the next chapters. Let us describe briefly our main contributions.

In Chapter 3 we study the sequence of polynomials $\{\Phi_n\}_{n\geqslant 0}$ which satisfies the following recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) + (-1)^{n+1}\alpha_{n+1}\Phi_n^*(z), \quad \alpha_{n+1} \in \mathbb{C}, \quad n \geqslant 0,$$

with the restriction $|\alpha_{n+1}| > 1$. The analysis of Perron-Carathéodory continued fractions shows that these polynomials satisfy the Szegő orthogonality with respect to a hermitian linear functional $\mathcal{L}$ in $\mathbb{P}$, which satisfies a special quasi-definite condition. In two particular cases, $\alpha_n > 0$ and $(-1)^n\alpha_n > 0$, respectively, zeros of the sequence of polynomials $\{\Phi_n\}_{n\geqslant 0}$ (real Szegő polynomials Vinet and Zhedanov [1999]) and associated quadrature rules are also studied. As a consequence of this study, we solve the following moment problem. Given a sequence $\{\mu_n\}_{n=0}^{\infty}$ of real numbers, we find necessary and sufficient conditions for the existence and uniqueness of a measure μ supported on $(1,\infty)$, such that

$$\mu_n = \int_1^\infty T_n(x)d\mu(x), \quad n \geqslant 0. \tag{1.10}$$

Here $\{T_n\}_{n\geqslant 0}$ are the Chebyshev polynomials of the first kind.

It is very well known that the Gram matrix of the bilinear form in $\mathbb{P}$ associated with a linear functional $\mathcal{L}$ in Λ, in terms of the canonical basis $\{z^n\}_{n\geq 0}$, is a Toeplitz matrix. In Chapter 4 we analyze two linear spectral transformations of $\mathcal{L}$ such that the corresponding Toeplitz matrix is the result of the addition of a constant in the main diagonal, i.e.,

$$\langle f,g\rangle_{\mathcal{L}_0} = \langle f,g\rangle_{\mathcal{L}} + m\int_{\mathbb{T}} f(z)\overline{g(z)}\frac{dz}{2\pi iz}, \quad f,g\in\mathbb{P}, \quad m\in\mathbb{R},$$

or in two symmetric sub-diagonals, i.e.,

$$\langle f,g\rangle_{\mathcal{L}_j} = \langle f,g\rangle_{\mathcal{L}} + \boldsymbol{m}\int_{\mathbb{T}} z^j f(z)\overline{g(z)}\frac{dz}{2\pi iz} + \overline{\boldsymbol{m}}\int_{\mathbb{T}} z^{-j} f(z)\overline{g(z)}\frac{dz}{2\pi iz}, \quad \boldsymbol{m}\in\mathbb{C}, \quad j>0,$$

of the initial Toeplitz matrix. We focus our attention on the analysis of the quasi-definite character of the perturbed linear functional, as well as in the explicit expressions of the new orthogonal polynomial sequence in terms of the first one. These transformations are known as local spectral transformations of the corresponding C-function (1.8); see Chapter 6. Analogous transformations for orthogonal polynomials on the real line, i.e., perturbations on the anti-diagonals of the corresponding Hankel matrix, are also considered. We define the modification of a quasi-definite functional $\mathcal{M}$ by the addition of derivatives of a real Dirac's delta linear functional, whose action results in such a perturbation, i.e.,

$$\left\langle \mathcal{M}_j, p\right\rangle = \left\langle \mathcal{M}_j, p\right\rangle + mp^{(j)}(a), \quad p\in\mathbb{P}, \quad m,a\in\mathbb{R}, \quad j\geq 0.$$

We establish necessary and sufficient conditions in order to preserve the quasi-definite character. A relation between the corresponding sequences of orthogonal polynomials is obtained, as well as the asymptotic behavior of their zeros. We also determine the relation between such perturbations and the so-called canonical linear spectral transformations.

In the first part of Chapter 5 we deal with a new example of linear spectral transformation associated with the influence of complex Dirac's deltas and their derivatives on the quasi-definiteness and the sequence of orthogonal polynomials associated with $\mathcal{L}$. This problem is related to the inverse polynomial modification Cantero et al. [2011], which is one of the generators of linear spectral transformations for the C-function (1.8), as we see in Chapter 6. We analyze the regularity conditions of a modification of the quasi-definite linear functional $\mathcal{L}$ by the addition of the first derivative of the complex Dirac's linear functional when its support is a point on the unit circle, i.e.,

$$\langle f,g\rangle_{\mathcal{L}_1} = \langle f,g\rangle_{\mathcal{L}} - im\left(\alpha f'(\alpha)\overline{g(\alpha)} - \overline{\alpha} f(\alpha)\overline{g'(\alpha)}\right), \quad m\in\mathbb{R}, \quad |\alpha|=1,$$

or two symmetric points with respect to the unit circle, i.e.,

$$\langle f,g\rangle_{\mathcal{L}_2} = \langle f,g\rangle_{\mathcal{L}} + i\boldsymbol{m}\left(\alpha^{-1} f(\alpha)\overline{g'(\overline{\alpha}^{-1})} - \alpha f'(\alpha)\overline{g(\overline{\alpha}^{-1})}\right)$$
$$+ i\overline{\boldsymbol{m}}\left(\overline{\alpha} f(\overline{\alpha}^{-1})\overline{g'(\alpha)} - \overline{\alpha}^{-1} p'(\overline{\alpha}^{-1})\overline{q(\alpha)}\right), \quad \boldsymbol{m},\alpha \in \mathbb{C}, \quad |\alpha| \neq 0,1.$$

Outer relative asymptotics for the new sequence of monic orthogonal polynomials in terms of the original ones are obtained.

In the second part of Chapter 5 we assume $\mathcal{L}$ is a positive definite linear functional associated with a positive measure σ. We study the relative asymptotics of the discrete Sobolev orthogonal polynomials. We focus our attention on the behavior of the zeros with respect to the particular case,

$$\langle f,g\rangle_{S_1} = \int_{\mathbb{T}} f(z)\overline{g(z)}d\sigma(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)}, \quad \alpha \in \mathbb{C}, \quad \lambda \in \mathbb{R}_+, \quad j \geq 0.$$

In Chapter 6 we obtain and study the set of generators for rational spectral transformations, which are related with the direct polynomial modification, i.e.,

$$\langle \mathcal{L}_R, f\rangle = \left\langle \mathcal{L}, \left(z-\alpha+z^{-1}-\overline{\alpha}\right) f(z)\right\rangle, \quad f \in \Lambda, \quad \alpha \in \mathbb{C},$$

and the inverse of a polynomial modification, i.e.,

$$\left\langle \mathcal{L}_{R^{(-1)}}, \left(z-\alpha+z^{-1}-\overline{\alpha}\right) f(z)\right\rangle = \langle \mathcal{L}, f\rangle, \quad f \in \Lambda, \quad \alpha \in \mathbb{C},$$

as well as with the $\pm k$ associated polynomials Peherstorfer [1996]. We deduce the relation between the corresponding C-functions and we study the regularity of the new linear functionals. We classify the spectral transformations of a C-function in terms of the moments associated with the linear functional $\mathcal{L}$. We also characterize the polynomial coefficients of a generic rational spectral transformation.

In Chapter 7 some concluding remarks which include indications of the direction of future work are presented. Finally, in Appendix A we consider the discrete Sobolev inner product associated with measures supported on the interval $(a,b) \subseteq \mathbb{R}$ (not necessary bounded), i.e.,

$$\langle p,q\rangle_{D_1} = \int_a^b p(x)q(x)d\mu(x) + \lambda p^{(j)}(\alpha)q^{(j)}(\alpha), \quad \alpha \notin (a,b), \quad \lambda > 0, \quad j \geq 0,$$

generalizing some known results concerning the asymptotic behavior of the zeros of the corresponding sequence of orthogonal polynomials. We also provide some numerical examples to illustrate the behavior of the zeros. Moreover, in Appendix B, the Uvarov perturbation of a quasi-definite linear functional by the addition of Dirac's linear functionals supported on r different points is studied.

Chapter 2

Orthogonal polynomials

What is true for OPRL [1] is even more true for orthogonal polynomials on the unit circle (OPUC).

— B. Simon. OPUC on one foot. *Boll. Amer. Math. Soc.*, 42:431-460, 2005

Orthogonal polynomials on the real line have attracted the interest of researchers for a long time. This subject is a classical one whose origins can be traced to Legendre's work Legendre [1785] on planetary motion. The study of the algebraic and analytic properties of orthogonal polynomials in the complex plane was initiated by Szegő in Szegő [1921a], and later continued by Szegő himself and several authors as Geronimus, Keldysh, Korovkin, Lavrentiev, and Smirnov. An overview of the developments until 1964, with more than 50 references on this subject, is due to Suetin Suetin [1966]. The complex analogue of the theory of orthogonal polynomials on the real line is naturally played by orthogonal polynomials on the unit circle. Following the works of Stieltjes, Hamburger, Toeplitz and others, Szegő investigated orthogonality on the unit circle in a series of papers around 1920 Szegő [1920, 1921b], where he introduced orthogonal polynomials, known in the literature as Szegő polynomials.

In this chapter we present a short introduction to the theory of orthogonal polynomials on the real line (especially for comparison purposes) and orthogonal polynomials on the unit circle. We discuss recurrence relations, reproducing kernel, associated moment problems, distribution of their zeros, quadrature rules, among other results that we need in the sequel. We also consider transformations of orthogonal polynomials using spectral transformations of the corresponding $\mathcal{S}$-functions and $\mathcal{C}$-functions, respectively. Finally, we establish the connection between measures on a bounded interval and on the unit circle by the so-called Szegő transformation. Most of the material is classical and available in different monographs as Chihara [1978], Freud [1971], Szegő [1975], Geronimus [1954, 1961], and the very recent monographs by Simon Simon [2005, 2011]. Therefore formal theorems and proofs are not given.

[1] Orthogonal Polynomials on the Real Line.

2.1 Orthogonal polynomials on the real line

Definition

Let $\mathcal{M}$ be a linear functional in the linear space $\mathbb{P}$ of the polynomials with complex coefficients. We define the moment of order n associated with $\mathcal{M}$ as the complex number

$$\mu_n = \langle \mathcal{M}, x^n \rangle, \quad n \geqslant 0. \tag{2.1}$$

The Gram matrix associated with the canonical basis $\{x^n\}_{n\geqslant 0}$ of $\mathbb{P}$ is given by

$$\mathbf{H} = \left[\left\langle \mathcal{M}, x^{i+j} \right\rangle\right]_{i,j\geq 0} = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n & \cdots \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}. \tag{2.2}$$

The matrices of this type, with constant values along anti-diagonals, are known as Hankel matrices Horn and Johnson [1990].

The moment functional (2.1) is said to be quasi-definite if the moment matrix $\mathbf{H}$ is strongly regular, or, equivalently, if the determinants of the principal leading submatrices $\mathbf{H}_n$ of order $(n+1)\times(n+1)$ are all different from 0 for every $n \geq 0$. In this case there exists a unique (up to an arbitrary non-zero factor) sequence $\{P_n\}_{n\geqslant 0}$ of monic orthogonal polynomials with respect to $\mathcal{M}$. We define the orthogonal monic polynomial, P_n, of degree n, by

$$\langle \mathcal{M}, P_n P_m \rangle = \gamma_n^{-2} \delta_{n,m}, \quad \gamma_n \neq 0.$$

Three-term recurrence relation

One of the most important characteristics of orthogonal polynomials on the real line is the fact that any three consecutive polynomials are connected by a simple relation which we can derive in a straightforward way. Indeed, let consider the polynomial $P_{n+1}(x) - xP_n(x)$, which is of degree at most n. Since $\{P_k\}_{k=0}^n$ is a basis for the linear space $\mathbb{P}_n$, we can write

$$xP_n(x) = P_{n+1}(x) + \sum_{k=0}^{n} \lambda_{n,k} P_k(x), \qquad \lambda_{n,k} = \frac{\langle \mathcal{M}, xP_nP_k \rangle}{\left\langle \mathcal{M}, P_k^2 \right\rangle}.$$

As P_n is orthogonal to every polynomial of degree at most $n-1$, we have $\lambda_{n,0} = \lambda_{n,1} = \cdots = \lambda_{n,n-2} = 0$ and

$$\lambda_{n,n-1} = \frac{\left\langle \mathcal{M}, P_n^2 \right\rangle}{\left\langle \mathcal{M}, P_{n-1}^2 \right\rangle}, \quad \lambda_{n,n} = \frac{\left\langle \mathcal{M}, xP_n^2 \right\rangle}{\left\langle \mathcal{M}, P_n^2 \right\rangle}.$$

We can thus find suitable complex numbers $b_0, b_1, \dots$ and $d_1, d_2, \dots$, such that

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + d_n P_{n-1}(x), \quad d_n \neq 0, \quad n \geqslant 0. \tag{2.3}$$

This three-term recurrence relation holds if we set $P_{-1} = 0$ and $P_0 = 1$ as initial conditions.

We next take up the important converse of the previous result. Let $\{b_n\}_{n\geqslant 0}$ and $\{d_n\}_{n\geqslant 1}$ be arbitrary sequences of complex numbers with $d_n \neq 0$, and let $\{P_n\}_{n\geqslant 0}$ be defined by the recurrence relation (2.3). Then, there is a unique functional $\mathcal{M}$ such that $\langle \mathcal{M}, 1\rangle = d_1$, and $\{P_n\}_{n\geqslant 0}$ is the sequence of monic orthogonal polynomials with respect to $\mathcal{M}$. We refer to this result as Favard's theorem Favard [1935].

Jacobi matrices

We can write the three-term recurrence relation (2.3) in matrix form,

$$x\mathbf{P}(x) = \mathbf{J}\mathbf{P}(x), \qquad \mathbf{P} = [P_0, P_1, \dots]^T,$$

where the semi-infinite tridiagonal matrix $\mathbf{J}$ is defined by

$$\mathbf{J} = \begin{bmatrix} b_0 & 1 & 0 & 0 & \cdots \\ d_1 & b_1 & 1 & 0 & \cdots \\ 0 & d_2 & b_2 & 1 & \ddots \\ 0 & 0 & d_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

$\mathbf{J}$ is said to be the monic Jacobi matrix Jacobi [1848] associated with the linear functional $\mathcal{M}$. A useful property of the matrix $\mathbf{J}$ is that the eigenvalues of its $n \times n$ leading principal submatrices $\mathbf{J}_n$ are the zeros of the polynomial P_n. Indeed, P_n is the characteristic polynomial of $\mathbf{J}_n$,

$$P_n(x) = \det(x\mathbf{I}_n - \mathbf{J}_n),$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

Integral representation

We can say that $\mathcal{M}$ is positive definite if and only if its moments are all real and $\det \mathbf{H}_n > 0$, $n \geqslant 0$. In this case there exists a unique sequence of orthonormal polynomials $\{p_n\}_{n\geqslant 0}$ with respect to $\mathcal{M}$, i.e., the following condition is satisfied,

$$\langle \mathcal{M}, p_n p_m \rangle = \delta_{n,m},$$

where

$$p_n(x) = \gamma_n x^n + \delta_n x^{n-1} + (\text{lower degree terms}), \quad \gamma_n > 0, \quad n \geqslant 0.$$

From the Riesz representation theorem Riesz [1909]; Rudin [1987], we know that every positive definite linear functional $\mathcal{M}$ has an integral representation (not necessarily unique)

$$\langle \mathcal{M}, x^n \rangle = \int_I x^n d\mu(x), \tag{2.4}$$

where μ denotes a non-trivial positive Borel measure supported on some infinite subset I of the real line. For orthonormal polynomials, (2.3) becomes

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad a_n^2 = d_n, \quad n \geqslant 0, \tag{2.5}$$

with initial conditions $p_{-1} = 0$, $p_0 = \mu_0^{-1/2}$, and the recurrence coefficients are given by

$$a_n = \int_I xp_{n-1}(x)p_n(x)d\mu(x) = \frac{\gamma_{n-1}}{\gamma_n} > 0,$$
$$b_n = \int_I xp_n^2(x)d\mu(x) = \frac{\delta_n}{\gamma_n} - \frac{\delta_{n+1}}{\gamma_{n+1}}.$$

Therefore,

$$p_n(x) = (a_n a_{n-1} \cdots a_1)^{-1} P_n(x) = \gamma_n P_n(x)$$

and the associated Jacobi matrix is

$$\mathbf{J}_\mu = \begin{bmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ 0 & 0 & a_3 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

There are explicit formulas for orthogonal polynomials in terms of determinants. The orthonormal polynomial of degree n is given by Heine's formula Heine [1878, 1881]

$$p_n(x) = \frac{1}{\sqrt{\det \mathbf{H}_n \det \mathbf{H}_{n-1}}} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix}, \tag{2.6}$$

where the leading coefficient γ_n is the ratio of two Hankel determinants,

$$\gamma_n = \sqrt{\frac{\det \mathbf{H}_{n-1}}{\det \mathbf{H}_n}}.$$

In the positive definite case, Favard's theorem can be rephrased as follows. If $d_{n+1} > 0$ and $b_n \in \mathbb{R}$, $n \geqslant 0$, there is a non-trivial positive Borel measure μ for which the Jacobi matrix is $\mathbf{J}_\mu$; equivalently, the corresponding sequence of orthogonal polynomials obeys (2.5). In general this measure is not unique, but a sufficient condition for uniqueness is that the recurrence coefficients are bounded or, equivalently, the moment problem is determinate.

Moment problem

Moment problems occur in different mathematical contexts like probability theory, mathematical physics, statistical mechanics, potential theory, constructive analysis or dynamical systems. An excellent account of the history of moment problems is given in Kjeldsen [1993]. In its simplest terms, a moment problem is related to the existence of a measure μ defined on an interval $I \subseteq \mathbb{R}$ for which all the moments

$$\mu_n = \int_I x^n d\mu(x), \quad n \geq 0, \tag{2.7}$$

exist. If the solution to the moment problem is unique, it is called determinate. Otherwise, the moment problem is said to be indeterminate. The monographs Akhiezer [1965] and Shohat and Tamarkin [1943] are the classical sources on moment problems; see also Simon [1998] from a different point of view using methods from the theory of finite difference operators.

There are many variations of a moment problem, depending on the interval I. In all of them, as suggested above, there are two questions to be answered, namely existence and uniqueness. Three particular cases of the general moment problem have come to be called classical moment problems, although strictly the term describes a much wider class. These are the following:

i) The Hamburger moment problem, where the measure is supported on $(-\infty, \infty)$.

ii) The Stieltjes moment problem, where the measure is supported on $(0, \infty)$.

iii) The Hausdorff moment problem, where the measure is supported on $(0, 1)$.

The Hausdorff moment problem is always determinate Hausdorff [1923]. Stieltjes, in his memoir Stieltjes [1894, 1895] introduced and solved the moment problem which was named after him by making extensive use of continued fractions. The necessary and sufficient conditions for determinacy of this moment problem are given by $\det \mathbf{H}_n > 0$, $\det \mathbf{H}_n^{(1)} > 0$, $n \geq 0$, where

$$\mathbf{H}_n^{(1)} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \ddots & \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{bmatrix}.$$

In Hamburger [1920, 1921, 1921], Hamburger solved the moment problem on the whole real line, showing that it was not just a trivial extension of Stieltjes' work. The Hamburger moment problem is determined if and only if $\det \mathbf{H}_n > 0$, $n \geq 0$.

More recent variations of these problems are the strong moment problems. In these cases, the sequence $\{\mu_n\}_{n\geqslant 0}$ is replaced by the bilateral sequence $\{\mu_n\}_{n\in\mathbb{Z}}$ of real numbers and the moment problem can be stated as follows. Given such a sequence $\{\mu_n\}_{n\in\mathbb{Z}}$ of real numbers, find a measure μ such that

$$\mu_n = \int_I x^n d\mu(x), \quad n \in \mathbb{Z}.$$

The strong Stieltjes and strong Hamburger moment problems can be formulated in the same way as the classical problems. The necessary and sufficient conditions are also given in terms of Hankel determinants involving the moments. Jones, Thron, and Waadeland Jones et al. [1980] proposed and solved the strong Stieltjes moment problem, while Jones, Njåstad, and Thron Jones et al. [1984] solved the strong Hamburger moment problem. In both cases, a central role was played by continued fractions.

Continued fractions

As Brezinski Brezinski [1991] points out, continued fractions were used implicitly for many centuries before their real discovery. An excellent text on the arithmetical and metrical properties of regular continued fractions is the classical work of Khintchine Khintchine [1963], which is the starting point for the most recent book by Rocket and Szüsz Rocket and Szüsz [1992]. In addition to these texts, the analytic theory of continued fractions is very well covered in Jones and Thron [1980]; Lorentzen and Waadeland [1992, 2008]; Wall [1948].

A continued fraction is a finite or infinite expansion of the form

$$q_0 + \cfrac{r_1}{q_1 + \cfrac{r_2}{q_2 + \cfrac{r_3}{q_3 + \dots}}} = q_0 + \frac{r_1|}{|q_1} + \frac{r_2|}{|q_2} + \frac{r_3|}{|q_3} + \cdots, \tag{2.8}$$

where $\{r_n\}_{n\geqslant 0}$ and $\{q_n\}_{n\geqslant 0}$ are real or complex numbers, or functions of real or complex variables. The finite continued fraction,

$$\frac{R_n}{Q_n} = q_0 + \frac{r_1|}{|q_1} + \frac{r_2|}{|q_2} + \frac{r_3|}{|q_3} + \cdots + \frac{r_n|}{|q_n},$$

obtained by truncation of (2.8), is called the n-th approximate of the continued fraction (2.8). The limit of R_n/Q_n when n tends to infinity is the value of the continued fraction.

The numerators R_n and denominators Q_n satisfy, respectively, the Wallis recurrence relations Wallis [1656]

$$R_{n+1} = q_{n+1}R_n + r_{n+1}R_{n-1}, \quad n \geqslant 1, \tag{2.9}$$

$$Q_{n+1} = q_{n+1}Q_n + r_{n+1}Q_{n-1}, \quad n \geqslant 1, \tag{2.10}$$

with $R_0 = q_0$, $Q_0 = 1$, $R_1 = q_0q_1 + r_1$, and $Q_1 = q_1$. These formulas lead directly to the connection between orthogonal polynomials and continued fractions. If we consider the following continued fraction

$$\frac{1}{|x-b_0} - \frac{a_1^2}{|x-b_1} - \frac{a_2^2}{|x-b_2} - \frac{a_3^2}{|x-b_3} - \cdots,$$

then $Q_n := P_n$ satisfies (2.3).

Christoffel-Darboux identity

In the literature, the polynomials

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y), \quad n \geqslant 0,$$

are usually called Kernel polynomials. The name comes from the fact that for any polynomial, q_n, of degree at most n, is given by

$$q_n(y) = \int_I q_n(x)K_n(x,y)d\mu(x).$$

The Kernel polynomial K_n can be represented in a simple way in terms of the polynomials p_n and p_{n+1} throughout the Christoffel-Darboux identity Chebyshev [1885]; Christoffel [1858]; Darboux [1878],

$$K_n(x,y) = a_{n+1}\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y},$$

that can be deduced in a straightforward way from the three-term recurrence relation (2.5). When y tends to x, we obtain its confluent form

$$K_n(x,x) = a_{n+1}\left(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right).$$

This last identity is used to prove two results that we show in this section, interlacing of zeros and Gauss-Jacobi quadrature formula.

Zeros

The fundamental theorem of algebra states that any polynomial of degree n has exactly n zeros (counting multiplicities). When dealing with orthogonal polynomials with respect to non-trivial probability measures supported on the real line, one can say much more about their localization. Two of the most relevant properties of zeros are the following:

i) The zeros of p_n are all real, simple and are located in the interior of the convex hull [1] of I.

ii) Suppose $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ are the zeros of p_n, then

$$x_{n,k} < x_{n-1,k} < x_{n,k+1}, \quad 1 \leqslant k \leqslant n-1.$$

[1] By convex hull of a set $E \subset \mathbb{C}$ we mean the smallest convex set containing E. $G \subset \mathbb{C}$ is convex if for each pair of points $x, y \in G$ the line connecting x and y is a subset of G.

The property *ii*) can also be proved using the Jacobi matrix $\mathbf{J}_\mu$ from the inclusion principle for the eigenvalues of a hermitian matrix Horn and Johnson [1990].

The following result is due to Wendroff Wendroff [1961]. Let P_{n+1} and P_n be two monic polynomials whose zeros are simple, real, and strictly interlacing. Then there is a positive Borel measure μ for which they are the corresponding orthogonal polynomials of degrees $n+1$ and n, respectively. All such measures have the same starting sequence $P_{n+1}, P_n, P_{n-1}, \ldots, P_0$.

Quadrature

A numerical quadrature consists of approximating the integral of a function $f : I \subset \mathbb{R} \to \mathbb{R}$ by a finite sum which uses only n function evaluations. For a positive Borel measure μ supported on I, an n-point quadrature rule is a set of points $x_1, x_2, \ldots, x_n$ and a set of associated numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that

$$\int_I f(x)d\mu(x) \sim \sum_{k=1}^{n} \lambda_k f(x_k)$$

in some sense for a large class of functions as possible.

A Gauss-Jacobi quadrature Jacobi [1859]; Stieltjes [1894, 1895] is a quadrature rule constructed to yield an exact result for polynomials of degree at most $2n-1$, by a suitable choice of the nodes and weights. If we choose the n nodes of a quadrature rule as the n zeros $x_{n,1}, x_{n,2}, \ldots, x_{n,n}$ of the orthogonal polynomial, P_n, with respect to μ supported on I and if we denote the corresponding so-called Cotes or Christoffel numbers by $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,n}$, then for every polynomial Q_{2n-1} of degree at most $2n-1$,

$$\sum_{k=1}^{n} \lambda_{n,k}\, Q_{2n-1}(x_{n,k}) = \int_I Q_{2n-1}(x)d\mu(x).$$

The Christoffel numbers are positive and are given by

$$\lambda_{n,k} = \left(\sum_{k=0}^{n-1} p_k^2(x_{n,k}) \right)^{-1}.$$

2.1.1 Classical orthogonal polynomials

The most important polynomials on the real line are the classical orthogonal polynomials Szegő [1975]. They are the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials (some special cases are the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials). These polynomials possess many properties that no other orthogonal polynomial system does. Among others, it is remarkable that the classical orthogonal polynomials satisfy a second order linear differential equation Bochner [1929]; Routh [1884]

$$\theta(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where θ is a polynomial of degree at most 2 and τ is a polynomial of degree 1, both independent of n. They also can be represented by a Rodrigues' distributional formula Cryer [1970]; Rasala [1981]

$$P_n(x) = \frac{1}{c_n\omega(x)}\frac{d^n}{dx^n}\left(\omega(x)\theta^n(x)\right),$$

where ω is the weight function and θ is a polynomial independent of n. Moreover, for every classical orthogonal polynomial sequence, their derivatives constitute also an orthogonal polynomial sequence on the same interval of orthogonality Cryer [1935]; Krall [1936]; Webster [1938].

Jacobi polynomials

The Jacobi polynomials Chihara [1978]; Szegő [1975], appear in the study of rotation groups [1]and in the solution to the equations of motion of the symmetric top McWeeny [2002]. They are orthogonal with respect to the absolutely continuous measure $d\mu(\alpha,\beta;x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, supported on $[-1,1]$ where for integrability reasons we need to take $\alpha,\beta > -1$. These polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)d\mu(\alpha,\beta;x) = \frac{2^{\alpha+\beta+1}}{n!(2n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\delta_{n,m},$$

where Γ is the Gamma function. From Rodrigues' formula we get

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{(-2)^n n!(1-x)^{\alpha}(1+x)^{\beta}}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right),$$

or, equivalently, solving the differential equation by Frobenius' methods, the Jacobi polynomials are defined via the hypergeometric function as follows

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{2^n(\alpha+1)_n^+}{(n+\alpha+\beta+1)_n^+}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right) \\ &= \binom{n+\alpha}{n}\sum_{j=0}^{n}\frac{(-n)_k^+(n+\alpha+\beta+1)_k^+}{(\alpha+1)_k^+ k!}\left(\frac{1-x}{2}\right)^2. \end{aligned}$$

The n-th Jacobi polynomial is the unique polynomial solution of the second order linear homogeneous differential equation

$$(x^2-1)y''(x) + ((2+\alpha+\beta)x+\alpha-\beta)y'(x) - n(n+1+\alpha+\beta)y(x) = 0.$$

Particular cases are $\alpha = \beta = -1/2$, given the Chebyshev polynomials of first kind,

$$T_n(x) = 2^{2n}\frac{(n!)^2}{(2n)!}P_n^{(-1/2,-1/2)}(x).$$

[1]A rotation group is a group in which the elements are orthogonal matrices with determinant 1. In the case of three-dimensional space, the rotation group is known as the special orthogonal group.

The change of variable $x = \cos\theta$ gives $T_n(x) = \cos(n\theta)$. The sequence $\{T_n\}_{n\geqslant 0}$ is used as an approximation to a least squares fit, and it is a special case of the Gegenbauer polynomial with $\alpha = 0$.

When $\alpha = \beta = -1/2$, we have the Chebyshev polynomial of second kind

$$U_n(x) = 2^{2n+1}\frac{((n+1)!)^2}{(2n+2)!}P_n^{(1/2,1/2)}(x).$$

With $x = \cos\theta$, we get $U_n(x) = \sin(n+1)\theta/\sin\theta$. The sequence $\{U_n\}_{n\geqslant 0}$ arises in the development of four-dimensional spherical harmonics in angular momentum theory. $\{U_n\}_{n\geqslant 0}$ is also a special case of the Gegenbauer polynomial with $\alpha = 1$.

Laguerre polynomials

The Laguerre polynomials Chihara [1978]; Szegő [1975] arise in quantum mechanics, as the radial part of the solution of the Schrödinger's equation for the hydrogen atom. They are orthogonal on the positive half of the real line, satisfying

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)d\mu(\alpha;x) = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{n,m},$$

where $d\mu(\alpha;x) = x^\alpha e^{-x}dx$ and $\alpha > -1$. Rodrigues' formula for them is

$$L_n^{(\alpha)}(x) = \frac{e^x}{n!x^\alpha}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}).$$

The polynomial $L_n^{(\alpha)}$ satisfies a second order linear differential equation that is a confluent hypergeometric equation

$$xy''(x) + (\alpha+1-x)y'(x) + ny(x) = 0,$$

and the Laguerre polynomials are a terminating confluent hypergeometric series

$$L_n^{(\alpha)}(x) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\,{}_1F_1(-n,\alpha+1,x) = \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-x)^j}{j!}.$$

Hermite polynomials

When $d\mu(x) = e^{-x^2}dx$ on the whole real line, we have the Hermite polynomials Chihara [1978]; Szegő [1975], satisfying the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)d\mu(x) = \sqrt{\pi}2^n n!\delta_{n,m}.$$

They arise in probability, such as the Edgeworth series, in numerical analysis as Gaussian quadrature, and in physics, where they give rise to the eigenstates of the quantum harmonic oscillator. From Rodrigues' formula,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and we can deduce their explicit formula in terms of hypergeometric functions

$$H_{2n}(x) = (-1)^n (1/2)_n^+ \, {}_1F_1\left(-n, \frac{1}{2}; x^2\right),$$
$$H_{2n+1}(x) = (-1)^n (3/2)_n^+ \, x \, {}_1F_1\left(-n, \frac{3}{2}; x^2\right).$$

The choice $\theta = 1$ and $\tau(x) = -2x$ gives their characterization as the polynomial eigenfunctions of the second order linear differential operator

$$L[y(x)] = y''(x) - 2xy'(x).$$

2.1.2 $\mathcal{S}$-functions and rational spectral transformations

$\mathcal{S}$-functions

The study of perturbations of the linear functional $\mathcal{M}$ introduced in (2.1), and their effects on the corresponding $\mathcal{S}$-function

$$S(x) = \left\langle \mathcal{M}, \frac{1}{x-y} \right\rangle, \tag{2.11}$$

where the functional $\mathcal{M}$ acts on the variable y, has a significant relevance in the theory of orthogonal polynomials on the real line. S admits, as a series expansion at infinity, the following equivalent representation

$$S(x) = \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}}, \tag{2.12}$$

i.e., it is a generating function of the sequence of moments for the linear functional $\mathcal{M}$ (questions of convergence are not considered). If the moments associated with μ are given by (2.7), the functions

$$S_n(x) = \int_I \frac{p_n(y)}{x-y} d\mu(y), \quad n \geqslant 0,$$

constitute a second (independent) solution of the difference equation

$$xy_n = a_{n+1}y_{n+1} + b_n y_n + a_n y_{n-1}, \quad n \geq 0.$$

They are called second kind functions associated with μ. In this case, the $\mathcal{S}$-function Stieltjes [1894, 1895] is given by

$$S_0(x) = S(x) = \int_I \frac{d\mu(y)}{x-y}.$$

One of the important properties of the $\mathcal{S}$-functions is its representation in terms of continued fractions, Stieltjes [1894, 1895]

$$S(x) = \frac{1}{\left| x-b_0 \right.} - \frac{a_1^2}{\left| x-b_1 \right.} - \frac{a_2^2}{\left| x-b_2 \right.} - \dots . \tag{2.13}$$

In fact, (2.13) was a starting point for the general theory of orthogonal polynomials in pioneering works by Chebyshev Chebyshev [1885] and Stieltjes Stieltjes [1894, 1895].

Spectral transformations

A rational spectral transformation Zhedanov [1997] of the $\mathcal{S}$-function S is a new $\mathcal{S}$-function defined by

$$\widetilde{S}(x) = \frac{a(x)S(x)+b(x)}{c(x)S(x)+d(x)}, \quad a(x)d(x)-b(x)c(x) \neq 0, \tag{2.14}$$

where a, b, c, and d are coprime polynomials. When $c=0$, the spectral transformation (2.14) is said to be linear. These polynomials should be chosen in such a way that the new $\mathcal{S}$-function, $\widetilde{S}$, has the same asymptotic behavior as initial (2.12),

$$\widetilde{S}(x) = \sum_{k=0}^{\infty} \frac{\widetilde{\mu}_k}{x^{k+1}},$$

where $\{\widetilde{\mu}_n\}_{n\geqslant 0}$ is the sequence of transformed moments. Hence, in general, the coefficients of the polynomials a, b, c, and d depend on the original moments $\{\mu_n\}_{n\geqslant 0}$. In particular, this means that the spectral transformations do not form a group. Indeed, for a given spectral transformation there exist many different reciprocal spectral transformations. Nevertheless, it is clear that one can always construct a composition of two spectral transformations, and moreover, for a given spectral transformation there is at least one reciprocal.

In terms of the moments, we can classify the spectral transformations of $\mathcal{S}$-functions as follows.

i) Local spectral transformations: spectral transformations under the modification of a finite number of moments.

ii) Global spectral transformations: spectral transformations under the modification of an infinite number of moments.

Notice that *i*) is a special case of linear spectral transformations related with perturbations on the anti-diagonals of the Hankel matrix (2.2). In general, *ii*) can be represented by the rational spectral transformation (2.14).

Linear spectral transformations

Without loss of generality, we can assume that the measure μ is normalized, i.e., $\mu_0 = 1$. The Christoffel transformation Szegő [1975] corresponds to a modification of the measure μ defined by

$$d\mu_c(x) = \frac{x-\beta}{\mu_1-\beta} d\mu(x), \quad \beta \notin I. \tag{2.15}$$

The sequence of orthogonal polynomials $\{P_n(\cdot;\mu_c)\}_{n\geqslant 0}$ associated with this transformation is given by

$$(x-\beta)P_n(x;\mu_c) = P_{n+1}(x) - \frac{P_{n+1}(\beta)}{P_n(\beta)}P_n(x), \quad n \geqslant 0.$$

Indeed, Christoffel transformation leads to the Kernel polynomial. Denoting the transformation (2.15) by $\mathcal{R}_C(\beta)$, the corresponding $\mathcal{S}$-function becomes

$$S_C(x) = \mathcal{R}_C(\beta)[S(x)] = \frac{(x-\beta)S(x)-1}{\mu_1-\beta}. \tag{2.16}$$

Conversely, if we start with a spectral transformation where a is a polynomial of first degree, b is constant, $d \equiv 1$, and $c \equiv 0$, the only choice for such a spectral transformation is (2.16). In general, a linear spectral transformation with $d \equiv 1$ is equivalent to a finite composition of Christoffel transformations Zhedanov [1997].

The reciprocal of a Christoffel transformation is the so-called Geronimus transformation Geronimus [1940a,b], consisting of a perturbation of μ such that

$$d\mu_g(x) = \frac{(\beta-x)^{-1}d\mu(x)+m\delta(x-\beta)}{m+S(\beta)}, \quad \beta \notin I, \quad m \in \mathbb{R}_+. \tag{2.17}$$

The sequence of orthogonal polynomials $\{P_n(\cdot;\mu_g)\}_{n\geqslant 0}$ with respect to (2.17) can be written as

$$P_n(x;\mu_g) = xP_n(x) - \frac{Q_n(\beta;m)}{Q_{n-1}(\beta,m)}P_{n-1}(x), \quad n \geqslant 0$$

where $P_0(\cdot;\mu_g) = 1$, and $Q_n(\beta,m)$ is a solution of the recurrence relation (2.3) with auxiliary parameter β,

$$Q_n(\beta,m) = S_n(\beta) + mP_n(\beta), \quad n \geqslant 0.$$

This transformation is denoted by $\mathcal{R}_G(\beta,m)$. The corresponding $\mathcal{S}$-function is

$$S_G(x) = \mathcal{R}_G(\beta,m)[S(x)] = \frac{S(\beta)+m-S(x)}{(x-\beta)(m+S(\beta))}. \tag{2.18}$$

We can see that the transformation (2.18) is reciprocal to (2.16). However, in contrast to the Christoffel transformation, (2.18) contains two free parameters, where the second free parameter defines the value of additional discrete mass as is seen is (2.17). In general, one can prove that a linear spectral transformation with $a \equiv 1$ is equivalent to a finite composition of Geronimus transformations Zhedanov [1997].

It is easy to see that for different values of β we have

$$\mathcal{R}_C(\beta_1)\circ\mathcal{R}_G(\beta_2,m) = \mathcal{R}_G(\beta_2,m)\circ\mathcal{R}_C(\beta_1).$$

However, for the same parameter we have the following relations

$$\mathcal{R}_C(\beta) \circ \mathcal{R}_G(\beta, m) = \mathcal{I} \quad \text{(Identity transformation)},$$
$$\mathcal{R}_G(\beta, m) \circ \mathcal{R}_C(\beta) = \mathcal{R}_U(\beta, m) \quad \text{(Uvarov transformation)}.$$

The Uvarov transformation Geronimus [1940a,b] consists of the addition of a real positive mass to the measure μ,

$$d\mu_u(x) = \frac{d\mu(x) + m\delta(x-\beta)}{1+m}, \quad \beta \notin I, \quad m \in \mathbb{R}_+.$$

The relation between the corresponding $\mathcal{S}$-functions is

$$S_U(x) = \mathcal{R}_U(\beta, m)[S(x)] = \frac{S(x) + m(x-\beta)^{-1}}{1+m}.$$

Rational spectral transformations

In Zhedanov [1997] it was proved that by means of $\pm k$ associated transformations we can reduce (2.14) to the linear form. Combining Christoffel transformation (2.16), Geronimus transformations (2.18), and the $\pm k$ associated transformations we get a wide class of rational spectral transformations (2.14).

From the sequence of monic orthogonal polynomials $\{P_n\}_{n\geqslant 0}$ we can define the sequence of associated monic polynomials Geronimus [1943], $\{P_n^{(k)}\}_{n\geqslant 0}$, $k \geqslant 1$, by means of the shifted recurrence relation

$$P_{n+1}^{(k)}(x) = (x - b_{n+k})P_n^{(k)}(x) - d_{n+k}P_{n-1}^{(k)}(x), \quad n \geqslant 0, \tag{2.19}$$

with $P_{-1}^{(k)} = 0$ and $P_0^{(k)} = 1$. The recurrence relation (2.19) can also be written in the matrix form

$$x\mathbf{P}^{(k)}(x) = \mathbf{J}^{(k)}\mathbf{P}^{(k)}(x), \qquad \mathbf{P}^{(k)} = \left[P_0^{(k)}, P_1^{(k)}, \ldots\right]^T,$$

where $\mathbf{J}^{(k)}$ is the tridiagonal matrix

$$\mathbf{J}^{(k)} = \begin{bmatrix} b_k & 1 & 0 & 0 & \cdots \\ d_{k+1} & b_{k+1} & 1 & 0 & \cdots \\ 0 & d_{k+2} & b_{k+2} & 1 & \ddots \\ 0 & 0 & d_{k+3} & b_{k+3} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

i.e., we have removed in the monic Jacobi matrix $\mathbf{J}$ the first k rows and columns.

The $\mathcal{S}$-function corresponding to the associated polynomials of order k, $S^{(k)}$, can be obtained using the formula

$$S^{(k)}(x) = \frac{S_k(x)}{d_1 S_{k-1}(x)},$$

where S_k is given by

$$S_k(x) = S(x)P_k(x) - P^{(1)}_{k-1}(x).$$

We denote this transformation of the Stieltjes functions by $\mathcal{R}^{(k)}[S(x)] = S^{(k)}(x)$.

If we are interested to characterize the sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ orthogonal with respect to $\mathcal{M}$, we consider the associated sequence of monic polynomials $\left\{P_n^{(1)}\right\}_{n\geqslant 0}$ and we have the following asymptotic expansion around infinity

$$P_n(x)S(x) - P^{(1)}_{n-1}(x) = O(x^{-n-1}).$$

It plays an important role in the theory of continued fractions. Returning to the Wallis recurrence relations (2.10), let notice that the numerators satisfy $P_n^{(1)}(x) = d_1 Q_{n+1}(x)$, $n \geqslant -1$.

On the other hand, let us consider a new family of orthogonal polynomials, $\{P_n^{(-k)}\}_{n\geqslant 0}$, which is obtained by pushing down k rows and columns in the Jacobi matrix $\mathbf{J}$, and by introducing in the upper left corner new coefficients b_{-i} $(i = k, k-1, ..., 1)$ on the diagonal, and d_{-i} $(i = k-1, k-2, ..., 0)$ on the lower sub-diagonal. The monic Jacobi matrix for the new sequence of polynomials is

$$\mathbf{J}^{(-k)} = \begin{bmatrix} b_{-k} & 1 & & & & & \\ d_{-k+1} & b_{-k+1} & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & b_0 & 1 & \\ & & & & d_1 & b_1 & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix},$$

These polynomials are called anti-associated polynomials of order k, and were analyzed in Ronveaux and Van Assche [1996].

Their corresponding Stieltjes formula can be obtained from Zhedanov [1997]

$$S^{(-k)}(x) = \frac{\widetilde{d}_k P^{(-k)}_{k-2}(x)S(x) - P^{(-k+1)}_{k-1}(x)}{\widetilde{d}_k P^{(-k)}_{k-1}(x)S(x) - P^{(-k+1)}_{k}(x)}. \tag{2.20}$$

If $k = 1$, then the anti-associated polynomials of the first kind appear and its corresponding Stieltjes function $S^{(-1)}$, given by (2.20), is

$$S^{(-1)}(x) = \frac{1}{x - \widetilde{b}_0 - \widetilde{d}_1 S(x)}, \tag{2.21}$$

where S is the Stieltjes function associated with μ, and $\widetilde{b}_0$ and $\widetilde{d}_1$ are free parameters. We denote $\mathcal{R}^{(-1)}_{(\widetilde{b}_0,\widetilde{d}_1)}[S(x)] = S^{(-1)}(x)$. Observe that $\mathcal{R}^{(-1)}$ is not a unique inverse of $\mathcal{R}^{(1)}$, because of its dependence on the free parameters $\widetilde{b}_0$ and $\widetilde{d}_1$.

2.2 Orthogonal polynomials on the unit circle

Definition

Let $\mathcal{L}$ be a linear functional in the linear space of Laurent polynomials with complex coefficients, Λ, satisfying

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \bar{c}_{-n}, \qquad n \in \mathbb{Z}. \tag{2.22}$$

$\mathcal{L}$ is said to be a hermitian linear functional. A bilinear functional associated with $\mathcal{L}$ can be introduced in $\mathbb{P}$ as follows

$$\langle f, g \rangle_{\mathcal{L}} = \left\langle \mathcal{L}, f(z)\bar{g}(z^{-1}) \right\rangle, \qquad f, g \in \mathbb{P}.$$

The complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ are said to be the moments associated with $\mathcal{L}$ and the infinite matrix

$$\mathbf{T} = \left[\left\langle z^i, z^j \right\rangle_{\mathcal{L}}\right]_{i,j\geq 0} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{2.23}$$

is the Gram matrix of the above bilinear functional in terms of the canonical basis $\{z^n\}_{n\geqslant 0}$ of $\mathbb{P}$. It is known in the literature as a Toeplitz matrix, a matrix in which each descending diagonal from left to right is constant Horn and Johnson [1990].

If $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is non-singular for every $n \geqslant 0$, $\mathcal{L}$ is said to be quasi-definite, and there exists a sequence of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$, orthogonal with respect to $\mathcal{L}$,

$$\langle \Phi_n, \Phi_m \rangle_{\mathcal{L}} = \mathbf{k}_n \delta_{n,m}, \quad \mathbf{k}_n \neq 0, \quad n \geqslant 0.$$

Szegő recurrence relations

We have seen that orthogonal polynomials on the real line satisfy a three-term recurrence relation. Such a recurrence relation does not hold for orthogonal polynomials on the unit circle, but there are also recurrence formulas. These polynomials satisfy the following forward and backward recurrence relations

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n \geqslant 0, \tag{2.24}$$
$$\Phi_{n+1}(z) = \left(1 - |\Phi_{n+1}(0)|^2\right) z\Phi_n(z) + \Phi_{n+1}(0)\Phi_{n+1}^*(z), \quad n \geqslant 0, \tag{2.25}$$

where $\Phi_n^*(z) = z^n\overline{\Phi}_n(z^{-1}) = z^n(\Phi_n)_*(z)$ is the so-called reversed polynomial, and the complex numbers $\{\Phi_n(0)\}_{n\geqslant 1}$, with

$$|\Phi_n(0)| \neq 1, \quad n \geqslant 1,$$

are known as Verblunsky, Schur or reflection coefficients. The monic orthogonal polynomials are therefore completely determined by the sequence $\{\Phi(0)\}_{n\geq 1}$. To obtain the recurrence formula, we take into account the fact that the reversed polynomial $\Phi_n^*(z)$ is the unique polynomial of degree at most n orthogonal to z^k, $1 \leqslant k \leqslant n$. (2.24) and (2.25) are called either the Szegő recurrence or Szegő difference relations. Moreover, we have

$$\langle \Phi_n, \Phi_n \rangle_{\mathcal{L}} = \mathbf{k}_n = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}}, \qquad n \geq 1, \quad \mathbf{k}_0 = c_0. \tag{2.26}$$

We can derive a recurrence formula which does not involve the reversed polynomials,

$$\Phi_n(0)\Phi_{n+1}(z) = (z\Phi_n(0) + \Phi_{n+1}(0))\Phi_n(z) - z\rho_n^2 \Phi_{n+1}(0)\Phi_{n-1}(z), \quad n \geqslant 0, \tag{2.27}$$

if we assume $\Phi_{-1} = 0$. The polynomials Φ_{n+1} can be found from Φ_{n-1} and Φ_n, if $\Phi_n(0) \neq 0$. This is an analogue of the three-term recurrence relation (2.3) for orthogonal polynomials on the real line, except for the factor z in the last term. In da Silva and Ranga [2005], the authors find bounds for complex zeros of polynomials generated by this kind of recurrence relations.

Integral representation

If $c_0 = 1$ and $\det \mathbf{T}_n > 0$, for every $n \geqslant 0$, $\mathcal{L}$ is said to be positive definite and it has the following integral representation

$$\langle \mathcal{L}, f \rangle = \int_{\mathbb{T}} f(z) d\sigma(z), \quad f \in \mathbb{P}, \tag{2.28}$$

where σ is a non-trivial probability measure supported on the unit circle $\mathbb{T}$. In such a case, there exists a unique sequence of polynomials $\{\phi_n\}_{n\geqslant 0}$ with positive leading coefficient, such that

$$\int_{\mathbb{T}} \phi_n(z)\overline{\phi_m(z)} d\sigma(z) = \delta_{m,n}.$$

$\{\phi_n\}_{n\geqslant 0}$ is said to be the sequence of orthonormal polynomials with respect to $d\sigma$. Denoting by κ_n the leading coefficient of ϕ_n, $\Phi_n = \kappa_n^{-1}\phi_n$ is the corresponding monic orthogonal polynomial of degree n. Moreover, $\langle \Phi_n, \Phi_n \rangle_{\mathcal{L}} = ||\Phi_n||_\sigma^2 = \mathbf{k}_n > 0$.

From the Pythagoras theorem, in (2.24) we get

$$\frac{||\Phi_n||_\sigma^2}{||\Phi_{n-1}||_\sigma^2} = 1 - |\Phi_n(0)|^2 > 0, \quad n \geqslant 1. \tag{2.29}$$

This shows that in the positive definite case the Verblunsky coefficients always satisfy

$$|\Phi_n(0)| < 1, \quad n \geqslant 1. \tag{2.30}$$

In this situation, we have an analogous of the Favard theorem Schur [1917, 1918]; Verblunsky [1935], formulated as follows. Any sequence of complex numbers obeying (2.30) arises as the Verblunsky coefficients of a unique non-trivial probability measure supported on the unit circle.

We use the notation $\rho_n = \sqrt{1-|\Phi_n(0)|^2} = \|\Phi_n\|_\sigma/\|\Phi_{n-1}\|_\sigma = \kappa_{n-1}/\kappa_n$. Hence, for the orthonormal polynomials ϕ_n, the recurrence relations (2.24)-(2.27) become

$$\begin{aligned} \rho_{n+1}\phi_{n+1}(z) &= z\phi_n(z) - \Phi_{n+1}(0)\phi_n^*(z), \quad n \geqslant 0, \\ \phi_{n+1}(z) &= \rho_{n+1}z\phi_n(z) + \Phi_{n+1}(0)\phi_{n+1}^*(z), \quad n \geqslant 0, \\ \rho_{n+1}\Phi_n(0)\phi_{n+1}(z) &= (z\Phi_n(0) + \Phi_{n+1}(0))\phi_n(z) - z\rho_n\Phi_{n+1}(0)\phi_{n-1}(z)(z), \quad n \geqslant 0. \end{aligned} \tag{2.31}$$

Kernel polynomials

In the case of orthogonal polynomials on the unit circle we have a simple expression for the reproducing kernel Akhiezer [1965]; Freud [1971]; Simon [2005], similar to the Christoffel-Darboux formula on the real line. The n-th polynomial kernel $K_n(z,y)$ associated with $\{\Phi_n\}_{n\geqslant 0}$ is defined by

$$\begin{aligned} K_n(z,y) = \sum_{j=0}^{n} \frac{\overline{\Phi_j(y)}\Phi_j(z)}{\mathbf{k}_j} &= \frac{\overline{\Phi_{n+1}^*(y)}\Phi_{n+1}^*(z) - \overline{\Phi_{n+1}(y)}\Phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\overline{y}z)} \\ &= \frac{\overline{\phi_{n+1}^*(y)}\phi_{n+1}^*(z) - \overline{\phi_{n+1}(y)}\phi_{n+1}(z)}{1-\overline{y}z}, \end{aligned} \tag{2.32}$$

and it satisfies the reproducing property,

$$\int_{\mathbb{T}} K_n(z,y)\overline{f(z)}d\sigma(z) = \overline{f(y)}, \tag{2.33}$$

for every polynomial f of degree at most n. Taking into account $\phi_{n+1}^*(0) = \kappa_{n+1}\Phi_{n+1}^*(0) = \kappa_{n+1}$, we find that

$$\Phi_n^*(z) = \frac{1}{\kappa_n^2}K_n(z,0) = \mathbf{k}_n K_n(z,0), \quad n \geq 0, \tag{2.34}$$

which is an expression for the reversed polynomials as a linear combination of the orthogonal polynomials up to degree n.

GGT matrices

Using (2.34) and the forward recurrence formula (2.24), we are able to express $z\phi_n(z)$ as a linear combination of $\{\phi_k\}_{k=0}^{n+1}$,

$$z\phi_n(z) = \frac{\kappa_n}{\kappa_{n+1}}\phi_{n+1}(z) - \frac{1}{\kappa_n}\Phi_{n+1}(0)\sum_{k=0}^{n}\kappa_k\overline{\Phi_k(0)}\phi_k(z),$$

or, in the matrix form,

$$z\boldsymbol{\phi}(z) = \mathbf{H}_\sigma\boldsymbol{\phi}(z),$$

where $\boldsymbol{\phi}(z) = [\phi_0(z), \phi_1(z), \dots]^T$, and the matrix $\mathbf{H}_\sigma$ is defined by

$$[\mathbf{H}_\sigma]_{i,j} = \left\langle z\phi_i, \phi_j \right\rangle_{\mathcal{L}} = \begin{cases} -\frac{\kappa_j}{\kappa_i}\Phi_{i+1}(0)\overline{\Phi_j(0)}, & j \leq i, \\ \frac{\kappa_i}{\kappa_{i+1}}, & j = i+1, \\ 0, & j > i+1. \end{cases}$$

This lower Hessenberg matrix Horn and Johnson [1990], where the j-th row has at most its first $j+1$ components non-zero, is called GGT representation of the multiplication by z, after Geronimus [1944]; Gragg [1993]; Teplyaev [1992].

In an analog way to the real line case, the zeros of the monic orthogonal polynomial Φ_n are the eigenvalues of $(\mathbf{H}_\sigma)_n$, the $n \times n$ principal leading sub-matrix of the GGT matrix $\mathbf{H}_\sigma$. Hence, Φ_n is the characteristic polynomial of $(\mathbf{H}_\sigma)_n$,

$$\Phi_n(z) = \det\left(z\mathbf{I}_n - (\mathbf{H}_\sigma)_n\right). \tag{2.35}$$

Szegő extremum problem and $\mathcal{S}$ class

The measure of orthogonality $d\sigma$ can be decomposed as the sum of a purely absolutely continuous measure with respect to the Lebesgue measure and a singular part. Thus, if we denote by σ', the Radon-Nikodym derivative Rudin [1987] of the measure σ supported in $[-\pi, \pi]$, then

$$d\sigma(\theta) = \sigma'(\theta)\frac{d\theta}{2\pi} + d\sigma_s, \tag{2.36}$$

where σ_s is the singular part of σ.

The Szegő extremum problem on the unit circle consists of finding

$$\lambda(z) = \lim_{n\to\infty} \lambda_n(z),$$

with

$$\lambda_n(z) = \inf_{f(z)=1} \left\{ \int_{-\pi}^{\pi} \left| f(e^{i\theta}) \right|^2 d\sigma(\theta);\ f \in \mathbb{P}_n \right\}.$$

$\lambda(z)$ is said to be the Christoffel function. The solution of this problem for $|z| < 1$ was given by Szegő in Szegő [1920, 1921b].

In the literature, an important class of measures is the Szegő class $\mathcal{S}$. We summarize some relevant characterizations to the $\mathcal{S}$ class. The following conditions are equivalent:

i) $\sigma \in \mathcal{S}$.

ii) $\displaystyle\int_{-\pi}^{\pi} \log \sigma'(\theta)\frac{d\theta}{2\pi} > -\infty.$

iii) $\displaystyle\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty.$

iv) $\displaystyle\lambda(0) = \prod_{n=0}^{\infty}(1 - |\Phi_{n+1}(0)|^2) < +\infty.$

From this we deduce that if the measure σ does not belong to the $\mathcal{S}$ class, the GGT matrix $\mathbf{H}_\sigma$ is unitary. In general, $\mathbf{H}_\sigma$ satisfies *i)* $\mathbf{H}_\sigma \mathbf{H}_\sigma^H = \mathbf{I}$; *ii)* $\mathbf{H}_\sigma^H \mathbf{H}_\sigma = \mathbf{I} - \lambda(0)\phi(0)\phi(0)^H$.

As a part of the analysis when $\sigma \in \mathcal{S}$, one can construct the Szegő function D, defined in $\mathbb{D}$ as

$$D(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \sigma'(\theta) d\theta\right), \quad z \in \mathbb{D}.$$

Thus, $|D|^2 = \sigma'$ almost everywhere on $\mathbb{T}$, and the solution of the Szegő extremum problem is given by

$$\lambda(z) = (1-|z|^2)|D(z)|^2, \quad z \in \mathbb{D}.$$

$\mathcal{N}$ class

We say that σ belongs to the Nevai class $\mathcal{N}$, if

$$\lim_{n\to\infty} \Phi_n(0) = \lim_{n\to\infty} \frac{\phi_n(0)}{\kappa_n} = 0.$$

The relation between the classes $\mathcal{S}$ and $\mathcal{N}$ can be viewed using the results in Martínez-Finkelshtein and Simon [2011]. If $\sigma \in \mathcal{S}$, then it has a normal L^2-derivative behavior, i.e.,

$$\lim_{n\to\infty}\left(\int_{-\pi}^{\pi} \frac{\left|\phi_n'(e^{i\theta})\right|^2}{n^2}\sigma'(\theta)d\theta\right)^{\frac{1}{2}} = 1,$$

and thus $\sigma \in \mathcal{N}$. Furthermore, if $\sigma \in \mathcal{N}$,

$$\left|\frac{\Phi_n(z)}{\Phi_{n-1}(z)} - z\right| \leqslant |\Phi_n(0)|, \qquad z \in \mathbb{C} \setminus \mathbb{D}.$$

Thus,

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{\Phi_{n-1}(z)} = z,$$

uniformly in compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$.

This result can be obtained under weaker conditions. A well known result of Rakhmanov Rakhmanov [1977, 1983] states that any probability measure σ with $\sigma' > 0$ almost everywhere on $\mathbb{T}$ belongs to the class $\mathcal{N}$.

CMV matrices

The GGT matrix has several constraints. If $\sigma \in \mathcal{S}$, $\{\phi_n\}_{n\geqslant 0}$ is not basis on Λ and the matrix $\mathbf{H}_\sigma$ is not unitary. Even more, all entries above the main diagonal and the first sub-diagonal are non-zero, and they depend on an unbounded number of Verblunsky coefficients. Consequently, the GGT matrix is somewhat difficult to manipulate. The more useful basis was discovered by Cantero, Moral, and Velazquez Cantero et al. [2003] (this result is one of the most interesting developments in the theory of orthogonal polynomials on the unit circle in recent years) as a matrix realization for the multiplication by z, with respect to the CMV orthonormal basis $\{\chi_i\}_{i\geq 0}$,

$$\left\langle z\chi_i, \chi_j \right\rangle_{\mathcal{L}} = 0, \quad |i-j| > k, \quad k \geq 0.$$

In this case $k = 2$ to be compared with $k = 1$ for the Jacobi matrices which correspond to the real line case. The CMV basis $\{\chi_n\}_{n\geqslant 0}$ is obtained by orthonormalizing $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ using the Gram-Schmidt process and the matrix, called the CMV matrix,

$$\mathbf{C} = \left[\left\langle z\chi_i, \chi_j \right\rangle_{\mathcal{L}}\right]_{i,j\geq 0},$$

is five-diagonal. Its origins are outlined in Watkins [1993] (see also Simon [2007]). Remarkably, the basis $\{\chi_n\}_{n\geqslant 0}$ can be expressed in terms of the sequences $\{\phi_n\}_{n\geqslant 0}$ and $\{\phi_n^*\}_{n\geqslant 0}$,

$$\chi_{2n}(z) = z^{-n}\phi_{2n}^*(z), \quad \chi_{2n+1}(z) = z^{-n}\phi_{2n+1}(z), \quad n \geqslant 0.$$

There is an important relation between CMV and monic orthogonal polynomials as (2.35) for the GGT representation,

$$\Phi_n(z) = \det(z\mathbf{I}_n - \mathbf{C}_n),$$

where $\mathbf{C}_n$ is the $n\times n$ principal leading sub-matrix of the CMV matrix $\mathbf{C}$.

The CMV matrices play the same role in the study of orthogonal polynomials on the unit circle that Jacobi matrices in orthogonal polynomials on the real line.

Zeros

If Φ_n is an orthogonal polynomial of degree n, all its zeros lie in the interior of the convex hull of the support of the measure of orthogonality Féjer [1922], and we recover the properties of zeros for orthogonal polynomials on the real line. From the Christoffel-Darboux formula we have for $z = y$,

$$K_n(z,z) = \sum_{k=0}^{n} |\phi_k(z)|^2 = \frac{|\phi_{n+1}^*(z)|^2 - |\phi_{n+1}(z)|^2}{1-|z|^2}. \tag{2.37}$$

If $z_{n,1}$ is a zero of ϕ_n with $|z_{n,1}| = 1$, then using (2.37) for $n-1$, $\phi_n^*(z_{n,1}) = 0$, and from the recurrence relation (2.31) we get $\phi_{n-1}(z_{n,1}) = 0$. Repeating this argument, we have $\phi_k(z_{n,1}) = 0$, $k \leq n$, but for $k = 0$, $\phi_0 = 1$, which gives a contradiction. Hence we conclude that ϕ_n has no zeros on the unit circle, and thus all the zeros of ϕ_n are in $\mathbb{D}$.

In Section 2.1 we see that the interlacing property for the zeros of two polynomials P_{n-1} and P_n, means that they are the $(n-1)$-st and n-th orthogonal polynomials associated with a measure $d\mu$ supported on the real line. In the case of the unit circle, we have an analogous result, which is known in the literature as the Schur-Cohn-Jury criterion Barnett [1983]. A monic polynomial f_n has its n zeros inside the unit circle if and only if the sequence of parameters $\{h_k\}_{k=0}^n$, defined by the following backward algorithm

$$
\begin{aligned}
g_n(z) &= f_n(z), \quad f_n(0) = h_n, \\
g_k(z) &= \frac{1}{z(1-|h_{k+1}|^2)}\left(g_{k+1}(z) - h_{k+1}g_{k+1}^*(z)\right), \quad k = n-1, n-2, \ldots, 0,
\end{aligned}
$$

satisfies $|h_k| < 1$, $k \geqslant 1$.

2.2.1 C-functions and rational spectral transformations

C-functions

In the sequel, we consider that $\sum_{k=0}^{\infty} c_{-k}z^k$ converges on $|z| < r$, $r > 0$, where $\{c_n\}_{n\geqslant 0}$ is the sequence of moments (2.22). Let $F : E \subset \mathbb{C} \to \mathbb{C}$ be a complex function associated with the linear functional $\mathcal{L}$, defined as follows

$$
F(z) = \left\langle \mathcal{L}, \frac{y+z}{y-z} \right\rangle.
$$

Here $\mathcal{L}$ acts on y. F is said to be a C-function associated with the linear functional $\mathcal{L}$. Since $\sum_{k=0}^{\infty} c_{-k}z^k$ converges on $|z| \leq r$, $F(z)$ is analytic in a neighborhood of $z = 0$, and we get the following representation of $F(z)$ as a series expansion at $z = 0$

$$
F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k}z^k, \quad |z| \leq r, \tag{2.38}
$$

where $c_0 \in \mathbb{R}$ and $c_{-k} \in \mathbb{C}$.

If $\mathcal{L}$ is positive definite and $c_0 = 1$, there is a probability measure σ supported on the unit circle such that F can be represented as a Riesz-Herglotz transformation Herglotz [1911]; Riesz [1911] of $d\sigma$ as follows

$$
F(z) = \int_{\mathbb{T}} \frac{y+z}{y-z} d\sigma(y). \tag{2.39}
$$

A complex function F which has a representation of the form (2.39) is called a Carathéodory function Carathéodory [1907]. It can be shown that F is a Carathéodory function if and only if $F(z)$ is analytic in $|z| < 1$ and $\Re F(z) > 0$ for $|z| < 1$; see Chapter 1. From this it follows immediately that F^{-1} is a Carathéodory function if and only if F is a Carathéodory function.

Characterization of orthogonal polynomials

For a given polynomial π of degree n with leading coefficient η, the polynomial of the second kind of π, Π, with respect to $\mathcal{L}$, is defined by

$$\Pi(z) = \begin{cases} \left\langle \mathcal{L}, \dfrac{z+y}{z-y}(\pi(y)-\pi(z)) \right\rangle, & \deg \pi > 0, \\ -\eta c_0, & \deg \pi = 0, \end{cases}$$

where $\mathcal{L}$ acts on y.

Next, we give necessary and sufficient conditions for a polynomial to be orthogonal with respect to a linear, not necessarily positive definite functional $\mathcal{L}$, with the help of the associated C-function F. The following statement holds Peherstorfer and Steinbauer [1995]:

$$\left\langle \Phi_n(z), z^k \right\rangle_{\mathcal{L}} = 0, \quad k = 0, \dots, n-1,$$

if and only if

$$\Phi_n(z)F(z) - \Omega_n(z) = O(z^n), \quad |z| < 1, \tag{2.40}$$

$$\Phi_n^*(z)F(z) + \Omega_n^*(z) = O\left(z^{n+1}\right), \quad |z| < 1, \tag{2.41}$$

where Ω_n is the polynomial of second kind associated with Φ_n with respect to $\mathcal{L}$.

There is an interesting way to rephrase the previous result, namely, given a linear functional $\mathcal{L}$ and its C-function F, and given Q_n, a monic polynomial of degree n, define Ω_n, a monic polynomial, by (2.40). Then, $Q_n = \Phi_n$ if and only if (2.41) holds.

Spectral transformations

As in the real line case, by a spectral transformation of F we mean a new C-function associated with the hermitian linear functional $\widetilde{\mathcal{L}}$, a modification of $\mathcal{L}$, such that

$$\widetilde{F}(z) = \widetilde{c}_0 + 2\sum_{k=1}^{\infty} \widetilde{c}_{-k} z^k, \quad |z| \leq r,$$

where $\{\widetilde{c}_n\}_{n\geqslant 0}$ is the sequence of transformed moments. We refer to rational spectral transformation as a transformation of a C-function F given by

$$\widetilde{F}(z) = \frac{A(z)F(z)+B(z)}{C(z)F(z)+D(z)}, \quad A(z)D(z) - B(z)C(z) \neq 0, \tag{2.42}$$

where A, B, C, and D are coprime polynomials. If $C = 0$, we have the subclass of linear spectral transformations. Following our classification in terms of the moments, the local spectral transformation is a special case of linear spectral transformation related to perturbations on two symmetric sub-diagonals of the Toeplitz matrix (2.23).

Linear spectral transformations

We use the $\mathcal{C}$-function F as a main tool in the investigation of spectral transformations of orthogonal polynomials. As an analog of the Christoffel transformation (2.15) for $\mathcal{S}$-functions, we consider a perturbation $\mathcal{L}_R$ of $\mathcal{L}$ defined by

$$\langle \mathcal{L}_R, f\rangle = \left\langle \mathcal{L}, \frac{1}{2}\left(z-\alpha+z^{-1}-\overline{\alpha}\right)f(z)\right\rangle, \quad f\in\Lambda, \quad \alpha\in\mathbb{C}. \tag{2.43}$$

It constitutes an example of linear spectral transformation and it was introduced in Cantero [1997]; Suárez [1993]. It is natural to analyze the existence of the inverse transformation, i.e., if there exists a linear functional $\mathcal{L}_{R^{(-1)}}$ such that

$$\left\langle \mathcal{L}_{R^{(-1)}}, \left(z-\alpha+z^{-1}-\overline{\alpha}\right)f(z)\right\rangle = \langle \mathcal{L}, f\rangle, \quad f\in\Lambda, \tag{2.44}$$

as well as to analyze if the quasi-definite character of the linear functional is preserved by such a transformation. Notice that this transformation does not define a unique linear functional $\mathcal{L}_{R^{(-1)}}$. The uniqueness depends on a free parameter. Recently, the spectral transformations (2.43) and (2.44) have been studied with a new approach in the framework of inverse problems for sequences of monic orthogonal polynomials Cantero et al. [2011].

For all values of α, such that $|\Re(\alpha)|>1$, the Laurent polynomial $z-\alpha+z^{-1}-\overline{\alpha}$ can be represented as a polynomial of the form $-\frac{1}{\beta}|z-\beta|^2$, where $\beta\in\mathbb{R}\setminus\{0\}$. The particular cases (2.43) and (2.44) with $|\Re(\alpha)|>1$ have been extensively covered in Garza [2008] and the references therein.

Rational spectral transformations

Two remarkable examples of rational spectral transformations are due to Peherstorfer Peherstorfer [1996]. We denote by $\{\Phi_n^{(k)}\}_{n\geqslant 0}$ the k-th associated sequence of polynomials of order $k\geqslant 1$ for the monic orthogonal sequence $\{\Phi_n\}_{n\geqslant 0}$ that constitutes the analog of the associated polynomials, satisfying (2.19). In this case they are generated by the recurrence relation

$$\Phi_{n+1}^{(k)}(z) = z\Phi_n^{(k)}(z) + \Phi_{n+k+1}(0)\left(\Phi_n^{(k)}(z)\right)^*, \qquad n\geq 0. \tag{2.45}$$

Notice that $\{\Phi_n^{(k)}\}_{n\geqslant 0}$ is again a sequence of orthogonal polynomials with respect to a new hermitian linear functional such that $F^{(k)}$ is the corresponding $\mathcal{C}$-function.

Denoting by $\mathcal{F}^{(k)}[F(z)] = F^{(k)}(z)$ the forward transformation of F, the corresponding $\mathcal{C}$-function is a rational spectral transformation given by

$$F^{(k)}(z) = \frac{\left(\Phi_k(z)+\Phi_k^*(z)\right)F(z)-\Omega_k(z)+\Omega_k^*(z)}{\left(\Phi_k(z)-\Phi_k^*(z)\right)F(z)-\Omega_k(z)-\Omega_k^*(z)},$$

where Ω_n is the polynomial of second kind of Φ_n with respect to $\mathcal{L}$. In other words, we remove the first k Verblunsky coefficients from the original sequence.

On the other hand, if we add complex numbers $z_1, z_2, \dots, z_k$ with $|z_i| \neq 1$, $1 \leqslant i \leqslant k$, to the original sequence of Verblunsky coefficients, we have the backward associated sequence of polynomials $\left\{\Phi_n^{(-k)}\right\}_{n\geqslant 0}$ as a sequence of monic orthogonal polynomials generated by $\{z_i\}_{i=1}^{k} \bigcup \{\phi_n(0)\}_{n\geq 1}$.

Denoting by $\mathcal{F}^{(-k)}[F(z)] = F^{(-k)}(z)$ the backward transformation of F, the corresponding C-function is a rational spectral transformation given by

$$F^{(-k)}(z) = \frac{-\left(\widetilde{\Omega}_k(z) + \widetilde{\Omega}_k^*(z)\right)F(z) - \widetilde{\Omega}_k^*(z) + \widetilde{\Omega}_k(z)}{\left(\widetilde{\Phi}_k^*(z) - \widetilde{\Phi}_k(z)\right)F(z) + \widetilde{\Phi}_k(z) + \widetilde{\Phi}_k^*(z)},$$

where $\widetilde{\Phi}_k$ (respectively $\widetilde{\Omega}_k$) is the k-th degree polynomial generated using the complex numbers $z_1, z_2, \dots, z_k$ (respectively $-z_1, -z_2, \dots, -z_k$) through the recurrence relation (2.45), i.e., $\widetilde{\Omega}_k$ is the polynomial of second kind associated with $\widetilde{\Phi}_k$. It is easily verified that $\mathcal{F}^{(k)} \circ \mathcal{F}^{(-k)} = \mathcal{I}$. Generally, the inverse is not always true since it depends on the choice of free parameters.

2.2.2 Connection with orthogonal polynomials on $[-1,1]$

Given a non-trivial probability measure μ, supported on the interval $[-1,1]$, then there exists a sequence of orthonormal polynomials $\{p_n\}_{n\geqslant 0}$ such that

$$\int_{-1}^{1} p_n(x)p_m(x)d\mu(x) = \delta_{n,m}.$$

We can define a measure σ supported on $[-\pi,\pi]$ such that

$$d\sigma(\theta) = \frac{1}{2}|d\mu(\cos\theta)|.$$

In particular, if μ is an absolutely continuous measure, i.e., $d\mu(x) = \omega(x)dx$, then

$$d\sigma(\theta) = \frac{1}{2}\omega(\cos\theta)|\sin\theta|d\theta.$$

This is the so-called Szegő transformation of probability measures supported on $[-1,1]$ to probability measures supported on $\mathbb{T}$.

If μ is a non-trivial probability measure on $[-1,1]$ (this is the reason why we introduced the factor $1/2$), σ is also a symmetric probability measure on the unit circle and, as a consequence, there exists a sequence of orthonormal polynomials $\{\phi_n\}_{n\geqslant 0}$ such that

$$\int_{-\pi}^{\pi} \phi_n(e^{i\theta})\overline{\phi_m(e^{i\theta})}d\sigma(\theta) = \delta_{n,m},$$

as well as the corresponding sequence of monic orthogonal polynomials. In this case,

$$\Phi_n(0) \in (-1,1), \quad n \geqslant 1.$$

There is a relation between the sequence of orthogonal polynomials associated with a measure μ supported on $[-1,1]$ and the sequence of orthogonal polynomials associated with the measure σ supported on the unit circle. The sequence of orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$ on the unit circle associated with the measure σ has real coefficients. In addition, if $2x = z + z^{-1}$, so that $x = \cos\theta$ corresponds to $z = e^{i\theta}$, then

$$p_n(x) = \frac{\kappa_{2n}}{\sqrt{2(1+\Phi_{2n}(0))}} \left(z^{-n}\Phi_{2n}(z) + z^n \Phi_{2n}(1/z)\right). \tag{2.46}$$

From (2.46) one can obtain a relation between the coefficients of the recurrence relations (2.5) and (2.24)-(2.25),

$$\begin{aligned} 2a_n &= \sqrt{(1-\Phi_{2n}(0))\left(1-\Phi_{2n-1}^2(0)\right)(1+\Phi_{2n-2}(0))}, \quad n \geqslant 1, \\ 2b_n &= \Phi_{2n-1}(0)\,(1-\Phi_{2n}(0)) - \Phi_{2n+1}(0)\,(1+\Phi_{2n}(0)), \quad n \geqslant 0. \end{aligned}$$

Conversely, if $R_n = P_{n+1}/P_n$, then

$$\Phi_{2n}(0) = R_n(1) - R_n(-1) - 1, \quad \Phi_{2n+1}(0) = \frac{R_n(1) + R_n(-1)}{R_n(1) - R_n(-1)}.$$

There is also a relation between the $\mathcal{S}$-function and C-function associated with μ and σ, respectively, as follows

$$F(z) = \frac{1-z^2}{2z} \int_{-1}^{1} \frac{d\mu(y)}{x-y} = \frac{1-z^2}{2z} S(x),$$

or, equivalently,

$$S(x) = \frac{F(z)}{\sqrt{x^2-1}},$$

with $2x = z + z^{-1}$ and $z = x - \sqrt{x^2-1}$.

Chapter 3

On special classes of Szegő polynomials

The notion of Schur parameters [1] is fundamental not only in the theory of orthogonal polynomials . . .

— A. Mukaihira and Y. Nakamura. Mukaihira and Nakamura [2002]

From Chapter 2 we know that the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geqslant 1}$ associated with a measure supported on the unit circle satisfy (2.30). Moreover, by the Favard theorem, any sequence obeying (2.30) arises as the Verblunsky coefficients of a unique non-trivial probability measure supported on the unit circle. An interesting question arises: What happens when $|\Phi_n(0)| > 1$ at most for some n? Clearly, these polynomials can not be orthogonal on the unit circle. In 1999, Vinet and Zhedanov Vinet and Zhedanov [1999] constructed special classes of Szegő polynomials when $|\Phi_n(0)| > 1$, $n \geqslant 1$. This situation is more interesting, because there are sequences $\{\Phi_n(0)\}_{n\geqslant 1}$ for which the moment problem is indeterminate. They consider two possible choices of the Verblunsky coefficients when the support of the associated measure lies on the real line. They note that if $\Phi_n(0) > 1$, $n \geqslant 1$, the corresponding orthogonality measure is supported on the negative half side of the real line. On the other hand, if $(-1)^n\Phi_n(0) > 1$, $n \geqslant 1$, the corresponding orthogonality measure is supported on the positive half side side of the real line. Their main tool is a mapping from symmetric polynomials on the real line to the Szegő polynomials Delsarte and Genin [1986, 1991]; see also Chapter 1.

The aim of this chapter is to study the properties of the sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$, which satisfy the same recurrence relation as orthogonal polynomials on the unit circle (Szegő orthogonality) with

$$\Phi_n(0) \in \mathbb{C}, \quad |\Phi_n(0)| > 1, \quad n \geqslant 1.$$

In our development a central role is played by continued fractions. We emphasize the natural parallelism

[1] Verblunsky coefficients.

existing between the theories of orthogonal polynomials on the real line and those on the unit circle. An analysis of the Perron-Carathéodory continued fractions Jones et al. [1986] shows that these polynomials satisfy the Szegő orthogonality, where the linear functional $\mathcal{L}$ defined in (2.22) satisfies

$$\bar{c}_{-n} = c_n, \quad (-1)^{n(n+1)/2} \det \mathbf{T}_n > 0, \quad n \geqslant 0.$$

The relations between the polynomials $\Phi_n(\omega; z) = \Phi_n(z) - \omega z \Phi_{n-1}(z)$, $\omega \neq 1$, and the para-orthogonal polynomials Jones et al. [1986], $\Phi_n(z) + \tau\Phi_n^*(z)$, $|\tau| = 1$ are also analysed. In the two particular cases considered in Vinet and Zhedanov [1999], the zeros of the Szegő polynomials, those of para-orthogonal polynomials, and the associated quadrature rules have been studied. As a consequence of this study, we solve the moment problem (1.10) associated with the Chebyshev polynomials of the first kind. This chapter could be considered in many aspects as a continuation of the introductory theory of orthogonal polynomials, using a different approach to the subject based on continued fractions and their modified approximates.

3.1 Special class of Szegő polynomials

The treatment of this section is similar to that is given in Jones et al. [1989], where the authors assume that $|\Phi_n(0)| \neq 1$, $n \geq 1$. The results given here, in addition to making the chapter self-contained, help the reader to see the specific properties satisfied by the associated Toeplitz and Hankel determinants when $|\Phi_n(0)| > 1$, $n \geq 1$.

3.1.1 Szegő polynomials from continued fractions

We start with the so-called Perron-Carathéodory continued fraction

$$\beta_0 - \frac{2\beta_0|}{|1} - \frac{1|}{|\bar{\alpha}_1 z} - \frac{\left(|\alpha_1|^2 - 1\right)z|}{|\alpha_1} - \frac{1|}{|\bar{\alpha}_2 z} - \cdots, \tag{3.1}$$

where we assume that $\beta_0 > 0$ and $|\alpha_n| > 1$, $n \geq 1$. This continued fraction was introduced in Jones et al. [1986]; see also Jones et al. [1987, 1989]. If we compare (3.1) with (2.8), then we have $q_0 = \beta_0$, $r_1 = -2\beta_0$, $q_1 = 1$, $r_{2n} = -1$, $r_{2n+1} = -\left(|\alpha_n|^2 - 1\right)z$, $q_{2n} = \bar{\alpha}_n z$, and $q_{2n+1} = \alpha_n$, $n \geqslant 1$.

Let $\{A_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$ be the sequence of numerator and denominator polynomials of (3.1), respectively. Then these polynomials satisfy the recurrence relations

$$\begin{bmatrix} A_{2n}(z) \\ B_{2n}(z) \end{bmatrix} = \bar{\alpha}_n z \begin{bmatrix} A_{2n-1}(z) \\ B_{2n-1}(z) \end{bmatrix} - \begin{bmatrix} A_{2n-2}(z) \\ B_{2n-2}(z) \end{bmatrix}, \quad n \geq 1, \tag{3.2}$$

$$\begin{bmatrix} A_{2n+1}(z) \\ B_{2n+1}(z) \end{bmatrix} = \alpha_n \begin{bmatrix} A_{2n}(z) \\ B_{2n}(z) \end{bmatrix} - \left(|\alpha_n|^2 - 1\right)z \begin{bmatrix} A_{2n-1}(z) \\ B_{2n-1}(z) \end{bmatrix}, \quad n \geq 1, \tag{3.3}$$

where $B_0 = 1$, $B_1 = 1$, $A_0 = \beta_0$, and $A_1 = -\beta_0$. From these recurrence relations, since $\alpha_n \neq 0$, $n \geq 1$, we get

$$\begin{bmatrix} A_{2n+2}(z) \\ B_{2n+2}(z) \end{bmatrix} = \left(\frac{\overline{\alpha}_{n+1}}{\overline{\alpha}_n} z - 1\right) \begin{bmatrix} A_{2n}(z) \\ B_{2n}(z) \end{bmatrix} - \frac{\overline{\alpha}_{n+1}}{\overline{\alpha}_n} \left(|\alpha_n|^2 - 1\right) z \begin{bmatrix} A_{2n-2}(z) \\ B_{2n-2}(z) \end{bmatrix}, \quad n \geq 1,$$

with $B_0 = 1$, $B_2(z) = \overline{\alpha}_1 z - 1$, $A_0 = \beta_0$, $A_2(z) = -\beta_0(\overline{\alpha}_1 z + 1)$, and

$$\begin{bmatrix} A_{2n+3}(z) \\ B_{2n+3}(z) \end{bmatrix} = \left(z - \frac{\alpha_{n+1}}{\alpha_n}\right) \begin{bmatrix} A_{2n+1}(z) \\ B_{2n+1}(z) \end{bmatrix} - \frac{\alpha_{n+1}}{\alpha_n} \left(|\alpha_n|^2 - 1\right) z \begin{bmatrix} A_{2n-1}(z) \\ B_{2n-1}(z) \end{bmatrix}, \quad n \geq 1.$$

Here $B_1 = 1$, $B_3(z) = z - \alpha_1$, $A_1 = -\beta_0$, and $A_3(z) = -\beta_0(z + \alpha_1)$. From these recurrence relations one can easily observe that both B_{2n} and B_{2n+1} are polynomials of exact degree n. Likewise, both A_{2n} and A_{2n+1} are polynomials of exact degree n. More precisely,

$$\begin{aligned} B_{2n}(z) &= \overline{\alpha}_n z^n + \ldots + (-1)^n, & B_{2n+1}(z) &= z^n + \ldots + (-1)^n \alpha_n, \\ A_{2n}(z) &= -\beta_0(\overline{\alpha}_n z^n + \ldots + (-1)^{n-1}), & A_{2n+1}(z) &= -\beta_0\left(z^n + \ldots + (-1)^{n-1}\alpha_n\right), \quad n \geq 1. \end{aligned}$$

Using the above recurrence relations one can also easily conclude that

$$A^*_{2n+1}(z) = z^n \overline{A_{2n+1}(1/\overline{z})} = z^n \overline{A}_{2n+1}(1/z) = (-1)^{n+1} A_{2n}(z), \quad n \geq 0, \tag{3.4}$$

$$B^*_{2n+1}(z) = z^n \overline{B_{2n+1}(1/\overline{z})} = z^n \overline{B}_{2n+1}(1/z) = (-1)^n B_{2n}(z), \quad n \geq 0. \tag{3.5}$$

Moreover, from these recurrence relations we obtain

$$\frac{A_{2n+2}(z)}{B_{2n+2}(z)} - \frac{A_{2n}(z)}{B_{2n}(z)} = \begin{cases} \left(2\beta_0 \overline{\alpha}_{n+1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{n+1} + O\left(z^{n+2}\right), \\ -\left(2\beta_0 \overline{\alpha}_n^{-1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{-n} + O\left(z^{-(n+1)}\right), \quad n \geq 1, \end{cases}$$

$$\frac{A_{2n+3}(z)}{B_{2n+3}(z)} - \frac{A_{2n+1}(z)}{B_{2n+1}(z)} = \begin{cases} \left(2\beta_0 \alpha_n^{-1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{n} + O\left(z^{n+1}\right), \\ -\left(2\beta_0 \alpha_{n+1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{-(n+1)} + O\left(z^{-(n+2)}\right), \quad n \geq 1, \end{cases}$$

$$\frac{A_{2n+1}(z)}{B_{2n+1}(z)} - \frac{A_{2n}(z)}{B_{2n}(z)} = \begin{cases} -\left(2\beta_0 \alpha_n^{-1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{n} + O\left(z^{n+1}\right), \\ -\left(2\beta_0 \overline{\alpha}_n^{-1} \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right) z^{-n} + O\left(z^{-(n+1)}\right), \quad n \geq 1. \end{cases}$$

Thus, there exists a pair of formal power series

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n, \quad |z| < 1, \qquad F_\infty(z) = -c_0 - 2\sum_{n=1}^{\infty} c_n z^{-n}, \quad |z| > 1, \tag{3.6}$$

with $c_0 = \beta_0$ and $\overline{c}_{-1} = c_1 = \beta_0\alpha_1$, such that

$$\begin{aligned}
F(z) - \frac{A_{2n}(z)}{B_{2n}(z)} &= \left(2\beta_0\overline{\alpha}_{n+1}\prod_{r=1}^{n}\Big(|\alpha_r|^2-1\Big)\right)z^{n+1} + O\Big(z^{n+2}\Big), \quad n \geq 1,\\
F_\infty(z) - \frac{A_{2n}(z)}{B_{2n}(z)} &= -\left(2\beta_0\overline{\alpha}_n^{-1}\prod_{r=1}^{n}\Big(|\alpha_r|^2-1\Big)\right)z^{-n} + O\Big(z^{-(n+1)}\Big), \quad n \geq 1,\\
F(z) - \frac{A_{2n+1}(z)}{B_{2n+1}(z)} &= \left(2\beta_0\alpha_n^{-1}\prod_{r=1}^{n}\Big(|\alpha_r|^2-1\Big)\right)z^{n} + O\Big(z^{n+1}\Big), \quad n \geq 1,\\
F_\infty(z) - \frac{A_{2n+1}(z)}{B_{2n+1}(z)} &= -\left(2\beta_0\alpha_{n+1}\prod_{r=1}^{n}\Big(|\alpha_r|^2-1\Big)\right)z^{-(n+1)} + O\Big(z^{-(n+2)}\Big), \quad n \geq 1.
\end{aligned} \tag{3.7}$$

That is, (3.1) corresponds to the formal power series expansions F and F_∞. Using the reciprocal properties (3.4)-(3.5), we also conclude that these formal power series expansions are such that

$$\overline{c}_{-n} = c_n, \quad n \geq 1. \tag{3.8}$$

If we write $B_{2n+1}(z) = \sum_{r=0}^{n} b_{n,r}z^r$ and $A_{2n+1}(z) = \sum_{r=0}^{n} a_{n,r}z^r$, then from (3.7),

$$\begin{aligned}
F(z)\sum_{r=0}^{n} b_{n,r}z^r - \sum_{r=0}^{n} a_{n,r}z^r &= \gamma_n z^n + O\Big(z^{n+1}\Big), \quad n \geq 1,\\
F_\infty(z)\sum_{r=0}^{n} b_{n,r}z^{r-n} - \sum_{r=0}^{n} a_{n,r}z^{r-n} &= O\Big(z^{-n-2}\Big), \quad n \geq 1,
\end{aligned}$$

where $\gamma_n = (-1)^n 2\beta_0 \prod_{r=1}^{n}(|\alpha_r|^2 - 1)$. This leads to two systems of $n+1$ linear equations and $n+1$ unknowns as follows

$$\mathbf{A}_1\mathbf{b} = \mathbf{a} + \mathbf{c}, \quad \mathbf{A}_2\mathbf{b} = -\mathbf{a},$$

where $\mathbf{a} = [a_{n,0}, a_{n,1}, \ldots, a_{n,n-1}, a_{n,n}]^T$, $\mathbf{b} = [b_{n,0}, b_{n,1}, \ldots, b_{n,n-1}, b_{n,n}]^T$, $\mathbf{c} = [0, 0, \ldots, 0, \gamma_n]^T$,

$$\mathbf{A}_1 = \begin{bmatrix} c_0 & 0 & \cdots & 0 & 0\\ 2c_{-1} & c_0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 2c_{-n+1} & 2c_{-n+2} & \cdots & c_0 & 0\\ 2c_{-n} & 2c_{-n+1} & \cdots & 2c_{-1} & c_0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} c_0 & 2c_1 & \cdots & 2c_{n-1} & 2c_n\\ 0 & c_0 & \cdots & 2c_{n-2} & 2c_{n-1}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & c_0 & 2c_1\\ 0 & 0 & \cdots & 0 & c_0 \end{bmatrix},$$

or, equivalently, the linear system

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_{-1} & c_0 & \cdots & c_{n-2} & c_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{-n+1} & c_{-n+2} & \cdots & c_0 & c_1 \\ c_{-n} & c_{-n+1} & \cdots & c_{-1} & c_0 \end{bmatrix} \begin{bmatrix} b_{n,0} \\ b_{n,1} \\ \vdots \\ b_{n,n-1} \\ b_{n,n} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma_n \end{bmatrix}. \tag{3.9}$$

Applying Cramer's rule for the coefficient $b_{n,n} = 1$ we get

$$\det \mathbf{T}_0 = c_0 = \beta_0, \quad \det \mathbf{T}_n = \left[(-1)^n \beta_0 \prod_{r=1}^{n} (|\alpha_r|^2 - 1)\right] \det \mathbf{T}_{n-1}, \quad n \geq 1.$$

Therefore,

$$\begin{aligned} \det \mathbf{T}_n &= \left((-1)^n \beta_0 \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right)\left((-1)^{n-1} \beta_0 \prod_{r=1}^{n-1} \left(|\alpha_r|^2 - 1\right)\right) \det \mathbf{T}_{n-1} = \ldots, \\ &= \left((-1)^n \beta_0 \prod_{r=1}^{n} \left(|\alpha_r|^2 - 1\right)\right)\left((-1)^{n-1} \beta_0 \prod_{r=1}^{n-1} \left(|\alpha_r|^2 - 1\right)\right) \ldots \left((-1)\beta_0 \prod_{r=1}^{1} \left(|\alpha_r|^2 - 1\right) \beta_0\right), \\ &= (-1)^{n(n+1)/2} \beta_0^{n+1} \prod_{m=1}^{n} \left(\prod_{r=1}^{m} \left(|\alpha_r|^2 - 1\right)\right). \end{aligned}$$

Hence, if $\beta_0 > 0$, then

$$\det \mathbf{T}_0 = \beta_0 > 0, \quad (-1)^{n(n+1)/2} \det \mathbf{T}_n = \beta_0^{n+1} \prod_{m=1}^{n} \left[\prod_{r=1}^{m} \left(|\alpha_r|^2 - 1\right)\right] > 0, \quad n \geq 1.$$

Since $(-1)^{n(n+1)/2} = (-1)^{\lfloor (n+1)/2 \rfloor}$, if we look at the Hankel determinants

$$\det \mathbf{H}_1^{(0)} = c_0, \quad \det \mathbf{H}_{n+1}^{(-n)} = \begin{vmatrix} c_{-n} & c_{-n+1} & \cdots & c_0 \\ c_{-n+1} & c_{-n+2} & \cdots & c_1 \\ \vdots & \vdots & & \vdots \\ c_0 & c_1 & \cdots & c_n \end{vmatrix},$$

which can be obtained by a rearrangement of rows of the Toeplitz determinants $\det \mathbf{T}_n$, then we have

$$\det \mathbf{H}_{n+1}^{(-n)} > 0, \quad n \geq 0. \tag{3.10}$$

Applying Cramer's rule in (3.9) for the coefficient $b_{n,0} = (-1)^n\alpha_n$, we obtain

$$\alpha_n = \frac{1}{\det \mathbf{T}_{n-1}} \begin{vmatrix} c_1 & \cdots & c_n \\ c_0 & \cdots & c_{n-1} \\ \vdots & & \vdots \\ c_{-n+2} & \cdots & c_1 \end{vmatrix} = \frac{1}{\det \mathbf{T}_{n-1}} \overline{\begin{vmatrix} c_{-1} & \cdots & c_{n-2} \\ c_{-2} & \cdots & c_{n-3} \\ \vdots & & \vdots \\ c_{-n} & \cdots & c_{-1} \end{vmatrix}}.$$

The above determinant expression follows by considering the transpose and then using (3.8). Hence, through an interchanging of rows in the determinants,

$$\det \mathbf{H}_n^{(-n+2)} = \overline{\det \mathbf{H}_n^{(-n)}} = \alpha_n \det \mathbf{H}_n^{(-n+1)}, \quad n \geq 1. \tag{3.11}$$

Since $|\alpha_n| > 1$, this means

$$\left|\det \mathbf{H}_n^{(-n+2)}\right| = \left|\det \mathbf{H}_n^{(-n)}\right| > \det \mathbf{H}_n^{(-n+1)} > 0, \quad n \geq 1.$$

Now, with respect to the formal power series expansions F and F_∞ in (3.6), we define the linear functional $\mathcal{L}$ on Λ. If $l_{r,s}(z) = \sum_{m=r}^{s} \lambda_m z^m$, $\lambda_m \in \mathbb{C}$ and $-\infty < r \leq s < \infty$, then

$$\langle \mathcal{L}, l_{r,s} \rangle = \sum_{m=r}^{s} \lambda_m c_m.$$

Therefore, from (3.9), we have for the monic polynomials $\Phi_n = B_{2n+1}$,

$$\langle \Phi_n(z), z^m \rangle_{\mathcal{L}} = \frac{1}{2}\gamma_n \delta_{n,m}, \quad 0 \leq m \leq n, \;\; n \geq 1.$$

Also, from (3.2) and (3.4)-(3.5),

$$\Phi_n(z) = z\Phi_{n-1}(z) + (-1)^n \alpha_n \Phi_{n-1}^*(z), \quad n \geq 1,$$

with $\Phi_0 = 1$, $\Phi_1(z) = z - \alpha_1$. Based on these facts, we refer to $\{B_{2n+1}\}_{n\geq 0}$ as the sequence of Szegő polynomials generated by the Perron-Carathéodory continued fraction (3.1).

3.1.2 Szegő polynomials from series expansions

Under the conditions (3.8) and (3.10), $\mathcal{L}$ is referred to as an sq-definite moment functional, meaning 'special quasi-definite moment functional'. When the moments are also real, the name rsq-definite moment functional is used.

The sequence of polynomials $\{\Phi_n\}_{n\geqslant 0}$ is defined by $\Phi_0 = 1$, and Φ_n a monic polynomial of degree n orthogonal with respect to the sq-definite linear functional $\mathcal{L}$. Then, with the condition (3.10), these polynomials always exist. In fact, we can easily obtain an analog result to (2.6),

$$\Phi_n(z) = \frac{1}{\det \mathbf{T}_{n-1}} \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_{-1} & c_0 & \cdots & c_{n-2} & c_{n-1} \\ \vdots & \vdots & & \vdots & \\ c_{-n+1} & c_{-n+2} & \cdots & c_0 & c_1 \\ 1 & z & \cdots & z^{n-1} & z^n \end{vmatrix}, \quad n \geq 1. \tag{3.12}$$

Moreover,

$$\langle \Phi_n(z), z^m \rangle_{\mathcal{L}} = \langle \Phi_n^*(z), z^{n-m} \rangle_{\mathcal{L}} = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}} \delta_{n,m}, \quad m \leq n, \quad n \geq 1. \tag{3.13}$$

The information on $\langle \Phi_n^*(z), z^{n-m} \rangle_{\mathcal{L}}$ follows from (3.8), and with this symmetric property we have

$$\overline{\langle l_{r,s}(z), z^m \rangle}_{\mathcal{L}} = \langle z^m, l_{r,s}(z) \rangle_{\mathcal{L}}, \quad l_{r,s} \in \Lambda. \tag{3.14}$$

We call $\{\Phi_n\}_{n\geqslant 0}$ the sequence of Szegő polynomials associated with the sq-definite moment functional $\mathcal{L}$.

Theorem 3.1.1. *Let $\alpha_n = (-1)^n \Phi_n(0)$, $n \geq 1$, where $\{\Phi_n\}_{n\geqslant 0}$ is the sequence of Szegő polynomials associated with the sq-definite moment functional $\mathcal{L}$. Then the following statements hold:*

i) The polynomials $\{\Phi_n\}_{n\geqslant 0}$ satisfy the recurrence relations

$$\begin{aligned} \Phi_n^*(z) &= (-1)^n \overline{\alpha}_n z \Phi_{n-1}(z) + \Phi_{n-1}^*(z), \quad n \geq 1, \\ \Phi_n(z) &= (-1)^n \alpha_n \Phi_n^*(z) - \left(|\alpha_n|^2 - 1\right) z \Phi_{n-1}(z), \quad n \geq 1. \end{aligned}$$

ii) $|\alpha_n| > 1$, $n \geq 1$.

iii) If we set $B_{2n}(z) = (-1)^n \Phi_n^*(z)$, $B_{2n+1}(z) = \Phi_n(z)$, $n \geq 0$, then $\{B_n\}_{n\geqslant 0}$ is the sequence of the denominator polynomials of (3.1).

Proof. Set

$$A(z) = \Phi_n^*(z) - \alpha_n z \Phi_{n-1}(z) - \Phi_{n-1}^*(z), \quad n \geq 1,$$

where $\alpha_n = -\left\langle \Phi_{n-1}^*(z), z^n \right\rangle_{\mathcal{L}} \big/ \left\langle \Phi_{n-1}(z), z^{n-1} \right\rangle_{\mathcal{L}}$. We consider A as a polynomial of degree n, and we will show that it is identically zero. Clearly,

$$\langle A, z^m \rangle_{\mathcal{L}} = 0, \quad 1 \leq m \leq n. \tag{3.15}$$

Since $A(0) = 0$, we can write $A^*(z) = \sum_{k=0}^{n-1} a_{n,k} \Phi_k(z)$, and hence

$$A(z) = \sum_{k=0}^{n-1} \overline{a}_{n,k} z^{n-k} \Phi_k^*(z).$$

Using here the results of (3.15) for $m = n$, $m = n-1$ until $m = 1$ and noting at each stage that $\langle \mathcal{L}, \Phi^*_{n-m} \rangle \neq 0$, we successively obtain that $a_0 = 0$, $a_1 = 0$ until $a_{n-1} = 0$. Hence, $A = 0$ and thus

$$\Phi^*_n(z) = \alpha_n z \Phi_{n-1}(z) + \Phi^*_{n-1}(z).$$

Comparing the coefficients of z^n, we obtain $\overline{\alpha}_n = \Phi_n(0)$, thus the first of the recurrence relations follows.

Now, to prove the second recurrence relation, we set

$$A(z) = \Phi_n(z) - (-1)^n \alpha_n \Phi^*_n(z) + \alpha_n z \Phi_{n-1}(z), \quad n \geq 1$$

with $\alpha_n = -\langle \Phi_n(z), z^n \rangle_{\mathcal{L}} \Big/ \left\langle \Phi_{n-1}(z), z^{n-1} \right\rangle_{\mathcal{L}}$. Hence,

$$\langle A, z^m \rangle_{\mathcal{L}} = 0, \quad 1 \leq m \leq n.$$

Since $A(0) = 0$, we can follow the same procedure as above and we get $A = 0$. Thus,

$$\Phi_n(z) = (-1)^n \alpha_n \Phi^*_n(z) - \alpha_n z \Phi_{n-1}(z).$$

Comparing the coefficients of z^n, we obtain $\overline{\alpha}_n = |\alpha_n|^2 - 1$, thus the second of the recurrence relations is deduced.

Since (2.26), we obtain from the above recurrence relation,

$$\begin{aligned} \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}} &= \langle \Phi_n(z), z^n \rangle_{\mathcal{L}} = (-1)^n \alpha_n \langle \Phi^*_n(z), z^n \rangle_{\mathcal{L}} - \left(|\alpha_n|^2 - 1\right) \left\langle \Phi_{n-1}(z), z^{n-1} \right\rangle_{\mathcal{L}} \\ &= -\left(|\alpha_n|^2 - 1\right) \frac{\det \mathbf{T}_{n-1}}{\det \mathbf{T}_{n-2}}, \quad n \geq 1. \end{aligned}$$

Thus,

$$|\alpha_n|^2 - 1 = -\frac{\det \mathbf{T}_n \det \mathbf{T}_{n-2}}{\det \mathbf{T}^2_{n-1}}, \quad n \geq 1.$$

By convention, $\det \mathbf{T}_{-1} = 1$. We can easily verify that to get (3.10) is sufficient to establish that $|\alpha_n|^2 - 1 > 0$, $n \geq 1$, and with this, *ii*).

To verify *iii*), let notice that by setting $B_{2n}(z) = (-1)^n \Phi^*_n(z)$, $B_{2n+1}(z) = \Phi_n(z)$, $n \geq 0$, we obtain from *i*),

$$\begin{aligned} B_{2n}(z) &= \overline{\alpha}_n z B_{2n-1}(z) - B_{2n-2}(z), \quad n \geq 1 \\ B_{2n+1}(z) &= \alpha_n B_{2n}(z) - (|\alpha_n|^2 - 1) z B_{2n-1}(z), \quad n \geq 1, \end{aligned}$$

with $B_0 = B_1 = 1$. Hence, from the theory of continued fractions the result follows. □

Since $\alpha_n = (-1)^n \Phi_n(0)$, from (2.26) we obtain, such as in (3.11),

$$\alpha_n = \frac{\det \mathbf{H}_n^{(-n+2)}}{\det \mathbf{H}_n^{(-n+1)}} = \frac{\overline{\det \mathbf{H}_n^{(-n)}}}{\det \mathbf{H}_n^{(-n+1)}}, \quad n \geq 1. \tag{3.16}$$

Since $\alpha_n \neq 0$, from *i*) of the above theorem, the following recurrence relation always holds,

$$\Phi_{n+1}(z) = (z - \beta_{n+1})\,\Phi_n(z) - \varsigma_{n+1} z \Phi_{n-1}(z), \quad n \geq 1, \tag{3.17}$$

where $\Phi_0 = 1$, $\Phi_1(z) = z - \beta_1$, $\beta_1 = \alpha_1$, $\varsigma_{n+1} = \beta_{n+1}\left(|\alpha_n|^2 - 1\right)$ and $\beta_{n+1} = \dfrac{\alpha_{n+1}}{\alpha_n}$, $n \geq 1$.

3.1.3 Polynomials of second kind

Let $\{\Omega_n\}_{n\geqslant 0}$ be the polynomials of the second kind of $\{\Phi_n\}_{n\geqslant 0}$. It is easily verified that Ω_n is a polynomial of exact degree n. More precisely,

$$\Omega_n(z) = -c_0\left(z^n + \ldots + (-1)^{n-1}\alpha_n\right).$$

Theorem 3.1.2. *Let $\{\Phi_n\}_{n\geqslant 0}$ be the Szegő polynomials associated with the sq-definite moment functional $\mathcal{L}$. Then, the polynomials of second kind Ω_n satisfy:*

i) $\Omega_n(z) = \left\langle \mathcal{L}, \dfrac{z+y}{z-y}\left(\dfrac{z^k}{y^k}\Phi_n(y) - \Phi_n(z)\right)\right\rangle, \quad 0 \leq k \leq n-1, \quad n \geq 1.$

ii) $-\Omega_n^*(z) = \left\langle \mathcal{L}, \dfrac{z+y}{z-y}\left(\dfrac{z^k}{y^k}\Phi_n^*(y) - \Phi_n^*(z)\right)\right\rangle, \quad 1 \leq k \leq n, \quad n \geq 1.$

iii) $-\Omega_n^*(z) = (-1)^n \overline{\alpha}_n z \Omega_{n-1}(z) - \Omega_{n-1}^*(z), \quad \Omega_n(z) = (-1)^{n+1}\alpha_n \Omega_n^*(z) - \left(|\alpha_n|^2 - 1\right) z \Omega_{n-1}(z), \quad n \geq 1.$

iv) For $n \geq 0$, if we set $A_{2n}(z) = (-1)^{n+1}\Omega_n^*(z)$, $A_{2n+1}(z) = \Omega_n(z)$, then $\{A_n\}_{n\geqslant 0}$ is the sequence of numerator polynomials of (3.1), with $\beta_0 = c_0$ and $\alpha_n = (-1)^n \Phi_n(0)$, $n \geq 1$.

Proof. The proof of the results in this theorem is the same as in Jones et al. [1989]. □

From the above theorem, it also follows that the polynomials $\{\Omega_n\}_{n\geqslant 0}$ satisfy the three term recurrence relation

$$\Omega_{n+1}(z) = (z - \beta_{n+1})\Omega_n(z) - \varsigma_{n+1} z \Omega_{n-1}(z), \quad n \geq 1, \tag{3.18}$$

with $R_0 = -c_0$ and $\Omega_1(z) = -c_0(z + \alpha_1)$. The values of ς_n and β_n are as in (3.17).

3.1.4 Para-orthogonal polynomials

For $\omega \in \mathbb{C}$, we consider the sequence of polynomials $\{\Phi_n(\omega;\cdot)\}_{n\geq 0}$ and $\{\Omega_n(\omega;\cdot)\}_{n\geq 0}$ given by $\Phi_0(\omega;\cdot) = 1$, $\Omega_0(\omega;\cdot) = -c_0$, and

$$\Phi_n(\omega; z) = \Phi_n(z) - \omega z \Phi_{n-1}(z), \quad \Omega_n(\omega; z) = \Omega_n(z) - \omega z \Omega_{n-1}(z), \quad n \geq 1.$$

Obviously, $\Phi_n(0;z) = \Phi_n(z)$ and for any ω,

$$\Phi_n(\omega;0) = \Phi_n(0) = (-1)^n\alpha_n \neq 0, \quad n \geq 1.$$

Theorem 3.1.3. *Let $\alpha_n = (-1)^n\Phi_n(0)$, $n \geq 1$, where $\{\Phi_n\}_{n\geqslant 0}$ is the sequence of Szegő polynomials associated with the sq-definite moment functional $\mathcal{L}$. Then, the following hold:*

i) $\langle \Phi_n(\omega;z), z^m\rangle_{\mathcal{L}} = 0, \quad 1 \leq m \leq n-1.$

ii) $\Omega_n(\omega;z) = \left\langle \mathcal{L}, \dfrac{z+y}{z-y}\left(\dfrac{z^k}{y^k}\Phi_n(\omega;y) - \Phi_n(\omega;z)\right)\right\rangle, \quad 1 \leq k \leq n-1, \quad n \geq 2.$

iii) $-\Omega_n^*(\omega;z) = \left\langle \mathcal{L}, \dfrac{z+y}{z-y}\left(\dfrac{z^k}{y^k}\Phi_n^*(\omega;y) - \Phi_n^*(\omega;z)\right)\right\rangle, \quad 1 \leq k \leq n-1, \quad n \geq 2.$

Proof. The proof of *i*) is immediate. To prove *ii*), we can write using *i*) of Theorem 3.1.2,

$$\begin{aligned}\Omega_n(\omega;z) &= \Omega_n(z) - \omega z\Omega_{n-1}(z), \quad n \geq 2,\\ &= \left\langle \mathcal{L}, \frac{z+y}{z-y}\left(\frac{z^k}{y^k}\Phi_n(y) - \Phi_n(z)\right)\right\rangle - \omega z\left\langle \mathcal{L}, \frac{z+y}{z-y}\left(\frac{z^{k-1}}{y^{k-1}}\Phi_{n-1}(y) - \Phi_{n-1}(z)\right)\right\rangle, \quad 1 \leq k \leq n-1.\end{aligned}$$

This leads us to the required result. Similarly, *iii*) follows from *ii*) of Theorem 3.1.2. □

For convenience, we write $\widehat{\Phi}_n(\omega;z) = \overline{\Phi}_n(z) - \overline{\omega}z\overline{\Phi}_{n-1}(z)$ and $\widehat{R}_n(\omega;z) = \overline{R}_n(z) - \overline{\omega}z\overline{R}_{n-1}(z)$, $n \geq 1$. Hence, observe that

$$\Phi_n^*(\omega;z) = z^n\overline{\Phi_n(\omega;1/\overline{z})} = z^n\widehat{S}_n(\omega;1/z) = \Phi_n^*(z) - \overline{\omega}\Phi_{n-1}^*(z), \quad n \geq 1.$$

From the recurrence relations for Φ_n in Theorem 3.1.1,

$$\Phi_n(1;z) = (-1)^n\alpha_n\Phi_{n-1}^*(0;z), \quad \Phi_n(-|\alpha_n|^2+1;z) = (-1)^n\alpha_n\Phi_n^*(0;z), \quad n \geq 1. \tag{3.19}$$

Theorem 3.1.4. *Let $\alpha_n = (-1)^n\Phi_n(0)$, $n \geq 1$, where $\{\Phi_n\}_{n\geqslant 0}$ are the Szegő polynomials associated with the sq-definite moment functional $\mathcal{L}$. If $\overline{\nu} = \dfrac{1-\omega-|\alpha_n|^2}{1-\omega}$, then*

i) $\Phi_n^*(\nu;z) = z^n\widehat{\Phi}_n(\nu;1/z) = \dfrac{1-\overline{\nu}}{(-1)^n\alpha_n}\Phi_n(\omega;z)$; *ii)* $\Omega_n^*(\nu;z) = z^n\widehat{R}_n(\nu;1/z) = \dfrac{1-\overline{\nu}}{(-1)^{n-1}\alpha_n}\Omega_n(\omega;z), \quad n \geq 1$;

iii) $\dfrac{d\Phi_n^*(\nu;z)}{dz} = nz^{n-1}\widehat{\Phi}_n(\nu;1/z) - z^{n-2}\widehat{\Phi}'_n(\nu;1/z) = \dfrac{1-\overline{\nu}}{(-1)^n\alpha_n}\Phi'_n(\omega;z), \quad n \geq 1.$

In particular, if $\omega_n = 1-\xi\overline{\alpha}_n$, with $|\xi| = 1$, then

iv) $\Phi_n^*(\omega_n;z) = z^n\widehat{\Phi}_n(\omega_n;1/z) = (-1)^n\overline{\xi}\Phi_n(\omega_n;z), \quad n \geq 1;$

v) $\Omega_n^*(\omega_n;z) = z^n\widehat{R}_n(\omega_n;1/z) = (-1)^{n-1}\overline{\xi}\Omega_n(\omega_n;z), \quad n \geq 1;$

vi) $nz^{n-1}\widehat{\Phi}_n(\omega_n;1/z) - z^{n-2}\widehat{\Phi}'_n(\omega_n;1/z) = (-1)^n\bar{\xi}\Phi'_n(\omega_n;z), \quad n \geq 1.$

Here, $\widehat{\Phi}'_n(\omega;z) = \dfrac{d\widehat{\Phi}_n(\omega;z)}{dz}$ *and* $\Phi'_n(\omega;z) = \dfrac{d\Phi_n(\omega;z)}{dz}$.

Proof. Since ω and ν are different from 1, the proof of *i)* follows from the relations in (3.19). *ii)*, especially when $n \geq 2$, follows from Theorem 3.1.3. *iii)* is obtained from *i)* by taking derivatives. To obtain the remaining parts, we let $\omega = \nu = \omega_n$ and this gives $|\omega_n - 1|^2 = |\alpha_n|^2$. □

Notice that we can also write

$$\Phi_n(\omega_n;z) = \Phi_n(1-\xi\overline{\alpha}_n;z) = \frac{|\alpha_n|^2 - \xi\overline{\alpha}_n}{|\alpha_n|^2-1}\left(\Phi_n(z) + \tau\Phi_n^*(z)\right), \tag{3.20}$$

where $\tau = (-1)^n \dfrac{\alpha_n}{\overline{\alpha}_n}\dfrac{1-\xi\overline{\alpha}_n}{\xi-\alpha_n}$. Again, following the notation used in the case of the classical Szegő polynomials Jones et al. [1989], we refer to the polynomials $\Phi_n(z)+\tau\Phi_n^*(z)$, when $|\tau|=1$, as the para-orthogonal polynomial of degree n associated with $\{\Phi_n\}_{n\geqslant 0}$. Since $|\tau|=1$, whenever $|\xi|=1$, the polynomials $\Phi_n(\omega_n;\cdot)$ are hence para-orthogonal polynomials multiplied by constant factors.

From *i)* of Theorem 3.1.3, we have for $n \geq 1$,

$$F(z) - \frac{\Omega_n(\omega;z)}{\Phi_n(\omega;z)} = O(z^n), \quad F_\infty(z) - \frac{\Omega_n(\omega;z)}{\Phi_n(\omega;z)} = \begin{cases} O\left(\dfrac{1}{z^{n+1}}\right), & \omega = 0, \\ O\left(\dfrac{1}{z^{n-1}}\right), & \omega = 1, \\ O\left(\dfrac{1}{z^{n}}\right), & \omega \neq \{0,1\}. \end{cases} \tag{3.21}$$

Let consider the sequence of polynomials $\{\chi_n^{(1)}(\omega;\cdot)\}_{n\geqslant 1}$ and $\{\chi_n^{(2)}(\omega;\cdot)\}_{n\geqslant 1}$,

$$\begin{aligned} \chi_n^{(1)}(\omega;z) &= \Omega_n(\omega;z)\Phi_{n-1}(z) - \Omega_{n-1}(z)\Phi_n(\omega;z), \quad n \geqslant 1, \\ \chi_n^{(2)}(\omega;z) &= \Phi'_n(\omega;z)\Phi_{n-1}(z) - \Phi'_{n-1}(z)\Phi_n(\omega;z), \quad n \geqslant 1. \end{aligned}$$

From the recurrence relations (3.17) and (3.18),

$$\begin{aligned} \chi_n^{(1)}(\omega;z) &= \Omega_n(z)\Phi_{n-1}(z) - \Omega_{n-1}(z)\Phi_n(z), \\ &= -\varsigma_n\varsigma_{n-1}z^2(\Omega_{n-2}(z)\Phi_{n-3}(z) + \Omega_{n-3}(z)\Phi_{n-2}(z)) = \ldots \\ &= \varsigma_n\varsigma_{n-1}\ldots\varsigma_3\varsigma_2 z^{n-1}(-\Omega_0(z)\Phi_1(z) + \Omega_1(z)\Phi_0(z)) = -2c_0\alpha_1\varsigma_2\cdots\varsigma_n z^{n-1}, \quad n \geq 2, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \chi_n^{(2)}(\omega;z) &= \alpha_n z\Big(\Phi_{n-2}(z)\Phi'_{n-1}(z) - \Phi'_{n-2}(z)\Phi_{n-1}(z)\Big) - \varsigma_n\Phi_{n-2}(z)\Phi_{n-1}(z) + (1-\omega)\Phi^2_{n-1}(z) \\ &= \varsigma_n\varsigma_{n-1}z^2\Big(\Phi'_{n-2}(z)\Phi_{n-3}(z) - \Phi'_{n-3}(z)\Phi_{n-2}(z)\Big) - \varsigma_n\Phi_{n-2}(z)\Phi_{n-1}(z) + \\ &\quad + (1-\omega)\Phi^2_{n-1}(z) + \varsigma_n z\Phi^2_{n-2}(z) - \varsigma_n\varsigma_{n-1}z\Phi_{n-2}(z)\Phi_{n-3}(z) \\ &= \varsigma_n\varsigma_{n-1}z^2\chi^{(2)}_{n-2}(0;z) + (1-\omega)\Phi^2_{n-1}(z) + \alpha_n\beta_{n-1}\Phi^2_{n-2}(z), \quad n \geq 2, \end{aligned} \tag{3.23}$$

with $\chi_1^{(1)}(\omega;z) = -2c_0\alpha_1$, $\alpha_1\chi_0^{(2)}(\omega;z) = 0$, and $\chi_1^{(2)}(\omega;z) = 1-\omega$.

The proof of the following result follows to the same method of the proof given in da Silva and Ranga [2005].

Theorem 3.1.5. *Let* $\alpha_n = (-1)^n\Phi_n(0)$, $n \geq 1$, *where* $\{\Phi_n\}_{n\geqslant 0}$ *are the Szegő polynomials associated with the sq-definite moment functional* $\mathcal{L}$. *Then, for* $\omega \neq 1$, *the zeros of* $\Phi_n(\omega;\cdot)$ *are the eigenvalues of the following lower Hessenberg matrix,*

$$\begin{bmatrix} \eta_1 & \varsigma_2 & 0 & \cdots & 0 & 0 \\ \eta_1 & \eta_2 & \varsigma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \varsigma_{n-1} & 0 \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \dfrac{\varsigma_n}{(1-\omega)} \\ \eta_1 & \eta_2 & \eta_3 & \cdots & \eta_{n-1} & \dfrac{\eta_n}{(1-\omega)} \end{bmatrix},$$

where $\eta_1 = \alpha_1$, $\eta_r = \alpha_r\bar{\alpha}_{r-1}$, *and* $\varsigma_r = \alpha_r\bar{\alpha}_{r-1} - \alpha_r/\alpha_{r-1}$, $r = 2,\ldots,n$.

Proof. From (3.17) and the definition of $\{\Phi_n(\omega,\cdot)\}_{n\geqslant 0}$, we have

$$\Phi_n(\omega;z) = \begin{vmatrix} z-\beta_1 & -\varsigma_2 & 0 & \cdots & 0 & 0 \\ -z & z-\beta_2 & -\varsigma_3 & \cdots & 0 & 0 \\ 0 & -z & z-\beta_3 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z-\beta_{n-1} & -\varsigma_n \\ 0 & 0 & 0 & \cdots & -z & (1-\omega)z-\beta_n \end{vmatrix},$$

or, equivalently, $\Phi_n(\omega;z) = \det(z\mathbf{A}_n - \mathbf{B}_n)$, where $\mathbf{A}_n$ and $\mathbf{B}_n$ are matrices of order n given by

$$\mathbf{A}_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1-\omega \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \beta_1 & \varsigma_2 & 0 & \cdots & 0 \\ 0 & \beta_2 & \varsigma_3 & \cdots & 0 \\ 0 & 0 & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_n \end{bmatrix}.$$

Since $\mathbf{A}_n$ is non-singular, $\Phi_n(\omega;z) = \det\mathbf{A}_n \det\left(z\mathbf{I}_n - \mathbf{A}_n^{-1}\mathbf{B}_n\right) = (1-\omega)\det\mathbf{G}_n$, where

$$\mathbf{G}_n = \begin{bmatrix} z-\beta_1 & \varsigma_2 & 0 & \cdots & 0 & 0 \\ \beta_1 & z-(\varsigma_2+\beta_2) & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \beta_1 & \varsigma_2+\beta_2 & \varsigma_3+\beta_3 & \cdots & z-(\varsigma_{n-1}+\beta_{n-1}) & \varsigma_n \\ \dfrac{\beta_1}{1-\omega} & \dfrac{\varsigma_2+\beta_2}{1-\omega} & \dfrac{\varsigma_3+\beta_3}{1-\omega} & \cdots & \dfrac{\varsigma_{n-1}+\beta_{n-1}}{1-\omega} & \dfrac{z(1-\omega)-(\varsigma_n+\beta_n)}{1-\omega} \end{bmatrix}.$$

Thus, the result follows. □

3.2 Polynomials with real zeros

Let $\mathcal{L}$ be an rsq-definite moment functional. That is, all the moments $\{c_n\}_{n\in\mathbb{Z}}$ are real and the properties (3.8) and (3.10) hold. Clearly, the associated Szegő polynomials $\{\Phi_n\}_{n\geqslant 0}$ are all real and, in particular, we can state the following theorem.

Theorem 3.2.1. *Let $\alpha_n = (-1)^n\Phi_n(0)$, $n \geq 1$, where $\{\Phi_n\}_{n\geqslant 0}$ are the Szegő polynomials associated with the rsq-definite moment functional $\mathcal{L}$. Then, the following statements are equivalent.*

i) $\det \mathbf{H}_n^{(-n)} > 0,\quad n \geq 1.$

ii) $\alpha_n > 1,\quad n \geq 1.$

iii) The zeros of the polynomials $\Phi_n(\omega;\cdot)$, $n \geq 1$, when $\omega < 1$, are simple and lie on the positive half side of the real line.

Proof. If $\mathcal{L}$ is an rsq-definite moment functional then their moments are real and $\det \mathbf{H}_n^{(-n+1)} > 0$, $n \geq 1$. Hence, from (3.16), $\det \mathbf{H}_n^{(-n)} > 0$, $n \geq 1$, and $\alpha_n > 1$, $n \geq 1$. On the other hand, if $\alpha_n > 1$, $n \geq 1$, then $\det \mathbf{H}_n^{(-n)} > 0$, $n \geq 1$, follows from (3.11). Therefore *i*) and *ii*) are equivalent.

We now prove the equivalence of *ii*) and *iii*). Suppose that all zeros of $\Phi_n(\omega;\cdot)$, $n \geq 1$, are positive. Since the leading coefficient $1-\omega$ of $\Phi_n(\omega;\cdot)$ is positive, $\alpha_n = (-1)^n\Phi_n(\omega;0) > 0$, $n \geq 1$. Combining this with the fact that $\mathcal{L}$ is an rsq-definite moment functional, *ii*) follows. On the other hand, if $\alpha_n > 1$, $n \geq 1$, then, from (3.23) and by induction, we can prove that $\chi_n^{(2)}(\omega;z) > 0$, $n \geq 1$, for $\omega < 1$ and for any real values of z. Using this with the additional conditions $(-1)^n\Phi_n(0) > 0$, $n \geq 1$, we easily establish that the zeros of $\Phi_n(\omega;\cdot)$ are positive and simple (and, even more, interlace with the zeros of Φ_{n-1}). This proves the equivalence of *ii*) and *iii*). Consequently, our theorem is proved. □

The recurrence relation (3.17), which corresponds to the conditions of the above theorem, has been studied, for instance, in Ranga et al. [1995].

Remark 3.2.1. *One can also establish in an analog way (or using the polynomials $\Psi_n(z) = (-1)^n\Phi_n(-z)$, $n \geq 1$) the equivalence of the statements.*

i) $(-1)^n \det \mathbf{H}_n^{(-n)} > 0\quad n \geq 1.$

ii) $(-1)^n\alpha_n > 1\quad n \geq 1.$

iii) The zeros of the polynomials $\Phi_n(\omega;\cdot)$, $n \geq 1$, when $\omega < 1$, are simple and lie on the negative half side of the real line.

In fact, when $(-1)^n\alpha_n > 1$, $n \geq 1$, the polynomials $\{\Phi_n\}_{n\geqslant 0}$ are the denominator polynomials of a positive continued fraction,

$$\frac{\lambda_1 z}{|1+\eta_1 z} + \frac{\lambda_2 z}{|1+\eta_2 z} + \frac{\lambda_3 z}{|1+\eta_3 z} + \cdots, \quad \lambda_n > 0, \quad \eta_n > 0, \quad n \geq 0.$$

This kind of continued fractions are known as $\mathcal{T}$-fractions. We can thus also use some known results on the denominator polynomials of a positive $\mathcal{T}$-fraction given in Jones et al. [1980]. From now on we restrict our analysis to the case when $\alpha_n > 1$, $n \geq 1$, and we refer to the associated moment functional as an rsq-definite moment functional on $(0,\infty)$.

The results given from here can be easily extended to the case when $(-1)^n\alpha_n > 1$, $n \geq 1$, that is, when the moment functional is an rsq-definite moment functional on $(-\infty,0)$.

Corollary 3.2.1. *Let $\{\Phi_n\}_{n\geqslant 0}$ be the sequence of Szegő polynomials associated with an rsq-definite moment functional $\mathcal{L}$ on $(0,\infty)$. Then, for the zeros $z_{n,r}(\omega)$ of $\Phi_n(\omega;\cdot)$, we have*

i) $0 < z_{n,1}(\omega) < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < z_{n,n}(\omega), \quad \omega < 1.$

ii) $z_{n,1}(\omega) < 0 < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < z_{n,n}(\omega), \quad \omega > 1.$

Proof. All we need to verify is that when $\omega > 1$, the zeros of $\Phi_n(\omega;\cdot)$ are still real and simple, and that one of the zeros is of the opposite sign. We verify this using the monic polynomial $(1-\omega)^{-1}\Phi_n(\omega;\cdot)$. Substituting $z = 0$ and $z = z_{n,r}(0)$, $r = 1,2,\ldots,n$, in the definition of $\Phi_n(\omega;\cdot)$,

$$\prod_{r=1}^{n} z_{n,r}(\omega) = (1-\omega)^{-1}\prod_{r=1}^{n} z_{n,r}(0)$$

and

$$(1-\omega)^{-1}\Phi_n(\omega; z_{n,r}(0)) = \frac{\omega}{\omega-1} z_{n,r}(0)\Phi_{n-1}(z_{n,r}(0)), \quad r = 1,2,\ldots,n.$$

Since the zeros of $\Phi_n(0;\cdot) = \Phi_n$ are positive, simple and interlace with the zeros of Φ_{n-1}, we can conclude that the zeros of $\Phi_n(\omega;\cdot)$ interlace with the zeros of Φ_n, proving that they are real and simple. Moreover, at the most one zero of $\Phi_n(\omega;\cdot)$ can have negative sign, which certainly happens if $(1-\omega)$ is negative. □

We can now state the following results on the quadrature rules associated with the polynomials $\{\Phi_n(\omega;\cdot)\}_{n\geq 0}$.

Theorem 3.2.2. *Let $\omega < 1$ and let $\{\Phi_n\}_{n\geqslant 0}$ be the Szegő polynomials associated with an rsq-definite moment functional $\mathcal{L}$ on $(0,\infty)$. Then, the following statements hold.*

i) $$\frac{\Omega_n(\omega;z)}{\Phi_n(\omega;z)} = \sum_{r=1}^{n}\frac{z_{n,r}(\omega)+z}{z_{n,r}(\omega)-z}\lambda_{n,r}(\omega), \quad \lambda_{n,r}(\omega) = \frac{\Omega_n(\omega; z_{n,r}(\omega))}{-2z_{n,r}(\omega)\Phi_n'(\omega; z_{n,r}(\omega))}, \quad n \geq 1.$$

ii) $$\lambda_{n,r}(\omega) > 0, \quad \sum_{r=1}^{n}\lambda_{n,r}(\omega) = c_0, \quad 1 \leq r \leq n.$$

iii) For any $n \geq 1$, if $l_{k,s}(z) = \sum_{j=k}^{s} a_j z^j$ is a Laurent polynomial such that $-n+1 \leq k \leq s \leq n-1$, then the quadrature rule holds

$$\langle \mathcal{L}, l_{k,s} \rangle = \sum_{r=1}^{n} \lambda_{n,r}(\omega) l_{k,s}(z_{n,r}(\omega)).$$

Proof. To prove *i*), first observe that the zeros of $\Phi_n(\omega;\cdot)$ are simple and lie within $(0,\infty)$. Hence, a partial fraction decomposition of the given type verifies *i*).

Notice that $\lambda_{n,r}(\omega)$ can be written as

$$\begin{aligned}
\lambda_{n,r}(\omega) &= \frac{-\Omega_n(\omega; z_{n,r}(\omega))\Phi_{n-1}(z)/z_{n,r}(\omega) + \Omega_{n-1}(z_{n,r}(\omega))\Phi_n(\omega; z_{n,r}(\omega))}{2\Phi_n'(\omega; z_{n,r}(\omega))\Phi_{n-1}(z) - \Phi_{n-1}'(z)\Phi_n(\omega; z_{n,r}(\omega))} \\
&= \frac{-\chi_n^{(1)}(\omega; z_{n,r}(\omega))/z_{n,r}(\omega)}{2\chi_n^{(2)}(\omega; z_{n,r}(\omega))}, \\
&= \frac{c_0(|\alpha_1|^2-1)(|\alpha_2|^2-1)\ldots(|\alpha_{n-1}|^2-1)\alpha_n z_{n-r}^{n-2}(\omega)}{\chi_n^{(2)}(\omega; z_{n,r}(\omega))},
\end{aligned}$$

where $\chi_n^{(1)}(\omega;\cdot)$ and $\chi_n^{(2)}(\omega;\cdot)$ are as in (3.22) and (3.23). Since $\alpha_n > 1$ for $n \geq 1$, one can easily verify that the numerator and denominator above are positive and, hence *ii*) is established.

When $n = 1$, *iii*) clearly follows from *ii*). To obtain *iii*) when $n \geq 2$, we note that $\widetilde{l}_{k,s}(y) = y^{n-1} l_{k,s}(y) \in \mathbb{P}_{2n-2}$. Hence, the interpolation on the zeros of $\Phi_n(\omega;\cdot)$ gives

$$\begin{aligned}
\widetilde{l}_{k,s}(y) = &\sum_{r=1}^{n} \frac{\Phi_n(\omega; y)}{(y - z_{n,r}(\omega))\Phi_n'(\omega; z_{n,r}(\omega))} \widetilde{l}_{k,s}(z_{n,r}(\omega)) \\
&+ \widetilde{l}_{k,s}(z_{n,1}(\omega), \ldots, z_{n,n}(\omega); y)\Phi_n(\omega; y).
\end{aligned}$$

Here, we use the divided difference $\widetilde{l}_{k,s}(z_{n,1}(\omega), \ldots, z_{n,n}(\omega); y) \in \mathbb{P}_{n-2}$, and therefore,

$$\left\langle \widetilde{l}_{k,s}(z_{n,1}(\omega), \ldots, z_{n,n}(\omega); y)\Phi_n(\omega; y), y^{n-1} \right\rangle_{\mathcal{L}} = 0.$$

This means,

$$\langle \mathcal{L}, l_{k,s} \rangle = \left\langle \widetilde{l}_{k,s}, y^{-n+1} \right\rangle_{\mathcal{L}} = \sum_{r=1}^{n} \widetilde{\lambda}_{n,r}(\omega)(z_{n,r}(\omega))^{n-1} l_{k,s}(z_{n,r}(\omega)),$$

where

$$\widetilde{\lambda}_{n,r}(\omega) = \left\langle \frac{\Phi_n(\omega; y)}{(y - z_{n,r}(\omega))\Phi_n'(\omega; z_{n,r}(\omega))}, y^{n-1} \right\rangle_{\mathcal{L}}.$$

Taking into account

$$\Omega_n(\omega; z) + 2z^n \left\langle \frac{\Phi_n(\omega; y)}{y - z}, y^{-n+1} \right\rangle_{\mathcal{L}} = \Phi_n(\omega; z) \left\langle \mathcal{L}, \frac{y+z}{y-z} \right\rangle,$$

we get the statement of our theorem. □

Moreover, using the results given in Theorem 3.1.4 and Theorem 3.2.2 the following results can also be easily verified. If $\omega < 1$ and $\nu = \dfrac{1-\omega-|\alpha_n|^2}{1-\omega}$, then for $n \geq 1$,

$$z_{n,r}(\omega) = 1/z_{n,n-r+1}(\nu), \qquad \lambda_{n,r}(\omega) = \lambda_{n,n-r+1}(\nu), \quad r = 1,2,\dots,n. \tag{3.24}$$

The results in Theorem 3.2.2 allow one to define the step function $\mu_n(\omega;.)$ by

$$\mu_n(\omega;y) = \begin{cases} 0, & 0 < y \leq z_{n,1}(\omega), \\ \displaystyle\sum_{r=1}^{s} \lambda_{n,r}(\omega), & z_{n,s}(\omega) < y \leq z_{n,s+1}(\omega), \\ c_0, & z_{n,n}(\omega) < y < \infty. \end{cases}$$

Then, from *i*) of Theorem 3.2.2 we get

$$\frac{\Omega_n(\omega;z)}{\Phi_n(\omega;z)} = \int_0^\infty \frac{y+z}{y-z} d\mu_n(\omega;y), \quad n \geq 1$$

and hence, from (3.21), for $n \geq 1$,

$$c_s = \int_0^\infty y^s d\mu_n(\omega;y), \quad -n+1 \leq s \leq n-1.$$

Helly's selection principle Chihara [1978] states that a sequence of functions which is locally of bounded total variation has a convergent subsequence. Hence, using the Helly selection theorem, there exists a subsequence $\{n_k\}$ such that $\{\mu_{n_k}(\omega)\}$ converges to a bounded non-decreasing function μ defined on the positive half side of the real line. The function μ is such that it has infinitely many points of increase in $(0,\infty)$ and

$$\lim_{k\to\infty} \frac{\Omega_{n_k}\omega;z)}{\Phi_{n_k}(\omega;z)} = \lim_{k\to\infty} \int_0^\infty \frac{y+z}{y-z} d\mu_{n_k}(\omega;t) = \int_0^\infty \frac{y+z}{y-z} d\mu(t), \quad 0 < \arg(z) < 2\pi.$$

The convergence is also uniform for compact subsets within $0 < \arg(z) < 2\pi$. The points of increase (support) of μ lie entirely on the positive half side of the real line. Moreover,

$$c_n = \int_0^\infty y^n d\mu(y), \quad n \in \mathbb{Z}, \tag{3.25}$$

and

$$\int_0^\infty y^{-m}\Phi_n(y)d\mu(y) = \int_0^\infty y^{-n+m}\Phi_n^*(y)d\mu(y) = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}} \delta_{n,m}, \quad 0 \leq m \leq n, \quad n \geq 1. \tag{3.26}$$

Assuming $\mathcal{L}$ to be an rsq-definite moment functional on $(0,\infty)$, we now consider the sequence of polynomials $\{\Phi_n(\widehat{\omega}_n;\cdot)\}_{n\geq 0}$, where $\widehat{\omega}_n = 1-\alpha_n$. The zeros of $\Phi_n(\widehat{\omega}_n;\cdot)$ are on the positive half side of the real line. Using the results given in Theorem 3.1.4, we can say more about these zeros and also the quadrature weights $\lambda_{n,r}(\widehat{\omega}_n)$ which follow from Theorem 3.2.2.

$$z_{n,r}(\widehat{\omega}_n) = 1/z_{n,n-r+1}(\widehat{\omega}_n), \quad \lambda_{n,r}(\widehat{\omega}_n) = \lambda_{n,n-r+1}(\widehat{\omega}_n), \quad r = 1,2,\ldots,n, \quad n \geq 1.$$

This means that the distribution given by the step function $\mu_n(\widehat{\omega}_n;\cdot)$ satisfies

$$d\mu_n(\widehat{\omega}_n;y) = -d\mu_n(\widehat{\omega}_n;1/y), \quad y \in (0,\infty).$$

Hence, applying the Helly selection theorem we can state the following result.

Theorem 3.2.3. *Let $\alpha_n = (-1)^n\Phi_n(0)$, where $\{\Phi_n\}_{n\geqslant 0}$ are the Szegő polynomials associated with $\mathcal{L}$, which is an rsq-definite moment functional on $(0,\infty)$. Then there exists a bounded non-decreasing function μ, with all its points of increase on $(0,\infty)$, such that $d\mu(y) = -d\mu(1/y)$. Moreover, (3.25) and (3.26) hold.*

Let $0 \leq a < b \leq \infty$. We say that the strong positive measure ν, defined on (a,b), belongs to the symmetric class $\mathcal{S}^3[\xi,\beta,b]$ if

$$\frac{d\nu(y)}{y^{\xi}} = -\frac{d\nu(\beta^2/y)}{(\beta^2/y)^{\xi}}, \quad y \in [a,b],$$

where $0 < \beta < b$, $a = \beta^2/b$ and $2\xi \in \mathbb{Z}$. The classification of the symmetry is according to the value of ξ Bracciali et al. [1999]; Common and McCabe [1996]; Ranga et al. [1995]. Notice that our orthogonality measure μ is classified as the class $\mathcal{S}^3(0,1,b)$, $1 < b \leq \infty$.

To be able to talk about Szegő polynomials on a positive interval $[a,b]$, it is important that the measure belongs to the class $\mathcal{S}^3(0,1,b)$. Otherwise, the monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ defined on $[a,b]$ by (3.26) do not satisfy *i*) of Theorem 3.1.1. For example, if the measure belongs to the class $\mathcal{S}^3(1/2,1,b)$, then for the monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ defined by (3.26) we would have, instead of *i* in Theorem 3.1.1,

$$\Phi_n(z) = \tilde{\delta}_n\Phi_n(z) = z\Phi_{n-1}(z) + \tilde{\delta}_n\Phi_{n-1}(z) - \tilde{\alpha}_n z\Phi_{n-2}(z), \quad n \geq 2,$$

where $\tilde{\delta}_n = \Phi_n(0)$ and $\tilde{\alpha}_n > 0$.

We now look at some properties of the sequences of polynomials $\{\Phi_n(\widehat{\omega}_n;\cdot)\}_{n\geq 0}$ and $\{\Phi_n(\check{\omega}_n;\cdot)\}_{n\geq 0}$, where $\widehat{\omega}_n = 1-\alpha_n$ as above, and $\check{\omega}_n = 1+\alpha_n$, when $\mathcal{L}$ is an rsq-definite moment functional on $(0,\infty)$. From Theorem 3.1.4 and Corollary 3.2.1, the only negative zero of $\Phi_n(\check{\omega}_n;\cdot)$ is -1.

In (3.20), letting $\xi = \alpha_n/|\alpha_n|$ and $\xi = -\alpha_n/|\alpha_n|$, we have for the monic polynomials $\Phi_n(\widehat{\omega}_n;z)/|\alpha_n|$ and $-\Phi_n(\check{\omega}_n;z)/|\alpha_n|$,

$$\frac{\Phi_n(\widehat{\omega}_n;z)}{|\alpha_n|} = \frac{\Phi_n(z)+\tau_n\Phi_n^*(z)}{1+|\alpha_n|}, \quad \frac{\Phi_n(\check{\omega}_n;z)}{-|\alpha_n|} = \frac{\Phi_n(z)-\tau_n\Phi_n^*(z)}{1-|\alpha_n|}, \quad n \geq 1,$$

where $\tau_n = (-1)^n \alpha_n/|\alpha_n|$. Hence, from *i)* of Theorem 3.1.1, we obtain

$$
\begin{aligned}
\frac{\Phi_{n+1}(\widehat{\omega}_{n+1};z)}{|\alpha_{n+1}|} &= \left(z - \frac{|\alpha_n|}{\alpha_n}\frac{\alpha_{n+1}}{|\alpha_{n+1}|}\right)\frac{\Phi_n(\widehat{\omega}_n;z)}{|\alpha_n|} \\
&\quad - \frac{|\alpha_n|}{\alpha_n}\frac{\alpha_{n+1}}{|\alpha_{n+1}|}(|\alpha_n|-1)(|\alpha_{n-1}|+1)z\frac{\Phi_{n-1}(z)+\widetilde{\tau}_{n-1}\Phi^*_{n-1}(z)}{1+|\alpha_{n-1}|}, \\
\frac{\Phi_{n+1}(\breve{\omega}_{n+1};z)}{-|\alpha_{n+1}|} &= \left(z - \frac{|\alpha_n|}{\alpha_n}\frac{\alpha_{n+1}}{|\alpha_{n+1}|}\right)\frac{\Phi_n(\breve{\omega}_n;z)}{-|\alpha_n|} \\
&\quad - \frac{|\alpha_n|}{\alpha_n}\frac{\alpha_{n+1}}{|\alpha_{n+1}|}(|\alpha_n|+1)(|\alpha_{n-1}|-1)z\frac{\Phi_{n-1}(z)-\widetilde{\tau}_{n-1}\Phi^*_{n-1}(z)}{1-|\alpha_{n-1}|},
\end{aligned}
$$

where $\widetilde{\tau}_{n-1} = (-1)^{n-1}\dfrac{\alpha_n^2}{|\alpha_n|^2}\dfrac{|\alpha_{n+1}|}{\alpha_{n+1}}$. Consequently, these lead to recurrence relations for $\dfrac{\Phi_n(\widehat{\omega}_n;z)}{|\alpha_n|}$ and $\dfrac{-\Phi_n(\breve{\omega}_n;z)}{|\alpha_n|}$, since $\widetilde{\tau}_{n-1} = \tau_{n-1}$ when $\alpha_n > 1$, $n \geq 1$.

Theorem 3.2.4. *Let $\alpha_n = (-1)^n\Phi_n(0)$, where $\{\Phi_n\}_{n\geqslant 0}$ are the Szegő polynomials associated with $\mathcal{L}$, which is an rsq-definite moment functional on $(0,\infty)$. Then,*

$$
\begin{aligned}
\frac{\Phi_{n+1}(\widehat{\omega}_{n+1};z)}{\alpha_{n+1}} &= (z-1)\frac{\Phi_n(\widehat{\omega}_n;z)}{\alpha_n} - (\alpha_n-1)(\alpha_{n-1}+1)z\frac{\Phi_{n-1}(\widehat{\omega}_{n-1};z)}{\alpha_{n-1}}, \quad n \geq 1, \\
\frac{\Phi_{n+1}(\breve{\omega}_{n+1};z)}{-\alpha_{n+1}} &= (z-1)\frac{\Phi_n(\breve{\omega}_n;z)}{-\alpha_n} - (\alpha_n+1)(\alpha_{n-1}-1)z\frac{\Phi_{n-1}(\breve{\omega}_{n-1};z)}{-\alpha_{n-1}}, \quad n \geq 1,
\end{aligned}
$$

with $\dfrac{\Phi_1(\widehat{\omega}_1;z)}{\alpha_1} = z-1$, $\dfrac{\Phi_0(\widehat{\omega}_0;z)}{\alpha_0} = 1$, $\dfrac{\Phi_1(\breve{\omega}_1;z)}{-\alpha_1} = z+1$, *and* $\dfrac{\Phi_0(\breve{\omega}_0;z)}{-\alpha_0} = 0$.

Let $\mathcal{L}$ be an rsq-definite moment functional on $(0,\infty)$ defined as in (3.8)-(3.10) and satisfying the conditions of Theorem 3.2.1. If Φ_n denotes the n-th Szegő polynomial associated with this moment functional, then the zeros $z_{n,r}$ of Φ_n are positive. We arrange them in an increasing order

$$0 < z_{n,1} < z_{n,2} < \cdots < z_{n,n-1} < z_{n,n}.$$

In what follows we denote by $z^*_{n,r}$ and $k_{n,r}$ the zeros of Φ^*_n and $K_n(\alpha,\cdot)$, respectively. Interesting inequalities between zeros of these polynomials follow from Theorem 3.1.1 and from (2.32). Our results read as follows.

Theorem 3.2.5. *The zeros of Φ_n and Φ^*_n interlace. More precisely,*

$$0 < z^*_{n,1} < z_{n,1} < z_{n-1,1} < \cdots < z^*_{n,n-1} < z_{n,n-1} < z_{n-1,n-1} < z^*_{n,n} < z_{n,n}.$$

Proof. By Theorem 3.1.1, we have

$$\Phi_n(z) = (-1)^n\alpha_n\Phi^*_n(z) - (|\alpha_n|^2-1)z\Phi_{n-1}(z), \quad n \geq 1.$$

Since the zeros of Φ_n and Φ_{n-1} interlace, we obtain from the above relation

$$\operatorname{sgn}\left((-1)^n \Phi_n^*(z_{n,r})\right) = \operatorname{sgn}\left(\Phi_{n-1}(z_{n,r})\right)$$

and

$$\operatorname{sgn}\left(\Phi_n(z_{n-1,r})\right) = \operatorname{sgn}\left((-1)^n \Phi_n^*(z_{n-1,r})\right).$$

Hence there exist $z_{n,r}^*$, $r = 1, \ldots, n$, zeros of Φ_n^*, such that

$$0 < z_{n,1}^* < z_{n,1} < z_{n-1,1} < \cdots < z_{n,n-1}^* < z_{n,n-1} < z_{n-1,n-1} < z_{n,n}^* < z_{n,n},$$

thus proving the theorem. □

Theorem 3.2.6. *If $\alpha < 0$, then the zeros of Φ_n and $K_n(\alpha, \cdot)$ satisfy the interlacing property*

$$0 < z_{n,1}^* < z_{n,1} < k_{n-1,1} < \cdots < z_{n,n-1}^* < z_{n,n-1} < k_{n-1,n-1} < z_{n,n}^* < z_{n,n}.$$

Proof. By (2.32)

$$\mathbf{k}_n(1 - \alpha z)K_{n-1}(z, \alpha) = \Phi_n^*(\alpha)\Phi_n^*(z) - \Phi_n(\alpha)\Phi_n(z).$$

Since $(-1)^{n(n+1)/2} \det \mathbf{T}_n > 0$ and (2.26), we have $\operatorname{sgn}(\mathbf{k}_n) = (-1)^n$. We also have $\operatorname{sgn}\left(\Phi_n(\alpha)\right) = (-1)^n$ and $\Phi_n^*(\alpha) > 0$. On the other hand, by Theorem 3.2.5, Φ_n and Φ_n^* have interlacing zeros. Therefore, from the above relation

$$(-1)^n \operatorname{sgn}\left(K_{n-1}(z_{n,r}, \alpha)\right) = \operatorname{sgn}\left(\Phi_n^*(z_{n,r})\right), \quad 1 \le r \le n,$$

and

$$\operatorname{sgn}\left(K_{n-1}(z_{n,r}^*, \alpha)\right) = -\operatorname{sgn}\left(\Phi_n(z_{n,r}^*)\right), \quad 1 \le r \le n.$$

Hence, there exist zeros $k_{n-1,r}$, $r = 1, \ldots, n-1$, of $K_{n-1}(z, \alpha)$, satisfying $0 < z_{n,1}^* < z_{n,1} < k_{n-1,1} < \cdots < z_{n,n-1}^* < z_{n,n-1} < k_{n-1,n-1} < z_{n,n}^* < z_{n,n}$. □

3.2.1 Associated moment problem

The necessary and sufficient conditions for the existence of a solution of the strong Stieltjes moment problem introduced in Chapter 1 are the following

$$\det \mathbf{H}_n^{(-n)} > 0, \quad \det \mathbf{H}_n^{(-n+1)} > 0, \quad n \ge 0.$$

From the Hamburger moment problem, these conditions become

$$\det \mathbf{H}_{2n}^{(-2n)} > 0, \quad \det \mathbf{H}_{2n+1}^{(-2n+1)} > 0, \quad n \ge 0.$$

The objective of this section is to solve the moment problem (1.10) formulated in the introduction. We can easily verify that the modified moments associated with the Chebyshev polynomials can be rewritten as follows

$$c_n = \int_0^\infty T_n(\cosh\,\theta)\, d\widetilde{\mu}(\cosh\,\theta) = -\frac{1}{2}\int_{-\infty}^0 e^{n\theta}\, d\widetilde{\mu}(\cosh\,\theta) + \frac{1}{2}\int_0^\infty e^{n\theta}\, d\widetilde{\mu}(\cosh\,\theta).$$

Substituting $x = e^\theta$, we obtain

$$c_n = -\frac{1}{2}\int_0^1 x^n d\widetilde{\mu}\left(\frac{x+x^{-1}}{2}\right) + \frac{1}{2}\int_1^\infty x^n d\widetilde{\mu}\left(\frac{x+x^{-1}}{2}\right) = \int_0^\infty x^n d\mu(x),$$

where μ is a non-decreasing distribution function supported on $(0,\infty)$

$$d\mu(x) = \begin{cases} -\dfrac{1}{2}\, d\widetilde{\mu}\left(\dfrac{x+x^{-1}}{2}\right), & 0 < x \le 1; \\ \dfrac{1}{2}\, d\widetilde{\mu}\left(\dfrac{x+x^{-1}}{2}\right), & 1 \le x < \infty. \end{cases}$$

Hence, our moment problem can be stated as follows. Given a sequence of real numbers $\{c_n\}_{n\geqslant 0}$ find necessary and sufficient conditions for the existence of a measure μ supported on $(0,\infty)$, with the symmetry $d\mu(x) = -d\mu(1/x)$, such that,

$$c_n = \int_0^\infty x^n\, d\mu(x), \quad n \ge 0. \tag{3.27}$$

Remark 3.2.2. *Consider the following moment problem: Given a sequence $\{c_n\}_{n\geqslant 0}$ of real numbers find necessary and sufficient conditions for the existence of a measure $\widehat{\mu}$ supported on the whole real line, with the symmetry $d\widehat{\mu}(\theta) = -d\widehat{\mu}(-\theta)$, such that*

$$\mu_n = \int_{-\infty}^\infty e^{n\theta} d\widehat{\mu}(\theta), \quad n \ge 0.$$

Notice that with the substitution $t = e^\theta$ and $x = \cosh(\theta)$ this moment problem is equivalent to the moment problem (3.27).

Theorem 3.2.7. *The moment problem associated with the Chebyshev polynomials of the first kind has at least one solution if and only if $c_n = c_{-n}$, and*

$$\det \mathbf{H}_{n+1}^{(-n)} > 0, \quad \det \mathbf{H}_{n+1}^{(-n-1)} > 0, \quad n \ge 0.$$

Proof. The existence of the measure μ follows immediately from Theorem 3.2.3. Conversely, let μ be a measure supported on $(0,\infty)$. Given the following quadratic form

$$J(n,l) = \sum_{i=-n}^{n}\sum_{j=-n}^{n} \mu_{i+j+l}\nu_i\nu_j, \quad l \ge 0, \quad n \ge 1,$$

it is easily verified that

$$J(n,l)=\sum_{i=-n}^{n}\sum_{j=-n}^{n}\Big(\int_0^{\infty}t^{i+j+l}d\mu(x)\Big)v_iv_j=\int_0^{\infty}t^l\Big(\sum_{i=-n}^{n}x^iv_i\Big)^2d\mu(x)>0.$$

Since $J(n,l)$ is positive definite, we can establish that

$$\det\mathbf{H}_{k+1}^{(-2n+l)}=\begin{vmatrix} c_{-2n+l} & c_{-2n+1+l} & \cdots & c_{-2n+k+l}\\ c_{-2n+1+l} & c_{-2n+2+l} & \cdots & c_{-2n+k+1+l}\\ \vdots & \vdots & \ddots & \vdots\\ c_{-2n+k+l} & c_{-2n+k+1+l} & \cdots & c_{-2n+2k+l}\end{vmatrix}>0,\quad k=0,1,2,\dots,2n.$$

Thus, for $l=0$ ($l=1$) with $k=2n$ and $k=2n-1$ ($k=2n-1$ and $k=2n-2$), the result follows. □

Theorem 3.2.8. *Let $\alpha_n=(-1)^n\Phi_n(0)$, where $\{\Phi_n\}_{n\geq 0}$ are the Szegő polynomials orthogonal with respect to the symmetric measure μ. The moment problem associated with the Chebyshev polynomials of the first kind is determinate if and only if*

$$\sum_{n=1}^{\infty}e_n=\infty,\quad or\quad \sum_{n=1}^{\infty}d_n=\infty, \tag{3.28}$$

where $e_1=\dfrac{1}{c_0\alpha_1}$, $d_1=\dfrac{1}{\mu_0}$,

$$d_{2n}=\frac{1}{\alpha_{2n}}\prod_{r=2}^{n}\left(\left(\frac{\alpha_{2r-1}^2}{\alpha_{2r-1}^2-1}\right)\left(\frac{\alpha_{2r-2}^2-1}{\alpha_{2r-2}^2}\right)\right)\frac{\alpha_1^2}{\alpha_1^2-1},\qquad e_{2n}=\frac{1}{\alpha_{2n-1}}\prod_{r=2}^{n}\left(\left(\frac{\alpha_{2r-1}^2}{\alpha_{2r-1}^2-1}\right)\left(\frac{\alpha_{2r-2}^2-1}{\alpha_{2r-2}^2}\right)\right)\frac{\alpha_1^2}{\alpha_1^2-1},$$

$$d_{2n+1}=\frac{1}{\alpha_{2n+1}}\prod_{r=1}^{n}\left(\left(\frac{\alpha_{2r-1}^2-1}{\alpha_{2r-1}^2}\right)\left(\frac{\alpha_{2r}^2}{\alpha_{2r}^2-1}\right)\right),\qquad e_{2n+1}=\frac{1}{\alpha_{2n}}\prod_{r=1}^{n}\left(\left(\frac{\alpha_{2r-1}^2-1}{\alpha_{2r-1}^2}\right)\left(\frac{\alpha_{2r}^2}{\alpha_{2r}^2-1}\right)\right).$$

Proof. From *ii*) of Theorem 3.1.1, and *iii*) of Theorem 3.1.2, we have the following Perron-Carathéodory continued fraction,

$$\frac{\Omega_n(z)}{\Phi_n(z)}=c_0-\frac{2c_0|}{|1}-\frac{1|}{|\alpha_1z}-\frac{(\alpha_1^2-1)z|}{|\alpha_1}-\cdots-\frac{1|}{|\alpha_{n-1}z}-\frac{(\alpha_n^2-1)z|}{|\alpha_n}. \tag{3.29}$$

The results in Theorem 3.2.2 allow us to define the step function μ_n by

$$\mu_n(x)=\begin{cases}0, & 0<x\leq z_{n,1},\\ \sum_{r=1}^{s}\lambda_{n,r}, & z_{n,s}<x\leq z_{n,s+1},\\ \mu_0, & z_{n,n}<x<\infty,\end{cases}$$

where $z_{n,r}$ are the zeros of the polynomial Φ_n. Then, from i) of Theorem 3.2.2,

$$\frac{\Omega_n(z)}{\Phi_n(z)} = \int_0^\infty \frac{x+z}{x-z} d\mu_n(x), \quad n \geq 1,$$

and

$$c_s = \int_0^\infty x^s d\mu_n(x), \quad -n+1 \leq s \leq n-1, \quad n \geq 1,$$

from the previous consideration, we have

$$\frac{w}{2}\left(\frac{c_0\Phi_n(-w)+\Omega_n(-w)}{\Phi_n(-w)}\right) = \int_0^\infty \frac{w}{x+w}\, d\widehat{\mu_n}(x),$$

where $d\widehat{\mu}(x) = xd\mu(x)$. Expanding $\displaystyle\int_0^\infty \frac{w}{x+w} d\widehat{\mu_n}(x)$, we can easily verify that

$$\frac{\widehat{\Omega}_n(w)}{\widehat{\Phi}_n(w)} = \begin{cases} -a_{-1}w - a_{-2}w^2 - \ldots - a_{-n}w^n + O\left(w^{(n+1)}\right), \\ a_0 + a_1 w^{-1} + \ldots + a_{n-1}w^{-(n-1)} + O(w^{-n}), \end{cases}$$

where

$$\widehat{\Omega}_n(w) = (-1)^n \frac{w}{2}\left(c_0\Phi_n(-w) + \Omega_n(-w)\right), \quad \widehat{\Phi}_n(w) = (-1)^n \Phi_n(-w).$$

From Theorem 3.2.7, $\det \mathbf{H}_{k+1}^{(-2n+l)} > 0$, $k = 0, 1, \ldots, 2n$, $l = 0, \pm 1, \pm 2, \ldots$, we can prove that the Hankel determinants

$$\det \widehat{\mathbf{H}}_n^{(r)} = \begin{vmatrix} a_r & a_{r+1} & \ldots & a_{r+n-1} \\ a_{r+1} & a_{r+2} & \ldots & a_{r+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+n-2} & a_{r+n-1} & \ldots & a_{r+2n-3} \\ a_{r+n-1} & a_{r+n} & \cdots & a_{r+2n-2} \end{vmatrix}, \quad n \geq 1, \tag{3.30}$$

satisfy the following conditions,

$$\begin{aligned} &\det \widehat{\mathbf{H}}_{n+1}^{(-n)} > 0, \quad n \geq 0, \\ \det \widehat{\mathbf{H}}_{2n}^{(-2n)} > 0, \quad &\det \widehat{\mathbf{H}}_{2n-1}^{(-(2n-1))} < 0, \quad n \geq 1. \end{aligned} \tag{3.31}$$

Therefore, the Chebyshev moment problem can be identified as a strong Stieltjes moment problem. Then there exists a solution if (3.31) holds Jones et al. [1980]. Obviously, $d\widehat{\mu}(x) = xd\mu(x)$ is a solution, taking into account that

$$a_n = \int_0^\infty (-x)^n d\widehat{\mu}(x), \quad n \in \mathbb{Z}.$$

Furthermore, we have

$$\int_0^\infty \frac{w}{x+w} d\widehat{\mu}_{2n}(x) = \frac{\widehat{\Omega}_{2n}(w)}{\widehat{\Phi}_{2n}(w)} < \int_0^\infty \frac{w}{x+w} d\widehat{\mu}(x) < \frac{\widehat{\Omega}_{2n+1}(w)}{\widehat{\Phi}_{2n+1}(w)} = \int_0^\infty \frac{w}{x+w} d\widehat{\mu}_{2n+1}(x), \quad w > 0.$$

Moreover, there also exist solutions $d\widehat{\mu}^{(0)}(x)$ and $d\widehat{\mu}^{(1)}(x)$, limits of $d\widehat{\mu}_{2n}(x)$ and $d\widehat{\mu}_{2n+1}(x)$, respectively, such that

$$\int_0^\infty \frac{w}{x+w} d\widehat{\mu}^{(0)}(x) \le \int_0^\infty \frac{w}{x+w} d\widehat{\mu}(t) \le \int_0^\infty \frac{w}{x+w} d\widehat{\mu}^{(1)}(x).$$

If $\widehat{\Omega}_n / \widehat{\Phi}_n$ converges, then the uniqueness of the solution is proved. Hence, we need to verify the convergence of the following Perron-Carathéodory continued fraction

$$\frac{\Omega_n(z)}{\Phi_n(z)} = \frac{\alpha_1 w}{\lvert w+\beta_1} + \frac{\alpha_2 w}{\lvert w+\beta_2} + \frac{\alpha_3 w}{\lvert w+\beta_3} + \cdots, \tag{3.32}$$

where $\alpha_1 = c_1, \beta_n = \alpha_n / \alpha_{n-1}$, and $\alpha_n = \beta_n(\alpha_{n-1}^2 - 1)$. Notice that (3.32) can be expressed as follows

$$\frac{\Omega_n(z)}{\Phi_n(z)} = \frac{w}{\lvert e_1 + d_1 \omega} + \frac{w}{\lvert e_2 + d_2 \omega} + \frac{w}{\lvert e_3 + d_3 \omega} + \cdots, \tag{3.33}$$

where

$$e_1 = \frac{1}{\mu_0 \alpha_1}, \quad d_1 = \frac{1}{\mu_0}, \quad d_{2n} = \prod_{r=1}^{n} \left(\frac{\alpha_{2r-1}}{\alpha_{2r}} \right), \quad e_{2n} = \beta_{2n} d_{2n},$$
$$d_{2n+1} = \frac{1}{\alpha_{2n+1} d_{2n}}, \qquad e_{2n+1} = \beta_{2n+1} d_{2n+1}, \quad n \ge 1.$$

By Jones et al. [1980], the Perron-Carathéodory continued fraction (3.33) converges uniformly on compact subsets of $\{w; |\arg(w)| < \pi\}$ if and only if (3.28) holds, as required. □

Chapter 4

Spectral transformations of moment matrices

Our hero is the intrepid, yet sensitive matrix $\mathbf{A}$.

Our villain is $\mathbf{E}$, who keeps perturbing $\mathbf{A}$.

When $\mathbf{A}$ is perturbed he puts on a crumpled hat: $\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{E}$.

— G. W. Stewart and J. -G. Sun. *Matrix perturbation theory*. Academic Press, New York, 1990

It is very well known that the Gram matrices of the bilinear functionals associated with (2.1) and (2.22) in the canonical basis $\{z^n\}_{n\geq 0}$ of $\mathbb{P}$ are Hankel and Toeplitz matrices, respectively. The main objective of this chapter is to study the perturbation of a fixed moment of the corresponding moment matrix. We refer to Álvarez-Nodarse and Petronilho [2004]; Álvarez-Nodarse et al. [1998]; Arvesú et al. [2004]; Belmehdi and Marcellán [1992]; Cachafeiro et al. [2003, 2007]; Godoy et al. [1997]; Krall [1940]; Marcellán and Maroni [1992]; Nevai [1979] for other contributions involving related perturbations of moment functionals.

This chapter is divided into two parts. In the first one, given a strongly regular Hankel matrix, and its associated sequence of moments – which defines a quasi-definite moment linear functional $\mathcal{M}$ – we study the perturbation on one anti-diagonal of the corresponding Hankel matrix. We define a linear functional $\mathcal{M}_j$, whose action results in such a perturbation and we establish necessary and sufficient conditions in order to preserve the quasi-definite character. A relation between the corresponding sequences of orthogonal polynomials is obtained, as well as the asymptotic behavior of their zeros. In the second one, we analyze a linear spectral transformation of $\mathcal{L}$, $\mathcal{L}_j$, such that the corresponding Toeplitz matrix is the result of the addition of a constant in two symmetric sub-diagonals of the initial Toeplitz matrix. We focus our attention on the analysis of the quasi-definite character of the perturbed linear functional, as well as on the explicit expressions of the new monic orthogonal polynomial sequence in terms of the first one. Some illustrative examples are pointed out.

4.1 Hankel matrices

Before introducing the problem to be analyzed in this section, let us briefly discuss one rather straightforward but interesting example where the moments are modified in a natural way. Instead of the canonical basis of $\mathbb{P}$, let us consider the basis $\{1,(x-a),(x-a)^2,\ldots\}$, where $a\in\mathbb{R}$. Then, the new sequence of moments $\{\upsilon_n\}_{n\geq 0}$ is given by

$$\upsilon_n=\langle\mathcal{M},(x-a)^n\rangle=\left\langle\mathcal{M},\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}a^{n-j}x^j\right\rangle=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}a^{n-j}\mu_j,\quad n\geq 0. \tag{4.1}$$

As a consequence, $\widehat{\mathbf{H}}$ the Hankel matrix associated with the new basis is

$$\widehat{\mathbf{H}}=\left[\left\langle\mathcal{M},(x-a)^{i+j}\right\rangle\right]_{i,j\geq 0}=\begin{bmatrix}\mu_0 & \mu_1+m_1 & \cdots & \mu_n+m_n & \cdots\\ \mu_1+m_1 & \mu_2+m_2 & \cdots & \mu_{n+1}+m_{n+1} & \cdots\\ \vdots & \vdots & \ddots & \vdots & \cdots\\ \mu_n+m_n & \mu_{n+1}+m_{n+1} & \cdots & \mu_{2n+1}+m_{2n+1} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix},$$

where $m_0=0$ and $m_i=\displaystyle\sum_{j=1}^{i-1}\binom{i}{j}(-1)^{i-j}a^{i-j}\mu_j$. Thus, if $\mathcal{M}$ is a quasi-definite moment linear functional, then the polynomials

$$\widehat{P}_n(x)=\frac{1}{\det\widehat{\mathbf{H}}_{n-1}}\begin{vmatrix}\upsilon_0 & \upsilon_1 & \upsilon_2 & \cdots & \upsilon_n\\ \upsilon_1 & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_{n+1}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ \upsilon_{n-1} & \upsilon_n & \upsilon_{n+1} & \cdots & \upsilon_{2n-1}\\ 1 & (x-a) & (x-a)^2 & \cdots & (x-a)^n\end{vmatrix},\quad n\geqslant 0, \tag{4.2}$$

constitute a sequence of monic polynomials orthogonal with respect to $\mathcal{M}$, using the new basis, where $\widehat{P}_n(x)=P_n(x)$. Notice that this simple change of the basis results in a perturbation on the anti-diagonals of the original moment matrix (2.2). Namely, the j-th anti-diagonal is perturbed by the addition of the constant m_j. In the remaining of the manuscript, we will use the basis $\{1,(x-a),(x-a)^2,\ldots\}$, since most of the required calculations can be performed in a most simple way. Now a natural question arises: Is there a linear functional $\mathcal{M}_j$ such that its action results on a perturbation of (only) the moment υ_j or, equivalently, the $(j+1)$-th anti-diagonal of the corresponding Hankel matrix $\widetilde{\mathbf{H}}$? In other words, we are interested in the properties of the functional $\mathcal{M}_j$ whose moments are given by

$$\widetilde{\upsilon}_n=\langle\mathcal{M}_j,(x-a)^n\rangle=\begin{cases}\upsilon_n, & n\neq j,\\ \upsilon_n+m, & n=j,\end{cases} \tag{4.3}$$

for some $m \in \mathbb{R}$.

4.1.1 Perturbation on the anti-diagonals of a Hankel matrix

In order to state our main result, we need some definitions. Given a moment linear functional $\mathcal{M}$, the usual distributional derivative $D\mathcal{M}$ Maroni [1991] is given by

$$\langle D\mathcal{M}, p\rangle = -\langle \mathcal{M}, p'\rangle, \quad p \in \mathbb{P}.$$

In particular, if j is a non-negative integer, then

$$\langle D^{(j)}\delta(x-a), p(x)\rangle = (-1)^j p^{(j)}(a).$$

In the last years, modifications by means of real Dirac's deltas have been considered by several authors from different points of view (hypergeometric character, holonomic equations that these polynomials satisfy, electrostatic interpretation of zeros, asymptotic properties, among others). In a pioneering result by Krall Krall [1940] were obtained three new classes of polynomials orthogonal with respect to measures which are not absolutely continuous with respect to the Lebesgue measure; the resulting polynomials satisfy a fourth-order linear differential equation. Nevai Nevai [1979] considered the addition of finitely many mass points to a positive measure supported on a bounded subset of the real line, and studied the asymptotic behavior of the corresponding orthogonal polynomials.

We begin with a simple but important remark. The moment linear functional $\mathcal{M}_j$ discussed in the introduction of this chapter is given by

$$\langle \mathcal{M}_j, p\rangle = \langle \mathcal{M}, p\rangle + (-1)^j \frac{m_j}{j!}\langle D^{(j)}\delta(x-a), p(x)\rangle = \langle \mathcal{M}, p\rangle + \frac{m_j}{j!} p^{(j)}(a), \quad a \in \mathbb{R}. \tag{4.4}$$

It is easy to see that all the moments associated with $\mathcal{M}_j$ are equal to the moments $\{\upsilon_n\}_{n\geqslant 0}$ of $\mathcal{M}$ in the basis $\{1, (x-a), (x-a)^2, \ldots\}$, except for the j-th one, which is equal to $\upsilon_j + m_j$. Notice that this perturbation is the simplest one that preserves the Hankel structure of the moment matrix. We can now state the result, which establishes necessary and sufficient conditions in order to the linear functional $\mathcal{M}_j$ preserves the quasi-definite character, and we provide the relation between the corresponding sequences of monic orthogonal polynomials.

Theorem 4.1.1. *Let $\mathcal{M}$ be a quasi-definite moment linear functional and $\{P_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Then, the following statements are equivalent:*

i) The moment linear functional $\mathcal{M}_j$, defined in (4.4), is quasi-definite.

ii) The matrix $\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1}$, *where*

$$\mathbf{D}_{j+1}=\frac{m}{j!}\begin{bmatrix}\binom{j}{j} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \binom{j}{0}\end{bmatrix},\quad \mathbf{K}_{j+1}=\begin{bmatrix}K_{n-1}^{(j,0)}(a,a) & K_{n-1}^{(j-1,0)}(a,a) & \cdots & K_{n-1}^{(0,0)}(a,a)\\ K_{n-1}^{(j,1)}(a,a) & K_{n-1}^{(j-1,1)}(a,a) & \cdots & K_{n-1}^{(0,1)}(a,a)\\ \vdots & \vdots & \ddots & \vdots\\ K_{n-1}^{(j,j)}(a,a) & K_{n-1}^{(j-1,j)}(a,a) & \cdots & K_{n-1}^{(0,j)}(a,a)\end{bmatrix},$$

is non-singular, and

$$\langle\mathcal{M}_j,P_n(j;\cdot)P_n\rangle=\langle\mathcal{M},P_n^2\rangle+\begin{bmatrix}P_n^{(j)}(a)\\ P_n^{(j-1)}(a)\\ \vdots\\ P_n(a)\end{bmatrix}^T\mathbf{D}_{j+1}(\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1})^{-1}\begin{bmatrix}P_n(a)\\ P_n^{(1)}(a)\\ \vdots\\ P_n^{(j)}(a)\end{bmatrix}\neq 0.$$

Moreover, if both $\mathcal{M}$ *and* $\mathcal{M}_j$ *are quasi-definite, and* $\{P_n(j;\cdot)\}_{n\geq 0}$ *is the sequence of monic orthogonal polynomials with respect to* $\mathcal{M}_j$, *then*

$$P_n(j;x)=P_n(x)-\begin{bmatrix}K_{n-1}^{(j,0)}(a,x)\\ K_{n-1}^{(j-1,0)}(a,x)\\ \vdots\\ K_{n-1}^{(0,0)}(a,x)\end{bmatrix}^T\mathbf{D}_{j+1}(\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1})^{-1}\begin{bmatrix}P_n(a)\\ P_n^{(1)}(a)\\ \vdots\\ P_n^{(j)}(a)\end{bmatrix}. \tag{4.5}$$

Proof. Suppose that $\mathcal{M}_j$ is a quasi-definite moment linear functional. Since $\{P_n\}_{n\geqslant 0}$ is the sequence of monic orthogonal polynomials with respect to $\mathcal{M}$, there exist constants $\lambda_{n,0},\ldots,\lambda_{n,n-1}$ such that

$$P_n(j;x)=P_n(x)+\sum_{k=0}^{n-1}\lambda_{n,k}P_k(x). \tag{4.6}$$

Thus, by the orthogonality property,

$$\lambda_{n,k}=-\frac{m_j}{j!}\frac{\sum_{l=0}^{r}\binom{j}{l}P_n^{(j-l)}(j;a)P_k^{(l)}(a)}{\langle\mathcal{M},P_k^2\rangle},\quad r=\min\{k,j\},\quad 0\leq k\leq n-1.$$

Substituting the above expression in (4.6), and taking the i-th derivative of $P_n(j;\cdot)$, we obtain

$$P_n^{(i)}(j;x)=P_n^{(i)}(x)-\frac{m_j}{j!}\sum_{l=0}^{j}\binom{j}{l}P_n^{(j-l)}(j;a)K_{n-1}^{(l,i)}(a,x),\quad i=0,1,\ldots,j. \tag{4.7}$$

Evaluating (4.7) at $x=a$, we have the following linear system

$$\left(\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1}\right)\begin{bmatrix} P_n(j;a) \\ P_n^{(1)}(j;a) \\ \vdots \\ P_n^{(j)}(j;a) \end{bmatrix} = \begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix}.$$

Since $\mathcal{M}_j$ is quasi-definite, there exists a unique sequence of monic orthogonal polynomials with respect to $\mathcal{M}_j$. Thus, the linear system has a unique solution and therefore the $(j+1)\times(j+1)$ matrix $\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1}$ is non-singular. Furthermore, (4.7) with $i=0$ reduces to (4.5).

On the other hand, notice that for $m=0,1,\ldots,n-1$, we have

$$\langle \mathcal{M}_j, P_n(j;\cdot)P_m\rangle = \langle \mathcal{M}, P_nP_m\rangle - \frac{m_j}{j!}\sum_{l=0}^{j}\binom{j}{l}P_n^{(j-l)}(j;a)P_m^{(l)}(a) + \frac{m_j}{j!}\left(P_n(j;x)P_m(x)\right)^{(j)}\Big|_{x=a} = 0,$$

and

$$0 \neq \langle \mathcal{M}_j, P_n(j;\cdot)P_n\rangle = \langle \mathcal{M}, P_n^2\rangle + \begin{bmatrix} P_n^{(j)}(a) \\ P_n^{(j-1)}(a) \\ \vdots \\ P_n(a) \end{bmatrix}^T \mathbf{D}_{j+1}(\mathbf{I}_{j+1}+\mathbf{K}_{j+1}\mathbf{D}_{j+1})^{-1}\begin{bmatrix} P_n(a) \\ P_n^{(1)}(a) \\ \vdots \\ P_n^{(j)}(a) \end{bmatrix}. \tag{4.8}$$

For the converse, let us assume that *ii*) holds, and define $\{P_n(j;\cdot)\}_{n\geq 0}$ as in (4.5). Then it is straightforward to show that $\{P_n(j;\cdot)\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to $\mathcal{M}_j$, and its quasi-definite character is proved. □

4.1.2 Zeros

In this subsection we assume that the linear functional $\mathcal{M}$ is positive definite, i.e., it has an associated positive measure supported on some interval $I\subseteq\mathbb{R}$, and that $\mathcal{M}_j$ is quasi-definite. We show some properties regarding the zeros of its corresponding sequence of monic orthogonal polynomials.

Let $x_1,\ldots,x_r$ be the zeros of $P_n(j;\cdot)$ on I with odd multiplicity, and define $Q_r(x)=(x-x_1)\ldots(x-x_r)$. Then, $P_n(j;x)Q_r(x)(x-a)^{2k}$, where k is the smallest integer such that $k\geqslant (j+1)/2$, is a polynomial that does not change sign on I and, furthermore, we have

$$\langle \mathcal{M}_j, P_n(j;x)Q_r(x)(x-a)^{2k}\rangle = \langle \mathcal{M}, P_n(j;x)Q_r(x)(x-a)^{2k}\rangle \neq 0.$$

From the orthogonality of $P_n(j;\cdot)$ with respect to $\mathcal{M}_j$ we have

Theorem 4.1.2. *For $n>2k$, the polynomial $P_n(j;\cdot)$ has at least $n-2k$ different zeros with odd multiplicity on I.*

We now analyze the asymptotic behavior of the zeros of $\{P_n(j;\cdot)\}_{n\geq 0}$ when the mass m_j tends to infinity. Notice that,

$$P_n(j;x)=\frac{1}{\det\widetilde{\mathbf{H}}_{n-1}}\begin{bmatrix} \upsilon_0 & \cdots & \upsilon_j+m_j & \cdots & \upsilon_n \\ \vdots & & \vdots & & \vdots \\ \upsilon_j+m_j & & \upsilon_{2j} & & \upsilon_{n+j} \\ \vdots & & \vdots & \ddots & \vdots \\ \upsilon_{n-1} & & \upsilon_{n+j-1} & & \upsilon_{2n-1} \\ 1 & \cdots & (x-a)^j & \cdots & (x-a)^n \end{bmatrix}. \tag{4.9}$$

On the other hand, let $\{R_n^k(a;\cdot)\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to the linear functional $\widehat{\mathcal{M}}$ defined by

$$\langle\widehat{\mathcal{M}},p\rangle=\langle\mathcal{M},(x-a)^k p(x)\rangle,\quad k\geq 0.$$

Here, we assume that a is not a zero of the polynomials $\{P_n\}_{n\geqslant 0}$ in order $\widehat{\mathcal{M}}$ to be quasi-definite. The Hankel matrix associated with $\widehat{\mathcal{M}}$ is

$$\mathbf{H}^{(k)}=\begin{bmatrix} \mu_k & \mu_{1+k} & \cdots & \mu_{n+k} & \cdots \\ \mu_{1+k} & \mu_{2+k} & \cdots & \mu_{n+1+k} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \mu_{n+k} & \mu_{n+1+k} & \cdots & \mu_{2n+k} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}.$$

From (4.2),

$$R_{n-j-1}^{2(j+1)}(a;x)=\frac{1}{\det\mathbf{H}_{n-j-2}^{(2j+2)}}\begin{bmatrix} \upsilon_{2j+2} & \upsilon_{2j+3} & \cdots & \upsilon_{n+j} & \upsilon_{n+j+1} \\ \upsilon_{2j+3} & \upsilon_{2j+4} & \cdots & \upsilon_{n+j+1} & \upsilon_{n+j+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \upsilon_{n+j} & \upsilon_{n+j+1} & \cdots & \upsilon_{2n-2} & \upsilon_{2n-1} \\ 1 & (x-a) & \cdots & (x-a)^{n-j-2} & (x-a)^{n-j-1} \end{bmatrix},\quad n>j+1,$$

If the matrix in (4.9) is block partitioned into $\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}$, where $\mathbf{A}$ is a $(j+1)\times(j+1)$ matrix, then

$$\begin{vmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{vmatrix}=\det\mathbf{D}\det(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}).$$

It is clear that $\det\mathbf{D}=\det\mathbf{H}_{n-j-2}^{(2j+2)}(x-a)^{j+1}R_{n-j-1}^{2(j+1)}(a;x)$. Moreover, $\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is a $(j+1)\times(j+1)$ matrix

Table 4.1: Zeros of $T_2(j;\cdot)$ for $a=3$, $j=1$ for some values of m

m	$x_{2,1}(j;m)$	$x_{2,2}(j;m)$
0	-0.707107	0.707107
0.1	-0.4034	1.61713
0.5	-0.317089	3.16098
1	-0.355225	3.74581
5	-0.95446	4.54373
10	-2.39228	4.881
10^2	$3.18582+1.08651i$	$3.18582-1.08651i$
10^3	$3.01522+0.317155i$	$3.01522-0.317155i$
10^4	$3.0015+0.0995593i$	$3.0015-0.0995593i$
10^5	$3.00015+0.0314605i$	$3.00015-0.0314605i$

that does not depend on m, and thus

$$P_n(j;x) = \frac{\det \mathbf{H}_{n-j-2}^{(2j+2)}(x-a)^{j+1}R_{n-j-1}^{2(j+1)}(a;x)Q(m_j)}{\det \mathbf{H}_{n-j-2}^{(2j+2)}R(m_j)}, \tag{4.10}$$

where $Q(m_j)$ and $R(m_j)$ are monic polynomials in m_j of degree $j+1$. Therefore,

$$\lim_{m_j\to\infty} P_n(j;x) = (x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;x), \tag{4.11}$$

and we conclude

Theorem 4.1.3. *The zeros $x_{n,k}(j;m_j)$, $k=1,\dots,n$, of the polynomial $P_n(j;\cdot)$ converge to the zeros of the polynomial $(x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;\cdot)$ when m_j tends to infinity.*

Observe that when m_j tends to infinity the mass point a attracts $j+1$ zeros of $P_n(j;\cdot)$.

A rather natural question is if $x_{n,k}(j;m_j)$, considered as functions of m_j, tend to the zeros of $(x-a)^{j+1}R_{n-(j+1)}^{2(j+1)}(a;x)$ in a monotonic way. For the particular case when $j=0$, it was proved Dimitrov et al. [2010,b] that the zeros of the so-called Laguerre and Jacobi type orthogonal polynomials, which are particular cases of the Uvarov spectral transformation, behave monotonically with respect to m_j. Unfortunately, this phenomenon does not occur for every positive integer j. We have performed some numerical experiments with specific classical measures. For example, if the initial measure is the one associated with the Laguerre polynomials $L_n^{(\alpha)}$, $j=1$, and $a=0$, then the zeros $x_{n,k}(1;m)$ of the corresponding polynomials $L_n(1;\cdot)$ converge to those of $x^2L_{n-2}^{\alpha+4}(x)$, although they are not monotonic functions of m_j when it varies in $(0,\infty)$.

We present some tables that show the behavior of the zeros of $T_n(j;\cdot)$ as a function of m_j, when the initial measure is $d\mu(-1/2,-1/2;x)$ (Chebyshev polynomials of the first kind; see Chapter 1) for $j=1$ and $n=2,3$.

Notice that there exist complex zeros depending on the values of the parameter m_j. It is also observed that two zeros of the polynomial approach the point $x=a$ as m_j increases, as established in Theorem 4.1.3.

Table 4.2: Zeros of $T_3(j;\cdot)$ for $a = 3$, $j = 1$ for some values of m

m	$x_{3,1}(j;m)$	$x_{3,2}(j;m)$	$x_{3,3}(j;m)$
0	−0.866025	0	0.866025
0.1	−0.801321	0.51227	3.61105
0.5	−1.18576	0.0479705	3.7638
1	−3.70458	−0.305553	3.86415
5	−0.510437	$3.38703 + 0.805069i$	$3.38703 - 0.805069i$
10	−0.525903	$3.15823 + 0.549961i$	$3.15823 - 0.549961i$
10^2	−0.538469	$3.01358 + 0.167363i$	$3.01358 - 0.167363i$
10^3	−0.539661	$3.00134 + 0.052716i$	$3.00134 - 0.052716i$
10^4	−0.539779	$3.00013 + 0.0166637i$	$3.00013 - 0.0166637i$
10^5	−0.539791	$3.00001 + 0.0052693i$	$3.00001 - 0.0052693i$

4.2 Toeplitz matrices

From the point of view of perturbations of positive definite hermitian Toeplitz matrices or, equivalently, probability measures supported on the unit circle, there is a wide literature Branquinho et al. [1999]; Daruis et al. [2007]; Geronimus [1954]; Godoy and Marcellán [1991, 1993]; Li and Marcellán [1999]; Marcellán et al. [1996, 1997], emphasizing the analytic properties of orthogonal polynomials with respect to the perturbed measures. In Cachafeiro et al. [2003] a spectral transformation associated with a modification of a measure on the unit circle by the addition of the normalized Lebesgue measure was introduced. The translation, in terms of the entries of the new Toeplitz matrix, means that we only perturb the main diagonal of the original Toeplitz matrix. In the introduction of their work the authors emphasized that this problem is strongly related with the method introduced by Pisarenko Pisarenko [1973] for retrieving harmonics from a covariance function. Four years later in Cachafeiro et al. [2007], the same authors generalized their previous result and studied the sequence of orthogonal polynomials associated with the sum of a measure supported on the unit circle in the class $\mathcal{S}$ and a Berstein-Szegő measure Simon [2005]. Indeed, they deduced that the corresponding measure belongs to the $\mathcal{S}$ class and obtained several properties about the norms of the associated sequence of orthogonal polynomials.

4.2.1 Diagonal perturbation of a Toeplitz matrix

Let $\mathcal{L}$ be the moment linear functional introduced in (2.22). We define a new linear functional $\mathcal{L}_0$ such that its associated bilinear form satisfies

$$\langle f,g\rangle_{\mathcal{L}_0} = \langle f,g\rangle_{\mathcal{L}} + m\int_{\mathbb{T}} f(z)\overline{g(z)}\frac{dz}{2\pi iz}, \quad f,g\in\mathbb{P}, \quad m\in\mathbb{R}. \tag{4.12}$$

Assume that $\mathcal{L}$ is a positive definite linear functional and let σ be its corresponding measure as defined in (2.28). Then, this transformation can be expressed in terms of the corresponding measure σ as

$$d\tilde{\sigma}(z) = d\sigma(z) + m\frac{dz}{2\pi iz}, \tag{4.13}$$

i.e., the addition of a Lebesgue measure to σ. We assume $m \in \mathbb{R}_+$ in order $\widetilde{\sigma}$ to be a positive Borel measure supported in $\mathbb{T}$. The moments $\{\widetilde{c}_k\}_{k\in\mathbb{Z}}$ associated with $\mathcal{L}_0$ are given by

$$\widetilde{c}_0 = c_0 + m, \quad \widetilde{c}_k = c_k, \quad k \in \mathbb{Z}. \tag{4.14}$$

As a consequence, the C-function of the linear functional $\mathcal{L}_0$ is

$$F_0(z) = F(z) + m. \tag{4.15}$$

Notice that $\widetilde{\mathbf{T}}$, the Toeplitz matrix associated with $\mathcal{L}_0$, is

$$\widetilde{\mathbf{T}} = \mathbf{T} + m\mathbf{I}, \tag{4.16}$$

i.e., a constant is added to the main diagonal of $\mathbf{T}$. We now proceed to obtain the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_0$.

Theorem 4.2.1. *Let $\mathcal{L}$ be a positive definite linear functional, and denote by $\{\Phi_n\}_{n\geqslant 0}$ its associated sequence of monic orthogonal polynomials. Then, $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_0$ defined by (4.12), is given by*

$$\Psi_n(z) = \Phi_n(z) - \mathbf{K}_{n-1}^T(z,0)(m^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^T)^{-1}\mathbf{\Phi}_n(0), \tag{4.17}$$

with $\mathbf{K}_{n-1}(z,0) = \left[K_{n-1}(z,0), K_{n-1}^{(0,1)}(z,0), \ldots, K_{n-1}^{(0,n-1)}(z,0)\right]^T$, $\mathbf{D}_n = diag\left\{\frac{1}{0!}, \ldots, \frac{1}{(n-1)!}\right\}$, $\mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(n-1)}(0)]^T$*, and*

$$\mathbf{P}_{n-1} = \begin{bmatrix} \phi_0(0) & \phi_1(0) & \cdots & \phi_{n-1}(0) \\ 0 & \phi_1'(0) & \cdots & \phi_{n-1}'(0) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \phi_{n-1}^{(n-1)}(0) \end{bmatrix}.$$

Proof. Set

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z),$$

where, for $0 \leqslant k \leqslant n-1$,

$$\lambda_{n,k} = \frac{1}{\mathbf{k}_k}\langle \Psi_n, \Phi_k\rangle_{\mathcal{L}_0} - m\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy} = -\frac{m}{\mathbf{k}_k}\int_{\mathbb{T}} \Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}.$$

Thus,

$$\Psi_n(z) = \Phi_n(z) - m\int_{\mathbb{T}} \Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} = \Phi_n(z) - m\sum_{j=0}^{n-1}\frac{\Psi_n^{(j)}(0)}{j!}\frac{K_{n-1}^{(0,j)}(z,0)}{j!}. \tag{4.18}$$

In particular, for $0 \leqslant i \leqslant n-1$, we get

$$\Psi_n^{(i)}(0) = \Phi_n^{(i)}(0) - m \sum_{j=0}^{n-1} \frac{\Psi_n^{(j)}(0)}{j!} \frac{K_{n-1}^{(i,j)}(0,0)}{j!}. \tag{4.19}$$

So, we have the following system of n linear equations and n unknowns

$$\Psi_n^{(i)}(0) = \Phi_n^{(i)}(0) - m \sum_{j=0}^{n-1} \frac{\Psi_n^{(j)}(0)}{j!} \frac{K_{n-1}^{(i,j)}(0,0)}{j!}, \quad i = 0,1,\ldots,n-1,$$

which reads as

$$(\mathbf{I}_n + m\mathbf{R}_{n-1}\mathbf{D}_n^2)\mathbf{\Psi}_n(0) = \mathbf{\Phi}_n(0), \tag{4.20}$$

where $\mathbf{\Psi}_n(0) = \left[\Psi_n(0), \ldots, \Psi_n^{(n-1)}(0)\right]^T$,

$$\mathbf{R}_{n-1} = \begin{bmatrix} K_{n-1}^{(0,0)}(0,0) & K_{n-1}^{(0,1)}(0,0) & \cdots & K_{n-1}^{(0,n-1)}(0,0) \\ K_{n-1}^{(1,0)}(0,0) & K_{n-1}^{(1,1)}(0,0) & \cdots & K_{n-1}^{(1,n-1)}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(n-1,0)}(0,0) & K_{n-1}^{(n-1,1)}(0,0) & \cdots & K_{n-1}^{(n-1,n-1)}(0,0) \end{bmatrix}, \tag{4.21}$$

As a consequence, (4.20) becomes $\mathbf{\Psi}_n(0) = m^{-1}(m^{-1}\mathbf{I}_n + \mathbf{R}_{n-1}\mathbf{D}_n^2)^{-1}\mathbf{\Phi}_n(0)$. Thus, (4.18) can be written

$$\Psi_n(z) = \Phi_n(z) - m\mathbf{K}_{n-1}^T(z,0)\mathbf{D}_n^2\mathbf{\Psi}_n(0) = \Phi_n(z) - \mathbf{K}_{n-1}^T(z,0)(m^{-1}\mathbf{D}_n^{-2} + \mathbf{R}_{n-1})^{-1}\mathbf{\Phi}_n(0),$$

which is (4.17), since $\mathbf{R}_{n-1} = \mathbf{P}_{n-1}\mathbf{P}_{n-1}^T$. $\square$

On the other hand, considering the derivatives of order j with respect to the variable z in the Christoffel-Darboux formula (2.32) we get

$$K_{n-1}^{(j,0)}(z,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\overline{y}z}\right)^{(j)} - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\overline{y}z}\right)^{(j)}.$$

Thus,

$$K_{n-1}^{(j,0)}(0,y) = \frac{\overline{\Phi_n^*(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n^*(z)}{1-\overline{y}z}\right)^{(j)}(0) - \frac{\overline{\Phi_n(y)}}{\mathbf{k}_n}\left(\frac{\Phi_n(z)}{1-\overline{y}z}\right)^{(j)}(0).$$

Now, by Leibniz rule

$$\left(\frac{\Phi_n(z)}{1-\overline{y}z}\right)^{(j)} = \sum_{k=0}^{j}\binom{j}{k}\Phi_n^{(j-k)}(z)\frac{k!\overline{y}^k}{(1-\overline{y}z)^{k+1}},$$

the evaluation at $z=0$ yields

$$j!\sum_{k=0}^{j}\frac{\Phi_n^{(j-k)}(0)}{(j-k)!}\overline{y}^k = j!\overline{\sum_{k=0}^{j}\frac{\overline{\Phi_n^{(j-k)}(0)}}{(j-k)!}y^k} = j!\overline{T_j^*(\Phi_n(y);0)},$$

where $T_j(p(y);0)$ denotes the $j-th$ Taylor polynomial of $p(y)$ around $y=0$. In an analog way,

$$j!\sum_{k=0}^{j}\frac{\Phi_n^{*(j-k)}(0)}{(j-k)!}\overline{y}^k = j!\overline{T_j^*(\Phi_n^*(y);0)}.$$

Therefore, we have proved

Theorem 4.2.2.
$$K_{n-1}^{(0,j)}(z,0) = j!\left(\frac{\Phi_n^*(z)}{\mathbf{k}_n}T_j^*(\Phi_n^*(z);0)-\frac{\Phi_n(z)}{\mathbf{k}_n}T_j^*(\Phi_n(z);0)\right).$$

From the previous theorem, if we denote

$$\mathbf{T}(\Phi_n(z);0) = \left[T_0^*(\Phi_n(z);0), T_1^*(\Phi_n(z);0),\ldots,T_{n-1}^*(\Phi_n(z);0)\right]^T,$$

then, (4.17) becomes

$$\begin{aligned}\Psi_n(z) = \Phi_n(z) &- \frac{\Phi_n(z)}{\mathbf{k}_n}\mathbf{T}^T(\Phi_n(z);0)\mathbf{D}_n^{-1}(m^{-1}\mathbf{D}_n^{-2}+\mathbf{P}_{n-1}\mathbf{P}_{n-1}^T)^{-1}\mathbf{\Phi}_n(0)\\ &-\frac{\Phi_n^*(z)}{\mathbf{k}_n}\mathbf{T}^T(\Phi_n^*(z);0)\mathbf{D}_n^{-1}(m^{-1}\mathbf{D}_n^{-2}+\mathbf{P}_{n-1}\mathbf{P}_{n-1}^T)^{-1}\mathbf{\Phi}_n(0) = a_n(z)\Phi_n(z)+b_n(z)\Phi_n^*(z),\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}a_n(z) &= 1-\frac{1}{\mathbf{k}_n}\mathbf{T}^T(\Phi_n(z);0)\mathbf{D}_n^{-1}(m^{-1}\mathbf{D}_n^{-2}+\mathbf{P}_{n-1}\mathbf{P}_{n-1}^T)^{-1}\mathbf{\Phi}_n(0),\\ b_n(z) &= -\frac{1}{\mathbf{k}_n}\mathbf{T}^T(\Phi_n^*(z);0)\mathbf{D}_n^{-1}(m^{-1}\mathbf{D}_n^{-2}+\mathbf{P}_{n-1}\mathbf{P}_{n-1}^T)^{-1}\mathbf{\Phi}_n(0).\end{aligned}$$

Example: Bernstein-Szegő polynomials

Let us consider the measure σ such that

$$d\sigma(\theta) = \frac{1-|\beta|^2}{|e^{i\theta}-\beta|^2}\frac{d\theta}{2\pi}, \quad |\beta|<1, \tag{4.23}$$

and apply the transformation studied in this section, i.e., let us define a new measure $\widetilde{\sigma}$

$$d\widetilde{\sigma}(\theta) = \frac{1-|\beta|^2}{|e^{i\theta}-\beta|^2}\frac{d\theta}{2\pi}+m\frac{d\theta}{2\pi}, \quad m\in\mathbb{R}_+. \tag{4.24}$$

Our aim is to find the sequence of monic polynomials orthogonal with respect to (4.24), that will be denoted by $\{\Psi_n\}_{n\geqslant 0}$. (4.23) is known in the literature as Bernstein-Szegő measure, and its corresponding sequence of monic orthogonal polynomials is Simon [2005], $\Phi_n(z)=z^{n-1}(z-\beta)$, $n\geqslant 1$. Furthermore, we have $\Phi_n^*(z) = 1-\overline{\beta}z$, $n\geqslant 1$. Notice that for $n=1$, from (4.22) we get

$$\Psi_1(z) = \left(1-\beta^2\left(1-|\beta|^2\right)\left(m+\left(1-|\beta|^2\right)^{-1}\right)\right)(z-\beta)-\left(\beta\left(1-|\beta|^2\right)\left(m+\left(1-|\beta|^2\right)^{-1}\right)\right)(1-\overline{\beta}z).$$

For $n \geqslant 2$, $T_j(\Phi_n(0);0) = 0$, $0 \leqslant j \leqslant n-2$, and $T_{n-1}(\Phi_n(z);0) = -\beta z^{n-1}$. Thus, $\mathbf{T}(\Phi_n(z);0) = \left[0,0,\ldots,-\overline{\beta}\right]^T$. On the other hand, $T_0(\Phi_n^*(0);0) = 1$, and $T_j(\Phi_n^*(z);0) = 1-\overline{\beta}z$, $1 \leqslant j \leqslant n-1$. Therefore, $\mathbf{T}(\Phi_n^*(z);0) = \left[1, z-\beta, z(z-\beta), \ldots, z^{n-2}(z-\beta)\right]^T$. Furthermore, in this case

$$\mathbf{P}_{n-1} = \frac{1}{\sqrt{1-|\beta|^2}}\mathbf{D}_n^{-1}\mathbf{B}_n,$$

where

$$\mathbf{B}_n = \begin{bmatrix} 1 & \beta & 0 & \cdots & 0 \\ 0 & 1 & \beta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & \beta \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}. \tag{4.25}$$

Denoting by

$$\mathbf{N}_n = \mathbf{D}_n^{-1}\left(m^{-1}\mathbf{D}_n^{-2} + \mathbf{P}_{n-1}\mathbf{P}_{n-1}^T\right)^{-1} = \left(m^{-1}\mathbf{I}_n + \frac{1}{1-|\beta|^2}\mathbf{B}_n\mathbf{B}_n^t\right)^{-1}\mathbf{D}_n,$$

and taking into account that $\mathbf{\Phi}_n(0) = [0,0,\cdots,-(n-1)!\beta]^T$, we get

$$a_n(z) = 1 - \frac{|\beta|^2}{1-|\beta|^2}(n-1)!n_{n-1,n-1}, \quad b_n(z) = -\frac{\beta}{1-|\beta|^2}(n-1)!\left(n_{0,n-1} + (z-\beta)\sum_{i=0}^{n-2} n_{i+1,n-1}z^i\right),$$

where $n_{i,j}$, $0 \leqslant i,j \leqslant n-1$, are the entries of $\mathbf{N}_n$. From this, all elements on (4.22) are known and we can compute Ψ_n. The corresponding Verblunsky coefficients for $n \geqslant 2$ are

$$\Psi_n(0) = b_n(0) = \frac{\beta}{1-|\beta|^2}(n-1)!(\beta n_{1,n-1} - n_{0,n-1}).$$

The first 20 Verblunsky coefficients for different values of m and β are shown in Figure 4.1(a), namely $m = 5, \beta = -0.2$ (blue discs) and $m = 10, \beta = -0.5$ (purple square).

Example: Chebyshev polynomials

Let us consider the measure

$$d\sigma(\theta) = |e^{i\theta} - 1|^2\frac{d\theta}{2\pi}, \tag{4.26}$$

and its corresponding sequence of monic orthogonal polynomials, given by Simon [2005]

$$\Phi_n(z) = \frac{1}{n+1}\sum_{k=0}^{n}(k+1)z^k, \quad n \geqslant 0. \tag{4.27}$$

We introduce the perturbation defined in this section and obtain

$$d\widetilde{\sigma}(\theta) = d\sigma(\theta) + m\frac{d\theta}{2\pi}. \tag{4.28}$$

We now proceed to get an explicit expression for the sequence of monic polynomials orthogonal with respect to $\widetilde{\sigma}$. Notice that

$$T_j(\Phi_n(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(k+1)z^k, \quad 0 \leqslant j \leqslant n-1,$$

and, as a consequence,

$$T_j^*(\Phi_n(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(k+1)z^{j-k}, \quad 0 \leqslant j \leqslant n-1.$$

On the other hand, since

$$\Phi_n^*(z) = \frac{1}{n+1}\sum_{k=0}^{n}(k+1)z^{n-k}, \quad n \geqslant 0,$$

we get

$$T_j(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^k, \quad 1 \leqslant j \leqslant n-1,$$

and, thus,

$$T_j^*(\Phi_n^*(z);0) = \frac{1}{n+1}\sum_{k=0}^{j}(n+1-k)z^{j-k}, \quad 1 \leqslant j \leqslant n-1.$$

Therefore, $\mathbf{T}(\Phi_n(0);0) = \frac{1}{n+1}[1,2,\ldots,n]^T$ and $\mathbf{T}(\Phi_n^*(0);0) = \frac{1}{n+1}[n+1,n,n-1,\ldots,2]^T$. Furthermore, since

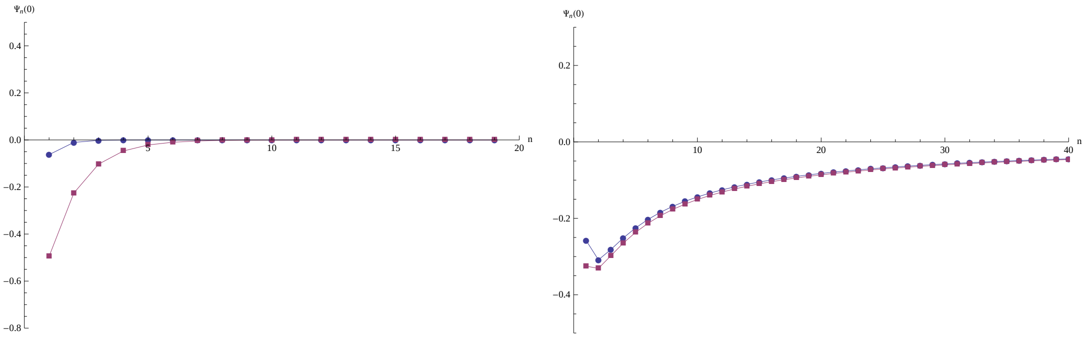
$$\phi_n(z) = \sqrt{\frac{2}{(n+1)(n+2)}}\sum_{k=0}^{n}(k+1)z^k, \qquad n \geqslant 0, \tag{4.29}$$

are the orthonormal polynomials with respect to σ, we obtain

$$\mathbf{P}_{n-1} = \mathbf{D}_n^{-1}\mathbf{A}_n\mathbf{\Lambda}_n \tag{4.30}$$

where

$$\mathbf{A}_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \mathbf{\Lambda}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{n-1} \end{bmatrix},$$



(a) (b)

Figure 4.1: Verblunsky coefficients for Bernstein-Szegő and Chebyshev polynomials

and $p_j = \sqrt{\dfrac{2}{(j+1)(j+2)}}$. Finally, we have $\mathbf{\Phi}_n(0) = \frac{1}{n+1}[1,2!,\cdots,n!]^T$, and we can obtain $\Psi_n(z)$ explicitly from (4.22). Figure 4.1(b) shows the behavior of the corresponding Verblunsky coefficients for different values of m, namely $m = 10$ (blue discs) and $m = 100$ (purple square).

This example can be generalized as follows. Let σ be an absolutely continuous measure whose Radon-Nikodyn derivative with respect to the Lebesgue measure is $\sigma' = |z-\alpha|^2$, $z = e^{i\theta}$, i.e., a positive trigonometric polynomial of degree 1. Applying the transformation introduced in this section, with $m \in \mathbb{R}_+$, and assuming $\alpha \in \mathbb{C}$, we obtain

$$m + |z-\alpha|^2 = m + 1 - \alpha z^{-1} - \overline{\alpha} z + |\alpha|^2, \quad z = e^{i\theta}, \tag{4.31}$$

i.e., another positive trigonometric polynomial that can be represented by $|\delta z - \gamma|^2$, where $\delta \in \mathbb{R}, \gamma \in \mathbb{C}$. Indeed, as

$$|\delta z - \gamma|^2 = \delta^2 - \delta\gamma z^{-1} - \delta\overline{\gamma} z + |\gamma|^2, \quad z = e^{i\theta},$$

the comparison of the coefficients with (4.31) yields $1 + |\alpha|^2 + m = \delta^2 + |\gamma|^2$ and $\alpha = \delta\gamma$. Thus,

$$1 + |\alpha|^2 + m = \frac{|\alpha|^2}{\delta^2} + \delta^2,$$

so we can get δ and γ in terms of m and α. In other words, in this case we can express the addition to a Chebyshev measure of a Lebesgue measure (multiplied by a constant m) as (4.26).

4.2.2 General perturbation of a Toeplitz matrix

In this subsection we generalize the previous perturbation, adding a mass $\boldsymbol{m}$ to any sub-diagonal of the Toeplitz matrix. Let $\mathcal{L}_j$ be a linear functional such that its associated bilinear functional satisfies

$$\langle f,g\rangle_{\mathcal{L}_j} = \langle f,g\rangle_{\mathcal{L}} + \boldsymbol{m}\int_{\mathbb{T}} z^j f(z)\overline{g(z)}\frac{dz}{2\pi iz} + \overline{\boldsymbol{m}}\int_{\mathbb{T}} z^{-j} f(z)\overline{g(z)}\frac{dz}{2\pi iz}, \quad \boldsymbol{m} \in \mathbb{C}, \quad j \geq 0. \tag{4.32}$$

Assume $\mathcal{L}$ is a positive definite linear functional. Then, in terms of the corresponding measures, the above transformation can be expressed as

$$d\widetilde{\sigma}(z) = d\sigma(z) + 2\Re(\boldsymbol{m}z^j)\frac{dz}{2\pi iz}. \tag{4.33}$$

From (4.32), if F_j is the C-function associated with $\mathcal{L}_j$, then

$$F_j(z) = F(z) + 2\boldsymbol{m}z^j, \tag{4.34}$$

i.e., a linear spectral transformation of F. The infinite Toeplitz matrix $\widetilde{\mathbf{T}}$ associated with $\mathcal{L}_j$, is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \boldsymbol{m}\mathbf{Z}^j + \overline{\boldsymbol{m}}\left(\mathbf{Z}^T\right)^j,$$

where $\mathbf{Z}$ is the shift matrix with ones on the first lower-diagonal and zeros on the remaining entries, and $\mathbf{Z}^T$ is its transpose. Equivalently,

$$\widetilde{\mathbf{T}} = \mathbf{T} + \begin{bmatrix} 0 & \cdots & \overline{\boldsymbol{m}} & 0 & \cdots \\ \vdots & 0 & \cdots & \overline{\boldsymbol{m}} & \cdots \\ \boldsymbol{m} & \vdots & \ddots & \vdots & \ddots \\ 0 & \boldsymbol{m} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

Assume that $\mathcal{L}$ is a positive definite linear functional, and denote by $\{\Phi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. We now proceed to determine necessary and sufficient conditions for $\mathcal{L}_j$ to be a quasi-definite functional, as well as the relation between $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Psi_n\}_{n\geqslant 0}$, the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_j$.

Theorem 4.2.3. *Let $\mathcal{L}$ be a positive definite moment linear functional and $\{\Phi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Then, the following statements are equivalent:*

i) $\mathcal{L}_j$ is a quasi-definite linear functional.

ii) The matrix $\mathbf{I}_n + \mathbf{S}_n$ is non-singular, and

$$\widetilde{\mathbf{k}}_n = \mathbf{k}_n + \mathbf{W}_n^T(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\overline{\mathbf{Y}_n(0)} + \overline{\boldsymbol{m}}\frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!} \neq 0, \quad n \geqslant 1, \tag{4.35}$$

with $\mathbf{Y}_n(0) = \left[\overline{\boldsymbol{m}}\frac{\Phi_n^{(j)}(0)}{j!}, \ldots, \overline{\boldsymbol{m}}\frac{\Phi_n^{(2j)}(0)}{(2j)!} + \boldsymbol{m}\frac{\Phi_n^{(0)}(0)}{(0)!}, \ldots, \overline{\boldsymbol{m}}\frac{\Phi_n^{(n)}(0)}{(n)!} + \boldsymbol{m}\frac{\Phi_n^{(n-2j)}(0)}{(n-2j)!}, \ldots, \boldsymbol{m}\frac{\Phi_n^{(n-j-1)}(0)}{(n-j-1)!}\right]^T,$

$$\mathbf{W}_n(0) = [\mathbf{\Phi}_n(0) - \overline{\boldsymbol{m}}n!\mathbf{C}_{(0,n-1;n)}],\ \mathbf{\Phi}_n(0) = [\Phi_n(0), \Phi_n'(0), \dots, \Phi_n^{(n-1)}(0)]^T,$$

$$\mathbf{S}_n = \left[\begin{array}{c|c|c} \boldsymbol{m}\mathbf{A}_{(0,j-1;0,j-1)} & \mathbf{B}_{(0,j-1;j,n-j-1)} & \overline{\boldsymbol{m}}\mathbf{C}_{(0,j-1;n-j,n-1)} \\ \hline \boldsymbol{m}\mathbf{A}_{(j,n-j-1;0,j-1)} & \mathbf{B}_{(j,n-j-1;j,n-j-1)} & \overline{\boldsymbol{m}}\mathbf{C}_{(j,n-j-1;n-j,n-1)} \\ \hline \boldsymbol{m}\mathbf{A}_{(n-j,n-1;0,j-1)} & \mathbf{B}_{(n-j,n-1;j,n-j-1)} & \overline{\boldsymbol{m}}\mathbf{C}_{(n-j,n-1;n-j,n-1)} \end{array}\right],$$

where $\mathbf{A}, \mathbf{B}$, *and* $\mathbf{C}$ *are matrices whose elements are given by*

$$a_{s,l} = \frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!}, \quad b_{s,l} = \boldsymbol{m}\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l)!(l+j)!} + \overline{\boldsymbol{m}}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!}, \quad c_{s,l} = \frac{K_{n-1}^{(s,l-j)}(0,0)}{(l)!(l-j)!}.$$

Moreover, $\{\Psi_n\}_{n\geqslant 0}$*, the corresponding sequence of monic orthogonal polynomials with respect to* $\mathcal{L}_j$*, is given by*

$$\Psi_n(z) = A_n(z)\Phi_n(z) + B_n(z)\Phi_n^*(z), \quad n \geqslant 1, \tag{4.36}$$

with

$$\begin{aligned} A_n(z) &= 1 + \frac{1}{\mathbf{k}_n}\mathbf{W}_n^T(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\mathbf{D}_n\mathcal{T}(\Phi_n(z);0) + \overline{\boldsymbol{m}}\frac{T_{n-j}^*(\Phi_n(z);0)}{\mathbf{k}_n}, \\ B_n(z) &= -\frac{1}{\mathbf{k}_n}\mathbf{W}_n^T(0)(\mathbf{I}_n + \mathbf{S}_n)^{-1}\mathbf{D}_n\mathcal{T}(\Phi_n^*(z);0) - \overline{\boldsymbol{m}}\frac{T_{n-j}^*(\Phi_n^*(z);0)}{\mathbf{k}_n}, \\ \mathcal{T}(\Phi_n(z);0) &= \Big[\boldsymbol{m}T_j^*(\Phi_n(z);0), \dots, \boldsymbol{m}T_{2j}^*(\Phi_n(z);0) + \overline{\boldsymbol{m}}T_0^*(\Phi_n(z);0), \dots, \boldsymbol{m}T_{n-1}^*(\Phi_n(z);0) + \\ &\quad \overline{\boldsymbol{m}}T_{n-2j-1}^*(\Phi_n(z);0), \dots, \overline{\boldsymbol{m}}T_{n-j-1}^*(\Phi_n(z);0)\Big]^T. \end{aligned}$$

Proof. Let us write $\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1}\lambda_{n,k}\Phi_k(z)$, where, for $0 \leqslant k \leqslant n-1$,

$$\lambda_{n,k} = -\frac{\boldsymbol{m}}{\mathbf{k}_k}\int_{\mathbb{T}} y^j\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy} - \frac{\overline{\boldsymbol{m}}}{\mathbf{k}_k}\int_{\mathbb{T}} y^{-j}\Psi_n(y)\overline{\Phi_k(y)}\frac{dy}{2\pi iy}.$$

Therefore,

$$\Psi_n(z) = \Phi_n(z) - \boldsymbol{m}\int_{\mathbb{T}} y^j\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} - \overline{\boldsymbol{m}}\int_{\mathbb{T}} y^{-j}\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}.$$

Taking into account that

$$y^j\Psi_n(y) = \sum_{l=0}^{n}\frac{\Psi_n^{(l)}(0)}{l!}y^{l+j} = \sum_{l=j}^{n+j}\frac{\Psi_n^{(l-j)}(0)}{(l-j)!}y^l,$$

and, for $|y|=1$, $K_{n-1}(z,y)=\displaystyle\sum_{l=0}^{n-1}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l}$, we obtain

$$\int_{\mathbb{T}} y^j\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}=\sum_{l=j}^{n-1}\frac{\Psi_n^{(l-j)}(0)}{(l-j)!}\frac{K_{n-1}^{(0,l)}(z,0)}{(l)!}=\sum_{l=0}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}.$$

In an analog way,

$$\int_{\mathbb{T}} y^{-j}\Psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}=\sum_{l=0}^{n-j}\frac{\Psi_n^{(l+j)}(0)}{(l+j)!}\frac{K_{n-1}^{(0,l)}(z,0)}{(l)!}=\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}.$$

Thus, we get

$$\Psi_n(z)=\Phi_n(z)-\boldsymbol{m}\sum_{l=0}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}-\overline{\boldsymbol{m}}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!},\tag{4.37}$$

or, equivalently,

$$\begin{aligned}\Psi_n(z)=\Phi_n(z)-\boldsymbol{m}\sum_{l=0}^{j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}-\overline{\boldsymbol{m}}\sum_{l=n-j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!},\\ -\sum_{l=j}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\left(\boldsymbol{m}\frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}+\overline{\boldsymbol{m}}\frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}\right).\end{aligned}\tag{4.38}$$

In particular, for $0\leqslant s\leqslant n$,

$$\begin{aligned}\Psi_n^{(s)}(0)=\Phi_n^{(s)}(0)-\boldsymbol{m}\sum_{l=0}^{j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l+j)!}-\overline{\boldsymbol{m}}\sum_{l=n-j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l-j)!},\\ -\sum_{l=j}^{n-j-1}\frac{\Psi_n^{(l)}(0)}{(l)!}\left(\boldsymbol{m}\frac{K_{n-1}^{(s,l+j)}(0,0)}{(l+j)!}+\overline{\boldsymbol{m}}\frac{K_{n-1}^{(s,l-j)}(0,0)}{(l-j)!}\right),\end{aligned}$$

i.e., we obtain a system of $n+1$ linear equations and $n+1$ unknowns as follows

$$\Psi_n^{(s)}(0)=\Phi_n^{(s)}(0)-\boldsymbol{m}\sum_{l=0}^{j-1}a_{s,l}\Psi_n^{(l)}(0)-\sum_{l=j}^{n-j-1}b_{s,l}\Psi_n^{(l)}(0)-\overline{\boldsymbol{m}}\sum_{l=n-j}^{n}c_{s,l}\Psi_n^{(l)}(0).$$

Thus, if $\mathbf{M}_{(s_1,s_2;l_1,l_2)}=[\boldsymbol{m}_{s,i}]_{s_1\leqslant s\leqslant s_2;l_1\leqslant l\leqslant l_2}$, then

$$(\mathbf{I}_{n+1}+\mathbf{S}_{n+1})\begin{bmatrix}\Psi_n^{(0)}(0)\\ \vdots\\ \Psi_n^{(n)}(0)\end{bmatrix}=\begin{bmatrix}\Phi_n^{(0)}(0)\\ \vdots\\ \Phi_n^{(n)}(0)\end{bmatrix}.$$

Notice that the entries in the last row of the above matrix vanish, which is consistent with the fact that $\Psi_n^{(n)}(0) = \Phi_n^{(n)}(0) = n!$. Therefore, if we denote by

$$\boldsymbol{\Psi}_n(0) = [\Psi_n(0), \Psi_n'(0), \ldots, \Psi_n^{(n-1)}(0)]^T,$$

then the above $(n+1)\times(n+1)$ linear system can be reduced to a $n\times n$ linear system as follows

$$(\mathbf{I}_n + \mathbf{S}_n)\boldsymbol{\Psi}_n(0) = \mathbf{W}_n(0). \tag{4.39}$$

Since $\mathcal{L}_j$ is a quasi-definite linear functional, there exists a unique family of monic polynomials orthogonal with respect to $\mathcal{L}_j$. Therefore, the matrix $\mathbf{I}_n + \mathbf{S}_n$ is non-singular, according to the existence and uniqueness of the solution of the above linear system. As a consequence, if

$$\mathbb{K}_{n-1}(z,0) = \begin{bmatrix} \boldsymbol{m}\frac{K_{n-1}^{(0,j)}(z,0)}{j!} \\ \vdots \\ \boldsymbol{m}\frac{K_{n-1}^{(0,2j)}(z,0)}{(2j)!} + \overline{\boldsymbol{m}}\frac{K_{n-1}^{(0,0)}(z,0)}{(0)!} \\ \vdots \\ \boldsymbol{m}\frac{K_{n-1}^{(0,n-1)}(z,0)}{(n-1)!} + \overline{\boldsymbol{m}}\frac{K_{n-1}^{(0,n-2j-1)}(z,0)}{(n-2j-1)!} \\ \vdots \\ \overline{\boldsymbol{m}}\frac{K_{n-1}^{(0,n-j-1)}(z,0)}{(n-j-1)!} \end{bmatrix}, \tag{4.40}$$

then (4.38) becomes

$$\Psi_n(z) = \Phi_n(z) - \boldsymbol{\Psi}_n^T(0)\mathbf{D}_n\mathbb{K}_{n-1}(z,0) - \overline{\boldsymbol{m}}\frac{K_{n-1}^{(0,n-j)}(z,0)}{(n-j)!}.$$

Thus, from (4.39) and Theorem 4.2.2, (4.36) holds. Furthermore,

$$\begin{aligned} 0 &\neq \langle \Psi_n, \Phi_n\rangle_{\mathcal{L}_j} \\ &= \mathbf{k}_n + \boldsymbol{m}\sum_{l=0}^{n-j}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l+j)}(0)}}{(l+j)!} + \overline{\boldsymbol{m}}\sum_{l=j}^{n}\frac{\Psi_n^{(l)}(0)}{(l)!}\frac{\overline{\Phi_n^{(l-j)}(0)}}{(l-j)!} = \mathbf{k}_n + \boldsymbol{\Psi}_n^T(0)\overline{\mathbf{Y}_n(0)} + \overline{\boldsymbol{m}}\frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!}, \end{aligned}$$

so (4.35) follows. For the converse, assume $\mathbf{I}_n + \mathbf{S}_n$ is non-singular for every $n \geq 1$ and define $\{\Psi_n\}_{n\geqslant 0}$ as in (4.36). We show that $\{\Psi_n\}_{n\geqslant 0}$ is orthogonal with respect to $\mathcal{L}_j$. Indeed, for $0 \leqslant k \leqslant n-1$ and taking into account (4.37), we get

$$\langle \Psi_n, \Phi_k \rangle_{\mathcal{L}_j} = \langle \Phi_n, \Phi_k \rangle_{\mathcal{L}} - \boldsymbol{m} \left\langle \sum_{l=0}^{n-j-1} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{K_{n-1}^{(0,l+j)}(z,0)}{(l+j)!}, \Phi_k(z) \right\rangle_{\mathcal{L}}$$
$$- \overline{\boldsymbol{m}} \left\langle \sum_{l=j}^{n} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{K_{n-1}^{(0,l-j)}(z,0)}{(l-j)!}, \Phi_k(z) \right\rangle_{\mathcal{L}} + \boldsymbol{m} \sum_{l=0}^{n-j-1} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{\overline{\Phi_k^{(l+j)}(0)}}{(l+j)!} + \overline{\boldsymbol{m}} \sum_{l=j}^{n} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{\overline{\Phi_k^{(l-j)}(0)}}{(l-j)!} = 0.$$

On the other hand,

$$\begin{aligned}\widetilde{\mathbf{k}}_n &= \langle \Psi_n, \Phi_n \rangle_{\mathcal{L}_j} \\ &= \mathbf{k}_n + \boldsymbol{m} \sum_{l=0}^{n-j} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{\overline{\Phi_n^{(l+j)}(0)}}{(l+j)!} + \overline{\boldsymbol{m}} \sum_{l=j}^{n} \frac{\Psi_n^{(l)}(0)}{(l)!} \frac{\overline{\Phi_n^{(l-j)}(0)}}{(l-j)!} = \mathbf{k}_n + \boldsymbol{\Psi}_n^T(0)\overline{\mathbf{Y}_n(0)} + \overline{\boldsymbol{m}} \frac{\overline{\Phi_n^{(n-j)}(0)}}{(n-j)!} \neq 0,\end{aligned}$$

since (4.35) is assumed. Thus, we conclude that $\mathcal{L}_j$ is quasi-definite. □

Remark 4.2.1. *The case $\mathcal{L}_0$ ($j = 0$) reduces to the linear functional analyzed in the previous subsection, with mass $\Re \boldsymbol{m}$. In such a case, $\widetilde{\mathbf{k}}_0 = \mathbf{k}_0 + \Re \boldsymbol{m}$. On the other hand, for $j \geqslant 1$, it follows from (4.32) that $\widetilde{\mathbf{k}}_l = \mathbf{k}_l$ for $0 \leqslant l \leqslant j-1$. In other words, we only need (4.35) for $n \geqslant j$. Notice that for a given j, the polynomials of degree $n < j$ remain unchanged. In such a case, (4.35) and (4.36) still hold, with the convention that the negative derivatives are zero.*

Finally, applying the Szegő transformation to (4.34), we get

$$\widetilde{S}(x) = S(x) + 2\boldsymbol{m} \frac{(x - \sqrt{x^2-1})^j}{\sqrt{x^2-1}},$$

and thus $\widetilde{S}$, the Stieltjes function for the corresponding perturbed measure on the real line, can not be expressed as a linear spectral transform of S, since square roots appear for any value of j. Therefore, we conclude that a perturbation on the moments of a measure supported on $\mathbb{T}$ does not yield a linear spectral transformation of the corresponding Stieltjes function. Conversely, if we consider a similar perturbation of the moments of a measure on the real line, then

$$\widetilde{S}(x) = S(x) + \frac{\boldsymbol{m}}{x^{j+1}}.$$

Applying the Szegő transformation,

$$F_0(z) = F(z) + \boldsymbol{m} \frac{1-z^2}{2z(x)^{j+1}} = F(z) + 2^j \boldsymbol{m} \frac{(1-z^2)z^j}{(z^2+1)^{j+1}},$$

which is a linear spectral transformation of F. In the special case when $j = 0$,

$$F_0(z) = F(z) - \boldsymbol{m} \frac{z^2-1}{z^2+1}.$$

As a conclusion, the study of linear spectral transformations on the unit circle is far more complicated than the real line case.

Example: Lebesgue polynomials

We present an example of the previous transformation when σ is the normalized Lebesgue measure and $j = 1$, i.e., the transformation

$$\langle f, g\rangle_{\mathcal{L}_1} = \int_{\mathbb{T}} f(z)\overline{g(z)}\frac{dz}{2\pi iz} + \boldsymbol{m}\int_{\mathbb{T}} zf(z)\overline{g(z)}\frac{dz}{2\pi iz} + \overline{\boldsymbol{m}}\int_{\mathbb{T}} zf(z)\overline{g(z)}\frac{dz}{2\pi iz}, \quad \boldsymbol{m}\in\mathbb{C}, \tag{4.41}$$

where $m \in \mathbb{C}$. Our purpose is to obtain necessary and sufficient conditions for $\mathcal{L}_1$ to be a positive definite (quasi-definite) functional. Consequently, we deduce its corresponding family of orthogonal polynomials, as well as the sequence of Verblunsky coefficients. Notice that in this case, $\Phi_n(z) = z^n$, $n \geqslant 0$, as well as $\mathbf{k}_n = 1$, $n \geqslant 0$. Thus,

$$K_{n-1}^{(0,l)}(z,0) = l!z^l, \quad 0 \leqslant l \leqslant n-1.$$

So, $\mathbb{K}_{n-1}(z,0) = \left[\boldsymbol{m}z, \boldsymbol{m}z^2 + \overline{\boldsymbol{m}}z^0, \ldots, \boldsymbol{m}z^{n-1} + \overline{\boldsymbol{m}}z^{n-3}, \overline{\boldsymbol{m}}z^{n-2}\right]^T$ and $\boldsymbol{\Phi}_n(0) = \mathbf{0}^T$, $n \geqslant 1$. On the other hand,

$$K_{n-1}^{(s,l)}(0,0) = \begin{cases} s!l! & \text{if} \quad s = l, \\ 0 & \text{otherwise,} \end{cases}$$

and, therefore,

$$\begin{aligned} a_{s,l} &= \frac{K_{n-1}^{(s,l+1)}(0,0)}{(l)!(l+1)!} = \delta_{s,s-1}, \quad b_{s,l} = \boldsymbol{m}\frac{K_{n-1}^{(s,l+1)}(0,0)}{(l)!(l+1)!} + \overline{\boldsymbol{m}}\frac{K_{n-1}^{(s,l-1)}(0,0)}{(l)!(l-1)!} = \boldsymbol{m}\delta_{s,s-1} + \overline{m}\delta_{s,s+1}, \\ c_{s,l} &= \frac{K_{n-1}^{(s,l-1)}(0,0)}{(l)!(l-1)!} = \delta_{s,s+1}, \end{aligned}$$

where $\delta_{s,l}$ is the Kronecker's delta. Thus,

$$\mathbf{I}_n + \mathbf{S}_n = \begin{bmatrix} 1 & \overline{\boldsymbol{m}} & & & \\ \boldsymbol{m} & 1 & \ddots & & \\ & \ddots & \ddots & \overline{\boldsymbol{m}} & \\ & & \boldsymbol{m} & 1 & \overline{\boldsymbol{m}} \\ & & & \boldsymbol{m} & 1 \end{bmatrix}, \quad n \geqslant 2.$$

Notice that for $n \geqslant 2$, $\mathbf{I}_n + \mathbf{S}_n$ is $\widetilde{\mathbf{T}}_{n-1}$, the $n \times n$ Toeplitz matrix associated with $\mathcal{L}_1$. We thus need to establish the conditions on m for $\widetilde{\mathbf{T}}_{n-1}$ be non-singular. Since $\widetilde{\mathbf{T}}_{n-1}$ is hermitian, their eigenvalues $\{\lambda_k\}_{k=1}^n$ are real numbers. Moreover, $\widetilde{\mathbf{T}}_{n-1}$ is quasi-definite if and only if $\lambda_k \neq 0$, for every $1 \leqslant k \leqslant n$ Horn and Johnson [1990]. From Theorem 2.4 in Böttcher and Grudsky [2005], the eigenvalues of $\widetilde{\mathbf{T}}_{n-1}$ are

$$\lambda_k = 1 + 2|\boldsymbol{m}|\cos\frac{\pi k}{n+1}, \quad k = 1,\ldots,n.$$

Thus, $\mathcal{L}_1$ is a quasi-definite linear functional if and only if

$$|\boldsymbol{m}| \neq -\left(2\cos\frac{\pi k}{n+1}\right)^{-1}, \quad k = 1,\ldots,n, \quad n \geq 1,$$

or, equivalently,

$$\frac{\pi k}{\cos^{-1}\left(-\frac{1}{2|\boldsymbol{m}|}\right)} \notin \mathbb{N}. \tag{4.42}$$

Assuming that (4.42) holds, $\{\Psi_n\}_{n\geqslant 0}$ can be obtained using (4.36), since all elements are known. Since $\mathbb{K}_{n-1}(0,0) = (0,\overline{\boldsymbol{m}},0,\ldots,0)^T$ and $\mathbf{W}_n = [0,\ldots,0,-\overline{\boldsymbol{m}}(n-1)!)]$, the sequence of Verblunsky coefficients can be computed using (A.1.2). It is not difficult to see that

$$\Psi_n(0) = -\overline{\boldsymbol{m}}^2(n-1)!\ell_{n,2},$$

where $\ell_{i,j} = (\mathbf{I}_n + \mathbf{S}_n)^{-1}_{i,j}$. An explicit expression for $\ell_{n,2}$ can be obtained using the method described in Usmani [1994]. Indeed,

$$\ell_{n,2} = \frac{(-1)^{n+2}\boldsymbol{m}^{n-2}}{\theta_n}$$

where θ_n is the solution of the recurrence relation

$$\theta_i = \theta_{i-1} - |\boldsymbol{m}|^2\theta_{i-2}, \qquad i = 2,\ldots,n,$$

with initial conditions $\theta_0 = \theta_1 = 1$. Thus,

$$\theta_n = \left(2^{-(n+1)} + \frac{2^{-(n+1)}}{\sqrt{1-|\boldsymbol{m}|^2}}\right)\left(\left(1-\sqrt{1-|\boldsymbol{m}|^2}\right)^n + \left(1+\sqrt{1-|\boldsymbol{m}|^2}\right)^n\right),$$

and therefore we get

$$\Psi_n(0) = \frac{(-1)^{n+3}\boldsymbol{m}^{n-2}\overline{\boldsymbol{m}}(n-1)!}{\left(2^{-(n+1)} + \frac{2^{-(n+1)}}{\sqrt{1-|\boldsymbol{m}|^2}}\right)\left(\left(1-\sqrt{1-|\boldsymbol{m}|^2}\right)^n + \left(1+\sqrt{1-|\boldsymbol{m}|^2}\right)^n\right)}. \tag{4.43}$$

From (4.43) notice that $|\Psi_n(0)|$ grows as n increases, and it grows faster for values $|m| > 1$. Using small values of $\boldsymbol{m}$, the first Verblunsky coefficients are small (close to zero), but then begin to grow as n increases. Since (4.43) is an increasing function, we deduce that the functional $\mathcal{L}_1$ defined by (4.41), for small values of $\boldsymbol{m}$, is quasi-definite.

Chapter 5

Spectral transformations associated with mass points

...then we get a μ (measure) for which mass points do not attract zeros ...

— E. B. Saff and V. Totik. Saff and Totik [1995]

In Chapter 4 we consider the addition of a Lebesgue measure to the hermitian linear functional $\mathcal{L}$ defined in (2.22). In other words, we perturb the main diagonal of the Toeplitz matrix (2.23). The generalization of the previous perturbations to affect any sub-diagonal of the Toeplitz matrix (2.23) is also considered. As we see, they are linear spectral transformations of the corresponding C-function (2.38).

One of the goals of this chapter is to show two new examples of linear spectral transformations associated with the first derivative of a complex Dirac's linear functional. The first one appears when the support of the Dirac's linear functional is a point on the unit circle. The second one corresponds to a Dirac's linear functional supported in two symmetric points with respect to the unit circle. Necessary and sufficient conditions for the quasi-definiteness of the new linear functional are obtained. Outer relative asymptotics for the new sequence of monic orthogonal polynomials in terms of the original ones are studied. We also prove that this spectral transformation can be decomposed as an iteration of particular cases of the canonical spectral transformations (2.43) and (2.44).

The last part of this chapter is devoted to the study of a relevant family of orthogonal polynomials associated with perturbations of the original orthogonality measure by means of mass points: discrete Sobolev orthogonal polynomials. We compare the discrete Sobolev orthogonal polynomials with the original ones. Finally, we analyze the behavior of their zeros and provide some numerical examples to illustrate it. An analogous inner product for measures supported on the real line is study in Appendix A.

5.1 Adding the derivative of a Dirac's delta

Let $\mathcal{L}$ be a hermitian linear functional given by (2.22). Its derivative $D\mathcal{L}$ Tasis [1989] is defined by

$$\langle D\mathcal{L}, f\rangle = -i\langle \mathcal{L}, zf'(z)\rangle, \quad f \in \Lambda.$$

In this section we first consider a perturbation of a linear functional $\mathcal{L}$ by the addition of a derivative of a Dirac's delta, i.e.,

$$\langle \mathcal{L}_1, f\rangle = \langle \mathcal{L}, f\rangle + m\langle D\delta_\alpha, f\rangle, \quad m \in \mathbb{R}, \quad |\alpha| = 1. \tag{5.1}$$

Let $\mathcal{L}_U$ be a linear functional such that

$$\langle \mathcal{L}_U, f\rangle = \langle \mathcal{L}, f\rangle + mf(\alpha), \quad m \in \mathbb{R}, \quad |\alpha| = 1.$$

We say that $\mathcal{L}_U$ is the Uvarov spectral transformation of the linear functional $\mathcal{L}$ Daruis et al. [2007]. The connection between the corresponding sequences of monic orthogonal polynomials as well as the associated GGT matrices using LU and QR factorization has been studied in Daruis et al. [2007]. The iteration of Uvarov transformations has been considered in Geronimus [1954]; Li and Marcellán [1999]; see also Appendix B. Asymptotic properties for the corresponding sequences of orthogonal polynomials have been studied in Wong [2009]. Notice that the addition of a Dirac's delta derivative (on a point of the unit circle) to a linear functional can be considered as the limit case of two Uvarov spectral transformations with equal masses and opposite sign, located on two nearby points on the unit circle $\alpha_1 = e^{i\theta_1}$ and $\alpha_2 = e^{i\theta_2}$, $0 \leqslant \theta_1, \theta_2 \leqslant 2\pi$, when $\theta_1 \to \theta_2$, but the difficulties to deal with them yield a different approach.

5.1.1 Mass point on the unit circle

In terms of the associated bilinear functional (5.1) becomes

$$\langle f, g\rangle_{\mathcal{L}_1} = \langle f, g\rangle_{\mathcal{L}} - im\Big(\alpha f'(\alpha)\overline{g(\alpha)} - \overline{\alpha} f(\alpha)\overline{g'(\alpha)}\Big). \tag{5.2}$$

In the next theorem we obtain necessary and sufficient conditions for $\mathcal{L}_1$ to be a quasi-definite linear functional, as well as an expression for its corresponding family of orthogonal polynomials.

Theorem 5.1.1. *Let us assume $\mathcal{L}$ is a quasi-definite linear functional and denote by $\{\Phi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Let consider $\mathcal{L}_1$ as in (5.2). Then, the following statements are equivalent:*

i) *$\mathcal{L}_1$ is quasi-definite.*

ii) *The matrix $\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)$, with*

$$\mathbb{K}_{n-1}(\alpha,\alpha) = \begin{bmatrix} K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) \\ K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha) \end{bmatrix}, \quad \mathbf{D}(\alpha) = \begin{bmatrix} 0 & -i\alpha \\ i\alpha^{-1} & 0 \end{bmatrix},$$

is non-singular, and

$$\mathbf{k}_n + m\mathbf{\Phi}_n(\alpha)^H \left(\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)\right)^{-1} \mathbf{\Phi}_n(\alpha) \neq 0, \quad n \geqslant 1. \tag{5.3}$$

Furthermore, the sequence $\{\Psi_n\}_{n\geqslant 0}$ *of monic orthogonal polynomials associated with* $\mathcal{L}_1$ *is given by*

$$\Psi_n(z) = \Phi_n(z) - m\begin{bmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha) \end{bmatrix}^T \left(\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)\right)^{-1} \mathbf{\Phi}_n(\alpha), \tag{5.4}$$

where $\mathbf{\Phi}_n(\alpha) = \left[\Phi_n(\alpha), \Phi_n'(\alpha)\right]^T$.

Proof. Assume $\mathcal{L}_1$ is quasi-definite and denote by $\{\Psi_n\}_{n\geqslant 0}$ its corresponding family of monic orthogonal polynomials. Let us consider the Fourier expansion

$$\Psi_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\Phi_k(z),$$

where, for $n \geq 1$,

$$\lambda_{n,k} = \frac{\langle \Psi_n(z), \Phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_k} = \frac{im\left(\alpha\Psi_n'(\alpha)\overline{\Phi_k(\alpha)} - \overline{\alpha}\Psi_n(\alpha)\overline{\Phi_k'(\alpha)}\right)}{\mathbf{k}_k}, \quad 0 \leq k \leq n-1.$$

Thus,

$$\begin{aligned} \Psi_n(z) &= \Phi_n(z) + \sum_{k=0}^{n-1} \frac{im\left(\alpha\Psi_n'(\alpha)\overline{\Phi_k(\alpha)} - \overline{\alpha}\Psi_n(\alpha)\overline{\Phi_k'(\alpha)}\right)}{\mathbf{k}_k}\Phi_k(z), \\ &= \Phi_n(z) + im\left(\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \overline{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right). \end{aligned} \tag{5.5}$$

Taking the derivative with respect to z in the previous expression and evaluating at $z = \alpha$, we obtain the linear system

$$\Psi_n^{(i)}(\alpha) = \Phi_n^{(i)}(\alpha) + im\left(\alpha\Psi_n'(\alpha)K_{n-1}^{(i,0)}(\alpha,\alpha) - \overline{\alpha}\Psi_n(\alpha)K_{n-1}^{(i,1)}(\alpha,\alpha)\right), \quad i = 0, 1,$$

which yields

$$\begin{bmatrix} \Phi_n(\alpha) \\ \Phi_n'(\alpha) \end{bmatrix} = \begin{bmatrix} 1 + im\overline{\alpha}K_{n-1}^{(0,1)}(\alpha,\alpha) & -im\alpha K_{n-1}(\alpha,\alpha) \\ im\overline{\alpha}K_{n-1}^{(1,1)}(\alpha,\alpha) & 1 - im\alpha K_{n-1}^{(1,0)}(\alpha,\alpha) \end{bmatrix} \begin{bmatrix} \Psi_n(\alpha) \\ \Psi_n'(\alpha) \end{bmatrix},$$

and denoting $\mathbf{Q} = [Q, Q']^T$, we get

$$\mathbf{\Phi}_n(\alpha) = \left(\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\alpha)\right)\mathbf{\Psi}_n(\alpha).$$

Thus, the necessary condition for regularity is that $\mathbf{I}_2 + m\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\alpha)$ must be non-singular. Taking into account $\mathbf{D}^{-1}(\alpha) = \mathbf{D}(\alpha)$ we have the first part of our statement. Furthermore, from (5.5),

$$\begin{aligned}\Psi_n(z) &= \Phi_n(z) + m\left(K_{n-1}(z,\alpha), K_{n-1}^{(0,1)}(z,\alpha)\right)\begin{bmatrix} 0 & i\alpha \\ -i\overline{\alpha} & 0 \end{bmatrix}\begin{bmatrix} \Psi_n(\alpha) \\ \Psi_n'(\alpha) \end{bmatrix} \\ &= \Phi_n(z) - m\begin{bmatrix} K_{n-1}(z,\alpha) \\ K_{n-1}^{(0,1)}(z,\alpha) \end{bmatrix}^T \left(\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)\right)^{-1} \mathbf{\Phi}_n(\alpha).\end{aligned}$$

This yields (5.4). Conversely, if $\{\Psi_n\}_{n\geqslant 0}$ is given by (5.5), then, for $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}\langle \Psi_n, \Psi_k \rangle_{\mathcal{L}_1} &= \left\langle \Phi_n(z) + im\left(\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \overline{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right), \Psi_k(z)\right\rangle_{\mathcal{L}_1} \\ &= \left\langle \Phi_n(z) + im\left(\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \overline{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right), \Psi_k(z)\right\rangle_{\mathcal{L}} \\ &\quad - im\left(\alpha\Psi_n'(\alpha)\overline{\Psi_k(\alpha)} - \overline{\alpha}\Psi_n(\alpha)\overline{\Psi_k'(\alpha)}\right) = 0.\end{aligned}$$

On the other hand, for $n \geqslant 1$,

$$\begin{aligned}\widetilde{\mathbf{k}}_n &= \langle \Psi_n(z), \Psi_n(z) \rangle_{\mathcal{L}_1} = \langle \Psi_n(z), \Phi_n(z) \rangle_{\mathcal{L}_1} \\ &= \left\langle \Phi_n(z) + im\left(\alpha\Psi_n'(\alpha)K_{n-1}(z,\alpha) - \overline{\alpha}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\alpha)\right), \Phi_n(z)\right\rangle_{\mathcal{L}} \\ &\quad - im\left(\alpha\Psi_n'(\alpha)\overline{\Phi_n(\alpha)} - \overline{\alpha}\Psi_n(\alpha)\overline{\Phi_n'(\alpha)}\right) \\ &= \mathbf{k}_n + m\mathbf{\Phi}_n(\alpha)^H \left(\mathbf{D}(\alpha) + m\mathbb{K}_{n-1}(\alpha,\alpha)\right)^{-1} \mathbf{\Phi}_n(\alpha) \neq 0,\end{aligned}$$

where we are using the reproducing property (2.33). As a conclusion, $\{\Psi_n\}_{n\geqslant 0}$ is the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_1$. □

Using the Christoffel-Darboux formula (2.32), another way to express (5.4) is the following.

Corollary 5.1.1. *Let $\{\Psi_n\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials associated to $\mathcal{L}_1$ defined as in (5.2). Then,*

$$(z-\alpha)^2\Psi_n(z) = A(z,n,\alpha)\Phi_n(z) + B(z,n,\alpha)\Phi_n^*(z), \tag{5.6}$$

where $A(z,n,\alpha)$ and $B(z,n,\alpha)$ are polynomials of degree 2 and 1, respectively, in the variable z, given by

$$\begin{aligned}A(z,n,\alpha) &= (z-\alpha)^2 - \frac{m\alpha}{\mathbf{k}_n\Delta_{n-1}}\left((Y_{1,1}\Phi_n(\alpha) + Y_{1,2}\Phi_n'(\alpha))\overline{\Phi_n(\alpha)}(z-\alpha)\right. \\ &\quad \left. + (Y_{2,1}\Phi_n(\alpha) + Y_{2,2}\Phi_n'(\alpha))(\Phi_n(\alpha)(z-\alpha) + \alpha\Phi_n(\alpha)z)\right), \\ B(z,n,\alpha) &= \frac{m\alpha}{\mathbf{k}_n\Delta_{n-1}}\left((Y_{1,1}\Phi_n(\alpha) + Y_{1,2}\Phi_n'(\alpha))\overline{\Phi_n^*(\alpha)}\right. \\ &\quad \left. + \left(Y_{2,1}\Phi_n(\alpha) + Y_{2,2}\Phi_n'(\alpha))(\overline{\Phi_n^*(\alpha)}'(z-\alpha) + \alpha\overline{\Phi_n^*(\alpha)}z\right)\right),\end{aligned}$$

where $Y_{1,1} = mK_{n-1}^{(1,1)}(\alpha,\alpha)$, $Y_{1,2} = im\alpha K_{n-1}^{(0,1)}(\alpha,\alpha)$, $Y_{2,1} = -im\overline{\alpha}K_{n-1}^{(1,0)}(\alpha,\alpha)$, $Y_{2,2} = m\alpha K_{n-1}(\alpha,\alpha)$, and Δ_{n-1} is the determinant of the matrix $\mathbf{D}(\alpha) + im\mathbb{K}_{n-1}(\alpha,\alpha)$.

5.1.1.1 Outer relative asymptotics

In this subsection we assume $\mathcal{L}$ is a positive definite linear functional, with an associated positive Borel measure σ. We are interested in the asymptotic behavior of the orthogonal polynomials associated with the addition of the derivative of a Dirac's delta on the unit circle given in (5.6). We assume that σ is regular in the sense of Stahl and Totik Stahl and Totik [1992], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1.$$

Regularity is a necessary and sufficient condition for the existence of n-th root asymptotics, i.e., $\lim_{n\to\infty} |\phi_n|^{1/n} < \infty$. It is easy to see that the existence of the ratio asymptotics $\lim_{n\to\infty} \phi_n/\phi_{n-1}$ implies the existence of the root asymptotics, and, in general, the converse is not true. Therefore, ratio asymptotics does not must hold for regular measures.

In particular, we study its ratio asymptotics with respect to $\{\Phi_n\}_{n\geqslant 0}$. First, we state some results that are useful in our study.

Theorem 5.1.2. Levin and Lubinsky [2007] *Let σ be a regular finite positive Borel measure supported on $(-\pi,\pi]$. Let $J \in (-\pi,\pi)$ be a compact subset such that σ is absolutely continuous in an open set containing J. Assume that σ' is positive and continuous at each point of J. Let i,j be non-negative integers. Then, uniformly for $\theta \in J$, $z = e^{i\theta}$,*

$$\lim_{n\to\infty} \frac{z^{i-j}}{n^{i+j}} \frac{K_n^{(i,j)}(z,z)}{K_n(z,z)} = \frac{1}{i+j+1}.$$

Lemma 5.1.1. Gonchar [1975] *Let f,g be two polynomials in $\mathbb{P}$ with degree at least j. Then*

$$\frac{f^{(j)}(z)}{g^{(j)}(z)} = \frac{g^{(j-1)}(z)}{g^{(j)}(z)}\left(\frac{f^{(j-1)}(z)}{g^{(j-1)}(z)}\right)' + \frac{f^{(j-1)}(z)}{g^{(j-1)}(z)}.$$

Using the previous lemma, the outer ratio asymptotics for the derivatives of orthonormal polynomials are deduced.

Lemma 5.1.2. *Assume $\mathcal{L}$ is a positive definite linear functional, with an associated positive Borel measure σ and denote by $\{\phi_n\}_{n\geqslant 0}$ its corresponding sequence of orthonormal polynomials. If $\sigma \in \mathcal{N}$, then uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$*

$$\lim_{n\to\infty} \frac{\phi_{n+1}^{(j)}(z)}{\phi_n^{(j)}(z)} = z, \qquad \lim_{n\to\infty} \frac{\phi_n^{(j)}(z)}{\phi_n^{(j+1)}(z)} = 0, \qquad j \geqslant 0.$$

Proof. According to Lemma 5.1.1,

$$\frac{\phi_{n+1}^{(j)}(z)}{\phi_n^{(j)}(z)} = \frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)}\left(\frac{\phi_{n+1}^{(j-1)}(z)}{\phi_n^{(j-1)}(z)}\right)' + \frac{\phi_{n+1}^{(j-1)}(z)}{\phi_n^{(j-1)}(z)}. \tag{5.7}$$

Using induction in j, we get uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$,

$$\lim_{n\to\infty}\left(\frac{\phi_{n+1}^{(j-1)}(z)}{\phi_n^{(j-1)}(z)}\right)' = 1, \quad \lim_{n\to\infty}\frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)} = 0.$$

Therefore, if n tends to infinity in (5.7), the result follows. ◻

Corollary 5.1.2. *If $\sigma \in \mathcal{N}$, then uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$*

$$\lim_{n\to\infty}\frac{\phi_n^{*(j)}(z)}{\phi_n^{(j)}(z)} = 0, \quad \lim_{n\to\infty}\frac{K_{n-1}^{(l,r)}(z,y)}{\phi_n^{(i)}(z)\overline{\phi_n^{(j)}(y)}} = 0, \quad 0 \leqslant l < i, 0 \leqslant r < j.$$

Proof. From Lemma 5.1.1, we have

$$\frac{\phi_n^{*(j)}(z)}{\phi_n^{(j)}(z)} = \frac{\phi_n^{(j-1)}(z)}{\phi_n^{(j)}(z)}\left(\frac{\phi_n^{*(j-1)}(z)}{\phi_n^{(j-1)}(z)}\right)' + \frac{\phi_n^{*(j-1)}(z)}{\phi_n^{(j-1)}(z)}. \tag{5.8}$$

Using a similar argument as in the proof of the previous lemma, the first statement follows. The second statement is a straightforward consequence of the first part of this corollary and Lemma 5.1.2. ◻

Theorem 5.1.3. *Let $\mathcal{L}$ be a positive definite linear functional, whose associated measure σ satisfies the conditions of Theorem 5.1.2. Let $\{\Psi_n\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials associated with $\mathcal{L}_1$ defined as in (5.2). Then, uniformly in $\mathbb{C}\setminus\overline{\mathbb{D}}$,*

$$\lim_{n\to\infty}\frac{\Psi_n(z)}{\Phi_n(z)} = 1.$$

Proof. From the expression (5.6),

$$\frac{\Psi_n(z)}{\Phi_n(z)} = \frac{A(z,n,\alpha)}{(z-\alpha)^2} + \frac{B(z,n,\alpha)}{(z-\alpha)^2}\frac{\Phi_n^*(z)}{\Phi_n(z)}.$$

Since, for $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$ by Corollary 5.1.2,

$$\lim_{n\to\infty}\frac{\Phi_n^*(z)}{\Phi_n(z)} = 0,$$

it suffices to show that, for $|\alpha| = 1$,

$$\lim_{n\to\infty}\frac{A(z,n,\alpha)}{(z-\alpha)^2} = 1.$$

Notice that $\lim_{n\to\infty}\Phi_n(\alpha) = O(1)$, $\lim_{n\to\infty}\Phi_n'(\alpha) = O(n)$, $\lim_{n\to\infty}\Phi_n^*(\alpha) = O(1)$, $\lim_{n\to\infty}\Phi_n^{*'}(\alpha) = O(n)$, and $\lim_{n\to\infty}K_n(\alpha,\alpha) = O(n)$.

On the other hand, dividing the numerator and denominator of $\dfrac{A(z,n,\alpha)}{(z-\alpha)^2} - 1$ by $n^2K_{n-1}(\alpha,\alpha)$, and using Theorem 5.1.2, we obtain

$$\lim_{n\to\infty} \frac{\Phi_n(\alpha)Y_{2,1}}{n^2K_{n-1}(\alpha,\alpha)} = O(1/n), \qquad \lim_{n\to\infty} \frac{\Phi_n'(\alpha)Y_{2,2}}{n^2K_{n-1}(\alpha,\alpha)} = O(1/n),$$

$$\lim_{n\to\infty} \frac{\Phi_n(\alpha)Y_{1,1}}{n^2K_{n-1}(\alpha,\alpha)} = O(1), \qquad \lim_{n\to\infty} \frac{\Phi_n'(\alpha)Y_{1,2}}{n^2K_{n-1}(\alpha,\alpha)} = O(1),$$

so that the numerator of $\dfrac{A(z,n,\alpha)}{(z-\alpha)^2} - 1$ behaves as $O(1)$. Similarily, one can shows that the denominator behaves as $O(n)$ and, therefore,

$$\lim_{n\to\infty} \frac{A(z,n,\alpha)}{(z-\alpha)^2} = 1.$$

The same arguments can be applied to $B(z,n,\alpha)$, what ensures the result. □

Example: Lebesgue polynomials

We now study one example that illustrates the behavior of the Verblunsky coefficients associated with the Lebesgue measure $d\sigma(\theta) = \dfrac{d\theta}{2\pi}$ and to the perturbation (5.2) given by

$$d\widetilde{\sigma}(\theta) = \frac{d\theta}{2\pi} + m\delta_\alpha',$$

where $m \in \mathbb{R}$ and $|\alpha| = 1$. It is very well known that $\Phi_n(z) = z^n$ is the n-th monic orthogonal polynomial with respect to $d\sigma$, and thus $\Psi_n(z)$, the n-th monic orthogonal polynomial with respect to $d\widetilde{\sigma}$, can be obtained using (5.6). Indeed, evaluating these polynomials at $z = 0$, for the special case $\alpha = 1$, is not difficult to show that

$$\Psi_n(0) = \left(\frac{n(n-1)(n+1)}{6} - \frac{in}{m}\right)\left(\frac{n^2(n-1)(n+1)}{12} - \frac{1}{m^2}\right)^{-1}. \tag{5.9}$$

From the last expression we are able to obtain the regularity condition in terms of the mass, by setting $|\Psi_n(0)| \neq 1$, $n \geqslant 1$. Notice that $|\Psi_n(0)| \to 0$, as it can be seen from (5.9). Thus, there exists a non-negative integer n_0, depending on m, such that $|\Psi_n(0)| < 1$ for $n \geqslant n_0$, but some of the preceding Verblunsky coefficients will be of modulus greater than 1, destroying the positivity of the perturbed functional. Indeed, from (5.3), we obtain that the positivity condition for this perturbation is

$$m^2 < \frac{12}{n^2(n^2-1)}, \quad n \geqslant 2.$$

Since the right side is a positive monotonic decreasing sequence, we only have a positive definite case if $m = 0$.

5.1.2 Mass points outside the unit circle

Now, consider a hermitian linear functional $\mathcal{L}_2$ such that its associated bilinear functional satisfies

$$\langle f,g\rangle_{\mathcal{L}_2} = \langle f,g\rangle_{\mathcal{L}} + im\left(\alpha^{-1}f(\alpha)\overline{g'(\overline{\alpha}^{-1})} - \alpha f'(\alpha)\overline{g(\overline{\alpha}^{-1})}\right) + i\overline{m}\left(\overline{\alpha}f(\overline{\alpha}^{-1})\overline{g'(\alpha)} - \overline{\alpha}^{-1}p'(\overline{\alpha}^{-1})\overline{q(\alpha)}\right),$$

with $m,\alpha \in \mathbb{C}$, $|\alpha| \neq 0$, and $|\alpha| \neq 1$. As in the previous section, we are interested in the regularity conditions for this linear functional and the corresponding family of orthogonal polynomials. Assuming that $\mathcal{L}_2$ is a quasi-definite linear functional and following the method used in the proof of Theorem 5.1.1, we get

$$\begin{aligned}\Psi_n(z) = \Phi_n(z) &+ im\left(\alpha\Psi_n'(\alpha)K_{n-1}(z,\overline{\alpha}^{-1}) - \alpha^{-1}\Psi_n(\alpha)K_{n-1}^{(0,1)}(z,\overline{\alpha}^{-1})\right)\\ &+ i\overline{m}\left(\overline{\alpha}^{-1}\Psi_n'(\overline{\alpha}^{-1})K_{n-1}(z,\alpha) - \overline{\alpha}\Psi_n(\overline{\alpha}^{-1})K_{n-1}^{(0,1)}(z,\alpha)\right).\end{aligned} \tag{5.10}$$

Evaluating the above expression and its first derivative in α and $\overline{\alpha}^{-1}$, we get the following linear systems

$$\begin{aligned}\begin{bmatrix}\Phi_n(\alpha)\\ \Phi_n'(\alpha)\end{bmatrix} &= \begin{bmatrix}1+im\alpha^{-1}K_{n-1}^{(0,1)}(\alpha,\overline{\alpha}^{-1}) & -im\alpha K_{n-1}(\alpha,\overline{\alpha}^{-1})\\ im\alpha^{-1}K_{n-1}^{(1,1)}(\alpha,\overline{\alpha}^{-1}) & 1-im\alpha K_{n-1}^{(1,0)}(\alpha,\overline{\alpha}^{-1})\end{bmatrix}\begin{bmatrix}\Psi_n(\alpha)\\ \Psi_n'(\alpha)\end{bmatrix}\\ &\quad+\begin{bmatrix}i\overline{m}\overline{\alpha}K_{n-1}^{(0,1)}(\alpha,\alpha) & -i\overline{m}\overline{\alpha}^{-1}K_{n-1}(\alpha,\alpha)\\ i\overline{m}\overline{\alpha}K_{n-1}^{(1,1)}(\alpha,\alpha) & -i\overline{m}\overline{\alpha}^{-1}K_{n-1}^{(1,0)}(\alpha,\alpha)\end{bmatrix}\begin{bmatrix}\Psi_n(\overline{\alpha}^{-1})\\ \Psi_n'(\overline{\alpha}^{-1})\end{bmatrix},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\begin{bmatrix}\Phi_n(\overline{\alpha}^{-1})\\ \Phi_n'(\overline{\alpha}^{-1})\end{bmatrix} &= \begin{bmatrix}im\alpha^{-1}K_{n-1}^{(0,1)}(\overline{\alpha}^{-1},\overline{\alpha}^{-1}) & -im\alpha K_{n-1}(\overline{\alpha}^{-1},\overline{\alpha}^{-1})\\ im\alpha^{-1}K_{n-1}^{(1,1)}(\overline{\alpha}^{-1},\overline{\alpha}^{-1}) & -im\alpha K_{n-1}^{(1,0)}(\overline{\alpha}^{-1},\overline{\alpha}^{-1})\end{bmatrix}\begin{bmatrix}\Psi_n(\alpha)\\ \Psi_n'(\alpha)\end{bmatrix}\\ &\quad+\begin{bmatrix}1+i\overline{m}\overline{\alpha}K_{n-1}^{(0,1)}(\overline{\alpha}^{-1},\alpha) & -i\overline{m}\overline{\alpha}^{-1}K_{n-1}(\overline{\alpha}^{-1},\alpha)\\ i\overline{m}\overline{\alpha}K_{n-1}^{(1,1)}(\overline{\alpha}^{-1},\alpha) & 1-i\overline{m}\overline{\alpha}^{-1}K_{n-1}^{(1,0)}(\overline{\alpha}^{-1},\alpha)\end{bmatrix}\begin{bmatrix}\Psi_n(\overline{\alpha}^{-1})\\ \Psi_n'(\overline{\alpha}^{-1})\end{bmatrix},\end{aligned} \tag{5.12}$$

which yields into the system of 4 linear equations with 4 unknowns

$$\begin{bmatrix}\mathbf{\Phi}_n(\alpha)\\ \mathbf{\Phi}_n(\overline{\alpha}^{-1})\end{bmatrix} = \begin{bmatrix}\mathbf{I}_2+m\mathbb{K}_{n-1}(\alpha,\overline{\alpha}^{-1})\mathbf{D}(\alpha) & \overline{m}\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\overline{\alpha}^{-1})\\ m\mathbb{K}_{n-1}(\overline{\alpha}^{-1},\overline{\alpha}^{-1})\mathbf{D}(\alpha) & \mathbf{I}_2+\overline{m}\mathbb{K}_{n-1}(\overline{\alpha}^{-1},\alpha)\mathbf{D}(\overline{\alpha}^{-1})\end{bmatrix}\begin{bmatrix}\mathbf{\Psi}_n(\alpha)\\ \mathbf{\Psi}_n(\overline{\alpha}^{-1})\end{bmatrix},$$

where $(\mathbf{Q},\mathbf{R})^T = (Q,Q',R,R')^T$. Thus, in order $\mathcal{L}_2$ to be a quasi-definite linear functional, we need that the 4×4 matrix defined as above must be non-singular. On the other hand,

$$\begin{bmatrix}\mathbf{\Psi}_n(\alpha)\\ \mathbf{\Psi}_n(\overline{\alpha}^{-1})\end{bmatrix} = \begin{bmatrix}\mathbf{I}_2+m\mathbb{K}_{n-1}(\alpha,\overline{\alpha}^{-1})\mathbf{D}(\alpha) & \overline{m}\mathbb{K}_{n-1}(\alpha,\alpha)\mathbf{D}(\overline{\alpha}^{-1})\\ m\mathbb{K}_{n-1}(\overline{\alpha}^{-1},\overline{\alpha}^{-1})\mathbf{D}(\alpha) & \mathbf{I}_2+\overline{m}\mathbb{K}_{n-1}(\overline{\alpha}^{-1},\alpha)\mathbf{D}(\overline{\alpha}^{-1})\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{\Phi}_n(\alpha)\\ \mathbf{\Phi}_n(\overline{\alpha}^{-1})\end{bmatrix}.$$

As a consequence, from (5.10), we get

$$\Psi_n(z) = \Phi_n(z) - m\begin{bmatrix}K_{n-1}(z,\overline{\alpha}^{-1})\\ K_{n-1}^{(0,1)}(z,\overline{\alpha}^{-1})\end{bmatrix}^T \mathbf{D}(\alpha)\mathbf{\Psi}_n(\alpha) - \overline{m}\begin{bmatrix}K_{n-1}(z,\alpha)\\ K_{n-1}^{(0,1)}(z,\alpha)\end{bmatrix}^T \mathbf{D}(\overline{\alpha}^{-1})\mathbf{\Psi}_n(\overline{\alpha}^{-1}), \tag{5.13}$$

where $\mathbf{\Psi}_n(\alpha)$ and $\mathbf{\Psi}_n(\overline{\alpha}^{-1})$ can be obtained from the above linear system. Assuming that the regularity conditions hold, and following the method used in the proof of Theorem 5.1.1, it is not difficult to show that $\{\Psi_n\}_{n\geqslant 0}$, defined as in (5.13), is the sequence of monic orthogonal polynomials with respect to $\mathcal{L}_2$.

5.1.2.1 Outer relative asymptotics

The following result was proved in Foulquié et al. [1999] using a different method, and it has been generalized for rectifiable Jordan curves or arcs in Branquinho et al. [2002]. We show here another proof of the same result.

Lemma 5.1.3. *If $\sigma \in \mathcal{N}$, then uniformly in $\mathbb{C}\backslash\overline{\mathbb{D}}$,*

$$\lim_{n\to\infty} \frac{K_{n-1}^{(i,j)}(z,y)}{\phi_n^{(i)}(z)\overline{\phi_n^{(j)}(y)}} = \frac{1}{z\overline{y}-1}, \qquad i,j \geqslant 0.$$

Proof. From the Christoffel-Darboux formula (2.32), we obtain

$$\phi_n^*(z)\overline{\phi_n^{*(j)}(y)} - \phi_n(z)\overline{\phi_n^{(j)}(y)} = (1-z\overline{y})K_{n-1}^{(0,j)}(z,y) - jzK_{n-1}^{(0,j-1)}(z,y),$$

and, as a consequence,

$$\begin{aligned}\phi_n^{*(i)}(z)\overline{\phi_n^{*(j)}(y)} - \phi_n^{(i)}(z)\overline{\phi_n^{(j)}(y)} &= (1-z\overline{y})K_{n-1}^{(i,j)}(z,y) - k\overline{y}K_{n-1}^{(i-1,j)}(z,y) \\ &\quad - j\left(zK_{n-1}^{(i,j-1)}(z,y) + kK_{n-1}^{(k-1,j-1)}(z,y)\right).\end{aligned}$$

Thus, dividing by $\phi_n^{(i)}(z)\overline{\phi_n^{(j)}(y)}$ and using Corollary 5.1.2 when n tends to infinity, the result follows. □

Remark 5.1.1. *Notice that*

$$\lim_{n\to\infty} \frac{K_{n-1}^{(j,j)}(\alpha,\alpha)}{|\phi_n^{(j)}(\alpha)|^2} = \frac{1}{|\alpha|^2-1}, \quad |\alpha|>1, \quad j\geqslant 1.$$

It is possible to obtain a generalization of Theorem 5.1.3 for the sequence of monic orthogonal polynomials associated with (5.10). As above, we can express (5.13) as in (5.6). Using the Christoffel-Darboux formula (2.32), we obtain

$$\Psi_n(z) = \left(1+\widetilde{A}(z,n,\alpha)\right)\Phi_n(z) + \widetilde{B}(z,n,\alpha)\Phi_n^*(z),$$

with

$$
\begin{aligned}
\widetilde{A}(z,n,\alpha) &= im\overline{\alpha}^{-1}\frac{\overline{\Phi_n'(\overline{\alpha}^{-1})}(1-\alpha^{-1}z)+z\overline{\Phi_n(\overline{\alpha}^{-1})}}{\mathbf{k}_n(1-\alpha^{-1})^2}\Psi_n(\alpha) - im\alpha\frac{\overline{\Phi_n(\overline{\alpha}^{-1})}}{\mathbf{k}_n(1-\alpha^{-1}z)}\Psi_n'(\alpha)\\
&+\overline{im}\alpha\frac{\overline{\Phi_n'(\alpha)}(1-\overline{\alpha}z)+z\overline{\Phi_n(\alpha)}}{\mathbf{k}_n(1-\overline{\alpha})^2}\Psi_n(\overline{\alpha}^{-1}) - \overline{im}\overline{\alpha}^{-1}\frac{\overline{\Phi_n(\alpha)}}{\mathbf{k}_n(1-\overline{\alpha}z)}\Psi_n'(\overline{\alpha}^{-1}),\\
\widetilde{B}(z,n,\alpha) &= im\alpha\frac{\overline{\Phi_n^*(\overline{\alpha}^{-1})}}{\mathbf{k}_n(1-\alpha^{-1}z)}\Psi_n'(\alpha) - im\overline{\alpha}^{-1}\frac{\overline{\Phi_n^{*\prime}(\overline{\alpha}^{-1})}(1-\alpha^{-1}z)+z\overline{\Phi_n^*(\overline{\alpha}^{-1})}}{\mathbf{k}_n(1-\alpha^{-1})^2}\Psi_n(\alpha)\\
&+\overline{im}\overline{\alpha}^{-1}\frac{\overline{\Phi_n^*(\alpha)}}{\mathbf{k}_n(1-\overline{\alpha}z)}\Psi_n'(\overline{\alpha}^{-1}) - \overline{im}\alpha\frac{\overline{\Phi_n^{*\prime}(\alpha)}(1-\overline{\alpha}z)+z\overline{\Phi_n^*(\alpha)}}{\mathbf{k}_n(1-\overline{\alpha})^2}\Psi_n(\overline{\alpha}^{-1}),
\end{aligned}
$$

where the values of $\Psi_n(\alpha), \Psi_n'(\alpha), \Psi_n(\overline{\alpha}^{-1})$, and $\Psi_n'(\overline{\alpha}^{-1})$ can be obtained by solving the 4×4 linear system shown above. Denoting the entries of the 2×2 matrices in (5.11) - (5.12) by $\{b_{i,j}\}, \{c_{i,j}\}, \{a_{i,j}\}$ and $\{d_{i,j}\}$, respectively, we get

$$
\begin{aligned}
\Psi_n(\alpha) &= \left(d_{1,1}\Phi_n(\alpha)+d_{1,2}\Phi_n'(\alpha)+c_{1,1}\Phi_n(\overline{\alpha}^{-1})+c_{1,2}\Phi_n'(\overline{\alpha}^{-1})\right)/\Delta,\\
\Psi_n'(\alpha) &= \left(d_{2,1}\Phi_n(\alpha)+d_{2,2}\Phi_n'(\alpha)+c_{2,1}\Phi_n(\overline{\alpha}^{-1})+c_{2,2}\Phi_n'(\overline{\alpha}^{-1})\right)/\Delta,\\
\Psi_n(\overline{\alpha}^{-1}) &= \left(a_{1,1}\Phi_n(\alpha)+a_{1,2}\Phi_n'(\alpha)+b_{1,1}\Phi_n(\overline{\alpha}^{-1})+b_{1,2}\Phi_n'(\overline{\alpha}^{-1})\right)/\Delta,\\
\Psi_n(\overline{\alpha}^{-1}) &= \left(a_{2,1}\Phi_n(\alpha)+a_{2,2}\Phi_n'(\alpha)+b_{2,1}\Phi_n(\overline{\alpha}^{-1})+b_{2,2}\Phi_n'(\overline{\alpha}^{-1})\right)/\Delta,
\end{aligned}
$$

where Δ is the determinant of the 4×4 matrix. To get the asymptotic result, it suffices to show that $\widetilde{A}(z,n,\alpha)\to 0$ and $\widetilde{B}(z,n,\alpha)\to 0$ as $n\to\infty$. First, notice that applying the corresponding derivatives to the Christoffel-Darboux formula (2.32), we obtain

$$
\begin{aligned}
K_{n-1}^{(0,1)}(z,y) &= \frac{\overline{\Phi_n^{*\prime}(y)}\Phi_n^*(z)-\overline{\Phi_n'(y)}\Phi_n(z)}{\mathbf{k}_n(1-\overline{y}z)}+\frac{zK_{n-1}(z,y)}{1-\overline{y}z},\\
K_{n-1}^{(1,0)}(z,y) &= \frac{\overline{\Phi_n^*(y)}\Phi_n^{*\prime}(z)-\overline{\Phi_n(y)}\Phi_n'(z)}{\mathbf{k}_n(1-\overline{y}z)}+\frac{\overline{y}K_{n-1}(z,y)}{1-\overline{y}z},\\
K_{n-1}^{(1,1)}(z,y) &= \frac{\overline{\Phi_n^{*\prime}(y)}\Phi_n^{*\prime}(z)-\overline{\Phi_n'(y)}\Phi_n'(z)}{\mathbf{k}_n(1-\overline{y}z)}+\frac{zK_{n-1}^{(1,0)}(z,y)+\overline{y}K_{n-1}^{(0,1)}(z,y)+K_{n-1}(z,y)}{1-\overline{y}z}.
\end{aligned}
$$

On the other hand, if $\mathcal{L}$ is positive definite, and its corresponding measure $\sigma\in\mathcal{N}$, then by Corollary 5.1.2 (see also Maté et al. [1987]) we have $\Phi_n(\alpha)=O(\alpha^n)$, $\Phi_n'(\alpha)=O(n\alpha^n)$, and

$$
\lim_{n\to\infty}\frac{\Phi_n(\alpha)}{\Phi_n^*(\alpha)}=0,\quad |\alpha|<1,\qquad \lim_{n\to\infty}\frac{\Phi_n^*(\alpha)}{\Phi_n(\alpha)}=0,\quad |\alpha|>1.
$$

Assume, without loss of generality, that $|\alpha|<1$. If $|\alpha|<1$ and $\sigma\in\mathcal{S}$, notice that $\Phi_n(\alpha)$ and $\Phi_n^*(\alpha)$ are $O(\alpha^n)$, then $\lim_{n\to\infty}K_n(\alpha,\alpha)<\infty$ and $K_n(\overline{\alpha}^{-1},\overline{\alpha}^{-1})=O(|\alpha|^{-2n})$, as well as $\overline{K_n(\alpha,\overline{\alpha}^{-1})}=K_n(\overline{\alpha}^{-1},\alpha)=O(n)$. Observe that, except for the entries containing $K_{n-1}(\alpha,\alpha)$ and their derivatives, all other entries of the 4×4 matrix diverge, and thus its determinant diverges much faster than any other term in the expressions for $\Psi_n(\alpha), \Psi'_n(\alpha), \Psi_n(\overline{\alpha}^{-1})$ and $\Psi'_n(\overline{\alpha}^{-1})$, so that $\widetilde{A}(z,n,\alpha) \to 0$ and $\widetilde{B}(z,n,\alpha) \to 0$ as n tends to ∞. As a consequence,

Theorem 5.1.4. *Let $\mathcal{L}$ be a positive definite linear functional, whose associated measure $\sigma \in \mathcal{S}$. Let $\{\Psi_n\}_{n\geqslant 0}$ the sequence of monic orthogonal polynomials associated to $\mathcal{L}_2$ defined as in (5.10). Then, uniformly in $\mathbb{C}\setminus\mathbb{T}$,*

$$\lim_{n\to\infty} \frac{\Psi_n(z)}{\Phi_n(z)} = 1.$$

5.1.3 C-functions and linear spectral transformations

First, we assume that $|\alpha| = 1$. Let us consider the moments associated with $\mathcal{L}_1$. Notice that $\widetilde{c}_0 = c_0$. For $k \geqslant 1$, we have $\widetilde{c}_k = \left\langle z^k, 1\right\rangle_{\mathcal{L}_1} = c_k - imk\alpha^k$. In a similar way, $\widetilde{c}_{-k} = c_{-k} + imk\overline{\alpha}^k$. Therefore,

$$\begin{aligned} F_1(z) &= \widetilde{c}_0 + 2\sum_{k=1}^{\infty} \widetilde{c}_{-k} z^k = c_0 + 2\sum_{k=1}^{\infty}(c_{-k} + imk\overline{\alpha}^k) z^k = F(z) + 2im\sum_{k=1}^{\infty} k\overline{\alpha}^k z^k \\ &= F(z) + \frac{2im\alpha}{z-\alpha} + \frac{2im\alpha^2}{(z-\alpha)^2}. \end{aligned}$$

This means that the resulting C-function is a perturbation of F by the addition of a rational function with a double pole at $z = \alpha$.

Now, we assume $|\alpha| > 1$, and let consider the moments associated with $\mathcal{L}_2$. Notice that $\widehat{c}_0 = c_0$. For $k \in \mathbb{N}$, we have from (5.10),

$$\widehat{c}_k = c_k - imk\alpha^k - \overline{im}k\overline{\alpha}^{-k}, \quad \widehat{c}_{-k} = c_{-k} + \overline{im}k\overline{\alpha}^k + imk\alpha^{-k},$$

and, as a consequence,

$$F_2(z) = \widehat{c}_0 + 2\sum_{k=1}^{\infty} \widehat{c}_{-k} z^k = F(z) - \frac{2im\alpha}{z-\alpha} + \frac{2im\alpha^2}{(z-\alpha)^2} - \frac{2\overline{im\alpha}^{-1}}{z-\overline{\alpha}^{-1}} + \frac{2\overline{im\alpha}^{-2}}{(z-\overline{\alpha}^{-1})^2}. \tag{5.14}$$

This means that the resulting C-function is a perturbation of the initial one by the addition of a rational function with two double poles at α and $\overline{\alpha}^{-1}$.

Connection with special linear transformations

We show that perturbations (5.10) can be expressed in terms of special cases of the spectral transformations (2.43) and (2.44). Let $\mathcal{F}_C(\alpha)$ and $\mathcal{F}_G(\alpha, m)$ be linear spectral transformations associated with the modification of the original functional $\mathcal{L}$ given by

$$\langle \mathcal{L}_C, f\rangle = \left\langle \mathcal{L}, |z-\alpha|^2 f(z)\right\rangle \tag{5.15}$$

and

$$\left\langle \mathcal{L}_G, |z-\alpha|^2 f(z)\right\rangle = \langle \mathcal{L}, f\rangle + m\delta_\alpha, \quad |\alpha| = 1, \quad m \in \mathbb{R}, \tag{5.16}$$

respectively Godoy and Marcellán [1991, 1993]. The polynomial coefficients associated with $\mathcal{F}_C(\alpha)$ and $\mathcal{F}_G(\alpha,m)$ are

$$A_C(z) = D_G(z) = (z-\alpha)(1-\overline{\alpha}z), \qquad D_C(z) = A_G(z) = z,$$
$$B_C(z) = -\overline{\alpha}c_0z^2 + (\alpha c_{-1} - \overline{\alpha}c_1)z + \alpha c_0, \qquad B_G(z) = \overline{\alpha}\widetilde{c}_0 z^2 + 2i\Im(q_0)z - \alpha\widetilde{c}_0,$$

where q_0 is a free parameter that depends on the mass used in (5.16). Now, consider the following product of transformations

$$\mathcal{F}_D = \mathcal{F}_{G_2}(\alpha,m_2)\circ\mathcal{F}_{G_1}(\alpha,m_1)\circ\mathcal{F}_{C_2}(\alpha)\circ\mathcal{F}_{C_1}(\alpha). \tag{5.17}$$

It is not difficult to show that F_D, the C-function associated with $\mathcal{F}_D$, is given by

$$\begin{aligned} F_D(z) &= F(z) + \frac{B_{C_1}(z)}{D_{G_1}(z)} + \frac{B_{G_2}(z)}{D_{G_2}(z)} + \frac{B_{C_2}(z)A_{G_1}(z)}{D_{G_1}(z)D_{G_2}(z)} + \frac{B_{G_1}(z)A_{G_2}(z)}{D_{G_1}(z)D_{G_2}(z)} \\ &= F(z) + \frac{B_{C_1}(z)+B_{G_2}(z)}{(z-\alpha)(1-\overline{\alpha}z)} + \frac{z(B_{C_2}(z)+B_{G_1}(z))}{(z-\alpha)^2(1-\overline{\alpha}z)^2}. \end{aligned}$$

Assuming that all transformations are normalized, i.e., all of the first moments are equal to 1, and denoting $K_1 = \alpha c_{-1} - \overline{\alpha}c_1 + 2i\Im(q_0^{(1)})$ and $K_2 = \alpha c_{-1} - \overline{\alpha}c_1 + 2i\Im(q_0^{(2)})$, where $q_0^{(1)}$ and $q_0^{(2)}$ are the free parameters associated with $\mathcal{F}_{G_1}$ and $\mathcal{F}_{G_2}$, respectively, we obtain

$$\begin{aligned} F_D(z) &= F(z) + \frac{K_2 z}{(z-\alpha)(1-\overline{\alpha}z)} + \frac{K_1 z^2}{(z-\alpha)^2(1-\overline{\alpha}z)^2} \\ &= F(z) + \frac{L_1}{(z-\alpha)} + \frac{L_2}{(z-\alpha)^2} + \frac{L_3}{(z-\overline{\alpha}^{-1})} + \frac{L_4}{(z-\overline{\alpha}^{-1})^2}, \end{aligned} \tag{5.18}$$

for some constants L_1, L_2, L_3, and L_4, satisfying

$$\begin{aligned} -\overline{\alpha}K_2 &= L_1 + L_3, \\ (1+|\alpha|^2)K_2 + K_1 &= -(\alpha+2\overline{\alpha}^{-1})L_1 + L_2 - (2\alpha+\overline{\alpha}^{-1})L_3 + L_4, \\ -\alpha K_2 &= (\overline{\alpha}^{-2} + 2\alpha\overline{\alpha}^{-1})L_1 - 2\overline{\alpha}^{-1}L_2 + (\alpha^2 + 2\overline{\alpha}^{-1}\alpha)L_3 - 2\alpha L_4, \\ 0 &= -\alpha\overline{\alpha}^{-2}L_1 + \overline{\alpha}^{-2}L_2 - \alpha^2\overline{\alpha}^{-1}L_3 + \alpha^2 L_4. \end{aligned}$$

Furthermore, comparing (5.14) and (5.18), we have $L_2 = -\alpha L_1$ and $L_4 = -\overline{\alpha}^{-1}L_3$. Solving the above system we arrive at

$$L_1 = \frac{\alpha|\alpha|^2}{1-|\alpha|^2}K_2, \quad L_3 = -\frac{\alpha}{1-|\alpha|^2}K_2,$$

and thus, we conclude that transformation (5.17) is equivalent to $\mathcal{F}_2(\alpha^{-1},m)$, the transformation associated with (5.10) , with

$$m = \frac{|\alpha|^2}{2i(1-|\alpha|^2)}K_2.$$

5.2 Non-standard inner products

In the last few years, some attention has been paid to the asymptotic properties of orthogonal polynomials with respect to non-standard inner products. In particular, the algebraic and analytic properties of orthogonal polynomials associated with a Sobolev inner product have attracted the interest of many researchers, see Marcellán and Ronveaux [2012] for an updated overview with more than 300 references.

A discrete Sobolev inner product in $\mathbb{C}\setminus\overline{\mathbb{D}}$ is given by

$$\langle f,g\rangle_S = \int_{\mathbb{T}} f(z)\overline{g(z)}d\sigma(z) + \mathbf{f}(Z)\,\mathbf{A}\,\mathbf{g}(Z)^H, \tag{5.19}$$

where

$$\mathbf{f}(Z) = \left(f(\alpha_1),\ldots,f^{(l_1)}(\alpha_1),\ldots,f(\alpha_m),\ldots,f^{(l_m)}(\alpha_m)\right),$$

$\mathbf{A}$ is an $M\times M$ positive semi-definite hermitian matrix, with $M = l_1+\ldots+l_m+m$, and $|\alpha_i|>1$, $i=1,\ldots,m$. Since $\mathbf{A}$ is positive semi-definite, the inner product (5.19) is positive definite. Therefore, there exists a sequence of polynomials $\{\psi_n\}_{n\geqslant 0}$,

$$\psi_n(z) = \gamma_n z^n + (\text{lower degree terms}), \qquad \gamma_n > 0,$$

which is orthonormal with respect to (5.19). We are interested in the outer relative asymptotic behavior of $\{\psi_n\}_{n\geqslant 0}$ with respect to the sequence $\{\phi_n\}_{n\geqslant 0}$ of orthonormal polynomials with respect to σ. We show that if $\sigma\in\mathcal{N}$ and $\mathbf{A}$ is positive definite, then this outer relative asymptotics follows. Similar results have been obtained for the case when the measure is supported on a bounded interval of the real line López et al. [1995]; Marcellán and Van Assche [1993].

5.2.1 Outer relative asymptotics

In Foulquié et al. [1999]; Li and Marcellán [1996]; Marcellán and Moral [2002], the relative asymptotic behavior of orthogonal polynomials with respect to a discrete Sobolev inner product on the unit circle was studied. In this section, we propose a slightly modified outline.

The nondiagonal structure of the matrix $\mathbf{A}$ makes the analysis of the situation much more difficult. First of all, let us prove an important result which gives precise information about the matrix $\mathbf{A}$.

Lemma 5.2.1. *The outer relative asymptotic behavior of orthogonal polynomials with respect to the inner product* (5.19) *does not depend on the matrix* $\boldsymbol{A}$.

Proof. Let $\left\{\widetilde{\psi}_n\right\}_{n\geqslant 0}$ be the sequence of orthonormal polynomials with respect to the inner product

$$\langle f,g\rangle_{\widetilde{S}} = \int_{\mathbb{T}} f(z)\overline{g(z)}d\sigma(z) + \mathbf{f}(Z)\,\mathbf{B}\,\mathbf{g}(Z)^H,$$

where $\boldsymbol{B}$ is an arbitrary positive definite hermitian matrix of order M. Expanding ψ_n in terms of $\{\phi_n\}_{n\geqslant 0}$,

we have

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\phi_k(z) \tag{5.20}$$

where

$$\lambda_{n,k} = \int_{\mathbb{T}} \psi_n(z)\overline{\phi(z)}d\sigma(z) = -\boldsymbol{\psi}_n(Z)\mathbf{A}\boldsymbol{\phi}_n(Z).$$

Substituting this expresion in (5.20), we obtain

$$\psi_n(z) = \frac{\gamma_n}{\kappa_n}\phi_n(z) - \boldsymbol{\psi}_n(Z)\ \mathbf{A}\ \mathbf{K}_n(z,Z)^T, \tag{5.21}$$

where $\mathbf{K}_n(z,Z) = (K_n(z,\alpha_1),\ldots,K_n^{(0,l_1)}(z,\alpha_1)),\ldots,K_n(z,\alpha_m),\ldots K_n^{(0,l_m)}(z,\alpha_m))$ and $K_n^{(i,j)}\ (z,y)$ denotes the i-th (resp. j-th) partial derivative of $K_n(z,y)$ with respect to the variable z (resp. y). In an analogous way, we get

$$\widetilde{\psi}_n(z) = \frac{\widetilde{\gamma}_n}{\kappa_n}\phi_n(z) - \widetilde{\boldsymbol{\psi}}_n(Z)\ \mathbf{B}\ \mathbf{K}_n(z,Z)^T, \tag{5.22}$$

where $\widetilde{\gamma}_n$ is the leading coefficient of ψ_n. From (5.21) and (5.22) and following the method used in the proof of Theorem 5.1.1, we get Foulquié et al. [1999]; Li and Marcellán [1996]

$$\frac{\widetilde{\gamma}_n}{\gamma_n}\frac{\widetilde{\psi}_n(z)}{\psi_n(z)} = \frac{\det(\mathbf{I}+\mathbf{A}\mathbb{T}_n)}{\det(\mathbf{I}+\mathbf{B}\mathbb{T}_n)}\frac{\det(\mathbf{I}+\mathbf{B}\mathbb{K}_n)}{\det(\mathbf{I}+\mathbf{A}\mathbb{K}_n)},$$

$$\left(\frac{\widetilde{\gamma}_n}{\gamma_n}\right)^2 = \frac{\det(\mathbf{I}+\mathbf{B}\mathbb{K}_n)}{\det(\mathbf{I}+\mathbf{A}\mathbb{K}_n)}\frac{\det(\mathbf{I}+\mathbf{A}\mathbb{K}_{n+1})}{\det(\mathbf{I}+\mathbf{B}\mathbb{K}_{n+1})},$$

where $\mathbb{K}_n$ is a positive definite matrix of order M, $n \geq M$, which can be described by blocks. The r, s block of $\mathbb{K}_n$ is the $(l_r+1)\times(l_s+1)$ matrix

$$\left(K_n^{(i,j)}(z_r,\overline{z_s})\right)_{i=0,\ldots,l_r}^{j=0,\ldots,l_s}, \qquad r,s = 0,\ldots,m.$$

$\mathbb{T}_n$ is obtained through the following equation $\mathbb{T}_n = \mathbb{K}_n + \mathbb{V}_n$, where $\mathbb{V}_n = -\dfrac{1}{\phi_n(z)}\mathbf{K}_n(z,Z)^T\boldsymbol{\phi}_n(Z)$. Since Foulquié et al. [1999]; Marcellán and Moral [2002]

$$\lim_{n\to\infty}\frac{\det(\mathbf{I}+\mathbf{A}\mathbb{K}_n)}{\det(\mathbf{I}+\mathbf{B}\mathbb{K}_n)} = \lim_{n\to\infty}\frac{\det(\mathbf{I}+\mathbf{A}\mathbb{T}_n)}{\det(\mathbf{I}+\mathbf{B}\mathbb{T}_n)} = \frac{\det\mathbf{A}}{\det\mathbf{B}},$$

we can deduce that

$$\lim_{n\to\infty}\frac{\widetilde{\gamma}_n}{\gamma_n}\frac{\widetilde{\psi}_n(z)}{\psi_n(z)} = 1, \quad \lim_{n\to\infty}\left(\frac{\widetilde{\gamma}_n}{\gamma_n}\right)^2 = 1,$$

and the lemma is proved. □

For the discrete Sobolev inner product with a single mass point associated with (5.19),

$$\langle f,g\rangle_{S_1} = \int_{\mathbb{T}} f(z)\overline{g(z)}d\sigma(z) + \lambda f^{(j)}(\alpha)\overline{g^{(j)}(\alpha)}, \quad |\alpha|>1, \tag{5.23}$$

we have

Lemma 5.2.2. *Let* $\{\psi_{n;1}\}_{n\geqslant 0}$, $\psi_{n;1} = \gamma_{n;1} z^n + $ *(lower degree terms) be the sequence of orthonormal polynomials with respect to* (5.23). *If* $\sigma \in \mathcal{N}$, *then*

$$\lim_{n\to\infty} \frac{\gamma_{n,1}}{\kappa_n} = \frac{1}{|\alpha|}.$$

Proof. From (5.21) we have

$$\psi_{n;1}(z) = \frac{\gamma_{n;1}}{\kappa_n}\phi_n(z) - \lambda\psi_{n;1}^{(j)}(\alpha)K_{n-1}^{(0,j)}(z,\alpha). \tag{5.24}$$

Taking derivatives in (5.24) and evaluating at $z = \alpha$, we get

$$\psi_{n;1}^{(j)}(\alpha) = \frac{\gamma_{n;1}/\kappa_n\, \phi_n^{(j)}(\alpha)}{1 + \lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}. \tag{5.25}$$

Thus, (5.25) yields

$$\left(\frac{\beta_n}{\alpha_n}\right)^2 = \frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)}.$$

Using the previous identity and Lemma 5.1.2,

$$\lim_{n\to\infty}\frac{\gamma_{n,1}^2}{\kappa_n^2} = \lim_{n\to\infty}\frac{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)}{1+\lambda K_n^{(j,j)}(\alpha,\alpha)} = \lim_{n\to\infty}\frac{|\phi_{n-1}^{(j)}(\alpha)|^2}{|\phi_n^{(j)}(\alpha)|^2} = \frac{1}{|\alpha|^2},$$

and the lemma is proved. □

Using the previous lemma, we prove the relative asymptotics in $\mathbb{C}\setminus\overline{\mathbb{D}}$.

Theorem 5.2.1. *If* $\sigma \in \mathcal{N}$, *then uniformly in* $\mathbb{C}\setminus\overline{\mathbb{D}}$

$$\lim_{n\to\infty}\frac{\psi_{n;1}(z)}{\phi_n(z)} = B(\alpha), \quad B(\alpha) = \frac{\overline{\alpha}(z-\alpha)}{|\alpha|(\overline{\alpha}z-1)}. \tag{5.26}$$

Proof. From (5.24), we have

$$\frac{\psi_{n;1}(z)}{\phi_n(z)} = \frac{\gamma_{n;1}}{\kappa_n} - \lambda\psi_{n;1}^{(j)}(\alpha)\overline{\phi_n^{(j)}(\alpha)}\frac{K_{n-1}^{(0,j)}(z,\alpha)}{\phi_n(z)\overline{\phi_n^{(j)}(\alpha)}}. \tag{5.27}$$

Using (5.25), we obtain

$$\lim_{n\to\infty}\lambda\psi_{n;1}^{(j)}(\alpha)\overline{\phi_n^{(j)}(\alpha)} = \left(|\alpha| - \frac{1}{|\alpha|}\right). \tag{5.28}$$

The outer relative asymptotics (5.26) follows letting n tends to infinity in (5.27), using Lemma 5.2.2, Lemma 5.1.3, and (5.28). □

From Theorem 5.2.1 we can see that the outer relative asymptotic behavior of orthogonal polynomials associated with (5.23) does not depend on the specific choice of j and λ.

Lemma 5.2.3. *$\sigma \in \mathcal{N}$, then $S_1 \in \mathcal{N}$.*

Proof. Assume, without loss of generality, that $j = 0$ and $\lambda = 1$. From (5.24) and (5.25) we get

$$\psi_{n;1}(z) = \frac{\gamma_{n;1}}{\kappa_n}\phi_n(z) - \frac{\phi_n(\alpha)}{1 + K_{n-1}(\alpha,\alpha)}K_{n-1}(z,\alpha). \tag{5.29}$$

The evaluation at $z = 0$ of this last expression yields

$$\frac{\psi_{n;1}(0)}{\gamma_{n;1}} = \frac{\phi_n(0)}{\kappa_n} - \frac{|\phi_n(\alpha)|^2}{1 + K_{n-1}(\alpha,\alpha)}\frac{K_{n-1}(0,\alpha)}{\gamma_{n;1}\overline{\phi_n(\alpha)}},$$

and using the Christoffel-Darboux formula (2.32), we obtain

$$\frac{K_{n-1}(0,\alpha)}{\gamma_{n;1}\,\overline{\phi_n(\alpha)}} = \frac{\kappa_n}{\gamma_{n;1}}\left(\frac{\overline{\phi_n^*(\alpha)}}{\overline{\phi_n(\alpha)}} - \frac{\phi_n(0)}{\kappa_n}\right). \tag{5.30}$$

From Corollary 5.1.2, under our conditions, the following limit holds $\lim\limits_{n\to\infty}\frac{\overline{\phi_n^*(\alpha)}}{\overline{\phi_n(\alpha)}} = 0$. Since $K_n(\alpha,\alpha)$ is an increasing sequence and $\lim\limits_{n\to\infty}\frac{1}{\phi_n(\alpha)} = 0$, applying the Stolz-Césaro criterion, we have

$$\lim_{n\to\infty}\frac{|\phi_n(\alpha)|^2}{1 + K_n(\alpha,\alpha)} = \left(1 - \frac{1}{|\alpha|^2}\right). \tag{5.31}$$

On the other hand, from (5.29) we can deduce the following identity

$$\frac{|\phi_n(\alpha)|^2}{1 + K_n(\alpha,\alpha)} = 1 - \frac{1 + K_{n-1}(\alpha,\alpha)}{1 + K_n(\alpha,\alpha)} = 1 - \left(\frac{\gamma_{n;1}}{\kappa_n}\right)^2. \tag{5.32}$$

Thus,

$$\lim_{n\to\infty}\frac{K_{n-1}(0,\alpha)}{\gamma_{n;1}\,\overline{\phi_n(\alpha)}} = 0$$

and the result follows. □

We are now in a position to sumarize the results obtained above, in the following statement.

Theorem 5.2.2. *Let $\{\psi_n\}_{n\geqslant 0}$ be the sequence of monic orthogonal polynomials associated with the inner product (5.19). Then, uniformly in $\mathbb{C}\setminus\mathbb{D}$,*

$$\lim_{n\to\infty}\frac{\psi_n(z)}{\phi_n(z)} = \prod_{i=1}^{m} B(\alpha_i)^{l_i+1}.$$

Proof. First of all, we prove the result for

$$\mathbf{f}(Z) = \mathbf{f}_m(Z) = \left(f^{(l_1)}(\alpha_1), \ldots, f^{(l_m)}(\alpha_m)\right),$$

and $\mathbf{A}_m$ a positive definite hermitian matrix of order m. Let $\{\psi_{n;m}\}_{n\geqslant 0}$ be the sequence of orthonormal polynomials with respect to (5.19) for $\mathbf{f}(Z) = \mathbf{f}_m(Z)$. We can assume, without loss of generality, $\mathbf{A}_m = \mathbf{I}_m$ by Lemma 5.2.1. Therefore, the relative asymptotics can be written as follows

$$\lim_{n\to\infty} \frac{\psi_{n;m}(z)}{\phi_n(z)} = \lim_{n\to\infty} \frac{\psi_{n;1}(z)}{\phi_n(z)} \prod_{i=2}^{m} \frac{\psi_{n;i}(z)}{\psi_{n;i-1}(z)},$$

which, using Lemma 5.2.3 and Theorem 5.2.1, immediately yields

$$\lim_{n\to\infty} \frac{\psi_{n;m}(z)}{\phi_n(z)} = \prod_{i=1}^{m} B(\alpha_i).$$

Finally, the proof for a general $\mathbf{f}(Z)$ is a straightforward consequence of the previous analysis. □

5.2.2 Zeros

In this subsection we study the asymptotic behavior of the zeros of orthogonal polynomials associated with the discrete Sobolev inner product (5.23). In contrast with the real line case Alfaro et al. [1996]; Bruin [1993]; Dimitrov et al. [2010,b]; Marcellán and Rafaeli [2011]; Pérez and Piñar [1993] (see also Appendix A), there is not a well developed theory for zeros of discrete Sobolev orthogonal polynomials on the unit circle.

The monic version of (5.27) is

$$\Psi_n(z) = \Phi_n(z) - \frac{\lambda \Phi_n^{(j)}(\alpha)}{1 + \lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} K_{n-1}^{(0,j)}(z,\alpha). \tag{5.33}$$

Thus,

$$\begin{bmatrix} \Psi_0(z) \\ \Psi_1(z) \\ \vdots \\ \Psi_{n-1}(z) \end{bmatrix} = \mathbf{L}_n \begin{bmatrix} \Phi_0(z) \\ \Phi_1(z) \\ \vdots \\ \Phi_{n-1}(z) \end{bmatrix},$$

where $\mathbf{L}_n$ is a $n \times n$ lower triangular matrix with 1 as entries in the main diagonal, and the remaining entries are given by (5.33), i.e.,

$$l_{m,k} = -\frac{1}{\|\Phi_k\|_\sigma^2} \frac{\lambda \Phi_m^{(j)}(\alpha)\overline{\Phi_k^{(j)}(\alpha)}}{(1 + \lambda K_{m-1}^{(j,j)}(\alpha,\alpha))}, \quad 1 \leqslant m \leqslant n, \quad 0 \leqslant k \leqslant m-1.$$

GGT matrices

One of our aim is to find a relation between $\mathbf{H}_\Psi$, the Hessenberg matrix associated with the monic orthogonal polynomials $\{\Psi_n\}_{n\geqslant 0}$, and $\mathbf{H}_\sigma$. In particular, we get

$$z\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}=(\mathbf{H}_\sigma)_n\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}+\Phi_n(z)\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix},$$

and, on the other hand,

$$z\begin{bmatrix}\Psi_0(z)\\ \Psi_1(z)\\ \vdots\\ \Psi_{n-1}(z)\end{bmatrix}=(\mathbf{H}_\Psi)_n\begin{bmatrix}\Psi_0(z)\\ \Psi_1(z)\\ \vdots\\ \Psi_{n-1}(z)\end{bmatrix}+\Psi_n(z)\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}. \tag{5.34}$$

Substituting in (5.34), we obtain

$$z\mathbf{L}_n\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}=(\mathbf{H}_\Psi)_n\mathbf{L}_n\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}+\Phi_n(z)\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}+\mathbf{A}_n\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix},$$

where

$$\mathbf{A}_n=\begin{bmatrix}0 & \dots & \dots & 0\\ \vdots & & & \vdots\\ 0 & \dots & \dots & 0\\ l_{n,0} & \dots & \dots & l_{n,n-1}\end{bmatrix}.$$

As a consequence,

$$z\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}=\left(\mathbf{L}_n^{-1}(\mathbf{H}_\Psi)_n\mathbf{L}_n+\mathbf{L}_n^{-1}\mathbf{A}_n\right)\begin{bmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{bmatrix}+\Phi_n(z)\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix},$$

so

$$(\mathbf{H}_\sigma)_n=\mathbf{L}_n^{-1}(\mathbf{H}_\Psi)_n\mathbf{L}_n+\mathbf{L}_n^{-1}\mathbf{A}_n$$

and therefore, since

$$\mathbf{L}_n^{-1}\mathbf{A}_n=\mathbf{A}_n,$$

we have

Theorem 5.2.3. *Let $(\mathbf{H}_\sigma)_n$ and $(\mathbf{H}_\Psi)_n$ be the $n\times n$ truncated GGT matrices associated with $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Psi_n\}_{n\geqslant 0}$, respectively. Then,*

$$(\mathbf{H}_\Psi)_n = \mathbf{L}_n((\mathbf{H}_\sigma)_n - \mathbf{A}_n)\mathbf{L}_n^{-1}.$$

As a consequence, the zeros of Ψ_{n+1} are the eigenvalues of the matrix $(\mathbf{H}_\sigma)_n - \mathbf{A}_n$, a rank one perturbation of the matrix $(\mathbf{H}_\sigma)_n$.

In the previous theorem we have characterized the eigenvalues of the GGT matrix associated with the discrete Sobolev polynomials as the eigenvalues of a rank one perturbation of the GGT matrix associated with the measure.

Notice that $\mathbf{A}_n = (0,\dots,0,1)^T(l_{n,0}, l_{n,1},\dots,l_{n,n-1})$ and, since $l_{n,k} = 0$ for $k < j$, then

$$\mathbf{A}_n = \frac{\lambda\Phi_n^{(j)}(\alpha)}{1+\lambda K_{n-1}^{(j,j)}(\alpha,\alpha)} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \left[0,\dots,0, \frac{\overline{\Phi_j^{(j)}(\alpha)}}{\|\Phi_j\|^2},\dots,\frac{\overline{\Phi_{n-1}^{(j)}(\alpha)}}{\|\Phi_{n-1}\|^2} \right].$$

As an example, if $d\sigma(\theta) = \dfrac{d\theta}{2\pi}$ is the Lebesgue measure, it is not difficult to see that in such a case, if $\alpha = 0$, then $\mathbf{A}_n = 0$, $n \neq j$, and

$$\mathbf{A}_j = \frac{\lambda(j!)^2}{1+\lambda(j!)^2} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0, & \dots, & 0, & 1, & 0, & \dots, & 0 \end{bmatrix},$$

where the one is in the position j. On the other hand, if $\alpha = 1$, then for $n \geqslant j$,

$$\mathbf{A}_n = \frac{\lambda\frac{(n)!}{(n-j)!}}{1+\lambda\displaystyle\sum_{k=j}^{n-1}\left(\frac{k!}{(k-j)!}\right)^2} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0, & \dots, & 0, & j!, & (j+1)!, & \dots, & \frac{(n-1)!}{(n-j-1)!} \end{bmatrix}.$$

Asymptotic behavior

First, denote by $\{\phi_n(\cdot; d\sigma_{j+1})\}_{n\geqslant 0}$ the corresponding sequence of orthonormal polynomial with respect to

$$d\sigma_j(z) = |z-\alpha|^{2(j+1)}d\sigma(z), \qquad j \geq 0,$$

i.e., the product of $j+1$ transformations as (5.15). For any $j \geq 0$, the relation between $\phi_n(\cdot; d\sigma_{j+1})$ and $\phi_n(\cdot, d\sigma)$ is given by Marcellán and Moral [2002]

$$(z-\alpha)^{j+1}\phi_{n-j-1}(z, d\sigma_{j+1}) = \frac{\eta_{n-j-1}}{\alpha_n}\left(\phi_n(z) - \sum_{k=0}^{j}\gamma_{n,k}K_{n-1}^{(0,k)}(z,\alpha)\right), \tag{5.35}$$

where η_n is the leading coefficient of $\phi_n(\cdot, d\sigma_{j+1})$, and $\gamma_{n,k}$ is the k-th component of the vector

$$\begin{bmatrix}\phi_n(\alpha) & \phi_n'(\alpha) & \dots & \phi_n^{(j)}(\alpha)\end{bmatrix}\begin{bmatrix} K_{n-1}(\alpha,\alpha) & K_{n-1}^{(0,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(0,j)}(\alpha,\alpha) \\ K_{n-1}^{(1,0)}(\alpha,\alpha) & K_{n-1}^{(1,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(1,j)}(\alpha,\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(j,0)}(\alpha,\alpha) & K_{n-1}^{(j,1)}(\alpha,\alpha) & \dots & K_{n-1}^{(j,j)}(\alpha,\alpha)\end{bmatrix}^{-1}.$$

If $\sigma \in \mathcal{N}$, then Marcellán and Moral [2002]

$$\lim_{n\to\infty}\frac{\phi_n(z; d\sigma_{j+1})}{\phi_{n+j+1}(z)} = \left(\frac{\overline{\alpha}}{|\alpha|}\frac{1}{\overline{\alpha}z-1}\right)^{j+1}, \tag{5.36}$$

holds uniformly in $|z| > 1$ if $|\alpha| \geqslant 1$, and in $|z| \geqslant 1$ if $|\alpha| > 1$.

On the other hand, by Theorem 5.2.1, for $|\alpha| > 1$,

$$\lim_{n\to\infty}\frac{\psi_n(z)}{\phi_n(z)} = \frac{\overline{\alpha}}{|\alpha|}\frac{z-\alpha}{\overline{\alpha}z-1}, \tag{5.37}$$

uniformly on every compact subset of $|z| > 1$. From (5.36) and (5.37), we have

$$\left(\frac{|\alpha|}{\overline{\alpha}}(\overline{\alpha}z-1)\right)^{j}(z-\alpha)\lim_{n\to\infty}\frac{\phi_n(z; d\sigma_{j+1})}{\phi_{n+j+1}(z)} = \lim_{n\to\infty}\frac{\psi_{n+j+1}(z)}{\phi_{n+j+1}(z)}.$$

Hence,

$$\lim_{n\to\infty}\frac{\psi_n(z)}{\phi_{n-j-1}(z; d\sigma_{j+1})} = \left(\frac{|\alpha|}{\overline{\alpha}}(\overline{\alpha}z-1)\right)^{j}(z-\alpha),$$

uniformly $|z| \geqslant 1$. The following result follows immediately from Hurwitz's Theorem Conway [1978].

Theorem 5.2.4. *There is a positive integer n_0 such that, for $n \geqslant n_0$, the n-th Sobolev monic orthogonal polynomial Ψ_n defined by (5.23), with $|\alpha| > 1$, has exactly 1 zero in $\mathbb{C}\setminus\overline{\mathbb{D}}$ accumulating in α, while the remaining zeros belong to $\mathbb{D}$.*

This result is analogous to the well known result of Meijer Meijer [1993] for Sobolev orthogonal polynomials on the real line; see also Appendix A. We now turn our attention to the case when λ tends to infinity. For a fixed n, $j = 0$ and λ tends to infinity, $n-1$ zeros of ψ_n tend to the zeros of $\phi_{n-1}(z, d\sigma_1)$, and the remaining zero tends to $z = \alpha$. On the other hand, for $j = 1$, the zeros of ψ_n tend to the zeros of a linear combination of $\Phi_n(z)$, $(z-\alpha)\Phi_{n-1}(z, d\sigma_1)$, and $(z-\alpha)^2\Phi_{n-2}(z, d\sigma_2)$ when $\lambda \to \infty$. This result can be generalized for arbitrary j. Indeed, from (5.35), notice that

$$-\gamma_{n,j}K_{n-1}^{(0,j)}(z,\alpha) = (z-\alpha)^{j+1}\phi_{n-j-1}(z,d\sigma_{j+1}) - \frac{\eta_{n-j-1}}{\alpha_n}\phi_n(z) + \sum_{k=0}^{j-1}\gamma_{n,k}K_{n-1}^{(0,k)}(z,\alpha).$$

Applying the last formula recursively for $k = 0,1,\ldots,j-1$, we obtain

Theorem 5.2.5. *Let $\{\psi_n\}_{n\geqslant 0}$ be the sequence of orthonormal polynomials with respect to (5.23), with $j \geq 0$. Then $\psi_n(z)$ is a linear combination of $\phi_n(z)$, $(z-\alpha)\phi_{n-1}(z,d\sigma_1)$, $\ldots$, $(z-\alpha)^{j+1}\phi_{n-j-1}(z,d\sigma_{j+1})$. As a consequence, the zeros of $\psi_n(z)$ tend to the zeros of such a linear combination when $\lambda \to \infty$.*

Now, we provide an extremal characterization for the limit discrete Sobolev polynomials, when the mass tends to infinity. Notice that when λ tends to infinity in (5.33), we get the limit polynomial

$$\widetilde{\Psi}_n(z) = \Phi_n(z) - \frac{\Phi_n^{(j)}(\alpha)}{K_{n-1}^{(j,j)}(\alpha,\alpha)}K_{n-1}^{(0,j)}(z,\alpha). \tag{5.38}$$

It is easily seen that $\widetilde{\Psi}_n^{(j)}(\alpha) = 0$, as well as $\widetilde{\Psi}_n$ is orthogonal to the linear space $\text{span}\{1, z-\alpha, \ldots, (z-\alpha)^{j-1}, (z-\alpha)^{j+1}, \ldots, (z-\alpha)^{n-1}\} \in \mathbb{P}_n$. Assume that $\widehat{\Psi}_n$ is a monic polynomial of degree n such that $\widehat{\Psi}_n^{(j)}(\alpha) = 0$. Then, we can write

$$\widehat{\Psi}_n(z) = \Phi_n(z) + \sum_{k=0}^{n-1}\lambda_{n,k}\phi_k(z), \tag{5.39}$$

for some (unique) complex numbers $\lambda_{n,k}$, and therefore

$$\Phi_n^{(j)}(\alpha) + \sum_{k=0}^{n-1}\lambda_{n,k}\phi_k^{(j)}(\alpha) = 0.$$

On the other hand, from Cauchy-Schwarz inequality, we get

$$\left|\Phi_n^{(j)}(\alpha)\right|^2 \leqslant \sum_{k=0}^{n-1}|\lambda_{n,k}|^2\sum_{k=0}^{n-1}|\phi_k^{(j)}(\alpha)|^2 = K_{n-1}^{(j,j)}(\alpha,\alpha)\sum_{k=0}^{n-1}|\lambda_{n,k}|^2,$$

and taking norms with respect to μ in (5.39), we obtain

$$\|\widehat{\Psi}_n\|_\sigma^2 = \|\Phi_n\|_\sigma^2 + \sum_{k=0}^{n-1}|\lambda_{n,k}|^2.$$

Thus,

$$\|\widehat{\Psi}_n\|_\sigma^2 = \|\Phi_n\|_\sigma^2 + \sum_{k=0}^{n-1}|\lambda_{n,k}|^2 \geqslant \|\Phi_n\|_\sigma^2 + \frac{|\Phi_n^{(j)}|^2}{K_{n-1}^{(j,j)}(\alpha,\alpha)}.$$

But the term in the right hand side is precisely $\|\widetilde{\Psi}_n\|_\sigma^2$. As a consequence, we have proved the following extremal characterization for the limit polynomial $\widetilde{\Psi}_n$.

Theorem 5.2.6. *Let $\widetilde{\Psi}_n$ be the limit monic polynomial of degree n defined by (5.38), then*

$$||\widetilde{\Psi}_n||^2_\sigma = \min\left\{\int_{\mathbb{T}} |\widehat{\Psi}_n(z)|^2 d\mu(z);\ \widehat{\Psi}_n(z) = z^n + (\textit{lower degree terms}),\ \widehat{\Psi}_n^{(j)}(\alpha) = 0\right\}.$$

Example: Lebesgue polynomials

For the normalized Lebesgue measure, it is very well known that its corresponding monic orthogonal polynomial sequence is $\Phi_n(z) = z^n$, $n \geqslant 0$; see example in Section 4.2.1. Thus,

$$\Phi_n^{(j)}(\alpha) = \frac{n!}{(n-j)!}\alpha^{n-j}, \quad K_{n-1}^{(0,j)}(z,\alpha) = \sum_{k=j}^{n-1} \frac{k!}{(k-j)!} z^k \bar{\alpha}^{k-j}, \quad K_n^{(j,j)}(\alpha,\alpha) = \sum_{k=j}^{n} \left(\frac{k!}{(k-j)!}\right)^2 |\alpha|^{2(k-j)}.$$

When n, j, and λ are fixed and α varies, we were able to identify some 'critical' values of α that change the behavior of the zeros of ψ_n. Of course, such values of α will depend on n, j, and λ. The following table illustrates such a situation when $n = 30$, $j = 2$, and $\lambda = 10$.

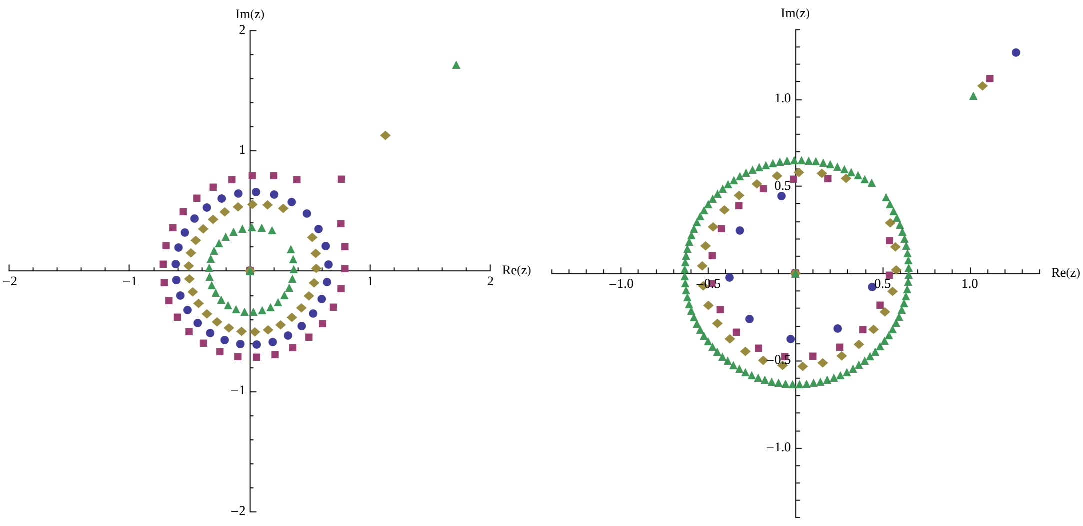
Table 5.1: Critical values for $n = 30$, $j = 2$, and $\lambda = 10$

α	Behavior of zeros
$0 < \lvert\alpha\rvert < 0.8202$	All zeros approximately aligned and increase with α
$\lvert\alpha\rvert \sim 0.8202$	One of the zeros (z_i) breaks the pattern
$0.8202 < \lvert\alpha\rvert < 1.3194$	z_i increases with α
$\lvert\alpha\rvert \sim 0.1.3194$	z_i changes sign
$1.1394 < \lvert\alpha\rvert < 1.6263$	z_i decreases with α
$\lvert\alpha\rvert \sim 1.6263$	z_i goes back to the aligned pattern
$\lvert\alpha\rvert > 1.6263$	All zeros approximately aligned and decrease with α

Figures 5.1(a) illustrates the information in Table 5.1, showing the location of the zeros of ψ_{30} for several values of α. Namely, the zeros corresponding to $\alpha = 0.4 + 0.4i$ (blue discs), $\alpha = 0.7 + 0.7i$ (purple square), $\alpha = 1.05 + 1.05i$ (yellow diamonds), and $\alpha = 1.6 + 1.6i$ (green triangles) were plotted. On the other hand, Figure 5.1(b) illustrates the behavior of the zeros of ψ_n as $n \to \infty$. The zeros corresponding to $n = 10$, $n = 20$, $n = 30$, and $n = 100$ were plotted. Notice that one of the zeros approaches the value of α as n increases, as stated in Theorem 5.2.4.

Example: Bernstein-Szegő polynomials

In order to analyze the behavior of the zeros according to the location of α, we present some numerical computations of such zeros for the orthogonal polynomials associated with perturbations of the form (5.23) for one special case of probability measures on the unit circle: the Bernstein-Szegő measures.



(a) $n = 15$ (b) $n = 500$

Figure 5.1: Zeros for Lebesgue polynomials with $n = 30$, $j = 2$, $\lambda = 10$

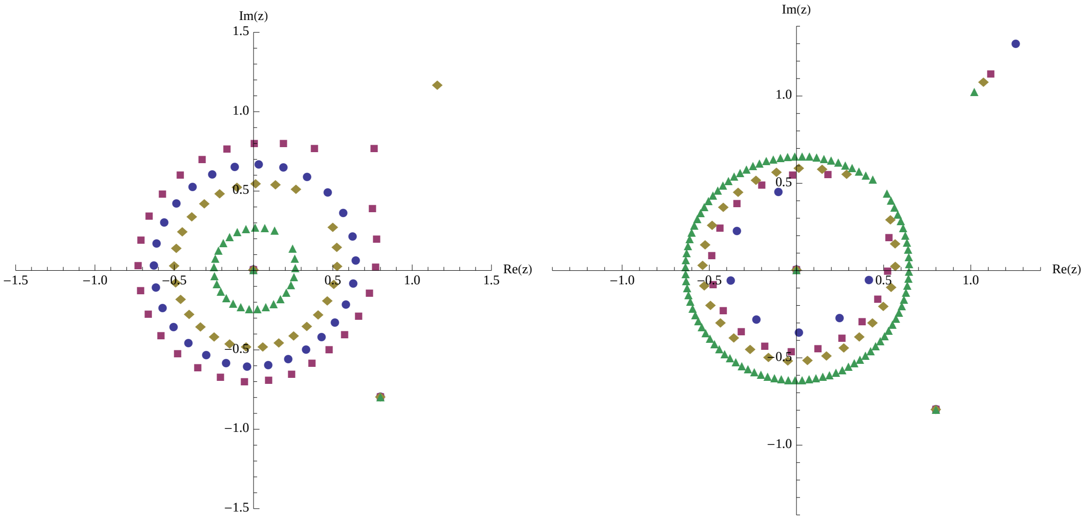
For the Bernstein-Szegő measure Simon [2005] $d\sigma(\theta) = \dfrac{1-|b|^2}{|1-bz|^2}\dfrac{d\theta}{2\pi}$ with $|b| < 1$ (see example in Section 4.2.1), we have $\phi_n(z) = z^{n-1}(z-\bar{b})$, $n \geqslant 1$. Thus, we get

$$
\begin{aligned}
\phi_n^{(j)}(\alpha) &= \frac{n!}{(n-j)!}\alpha^{n-j} - \bar{b}\frac{(n-1)!}{(n-j-1)!}\alpha^{n-j-1},\\
K_{n-1}^{(0,j)}(z,\alpha) &= \sum_{k=j}^{n-1}(z^k - \bar{b}z^{k-1})\left(\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right),\\
K_n^{(j,j)}(\alpha,\alpha) &= \sum_{k=j}^{n}\left|\frac{k!}{(k-j)!}\bar{\alpha}^{k-j} - \bar{b}\frac{(k-1)!}{(k-j-1)!}\bar{\alpha}^{k-j-1}\right|^2,
\end{aligned}
$$

and, we obtain again the expression of $\psi_n(z)$ using (5.24). We perform a similar numerical analysis of the zeros of ψ_{30} as a function of α as in the Lebesgue case. Figure 5.2(a) shows the behavior of the zeros of ψ_{30} for fixed j, λ, and b, and several values of α, namely $\alpha = 0.4+0.4i$ (blue discs), $\alpha = 0.7+0.7i$ (purple square), $\alpha = 1.08+1.08i$ (yellow diamonds), and $\alpha = 2.2+2.2i$ (green triangles). As before, the behavior of the zeros as $n \to \infty$ is illustrated in Figure 5.2(b), using the same values of n plotted in the Lebesgue case.

On the other hand, the behavior of the zeros of ψ_n when $\lambda \to \infty$ is shown in Figure 5.2.2. According to Theorem 5.2.5, the zeros of ψ_n tend to a linear combination of $\phi_n(z)$, $(z-\alpha)\phi_{n-1}(z, d\mu_1)$ and $(z-\alpha)\phi_{n-2}(z, d\mu_2)$, when $j = 1$. We computed the zeros of such polynomials for the Lebesgue (Fig. 5.3(a)) and Bernstein-Szegő (Fig. 5.3(b)) cases.

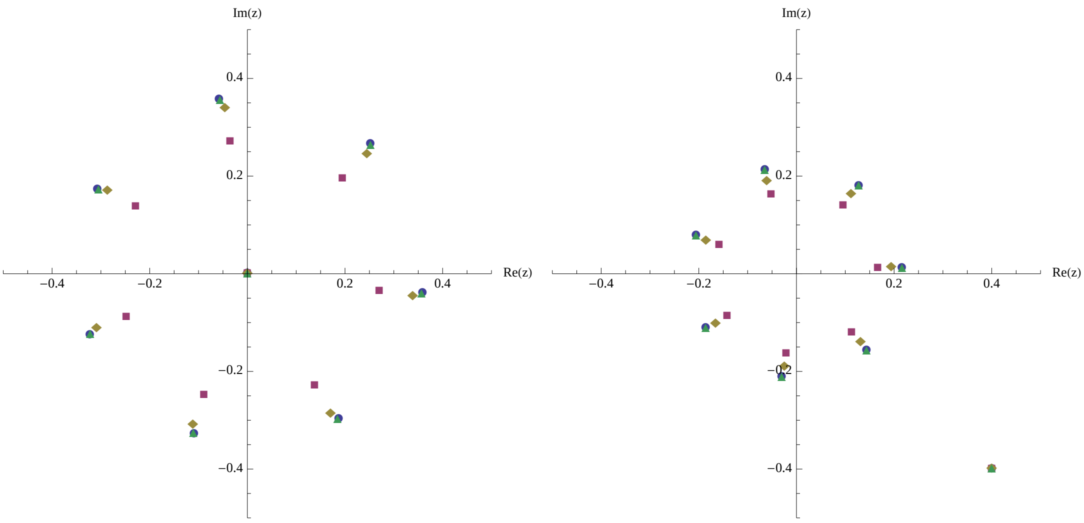
The points on the outer diameter correspond to the zeros of the above mentioned linear combination. Notice that when λ increases, the zeros of ψ_n approach the outer diameter, as per Theorem 5.2.5.



(a) $n = 30$

(b) $n = 30, \alpha = 1 + i$

Figure 5.2: Zeros of Bernstein-Szegő polynomials with $b = 0.8 + 0.8i$, $j = 2$, and $\lambda = 1$



(a) $\alpha = 0.2 + 0.2i, \lambda = 0.1, 1, 10.$

(b) $b = 0.4 + 0.4i, \alpha = 0.1 + 0.1i, \lambda = 0.1, 1, 5.$

Figure 5.3: Zeros of Lebesgue and Bernstein-Szegő polynomials with $n = 8$, $j = 1$

Chapter 6

Generators of rational spectral transformations for C-functions

...Stieltjes function(s) [1] with polynomial coefficients can be presented as a finite superposition of four fundamental elementary transforms ...

— A. Zhedanov. Zhedanov [1997]

In this chapter we deal with transformations of sequences of orthogonal polynomials associated with the linear functional $\mathcal{L}$ using spectral transformations of the corresponding C-function F. First, we study the modifications obtained by multiplying a hermitian functional by a polynomial of any degree, in short, polynomial modifications. We characterize when two functionals are related by a polynomial modification. We are interested in those modifications which preserve their hermitian character. The possibility of considering modifications that do not preserve the hermitian character of the functional leads to left and right orthogonality Baxter [1961]. Hence, the preservation of the hermiticity calls for the use of Laurent polynomials as perturbations. Laurent polynomial modifications of hermitian linear functional have been previously considered in Cantero [1997]; Daruis et al. [2007]; Garza [2008]; Godoy and Marcellán [1991, 1993]; Marcellán and Hernández [2008]; Suárez [1993].

Next, we distinguish two related problems: to characterize the quasi-definiteness character of the direct polynomial modification (2.43) and the inverse polynomial modification (2.44) from the original functional. Due to the non-uniqueness of the inverse problem we pay special attention to this one. Finally, we show that a linear spectral transformation of F can be obtained as a finite composition of spectral transformations (2.43)-(2.44), and also that any rational spectral transformations can be obtained as a finite composition of linear and $\pm k$ associated spectral transformations.

[1] $\mathcal{S}$- functions.

6.1 Hermitian polynomial transformation

Let $\mathcal{L}$ be the hermitian linear functional introduced in (2.22). The polynomial modification $f\mathcal{L}$ is defined by

$$\langle f\mathcal{L}, g\rangle = \langle \mathcal{L}, fg\rangle, \qquad f, g \in \Lambda.$$

This polynomial modification is hermitian if and only if f is a hermitian Laurent polynomial, i.e., $f = f_*$, which is equivalent to state that $f = p + p_*$, $p \in \mathbb{P}$. Such a polynomial p can be uniquely determined by f simply requiring $p(0) \in \mathbb{R}$.

Féjer-Riesz's Theorem Féjer [1915]; Riesz [1915] states that any Laurent polynomial f which is non-negative on $\mathbb{T}$ can be factorized $f(e^{i\theta}) = |p(e^{i\theta})|^2$, where p is a polynomial whose zeros all lie in $\overline{\mathbb{D}}$ or, in other words, one can find p whose zeros are all in $\mathbb{C}\setminus\mathbb{D}$. By analyticity, $f(z) = p(z)\overline{p(1/\overline{z})}$. Thus, $f\mathcal{L}$ is positive definite for a positive definite linear functional $\mathcal{L}$ if f also satisfies Féjer-Riesz's condition. Its analog on the real line, that is, $P(x) \geqslant 0$ on $\mathbb{R}$ implies $P(x) = Q(x)\overline{Q(\overline{x})}$ where Q has all its zeros in $\overline{\mathbb{C}}_+$ it is well known.

Another way of characterizing a hermitian Laurent polynomial modification is through the polynomial $g = z^{\deg p} f$ of degree $2\deg p$. The condition $f = f_*$ means that g is self-reciprocal, i.e., $g = g^*$. Therefore, hermitian polynomial modifications are related with self-reciprocal polynomials of even degree. The zeros z_i of a self-reciprocal polynomial lie on the unit circle or appear in symmetric pairs z_i, $1/\overline{z_i}$. Indeed, this property characterizes the self-reciprocal polynomials up to numerical factors. This implies that any self-reciprocal polynomial of even degree factorizes into a product of self-reciprocal polynomials of degree 2. As a consequence, an arbitrary hermitian polynomial modification $f\mathcal{L}$ can be factorized as a product of elementary ones of degree one. In the sequel we assume for simplicity that the polynomial modification f is a monic hermitian Laurent polynomial from which we can deduce immediately the more general case.

We denote by $W^{\perp_n}$ [1]the orthogonal complement in $\mathbb{P}_n$ of a subspace $W \subset \mathbb{P}_n$.

Theorem 6.1.1. Tasis [1989] *Let $\mathcal{L}$ be a hermitian linear functional such that the corresponding sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$ exists. Then, $\left\{z^k\Phi_n(z)\right\}_{k=1}^{r} \cup \left\{z^k\Phi_n^*(z)\right\}_{k=1}^{r}$ is a basis of $(z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$ for $n \geq r \geq 1$ and a generator system of $\mathbb{P}_{n+r}$ for $r > n \geq 0$.*

As an immediate consequence of Theorem 6.1.1 we have the following result.

Corollary 6.1.1. Cantero et al. [2011] *Let $\mathcal{L}$ be a hermitian linear functional such that there exists the corresponding sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$. Then, every polynomial $\Psi_n(z) \in (z^r\mathbb{P}_{n-r-1})^{\perp_{n+r}}$ has a unique decomposition, $\Psi_n(z) = C(z)\Phi_n(z) + D(z)\Phi_n^*(z)$, $C \in \mathbb{P}_r$, $D \in \mathbb{P}_{r-1}$, for $n \geq r \geq 1$, and every polynomial $\Psi_n \in \mathbb{P}_{n+r}$ has infinitely many decompositions for $r > n \geq 0$.*

To study the hermitian polynomial modifications we have the next theorem.

[1]The orthogonal complement $W^{\perp_n}$ of a subspace W of an inner product space V is the set of all vectors in V that are orthogonal to every vector in W.

Theorem 6.1.2. Cantero et al. [2011] *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be hermitian linear functionals with finite segments of monic orthogonal polynomials $\{\Phi_j\}_{j=0}^{n}$, $\{\Psi_j\}_{j=0}^{n+r}$, respectively, and let $f(z) = p(z) + p_*(z) = z^{-r}g(z)$ with f be a polynomial of degree r. Then, the following statements are equivalent:*

i) $\widetilde{\mathcal{L}} = f\mathcal{L}$ *in* $\mathbb{P}_n$.

ii) There exist $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ with $C_j(0) \neq 0$ such that

$$g(z)\Psi_j = C_j(z)\Phi_{j+r}(z) + D_j(z)\Phi^*_{j+r}(z), \quad j \geqslant 0.$$

iii) There exist $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ with $C_j(0) \neq 0$ such that

$$g(z)\Psi^*_j(z) = zD^*_j(z)\Phi_{j+r}(z) + C^*_j(z)\Phi^*_{j+r}(z), \quad D^*_j(z) = D_j^{*r-1}(z), \quad j \geqslant 0.$$

The polynomials $C_j \in \mathbb{P}_r$, $D_j \in \mathbb{P}_{r-1}$ satisfying ii) or iii) are unique, $\deg C_j = r$, $C_j(0) \in \mathbb{R}$ and $C^*_j(0) = g(0)$.

The above theorem has the following consequence for quasi-definite functionals.

Corollary 6.1.2. Cantero et al. [2011] *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be quasi-definite hermitian linear functionals with sequence of monic orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$, $\{\Psi_n\}_{n\geqslant 0}$, respectively, and let $f(z) = p(z) + p(z)_* = z^{-r}g(z)$ with f be a polynomial of degree r. Then, $\mathcal{L} = p\widetilde{\mathcal{L}}$ if and only if there exist polynomials $C_n \in \mathbb{P}_r$, $D_n \in \mathbb{P}_{r-1}$ with $C_n(0) \neq 0$ such that*

$$g(z)\Psi_n(z) = C_n(z)\Phi_{n+r}(z) + D_n(z)\Phi^*_{n+r}(z), \qquad n \geq 0,$$

or, equivalently,

$$g(z)\Psi^*_n(z) = zD^*_n(z)\Phi_{n+r}(z) + C^*_n(z)\Phi^*_{n+r}(z), \qquad n \geq 0.$$

6.2 Laurent polynomial transformation: Direct problem

The direct problem deals with the case where we suppose that a hermitian linear functional $\mathcal{L}$ associated with the sequence of orthogonal polynomials $\{\Phi_n\}_{n\geqslant 0}$, and a hermitian polynomial of degree 1 are given, i.e., we consider the spectral transformation (2.43). Then, we obtain information about the functional $\mathcal{L}_R$ and its sequence of monic orthogonal polynomials $\{\Psi_n\}_{n\geqslant 0}$.

From (2.44), we get

$$\widetilde{c}_{-k} = c_{-(k+1)} + c_{-(k-1)} - (\alpha + \overline{\alpha})c_{-k}, \tag{6.1}$$

where $\{\widetilde{c}_n\}_{n\geqslant 0}$ is the sequence of moments associated with $\mathcal{L}_R$. Notice that if $a = -2\Re(\alpha)$, and using (6.1), we obtain

$$\widetilde{\mathbf{T}} = \mathbf{Z}\mathbf{T} + a\mathbf{T} + \mathbf{T}\mathbf{Z}^T,$$

where $\widetilde{\mathbf{T}}$ is the Toeplitz matrix associated with $\mathcal{L}_R$, and $\mathbf{Z}$ is the shift matrix with ones on the first upper-diagonal and zeros on the remaining entries.

6.2.1 Regularity conditions and Verblunsky coefficients

If $\mathcal{L}$ is quasi-definite, necessary and sufficient conditions for $\mathcal{L}_R$ to be also quasi-definite have been studied. Moreover, the explicit expression for sequences of monic polynomials orthogonal with respect to $\mathcal{L}_R$ have been obtained Cantero [1997]; Suárez [1993].

Theorem 6.2.1. Suárez [1993]

i) If $|\Re\alpha| \neq 1$, and b_1, b_2 are zeros of the polynomial $z^2-(\alpha+\bar{\alpha})z+1$, then $\mathcal{L}_R$ is quasi-definite if and only if $K_n^*(b_1,b_2) \neq 0$, $n \geqslant 0$. In addition, if $\{\Psi_n\}_{n\geqslant 0}$ denotes the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_R$, then

$$\Psi_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(b_1,b_2) - K_{n-1}^*(z,b_2)\Phi_n(b_1)}{K_{n-1}^*(b_1,b_2)(z-b_1)}, \quad n \geqslant 1, \tag{6.2}$$

and

$$\Psi_{n-1}(0) = \frac{\Phi_n(b_1)\Phi_{n-1}(b_2) - \Phi_n(b_2)\Phi_{n-1}(b_1)}{K_{n-1}^*(b_1,b_2)(b_1-b_2)\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{6.3}$$

ii) If $|\Re\alpha| = 1$, and b is the double zero of the polynomial $z^2-(\alpha+\bar{\alpha})z+1$, then $\mathcal{L}_R$ is quasi-definite if and only if $K_n^*(b,b) \neq 0$, $n \geqslant 0$. In addition,

$$\Psi_{n-1}(z) = \frac{\Phi_n(z)K_{n-1}^*(b,b) - K_{n-1}^*(z,b)\Phi_n(b)}{K_{n-1}^*(b,b)(z-b)}, \quad n \geqslant 1, \tag{6.4}$$

and

$$\Psi_{n-1}(0) = -b\frac{\Phi_n(0)K_{n-1}^*(b,b)\mathbf{k}_{n-1} - \Phi_{n-1}(b)\Phi_n(b)}{K_{n-1}^*(b,b)\mathbf{k}_{n-1}}, \quad n \geqslant 1. \tag{6.5}$$

According to Corollary 6.1.2, (6.2) and (6.4) can be written as follows.

Corollary 6.2.1.

i) If $|\Re\alpha| \neq 1$, and b_1, b_2 are the zeros of the polynomial $z^2-(\alpha+\bar{\alpha})z+1$, then

$$\begin{aligned}(z-b_1)(z-b_2)\Psi_n(z) &= z\left(z-b_2-\frac{\Phi_{n+1}(b_1)\Phi_n^*(b_2)}{\mathbf{k}_nK_n^*(b_1,b_2)}\right)\Phi_n(z)\\ &\quad+\left((z-b_2)\Phi_{n+1}(0)+\frac{b_2\Phi_{n+1}(b_1)\Phi_n(b_2)}{\mathbf{k}_nK_n^*(b_1,b_2)}\right)\Phi_n^*(z).\end{aligned}$$

ii) If $|\Re\alpha| = 1$, and b is the double zero of the polynomial $z^2-(\alpha+\bar{\alpha})z+1$, then

$$\begin{aligned}(z-b)^2\Psi_n(z) &= z\left(z-b-\frac{\Phi_{n+1}(b)\Phi_n^*(b)}{\mathbf{k}_nK_n^*(b,b)}\right)\Phi_n(z)\\ &\quad+\left((z-b)\Phi_{n+1}(0)+\frac{b\Phi_{n+1}(b)\Phi_n(b)}{\mathbf{k}_nK_n^*(b,b)}\right)\Phi_n^*(z).\end{aligned}$$

There is another equivalent condition for the quasi-definiteness of $\mathcal{L}_R$ and, consequently, an expression for the corresponding Verblunsky coefficients.

Theorem 6.2.2. Cantero [1997] *The linear functional $\mathcal{L}_R$ is quasi-definite if and only if $\Pi_n(b_1) \neq 0$, $n \geqslant 0$, where*

$$\Pi_n(x) = \begin{vmatrix} x\Phi_n(x) & \Phi_n^*(x) \\ x^{-1}\Phi_n(x^{-1}) & \Phi_n^*(x^{-1}) \end{vmatrix}.$$

Moreover, the families of Verblunsky coefficients $\{\Psi_n(0)\}_{n\geqslant 1}$ are given by

$$Y_n(0) = \left(b_1 - b_1^{-1}\right) \frac{\Phi_n(b_1)\Phi_n(b_1^{-1})}{\Pi_n(b_1)}, \quad n \geqslant 1. \tag{6.6}$$

In Garza [2008] it was obtained the relation between the corresponding GGT matrices as well as an explicit expression for the Verblunsky coefficients of associated with this perturbation is given. Thus, the invariance of the Szegő class of bounded variation measures follows.

6.2.2 C-functions

From the relation between the moments (6.1) we get that the C-function associated with $\mathcal{L}_R$ is

$$F_R(z) = \frac{1}{2}\left(\left(z + z^{-1} - (\alpha + \overline{\alpha})\right)F(z) + c_0\left(z - z^{-1}\right) + c_1 - c_{-1}\right). \tag{6.7}$$

In a more general situation, if we consider a finite composition of $\mathcal{L}_R$ with order $k \geq 0$ defined by

$$\mathcal{L}_{R^{(k)}} = \Re\left(\prod_{i=1}^{k}(z - \alpha_i)\right)\mathcal{L}, \qquad \alpha_i \in \mathbb{C}, \tag{6.8}$$

we can prove the following result.

Theorem 6.2.3. *A generic rational spectral transformation with $C = 0$ and $D = 1$ is equivalent to (6.8). Furthermore,*

$$A(z) = \frac{1}{2}\Re p(z), \qquad B(z) = P(z) - P_*(z), \tag{6.9}$$

where P is the polynomial of second kind associated with

$$p(z) = \prod_{i=1}^{k}(z - \alpha_i)$$

with respect to the linear functional (6.8).

Proof. From (6.7) we get that $\mathcal{L}_R$ is a spectral transformation with $C = 0$ and $D = 1$. On the other hand, if we start with a generic spectral transformation where $C = 0$, B is a hermitian Laurent polynomial of degree one, and $D = 1$, A should be a hermitian Laurent polynomial of degree one (see Section 6.4), and the only choice for this spectral transformation is (6.7) containing one free parameter α.

If we consider the linear functional (6.8), from (6.7) we have

$$\widetilde{F}(z) = A(z)F(z) + B(z), \tag{6.10}$$

where

$$A(z) = (z + z^{-1} - (\alpha_1 + \overline{\alpha}_1)) \cdots (z + z^{-1} - (\alpha_k + \overline{\alpha}_k))$$

is a hermitian Laurent polynomial of degree k. This transformation contains k free parameters α_k. B is a polynomial of the same type and degree as A.

Conversely, if we start with a generic spectral transformation where

$$A = (z + z^{-1} - (\alpha_1 + \overline{\alpha}_1)) \cdots (z + z^{-1} - (\alpha_k + \overline{\alpha}_k))$$

is a hermitian Laurent polynomial of degree k, $C \equiv 0$, and $D \equiv 1$, then B should be a hermitian Laurent polynomial of same degree as A, in order to satisfy $A \equiv A_*$ (see Section 6.4). Moreover, it is easily seen that for (6.8) the polynomial B is uniquely determined by means of the sequence of moments $\{c_n\}_{n\in\mathbb{Z}}$ associated with the linear functional $\mathcal{L}$. Furthermore, (6.9) follows immediately from the definition of (6.8). □

6.3 Laurent polynomial transformations: Inverse problem

In this section we study the inverse polynomial modification given by (2.44). More precisely, given a hermitian functional $\mathcal{L}$ whose corresponding sequence of monic orthogonal polynomials is denoted by $\{\Phi_n\}_{n\geqslant 0}$ and a hermitian Laurent polynomial of degree 1, we obtain information about the hermitian solutions $\mathcal{L}_{R^{(-1)}}$ of (2.44) and their sequence of monic orthogonal polynomials $\{\Psi_n\}_{n\geqslant 0}$. If $\mathcal{L}$ is a positive definite linear functional, necessary and sufficient conditions in order $\mathcal{L}_{R^{(-1)}}$ to be a quasi-definite linear functional are given. The relation between the corresponding sequences of monic orthogonal polynomials is presented. We also obtain the relations between the corresponding C-functions in such a way that a linear spectral transformation appears.

We can describe the hermitian solutions of $\mathcal{L}_{R^{(-1)}}$ starting from a particular one $\mathcal{L}_{R_0^{(-1)}}$ Cantero et al. [2011]. This approach shows that the inverse problem is related to the study of the influence of Dirac's deltas and their derivatives on the quasi-definiteness and the sequence of monic orthogonal polynomials of a hermitian functional.

We are only interested in those values of α such that $0 < |\Re(\alpha)| < 1$. However, in this case the zeros b and $\overline{b}$ of $z^2 - (\alpha + \overline{\alpha})z + 1$ are complex conjugate and, furthermore, $|b| = 1$. We denote by σ and $\sigma_R^{(-1)}$ the measures associated with the positive definite case of $\mathcal{L}$ and $\mathcal{L}_{R^{(-1)}}$, respectively, i.e.,

$$d\sigma_R^{(-1)}(z) = \frac{d\sigma(z)}{2\Re(z-\alpha)} + \boldsymbol{m}_1\delta(z-b) + \boldsymbol{m}_2\delta(z-\overline{b}), \quad \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathbb{R}.$$

Here, σ is a non-trivial probability measure supported on $\mathbb{T}$, which can be decomposed as in (2.36).

Thus, if $\sigma_s = 0$, then the integral

$$\widetilde{c}_n = \int_0^{2\pi} \frac{e^{in\theta}\omega(\theta)}{z+z^{-1}-(\alpha+\overline{\alpha})}\frac{d\theta}{2\pi} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^n\omega(z)}{z^2-(\alpha+\overline{\alpha})z+1}dz,$$

has singularities in $z = b$ and $z = \overline{b}$. These singularities can be removed if we consider

$$\begin{aligned}\widetilde{c}_n &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^n\omega(\theta)}{z^2-(\alpha+\overline{\alpha})z+1}dz \\ &= \frac{1}{2\pi i(b-\overline{b})}\left(\int_{\mathbb{T}} \frac{z^n(\omega(z)-\omega(b))}{z-b}dz - \int_{\mathbb{T}} \frac{z^n\left(\omega(z)-\omega(\overline{b})\right)}{z-\overline{b}}dz + \frac{1}{2}b^n\omega(b) - \frac{1}{2}\overline{b}^n\omega(\overline{b})\right),\end{aligned} \tag{6.11}$$

assuming that $\omega \in C^{2+}$; that is, ω satisfies the Lipschitz condition [1] of order τ $(0 < \tau \leqslant 1)$ on $\mathbb{T}$ Volkovyski et al. [1972]. Notice that this is also valid if $\sigma_s \neq 0$, as long as σ_s has a finite number of mass points different from b and $\overline{b}$.

From (2.44) we get

$$c_{-k} = \widetilde{c}_{-(k+1)} + \widetilde{c}_{-(k-1)} - (\alpha+\overline{\alpha})\widetilde{c}_{-k}. \tag{6.12}$$

If $a = -2\Re(\alpha)$, and using (6.12), we obtain

$$\mathbf{T} = \mathbf{Z}\widetilde{\mathbf{T}} + a\widetilde{\mathbf{T}} + \widetilde{\mathbf{T}}\mathbf{Z}^T$$

Furthermore, the hermitian Toeplitz matrices can be characterized as $\mathbf{T} = \mathbf{T}^*$ together with $\mathbf{Z}\mathbf{T}\mathbf{Z}^T = \mathbf{T}$, and, therefore,

$$\mathbf{T}\mathbf{Z}^T = \widetilde{\mathbf{T}}\mathbf{B},$$

where $\mathbf{B} = \mathbf{I} + a\mathbf{Z}^T + (\mathbf{Z}^T)^2$ is an infinite lower triangular matrix with ones in the main diagonal, with the following structure

$$\mathbf{B} = \left[\begin{array}{c|c|c} \mathbf{A} & 0 & \cdots \\ \hline \mathbf{A}^T & \mathbf{A} & \ddots \\ \hline 0 & \ddots & \ddots \end{array}\right],$$

where $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$. On the other hand, is not difficult to show that

$$\mathbf{B}^{-1} = \left[\begin{array}{c|c|c|c} \mathbf{A}_1 & 0 & 0 & \cdots \\ \hline \mathbf{A}_2 & \mathbf{A}_1 & 0 & \ddots \\ \hline \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right],$$

[1] $|\omega(z)-\omega(y)| \leqslant C|x-y|^\tau$, $C > 0$, $z, y \in \mathbb{T}$.

where $\mathbf{A}_1 = \mathbf{A}^{-1}$, $\mathbf{A}_k = (-1)^{k-1}\mathbf{A}^{-1}\mathbf{M}^{k-1}$, $k \geqslant 2$, and

$$\mathbf{M} = \mathbf{A}^{-1}\mathbf{A}^T = \begin{bmatrix} 1 & a \\ -a & 1-a^2 \end{bmatrix}.$$

In other words, $\mathbf{B}^{-1}$ is a lower triangular block matrix, with Toeplitz structure. Finally,

$$\mathbf{TS} = \widetilde{\mathbf{T}},$$

where $\mathbf{S}$ is given by

$$\mathbf{S} = \mathbf{Z}^T\mathbf{B}^{-1} = \left[\begin{array}{c|c|c|c} \mathbf{Z}_1^T & 0 & 0 & \dots \\ \hline \mathbf{Z}_1 & \mathbf{Z}_1^T & 0 & \ddots \\ \hline 0 & \mathbf{Z}_1 & \mathbf{Z}_1^T & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right]\left[\begin{array}{c|c|c|c} \mathbf{A}_1 & 0 & 0 & \dots \\ \hline \mathbf{A}_2 & \mathbf{A}_1 & 0 & \ddots \\ \hline \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right],$$

with $\mathbf{Z}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, i.e., $\mathbf{S}$ is also a lower triangular block matrix with Toeplitz structure.

6.3.1 Regularity conditions and Verblunsky coefficients

Assume that $\mathcal{L}_{R^{(-1)}}$ is quasi-definite and let $\{\Psi_n\}_{n\geqslant 0}$ be its corresponding sequence of monic orthogonal polynomials with leading coefficients $\widetilde{\kappa}_n$. We next state the relation between $\{\Psi_n\}_{n\geqslant 0}$ and $\{\Phi_n\}_{n\geqslant 0}$.

Theorem 6.3.1. *Let $\mathcal{L}$ be a positive definite linear functional. If $\mathcal{L}_{R^{(-1)}}$, given as in (2.44), is a quasi-definite linear functional, then Ψ_n, the n-th monic polynomial orthogonal with respect to $\mathcal{L}_{R^{(-1)}}$, is*

$$\Psi_n(z) = \left(z + \frac{\widetilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Phi_n(0) - \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\Phi^*_{n-1}(z). \tag{6.13}$$

Conversely, if $\{\Psi_n\}_{n\geqslant 0}$ is given by (6.13) and assuming that $|\Psi_n(0)| \neq 1$, $n \geqslant 1$, then $\{\Psi_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_{R^{(-1)}}$.

Proof. Let

$$\Psi_n(z) = \Phi_n(z) + \sum_{m=0}^{n-1}\lambda_{n,m}\Phi_m(z). \tag{6.14}$$

Multiplying the above expression by $\overline{\Phi_m}$ and applying $\mathcal{L}$, for $0 \leqslant m \leqslant n-1$, we get

$$\left\langle \mathcal{L}, \Psi_n\overline{\Phi}_m \right\rangle = \lambda_{n,m}k_m,$$

or, equivalently,

$$\left\langle \mathcal{L}_{R^{(-1)}}, \left(z + z^{-1} - (\alpha+\overline{\alpha})\right)\Psi_n(z)\overline{\Phi_m(z)} \right\rangle = \lambda_{n,m}k_m.$$

Thus,

$$\lambda_{n,m} = \frac{1}{k_m}\left\langle \mathcal{L}_{R^{(-1)}}, \left(z+z^{-1}-(\alpha+\overline{\alpha})\right)\Psi_n(z)\overline{\Phi_m(z)}\right\rangle, \quad 0 \leqslant m \leqslant n-1.$$

If $m = n-1$, then

$$\lambda_{n,n-1} = \frac{1}{k_{n-1}}\left(\left\langle \mathcal{L}_{R^{(-1)}}, z\Psi_n(z)\overline{\Phi_{n-1}(z)}\right\rangle + \left\langle \mathcal{L}_{R^{(-1)}}, z^{-1}\Psi_n(z)\overline{\Phi_{n-1}(z)}\right\rangle\right) = \frac{\widetilde{k}_n}{k_{n-1}}\left(1-\Psi_{n+1}(0)\overline{\Phi_{n-1}(0)}\right).$$

On the other hand, for $0 \leqslant m \leqslant n-2$,

$$\lambda_{n,m} = \frac{1}{k_m}\left\langle \mathcal{L}_{R^{(-1)}}, z\Psi_n(z)\overline{\Phi_m(z)}\right\rangle = -\frac{\widetilde{k}_n}{k_m}\Psi_{n+1}(0)\overline{\Phi_m(0)}.$$

Substituting these values into (6.14), we obtain

$$\begin{aligned}\Psi_n(z) &= \Phi_n(z) + \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \widetilde{k}_n\Psi_{n+1}(0)\sum_{m=0}^{n-1}\frac{\overline{\Phi_m(0)}\Phi_m(z)}{k_m} \\ &= \Phi_n(z) + \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\Phi^*_{n-1}(z).\end{aligned} \tag{6.15}$$

Using the recurrence relation, we get

$$\begin{aligned}\Psi_n(z) &= z\Phi_{n-1}(z) + \Phi_n(0)\Phi^*_{n-1}(z) + \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(z) - \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\Phi^*_{n-1}(z) \\ &= \left(z + \frac{\widetilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Phi_n(0) - \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\Phi^*_{n-1}(z),\end{aligned} \tag{6.16}$$

which proves the first statement of the theorem.

Notice that evaluating (6.15) at $z = 0$, we get

$$\Psi_n(0) = \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(0) + \Phi_n(0) - \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0), \tag{6.17}$$

and thus (6.16) becomes

$$\Psi_n(z) = \left(z + \frac{\widetilde{k}_n}{k_{n-1}}\right)\Phi_{n-1}(z) + \left(\Psi_n(0) - \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(0)\right)\Phi^*_{n-1}(z).$$

If we denote $\nu_n = \widetilde{k}_{n+1}/k_n$ and $l_n = \Psi_{n+1}(0) - \nu_n\Phi_n(0)$, considering the reversed polynomial of Ψ_{n+1}, we obtain the following linear transfer equation

$$\begin{bmatrix}\Psi_{n+1}(z)\\ \Psi^*_{n+1}(z)\end{bmatrix} = \begin{bmatrix} z+\nu_n & l_n \\ \overline{l}_n z & \nu_n z + 1\end{bmatrix}\begin{bmatrix}\Phi_n(z)\\ \Phi^*_n(z)\end{bmatrix}.$$

Notice that the determinant of the above transfer matrix is

$$\begin{aligned}(z+\nu_n)(\nu_n z+1)-|l_n|^2 z &= \nu_n z^2+\left(\nu_n^2+1-|l_n|^2\right)z+\nu_n\\ &= \nu_n(z^2+1)+\left(\nu_n^2(1-|\Phi_n(0)|^2)+1-|\Psi_{n+1}(0)|^2\right)z\\ &+\nu_n\left(\Psi_{n+1}(0)\overline{\Phi_n(0)}+\overline{\Psi_{n+1}(0)}\Phi_n(0)\right)z\\ &= \nu_n(z^2-(\alpha+\overline{\alpha})z+1),\end{aligned}$$

where the last equality becomes clear looking at (6.18) in the following theorem. Furthermore, we get

$$\Phi_n(z)=\frac{(\nu_n z+1)\Psi_{n+1}(z)-l_n\Psi^*_{n+1}(z)}{\nu_n\left(z^2-(\alpha+\overline{\alpha})z+1\right)},\quad \Phi^*_n(z)=\frac{(z+\nu_n)\Psi^*_{n+1}(z)-\bar{l}_n z\Psi_{n+1}(z)}{\nu_n\left(z^2-(\alpha+\overline{\alpha})z+1\right)},$$

and thus we obtain the following alternative expression that relates both sequences of polynomials

$$\left(z^2-(\alpha+\overline{\alpha})z+1\right)\Phi_n(z)=\Psi_{n+2}(z)+\nu_n^{-1}\left(\Psi_{n+1}(z)-\Phi_{n+1}(0)\Psi^*_{n+1}(z)\right).$$

We now prove that the sequence of monic polynomials $\{\Psi_n\}_{n\geqslant 0}$ given in (6.13) is orthogonal with respect to $\mathcal{L}_{R^{(-1)}}$. Notice that $\Psi_{n+1}(z)-\Psi_{n+1}(0)\Psi^*_n(z)$ is a polynomial of degree $n+1$ that vanishes at $z=0$ and, thus, $\Psi_{n+1}(z)-\Psi_{n+1}(0)\Psi^*_n(z)=zp(z)$, where $p(z)$ is a polynomial of degree n. Then,

$$\begin{aligned}zp(z)&=(z+\nu_n)\Phi_n(z)+l_n\Phi^*_n(z)-\Psi_{n+1}(0)\left(\overline{l_{n-1}}z\Phi_{n-1}(z)+(\nu_{n-1}z+1)\Phi^*_{n-1}(z)\right)\\ &=z(z+\nu_{n-1})\Phi_{n-1}(z)+zl_{n-1}\Phi^*_{n-1}(z)=z\Psi_n(z),\end{aligned}$$

where the fourth equality follows from (2.29) and (6.17). That is, $\{\Psi_n\}_{n\geqslant 0}$ satisfies a recurrence relation like (2.24), and it is therefore an orthogonal sequence with respect to some linear functional $\widetilde{\mathcal{L}}$. We will prove that $\widetilde{\mathcal{L}}=\mathcal{L}_{R^{(-1)}}$. For $0\leqslant k\leqslant n-1$, consider

$$\begin{aligned}\left\langle\widetilde{\mathcal{L}},\left(z+z^{-1}-(\alpha+\overline{\alpha})\right)\Phi_n(z)\overline{z}^k\right\rangle&=\left\langle\widetilde{\mathcal{L}},\left(z^2-(\alpha+\overline{\alpha})z+1\right)\Phi_n(z)\overline{z}^{k+1}\right\rangle\\ &=\left\langle\widetilde{\mathcal{L}},\Psi_{n+2}(z)\overline{z}^{k+1}\right\rangle+\nu_n^{-1}\left\langle\widetilde{\mathcal{L}},\left(\Psi_{n+1}(z)-\Phi_{n+1}(0)\Psi^*_{n+1}(z)\right)\overline{z}^{k+1}\right\rangle=0.\end{aligned}$$

On the other hand, for $k=n$ we get

$$\left\langle\widetilde{\mathcal{L}},\Psi_{n+2}(z)+\nu_n^{-1}\left(\Psi_{n+1}(z)-\Phi_{n+1}(0)\Psi^*_{n+1}(z)\right)\overline{z}^{k+1}\right\rangle=\nu_n^{-1}\widetilde{k}_{n+1}=k_n.$$

Thus, $\{\Phi_n\}_{n\geqslant 0}$ is the sequence of monic polynomials orthogonal with respect to $\left(z+z^{-1}-(\alpha+\overline{\alpha})\right)\widetilde{\mathcal{L}}$. But then $\left(z+z^{-1}-(\alpha+\overline{\alpha})\right)\widetilde{\mathcal{L}}=\mathcal{L}$ and, therefore, $\widetilde{\mathcal{L}}=\mathcal{L}_{R^{(-1)}}$. □

Theorem 6.3.2. *Let $\mathcal{L}$ be a positive definite linear functional and σ its associated measure. If $\mathcal{L}_{R^{(-1)}}$ is a quasi-definite linear functional, then*

i) $(\Im(\widetilde{c}_1))^2\neq(1-(\Re(\alpha))^2)\widetilde{c}_0^2-\Re(\alpha)\widetilde{c}_0-\dfrac{1}{4}.$

ii) $\left(1-|\Phi_n(0)|^2\right)\nu_n^2 + A_{n+1}\nu_n + 1 - |\Psi_{n+1}(0)|^2 = 0,\ n \geqslant 1,$

where $A_n = \overline{\Psi_n(0)}\Phi_{n-1}(0) + \Psi_n(0)\overline{\Phi_{n-1}(0)} + \alpha + \overline{\alpha}$.

Proof. From (6.12), for $k=0$ and assuming that $c_0 = 1$, we have

$$\Re(\widetilde{c}_1) = \frac{1}{2} + \Re(\alpha)\widetilde{c}_0.$$

In addition, in order to be $\mathcal{L}_{R^{(-1)}}$ a quasi-definite functional, we need

$$\det \widetilde{\mathbf{T}}_1 = \begin{vmatrix} \widetilde{c}_0 & \widetilde{c}_1 \\ \widetilde{c}_{-1} & \widetilde{c}_0 \end{vmatrix} = \widetilde{c}_0^2 - (\Re(\widetilde{c}_1))^2 - (\Im(\widetilde{c}_1))^2 \neq 0,$$

where $\widetilde{\mathbf{T}}$ is the Toeplitz matrix associated with $\mathcal{L}_{R^{(-1)}}$, and $\widetilde{\mathbf{T}}_n$ is its corresponding $(n+1)\times(n+1)$ leading principal submatrix. Therefore, for the choice of α, we get

$$(\Im(\widetilde{c}_1))^2 \neq \widetilde{c}_0^2 - \left(\frac{1}{2} + \Re(\alpha)\widetilde{c}_0\right)^2,$$

which is *i*). Thus, $\widetilde{c}_0$ and $\Im(\widetilde{c}_1)$ are free parameters, while $\Re(\widetilde{c}_1)$ is determined by $\widetilde{c}_0$ and the choice of α.

Furthermore, we have

$$\begin{aligned} k_n = \langle \Phi_n, \Phi_n \rangle_{\mathcal{L}} = \langle \Psi_n, \Phi_n \rangle_{\mathcal{L}} &= \left\langle \left(z + z^{-1} - (\alpha + \overline{\alpha})\right)\Psi_n(z), \Phi_n(z) \right\rangle_{\mathcal{L}_{R^{(-1)}}} \\ &= -\left(\Psi_{n+1}(0)\overline{\Phi_n(0)} + \alpha + \overline{\alpha}\right)\widetilde{k}_n + \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{(-1)}}}. \end{aligned}$$

On the other hand, from (6.15),

$$\langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{(-1)}}} = -\overline{\Psi_{n+1}(0)}\Psi_n(0)\widetilde{k}_n - \frac{\widetilde{k}_n}{k_{n-1}}\widetilde{k}_n + \frac{\widetilde{k}_n}{k_{n-1}}\overline{\Psi_{n+1}(0)}\Phi_{n-1}(0)\widetilde{k}_n,$$

and from (6.17),

$$\begin{aligned} \langle \Psi_n(z), z\Phi_n(z)\rangle_{\mathcal{L}_{R^{(-1)}}} &= -\overline{\Psi_{n+1}(0)}\Psi_n(0)\widetilde{k}_n - \frac{\widetilde{k}_n}{k_{n-1}}\widetilde{k}_n \\ &\quad + \left(\Psi_n(0) - \Phi_n(0) + \frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right)\overline{\Psi_{n+1}(0)}\widetilde{k}_n \\ &= -\overline{\Psi_{n+1}(0)}\Phi_n(0)\widetilde{k}_n - \frac{\widetilde{k}_n}{k_{n-1}}\widetilde{k}_n + \frac{\widetilde{k}_n}{k_{n-1}}|\Psi_{n+1}(0)|^2\widetilde{k}_n. \end{aligned}$$

Thus, if $A_{n+1} = \overline{\Psi_{n+1}(0)}\Phi_n(0) + \Psi_{n+1}(0)\overline{\Phi_n(0)} + \alpha + \overline{\alpha}$, then

$$k_n = -A_{n+1}\widetilde{k}_n + \left(|\Psi_{n+1}(0)|^2 - 1\right)\frac{\widetilde{k}_n}{k_{n-1}}\widetilde{k}_n,$$
$$1 - |\Phi_n(0)|^2 = -A_{n+1}\frac{\widetilde{k}_n}{k_{n-1}} + \left|\frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0)\right|^2 - \left(\frac{\widetilde{k}_n}{k_{n-1}}\right)^2.$$

Since, from (6.17), $\frac{\widetilde{k}_n}{k_{n-1}}\Psi_{n+1}(0) = \frac{\widetilde{k}_n}{k_{n-1}}\Phi_{n-1}(0) + \Phi_n(0) - \Psi_n(0)$, we obtain

$$1 - |\Psi_n(0)|^2 = -A_n\frac{\widetilde{k}_n}{k_{n-1}} - \left(1 - |\Phi_{n-1}(0)|^2\right)\left(\frac{\widetilde{k}_n}{k_{n-1}}\right)^2.$$

Therefore,

$$\left(1 - |\Phi_{n-1}(0)|^2\right)\left(\frac{\widetilde{k}_n}{k_{n-1}}\right)^2 + A_n\frac{\widetilde{k}_n}{k_{n-1}} + 1 - |\Psi_n(0)|^2 = 0, \tag{6.18}$$

which is *ii*). □

Now, from (6.17),

$$\Psi_{n+1}(0) = \Phi_{n-1}(0) + [\Phi_n(0) - \Psi_n(0)]\frac{k_{n-1}}{\widetilde{k}_n} = \frac{(\Phi_n(0) - \Psi_n(0))\prod_{k=1}^{n-1}\left(1 - |\Phi_k(0)|^2\right)}{\prod_{k=1}^{n}\left(1 - |\Psi_k(0)|^2\right)\widetilde{c}_0} + \Phi_{n-1}(0), \quad n \geqslant 1.$$

We can thus build an algorithm to compute recursively the sequence $\{\Psi_{n+1}(0)\}_{n\geqslant 1}$, starting from $\Psi_1(0)$; see Algorithm 6.1.

Example: Chebyshev polynomials

Let $d\sigma(\theta)$ the measure associated with the Chebyshev polynomials defined in (4.26). It is well known Simon [2005] that the family of Verblunsky coefficients associated with $d\sigma$ is

$$\Phi_n(0) = \frac{1}{n+1}, \quad n \geqslant 1.$$

Now, let us consider the perturbation

$$d\widetilde{\sigma}(\theta) = \frac{|z-1|^2}{z + z^{-1} - (\alpha + \overline{\alpha})}\frac{d\theta}{2\pi}, \quad |z| = 1,$$

Algorithm 6.1 Check Regularity

input $\alpha, \widetilde{c}_0, \{\Phi_n(0)\}_{n\geqslant 1}$

$\Re(\widetilde{c_1}) = \frac{1}{2} + \Re(\alpha)\widetilde{c}_0$

if Theorem 6.3.2 *i*) **then**

$\quad \Psi_1(0) = -\dfrac{\widetilde{c}_1}{\widetilde{c}_0}$

end if

for $n = 1, 2, \ldots$ **do**

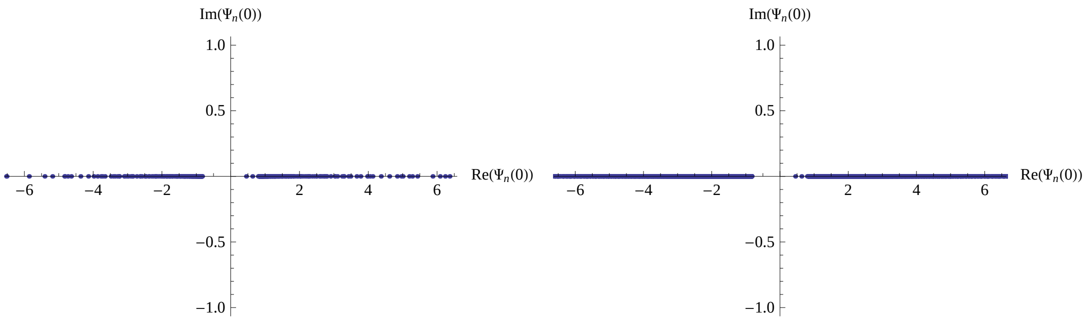
$$\Psi_{n+1}(0) = \frac{(\Phi_n(0) - \Psi_n(0)) \displaystyle\prod_{k=1}^{n-1}\left(1 - |\Phi_k(0)|^2\right)}{\displaystyle\prod_{k=1}^{n}\left(1 - |\Psi_k(0)|^2\right)\widetilde{c}_0} + \Phi_{n-1}(0)$$

$\quad$ **if** $|\Psi_{n+1}(0)| = 1$ **then**

$\quad\quad$ break

$\quad$ **end if**

end for



(a) $n = 500$ (b) $n = 10^3$

Figure 6.1: Verblunsky coefficients for Chebyshev polynomials with $\Re\alpha = 0.8$ and $a = 0.5i$

where $\Re(\alpha) = 0.6$. Notice that $b = 0.6 + 0.8i$. Then, according to (6.11),

$$\tilde{c}_0 = \frac{1}{1.6i}\left(\int_0^{2\pi} \frac{|e^{i\theta} - 1|^2 - 0.8}{1 - (0.6 + 0.8i)e^{-i\theta}} \frac{d\theta}{2\pi} - \int_0^{2\pi} \frac{|e^{i\theta} - 1|^2 - 0.8}{1 - (0.6 - 0.8i)e^{-i\theta}} \frac{d\theta}{2\pi}\right) = -1,$$

and

$$\begin{aligned}\tilde{c}_1 = \frac{1}{1.6i}\Bigg(&\int_0^{2\pi} \frac{(|e^{i\theta} - 1|^2 - 0.8)e^{i\theta}}{1 - (0.6 + 0.8i)e^{-i\theta}} \frac{d\theta}{2\pi} - \int_0^{2\pi} \frac{(|e^{i\theta} - 1|^2 - 0.8)e^{i\theta}}{1 - (0.6 - 0.8i)e^{-i\theta}} \frac{d\theta}{2\pi} \\ &+ \frac{1}{2}(0.6 + 0.8i)(0.8) - \frac{1}{2}(0.6 - 0.8i)(0.8)\Bigg) = 0.4.\end{aligned}$$

Observe that part *i*) of Theorem 6.3.2 holds. Applying the algorithm, the first 500 Verblunsky coefficients are shown in Figure 6.1(a). Notice that all of the new Verblunsky coefficients are real. They are distributed on both sides of the origin, in nearly symmetric intervals. If we repeat the computation for $n = 1500$, the values accumulate over such intervals. This is shown in Figure 6.1(b).

Example: Geronimus polynomials

In this subsection we consider a linear functional such that the corresponding measure is supported on an arc of the unit circle. Such a situation appears Geronimus [1954]; Simon [2005] when $\Phi_n(0) = a$, $n \geqslant 1$, with $0 < |a| < 1$. Here the measure σ associated with $\{\Phi_n(0)\}_{n\geqslant 1}$ is supported on the arc

$$\Delta_\nu = \{e^{i\theta} : \nu \leqslant \theta \leqslant 2\pi - \nu\},$$

with $\cos(\nu/2) = \sqrt{1-|a|^2}$, but it can have a mass point located on $\mathbb{T}$. The orthogonality measure σ is given by

$$d\sigma(\theta) = \frac{\sqrt{\sin\left(\frac{\theta+\nu}{2}\right)\sin\left(\frac{\theta-\nu}{2}\right)}}{2\pi\sin\left(\frac{\theta-\tau}{2}\right)}\,d\theta + \boldsymbol{m}_\tau\delta(z-e^{i\tau}), \tag{6.19}$$

where $e^{i\tau} = \dfrac{1-a}{1-\overline{a}}$ and

$$\boldsymbol{m}_\tau = \begin{cases} \dfrac{2|a|^2-a-\overline{a}}{|1-a|}, & \text{if } |1-2a| > 1, \\ 0, & \text{if } |1-2a| \leqslant 1. \end{cases}$$

Moreover, the orthonormal polynomials associated with σ are given by

$$\varphi_n(z) = \frac{1}{(1-|a|^2)^{n/2}}\left((z+a)\frac{z_1^n - z_2^n}{z_1-z_2} - z\left(1-|a|^2\right)\frac{z_1^{n-1}-z_2^{n-1}}{z_1-z_2}\right), \quad n\in\mathbb{N},$$

with

$$z_1 = \frac{z+1+\sqrt{(z-e^{i\nu})(z-e^{-i\nu})}}{2}, \quad z_2 = \frac{z+1-\sqrt{(z-e^{i\nu})(z-e^{-i\nu})}}{2}.$$

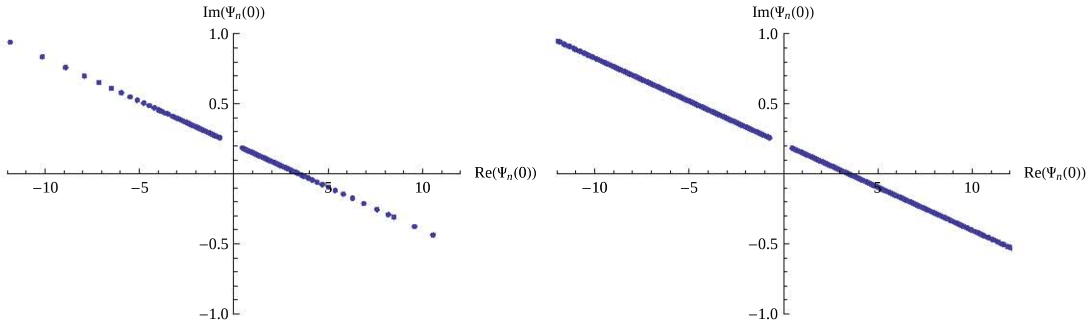
Consider a perturbation of (6.19) given by

$$d\widetilde{\sigma}(z) = \frac{d\sigma(z)}{z+z^{-1}-(\alpha+\overline{\alpha})},$$

with $\Re\alpha = 0.8$ and $a = 0.5i$. Notice that in this case $b = 0.8+0.6i$ and thus $b \notin \Delta_\nu$. Then,

$$\widetilde{c}_0 = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{\sqrt{\sin\left(\frac{1}{2}\theta+\frac{1}{6}\pi\right)\sin\left(\frac{1}{2}\theta-\frac{1}{6}\pi\right)}}{2(\cos\theta-0.8)\pi\sin(\theta/2)}\,d\theta = -0.45876,$$

$$\widetilde{c}_1 = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \frac{(\cos\theta+i\sin\theta)\sqrt{\sin\left(\frac{1}{2}\theta+\frac{1}{6}\pi\right)\sin\left(\frac{1}{2}\theta-\frac{1}{6}\pi\right)}}{2(\cos\theta-0.8)\pi\sin(\theta/2)}\,d\theta = 0.13299,$$



(a) $n = 500$ (b) $n = 10^3$

Figure 6.2: Verblunsky coefficients for Geronimus polynomials with $\Re\alpha = 0.8$ and $a = 0.5i$

and i) holds. In such a situation, the algorithm becomes

$$\Psi_{n+1}(0) = \frac{[a - \Psi_n(0)](1 - |a|^2)^{n-1}}{\displaystyle\prod_{k=1}^{n}(1 - |\Psi_k(0)|^2)\widetilde{c}_0} + a, \quad n \geqslant 1,$$

and the first 500 Verblunsky coefficients are shown in Figure 6.2(a). As it is shown in this figure, the Verblunsky coefficients associated with the modified measure have the same argument with respect to a certain point (the value of a). That is, they are located on a straight line, on both sides of a. When n increases, the density of the points on the line increases, as shown in Figure 6.2(b).

6.3.2 C-functions

Assuming that $\mathcal{L}_{R^{(-1)}}$ is a quasi-definite linear functional, we denote its associated C-function by $F_{R^{(-1)}}$. Multiplying (6.12) by z^k, $k \geqslant 1$, and replacing in (2.38), we get

$$\sum_{k=1}^{\infty} c_{-k} z^k = \sum_{k=1}^{\infty} \widetilde{c}_{-(k+1)} z^k + \sum_{k=1}^{\infty} \widetilde{c}_{-(k-1)} z^k - (\alpha + \overline{\alpha}) \sum_{k=1}^{\infty} \widetilde{c}_{-k} z^k,$$
$$F(z) - 1 = \left(z + z^{-1} - (\alpha + \overline{\alpha})\right) F_{R^{(-1)}}(z) + \widetilde{c}_0 \left(z - z^{-1} + (\alpha + \overline{\alpha})\right) - 2\widetilde{c}_{-1}.$$

Therefore

$$F_{R^{(-1)}}(z) = \frac{F(z) + (z^{-1} - z - (\alpha + \overline{\alpha}))\widetilde{c}_0 + 2\widetilde{c}_{-1} - 1}{z + z^{-1} - (\alpha + \overline{\alpha})}.$$

Notice that from (6.12), $1 + (\alpha + \overline{\alpha})\widetilde{c}_0 = \widetilde{c}_1 + \widetilde{c}_{-1}$, and thus

$$F_{R^{(-1)}}(z) = \frac{zF(z) - \widetilde{c}_0 z^2 + (\widetilde{c}_{-1} - \widetilde{c}_1)z + \widetilde{c}_0}{z^2 - (\alpha + \overline{\alpha})z + 1}, \tag{6.20}$$

which is equivalent to

$$F_{R^{(-1)}}(z) = \frac{zF(z)}{z^2-(\alpha+\overline{\alpha})z+1} + \boldsymbol{m}_1\frac{z+b}{z-b} + \boldsymbol{m}_2\frac{z+\overline{b}}{z-\overline{b}}, \tag{6.21}$$

where b, $\overline{b}$ are the zeros of $z^2-(\alpha+\overline{\alpha})z+1$, with $|b|=1$, and

$$\boldsymbol{m}_1 = -\frac{1}{2}\left(\widetilde{c}_0 + \frac{\Im(\widetilde{c}_1)}{\Im(b)}\right), \quad \boldsymbol{m}_2 = -\frac{1}{2}\left(\widetilde{c}_0 - \frac{\Im(\widetilde{c}_1)}{\Im(b)}\right).$$

On the other hand, from (6.21)

$$\begin{aligned} F_{R^{(-1)}}(z) &= \left(\frac{b}{(b-\overline{b})(z-b)} - \frac{\overline{b}}{(b-\overline{b})(z-\overline{b})}\right)F(z) - \boldsymbol{m}_1(1+\overline{b}z)\sum_{k=0}^{\infty}\frac{z^k}{b^k} - \boldsymbol{m}_2(1+bz)\sum_{k=0}^{\infty}b^kz^k \\ &= \left(\sum_{k=1}^{\infty}\frac{b^k-\overline{b}^k}{b-\overline{b}}z^k\right)F(z) - \boldsymbol{m}_1\left(1+2\sum_{k=1}^{\infty}\overline{b}^kz^k\right) - \boldsymbol{m}_2\left(1+2\sum_{k=1}^{\infty}b^kz^k\right), \end{aligned}$$

and thus

$$\widetilde{c}_0 + 2\sum_{k=1}^{\infty}\widetilde{c}_{-k}z^k = \left(\sum_{k=1}^{\infty}\frac{b^k-\overline{b}^k}{b-\overline{b}}z^k\right)\left(1+2\sum_{k=1}^{\infty}c_{-k}z^k\right) - \boldsymbol{m}_1\left(1+2\sum_{k=1}^{\infty}\overline{b}^kz^k\right) - \boldsymbol{m}_2\left(1+2\sum_{k=1}^{\infty}b^kz^k\right).$$

Therefore, comparing coefficients of z^n on both sides of the last expression, we have for $n \geqslant 2$,

$$\widetilde{c}_{-n} = \sum_{k=1}^{n-1}\frac{b^k-\overline{b}^k}{b-\overline{b}}c_{-(n-k)} + \frac{1}{2}\frac{b^n-\overline{b}^n}{b-\overline{b}} - \boldsymbol{m}_1\overline{b}^n - \boldsymbol{m}_2b^n.$$

Comparing the independent terms and the coefficients for z we can deduce (6.12) for $n=0$ and $n=1$. Furthermore, denoting this transformation by $\mathcal{F}_{R^{(-1)}}$,

$$\mathcal{F}_R(\alpha)\circ\mathcal{F}_{R^{(-1)}}(\alpha) = \mathcal{I}, \qquad \mathcal{F}_{R^{(-1)}}(\alpha)\circ\mathcal{F}_R(\alpha) = \mathcal{F}_U(b,\widehat{\boldsymbol{m}}_1)\circ\mathcal{F}_U(\overline{b},\widehat{\boldsymbol{m}}_2),$$

that follows in a straightforward way from the definition of $\mathcal{F}_R$ and $\mathcal{F}_{R^{(-1)}}$.

Denoting $H(z) = (\mathcal{F}_{R^{(-1)}}(\alpha)\circ\mathcal{F}_R(\alpha))[F(z)]$,

$$H(z) = \frac{zF_R(z) - \widetilde{c}_0z^2 + (\widetilde{c}_{-1}-\widetilde{c}_1)z + \widetilde{c}_0}{z^2-(\alpha+\overline{\alpha})z+1} = F(z) + \widehat{\boldsymbol{m}}_1\frac{z+b}{z-b} + \widehat{\boldsymbol{m}}_2\frac{z+\overline{b}}{z-\overline{b}},$$

with $\widehat{\boldsymbol{m}}_1 = \widetilde{\boldsymbol{m}}_1 + \boldsymbol{m}_1$, $\widehat{\boldsymbol{m}}_2 = \widetilde{\boldsymbol{m}}_2 + \boldsymbol{m}_2$, and

$$\widetilde{\boldsymbol{m}}_1 = \frac{1}{2}\left(1 + \frac{\Im(c_1)}{\Im(b)}\right), \quad \widetilde{\boldsymbol{m}}_2 = \frac{1}{2}\left(1 - \frac{\Im(c_1)}{\Im(b)}\right).$$

As in (6.8), given a finite composition of order $k \in \mathbb{N}$ of $\mathcal{L}_{R^{(-1)}}$ defined by

$$\mathcal{L} = \Re\left(\prod_{i=1}^{k}(z-\alpha_i)\right)\mathcal{L}_{R^{(-k)}}, \qquad \alpha_i \in \mathbb{C}, \tag{6.22}$$

we can deduce

Theorem 6.3.3. *A generic spectral transformation with $A = 1$ and $C = 0$ is equivalent to (6.22). Furthermore, B and D are given by*

$$B(z) = \widetilde{P}(z) - \widetilde{P}_*(z), \qquad D(z) = \Re p(z), \tag{6.23}$$

where $\widetilde{P}$ is the polynomial of second kind of

$$p(z) = \prod_{i=1}^{k}(z-\alpha_i)$$

with respect to the linear functional (6.22).

Proof. From Theorem 6.23 it is straightforward to show that the C-function $F_{\mathcal{L}_{R^{(-1)}}}$ is the reciprocal of $F_{\mathcal{L}_r}$. However, in contrast to F_R, $F_{R^{(-1)}}$ contains three free parameters α, c_0 and $\Im c_1$, while $\Re c_1$ is determined by c_0 and the choice of α.

Conversely, if we start with a spectral transformation with $A = 1$, B is a hermitian Laurent polynomial of degree one and $C = 0$, then D is a hermitian Laurent polynomial of degree one with three restrictions for their coefficients. We hence get just (6.20). Using an analog of Theorem 6.23 we can complete the proof. $\square$

Notice that for different values $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\beta_1, \beta_2, \dots, \beta_r$, $\mathcal{F}_{R^{(k)}}[F] = F_{R^{(k)}}$ and $\mathcal{F}_{R^{(-k)}}[F] = F_{R^{(-k)}}$ we get

$$\mathcal{F}_{R^{(k)}}(\alpha_1, \alpha_2, \dots, \alpha_r) \circ \mathcal{F}_{R^{(-k)}}(\beta_1, \beta_2, \dots, \beta_r) = \mathcal{F}_{R^{(-k)}}(\beta_1, \beta_2, \dots, \beta_r) \circ \mathcal{F}_{R^{(k)}}(\alpha_1, \alpha_2, \dots, \alpha_r).$$

Moreover, for the same parameters, we have the following relations

$$\begin{aligned}
&(\mathcal{F}_{R^{(-k)}} \circ \mathcal{F}_{R^{(k)}})(\alpha_1, \alpha_2, \dots, \alpha_r) = \mathcal{I} \\
&(\mathcal{F}_{R^{(k)}} \circ \mathcal{F}_{R^{(-k)}})(\alpha_1, \alpha_2, \dots, \alpha_r) = \mathcal{F}_U(\alpha_1, \dots, \alpha_r) \circ \mathcal{F}_U(\bar{\alpha}_1, \dots, \bar{\alpha}_r),
\end{aligned}$$

where $\mathcal{F}_U(\alpha_1, \dots, \alpha_k)$ is the so-called general Uvarov spectral transformation as the result of the addition of masses at the points $z = \alpha_1, z = \alpha_2, \dots, z = \alpha_r$; see Appendix B.

6.4 Rational spectral transformations

Spectral transformations under the modification of a finite number of moments are given by

$$\widetilde{F}_j(z) = F(z) + E(z),$$

where $E(z) = \sum_{j\in G} m_j z^j$ for $m_j \in \mathbb{C}$ and G a finite subset of non-negative integer numbers. Hence, a generator system of local spectral transformations follows immediately from Chapter 4.

Theorem 6.4.1. *A local spectral transformation can be obtained as a finite composition of spectral transformations associated with the linear functionals $\mathcal{L}_j$ defined in (4.32).*

In general, a global spectral transformation can be represented by the following rational expression (2.42). In order to prove our main results in the following lemma we characterize the polynomial coefficients of (2.42).

Lemma 6.4.1. *Only one of the following two statements holds:*

i) The polynomial coefficients in (2.42) are hermitian Laurent polynomials of the same degree such that

$$A_*(z) = A(z), \quad B_*(z) = -B(z), \quad C_*(z) = -C(z), \quad D_*(z) = D(z).$$

ii) The polynomial coefficients in (2.42) are self-reciprocal polynomials of the same degree such that

$$A^*(z) = A, \quad B^*(z) = -B(z), \quad C^*(z) = -C(z), \quad D^*(z) = D(z).$$

Proof. Since F is the rational spectral transformation associated with (2.42) given by

$$F(z) = \frac{-D(z)\widetilde{F}(z) + B(z)}{C(z)\widetilde{F}(z) - A(z)}, \tag{6.24}$$

multiplying and dividing (6.24) for z^l, where l is the minimum non-negative integer number such that $z^l A(z)$, $z^l B(z)$, $z^l C(z)$, and $z^l D(z)$ are polynomials, and using the characterization of orthogonal polynomials with respect to the functional $\mathcal{L}$ (2.40)-(2.41), we immediately get

$$z^l(-\Phi_n(z)D(z) - \Omega_n(z)C(z))\widetilde{F}(z) - z^l(-\Phi_n(z)B(z) - \Omega_n(z)A(z)) = O(z^{n+\nu}),$$
$$z^l(-\Phi_n^*(z)D(z) + \Omega_n^*(z)C(z))\widetilde{F}(z) + z^l(\Phi_n^*(z)B(z) - \Omega_n^*(z)A(z)) = O(z^{n+1+\nu}),$$

where ν is a positive integer such that

$$z^l(A(z) - C(z)\widetilde{F}(z)) = O(z^\nu).$$

Therefore, we can deduce that the new polynomial coefficients $z^l A(z)$, $z^l B(z)$, $z^l C(z)$, and $z^l D(z)$ satisfy the following relations

$$(z^l A(z))^* = z^l A(z),\ (z^l B(z))^* = -z^l B(z),\ (z^l C(z))^* = -z^l C(z),\ \text{and}\ (z^l D(z))^* = z^l D(z).$$

Suppose that $l \neq 0$. Then, A, B, C, and D are Laurent hermitian polynomials of the same degree l, which prove *i*). On the other hand, if we suppose that $l = 0$, then A, B, C, and D are self-reciprocal polynomials of the same degree. Thus, *ii*) follows. □

If we have a generic rational spectral transformation with self-reciprocal polynomial coefficients of odd degree, then it can not be transformed into a equivalent rational spectral transformation with hermitian Laurent polynomial coefficients. We use the following result concerning the symmetrization of sequence of orthogonal polynomials to study this problem.

Theorem 6.4.2. Marcellán and Sansigre [1991] *There exists one and only one sequence of monic polynomials $\{\Psi_n\}_{n\geqslant 0}$ orthogonal with respect to a hermitian linear functional such that*

$$\Psi_{2n}(z) = \Phi_n(z^2), \quad n \geqslant 0.$$

Furthermore,

$$\Psi_{2n+1}(z) = z\Phi_n(z^2), \quad n \geqslant 0.$$

Notice that the odd Verblunsky coefficients for the new sequence, $\{\Psi(0)_n\}_{n\geqslant 0}$, are zero. A linear functional $\mathcal{L}$ is said to be symmetric if all its moments of odd order are 0, i.e.,

$$\left\langle \mathcal{L}, z^{2n+1} \right\rangle = 0, \quad n \geq 0.$$

It is interesting to recall that in the real line case if we look for $\{Q_n\}_{n\geqslant 0}$, which is a sequence of monic orthogonal polynomials and such that $Q_{2n+1}(x) = xP_n(x^2)$, there is not a unique solution, i.e., we can find an infinity number of sequences of polynomials $\{R_n\}_{n\geqslant 0}$ such that $Q_{2n}(x) = R_n(x^2)$.

Theorem 6.4.3. *A generic rational spectral transformation with self-reciprocal polynomial coefficients of odd degree has symmetric generators to the corresponding symmetric rational spectral transformation which has hermitian Laurent polynomial coefficients.*

Proof. Let $\left\{\widetilde{\Phi}_n\right\}_{n\geqslant 0}$ be the sequence of monic orthogonal polynomials associated with the C-functions, $\widetilde{F}$, a rational spectral transformation of F given by (2.42). We assume that A, B, C, and D are self-reciprocal polynomials of odd degree. From Theorem 6.4.2 there is an unique sequence of monic orthogonal polynomials $\{\psi_n\}_{n\geqslant 0}$ (respectively $\left\{\widetilde{\psi}_n\right\}_{n\geqslant 0}$) such that

$$\widetilde{\psi}_{2n}(z) = \widetilde{\phi}_n(z^2), \qquad \psi_{2n}(z) = \phi_n(z^2), \qquad n \geq 0,$$

with respect to the symmetric linear functional $\mathcal{D}$ (respectively $\widetilde{\mathcal{D}}$).

On the other hand, the corresponding symmetric C-functions associated with the symmetric functionals $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are $F_{\mathcal{D}}(z) = F(z^2)$ and $F_{\widetilde{\mathcal{D}}}(z) = \widetilde{F}(z^2)$, respectively. Therefore,

$$F_{\widetilde{\mathcal{D}}}(z) = \frac{A(z^2)F_{\mathcal{D}}(z) + B(z^2)}{C(z^2)F_{\mathcal{D}}(z) + D(z^2)}$$

is a rational spectral transformation associated with $F_{\mathcal{D}}$ with self-reciprocal polynomial coefficients of even degree. Thus, $F_{\widetilde{\mathcal{D}}}$ is equivalent to a rational spectral transformation with hermitian Laurent polynomial coefficients. □

According to the previous theorem, throughout this chapter we consider only hermitian Laurent polynomial coefficients.

Example: Aleksandrov transformation

Notice that if the coefficients A, B, C, and D of (2.42) are constant, then A and D are real numbers and B and C are pure imaginary numbers. An example of this is the Aleksandrov transformation Simon [2005].

Let $\{\Phi_n(0)\}_{n\geqslant 1}$ be the family of Verblunsky coefficients, and let λ be a complex number with $|\lambda| = 1$. When we consider a new family of Verblunsky coefficients defined by $\{\Phi_n^{\lambda}(0)\}_{n\geqslant 1}$, with $\Phi_n^{\lambda}(0) = \lambda\Phi_n(0)$, $n \geq 0$, the resulting transformation is called Aleksandrov transformation. The C-functions are related by

$$F^{\lambda}(z) = \frac{(\lambda+1)F(z) + \lambda - 1}{(\lambda-1)F(z) + \lambda + 1}.$$

6.4.1 Generator system for rational spectral transformations

Theorem 6.4.4. *A generic linear spectral transformation can be obtained as a finite composition of spectral transformations associated with a modification of the functional by the real part of a polynomial and its inverse.*

Proof. Let $H = \mathcal{F}[F]$ be the C-function obtained from (2.38) after a generic linear spectral transformation with hermitian Laurent polynomial coefficients. Hence, we can apply to F the finite composition of $\mathcal{L}_R$ given in (6.23). Thus, we get a new C-function as a result of the composition

$$\mathcal{F}_{R^{(k)}}[H(z)] = \frac{1/2(\Re p(z))A(z)F(z) + 1/2(\Re p(z))B(z) + (P(z) - P_*(z))D(z)}{D(z)}$$

with some polynomials such that the zeros of the polynomial p can be chosen as the parameters of the spectral transformation. Hence, we can always choose $\Re p = \tau D$, where τ is a constant. Thus, we get

$$\mathcal{F}_{R^{(k)}}[H(z)] = \tau A(z)F(z) + \tau B(z) + P(z) - P_*(z). \tag{6.25}$$

This transformation is reduced to the case considered in (6.10). Therefore, from Theorem 6.23 the C-function $\mathcal{F}_{R^{(k)}}[H]$ is obtained from (2.38) by means of a finite composition of (2.43).

On the other hand, from (6.25) and using a finite composition of (2.44), we get the C-function

$$H(z) = \mathcal{F}_{R^{(-k)}}\left[\mathcal{F}_{R^{(k)}}\left[\mathcal{F}[F(z)]\right]\right],$$

and so our statement holds. □

Theorem 6.4.5. *A generic rational spectral transformation can be obtained as a finite composition of linear and associated canonical spectral transformations.*

Proof. Assuming hermitian Laurent polynomial as coefficients of the perturbed linear functional, the application of backward transformations with even degree $2k$ to the real spectral transformation (2.42) of degree k yields a new rational spectral transformation where the transformed Laurent polynomial $\widetilde{C}$ is hermitian and is given by

$$\widetilde{C}(z) = z^k\left(A(z)+C(z)\right)(\widetilde{\phi}_{2k})_*(z) + z^{-k}\left(C(z)-A(z)\right)\widetilde{\phi}_{2k}(z).$$

Notice that the polynomial $\widetilde{\phi}_{2k}$ can be chosen in an arbitrary way. Indeed, instead of choosing arbitrary $2k$ Verblunsky parameters we can choose the polynomial $\widetilde{\phi}_{2k}$ satisfying $|\widetilde{\phi}_{2k}(0)| \neq 1$, and from the Schur-Cohn-Jury criterion we obtain a sequence of complex numbers $\widetilde{\phi}_{2k}(0), \ldots, \widetilde{\phi}_1(0)$ with modulus different of 1. Let $(\phi_n(0))_{n\geq 1}$ be the Verblunsky coefficients of the hermitian linear functional associated with (2.42). Then, by Favard's theorem $(\widetilde{\phi}_i(0))_{i=1}^{2k} \bigcup (\phi_n(0))_{n\geq 1}$ arises as a new sequence of Verblunsky coefficients of a hermitian linear functional. Notice that it is unique.

On the other hand, in order to preserve the hermitian character of $\widetilde{C}$, the principal leading coefficients of A and C are different and not symmetric with respect to the origin. Thus,

$$\deg\left(A(z)+C(z)\right) = \deg(C(z)-A(z)) = k.$$

Moreover, without loss of generality, the polynomial $z^k(A(z)+C(z))$ evaluated at $z=0$ can be chosen in such a way that its modulus is different from one. Therefore, we can choose the polynomial $\widetilde{\phi}_{2k}$ such that

$$\widetilde{\phi}_{2k}(z) = -z^k(C(z)+A(z)).$$

Using Lemma 6.4.1 we obtain the reciprocal polynomial

$$(\widetilde{\phi}_{2k})_*(z) = z^{-k}(C(z)-A(z)),$$

leading to $\widetilde{C} = 0$. But this means that we can reduce our rational spectral transformation to a linear spectral transformation and the result follows. □

Chapter 7

Conclusions and open problems

'Would you tell me, please, which way I ought to go from here?'

'That depends a good deal on where you want to get to,' said the Cat.

'I don't much care where —' said Alice.

'Then it doesn't matter which way you go,' said the Cat.

'— so long as I get SOMEWHERE,' Alice added as an explanation.

'Oh, you're sure to do that,' said the Cat, 'if you only walk long enough.'

— L. Carroll. *Alice's Adventures in the Wonderland.* Random House, New York, 1865

7.1 Conclusions

The results showed in this work have been focussed on the study of spectral transformations of linear functionals.

We have presented an alternative approach to the theory of orthogonal polynomials, giving a central role to continued fractions. The results have been proved under the assumption that the Verblunsky coefficients $\{\Phi_n(0)\}_{n\geq 0}$ satisfy $|\Phi_n(0)| > 1$. As a consequence, we have obtained specific properties of the corresponding Hankel and Toeplitz determinants which we have used to deduce necessary and sufficient conditions for the existence of the moment problem associated with Chebyshev polynomials of the first kind.

We also proved that the zeros of orthogonal polynomial $\Phi_n(\omega;\cdot)$ satisfy, for $\Phi_n(0) > 1$, $n \geq 1$,

$$0 < z_{n,1}(\omega) < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < z_{n,n}(\omega), \quad \omega < 1,$$

$$z_{n,1}(\omega) < 0 < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < z_{n,n}(\omega), \quad \omega > 1.$$

and, for $\Phi_n(0) < 1$, $n \geq 1$,

$$z_{n,1}(\omega) < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < z_{n,n}(\omega) < 0, \quad \omega < 1,$$

$$z_{n,1}(\omega) < z_{n,2}(\omega) < \ldots < z_{n,n-1}(\omega) < 0 < z_{n,n}(\omega), \quad \omega > 1.$$

The properties of the zeros allowed us to define a step function, and by using Helly's selection principle we have showed that an rsq-definite linear functional has an integral representation in terms of a measure that belongs to the class $\mathcal{S}^3(0,1,b)$. This result played a key role to find conditions for the determinacy of our moment problem.

Motivated by applications in integrable systems, we have obtained explicit expressions of the orthogonal polynomials associated with the perturbations of a hermitian linear functional in terms of the first one. In the same direction, we defined a perturbation of a quasi-definite linear functional by the addition of the first derivative of the Dirac linear functional when its support is a point on the unit circle or two points symmetric with respect to the unit circle. In both cases we get

$$\lim_{n\to\infty} \frac{\Psi_n(z)}{\Phi_n(z)} = 1, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

where $\{\Psi_n\}_{n\geqslant 0}$ is the sequence of orthogonal polynomials associated with the corresponding perturbed functional. Nevertheless, we are far from achieving our initial goals.

We have analyzed the previous results for asymptotics of the discrete Sobolev polynomials and their zeros when the mass tends to infinity. The study of these systems was motivated by the search of efficient algorithms for computing Fourier expansions of a function in terms of discrete Sobolev orthogonal polynomials. In this direction very few results are known in the literature. We have stated that the resulting polynomials tend to a linear combination of polynomial perturbations of several orders, i.e.,

$$\phi_n(z), (z-\alpha)\phi_{n-1}(z, d\sigma_1), \ldots, (z-\alpha)^{j+1}\phi_{n-j-1}(z, d\sigma_{j+1}).$$

We have characterized the eigenvalues of the Hessenberg matrix $(\mathbf{H}_\Psi)_n$ associated with the discrete Sobolev polynomials, i.e., the zeros of the discrete Sobolev polynomials, as the eigenvalues of a rank one perturbation of the Hessenberg matrix associated with the measure σ, i.e.,

$$(\mathbf{H}_\Psi)_n = \mathbf{L}_n(\mathbf{H}_n - \mathbf{A}_n)\mathbf{L}_n^{-1},$$

where $\mathbf{A}_n$ and $\mathbf{L}_n$ are known. As a consequence, the zeros of Ψ_n are the eigenvalues of the matrix $\mathbf{H}_n - \mathbf{A}_n$, a rank one perturbation of the matrix $\mathbf{H}_n$. We also proved a characterization for the limit of these discrete Sobolev polynomials, when the mass tends to infinity as extremal polynomials. The results have been deduced for measures on the Nevai class, so in this sense they are sharp. We also showed that, as an analog of the real line case, a generic rational spectral transformation of the C-function is a finite composition of four canonical spectral transformations which we have studied in detail. To obtain this result it was essential our classification and characterization of the polynomial coefficients of rational spectral transformations obtained from the characterization of orthogonal polynomials with respect to a functional given by Peherstorfer and Steinbauer. All our results are based on the strong relationship between the theory of orthogonal polynomials on the real line and on the unit circle.

7.2 Open problems

In this section we formulate some open questions which have arisen during our research.

Zeros of Szegő-type polynomials

The Szegő-type polynomials Lamblém [2011]; Lamblém et al. [2010] are orthogonal with respect to a moment linear functional $\mathcal{T}$, such that their moments

$$c_n = \langle \mathcal{T}, z^n \rangle = c_{-n}, \quad n \geqslant 0,$$

are all complex. If the linear functional $\mathcal{T}$ is such that $\{c_n\}_{n\in\mathbb{Z}}$ is real and $(-1)^{n(n+1)/2}\det \mathbf{T}_n > 0$, $n \geqslant 0$, then the zeros of associated polynomials have been studied in Chapter 3. We were unable to obtain the localization and asymptotic behavior of the zeros of Szegő-type orthogonal polynomials. In Lamblém [2011]; Lamblém et al. [2010] the authors establish that the hypergeometric functions

$$\Phi_n(b;z) = \frac{(2b+1)_n^+}{(b+1)_n^+} {}_2F_1(-n, b+1; 2b+2; 1-z), \quad 2b \neq \mathbb{Z}_-, \quad n \geq 0,$$

are the Szegő-type polynomials with respect to a moment functional with moments $\{c_n(b)\}_{n\geq 0}$,

$$c_0(b) = 2, \quad c_n(b) = 2\frac{(-b)_n^+}{(b+1)_n^+} = c_{-n}(b), \quad n \geq 1.$$

Figure 7.1 shows the behavior of the zeros of $\Phi_n(b;\cdot)$ for fixed $n = 100$ and several values of the parameter b, namely $b = 1000$ (green triangles) $b = 100i$ (blue discs), $b = 5+6i$ (purple square), and $b = 11$ (yellow diamonds).

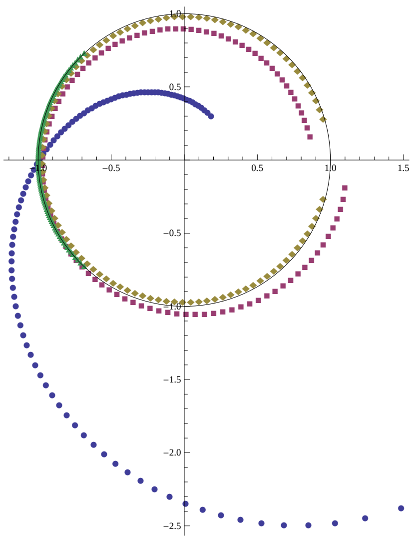


Figure 7.1: Zeros of the Szegő-type polynomial ${}_2F_1(-100, b+1; 2b+2; 1-z)$

Perturbations on anti-diagonals of Hankel matrices

Given $j \geq 0$ it is natural to ask if the corresponding perturbation $\mathcal{M}_j$, studied in Chapter 4, preserves the positive definiteness of $\mathcal{M}$. Of course, the necessary and sufficient conditions are given in Theorem 4.1.1. However, if one is interested in the existence of a neighborhood (τ_1, τ_2) such that the functional $\mathcal{M}_j$ is positive definite for every $m_j \in (\tau_1, \tau_2)$, then to determine such an interval can be very complicated. An open problem is to analyze if there exists a different approach that allows one to determine the values of m_j such that $\mathcal{M}_j$ is positive definite. Certainly, the interval (τ_1, τ_2) should depend essentially on the initial functional $\mathcal{M}$ and the point a.

Another question that might be of interest is if given two positive definite moment functionals $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, then there exists a sequence of perturbations $\mathcal{M}_{j_k}$ such that we can obtain $\widetilde{\mathcal{M}}$ from the consecutive applications of those perturbations to $\mathcal{M}$, i.e.,

$$\mathcal{M} \xrightarrow{\mathcal{M}_{j_1}} \mathcal{M}^{(1)} \xrightarrow{\mathcal{M}_{j_2}} \mathcal{M}^{(2)} \xrightarrow{\mathcal{M}_{j_3}} \cdots \xrightarrow{\mathcal{M}_{j_k}} \mathcal{M}^{(k)} \to \widetilde{\mathcal{M}},$$

with the condition that the positiveness must be preserved in each step. As an example, consider the linear functionals $\mathcal{M}^1$ and $\mathcal{M}^2$, associated with the Chebyshev polynomials of first and second kind, respectively. When using the basis $\{1, (x-1), (x-1)^2, \ldots\}$, it is easy to see that one of the sequences of moments can be obtained from the other one by means of a shift. Thus, as explained in Chapter 4, one can go from $\mathcal{M}^1$ to $\mathcal{M}^2$ applying a sequence of perturbations $\mathcal{M}_{j_k} = \mathcal{M}_k$, $k = 0, 1, \ldots$, with $a = 1$, and m_{j_k}. However, proceeding in such a way, the positive definiteness would be lost after second step. Nevertheless, another sequence $\mathcal{M}_{j_k}$ which preserves the positive definiteness may still exist.

Zeros of Sobolev orthogonal polynomials on the real line

Several people have studied the zeros of the polynomials $S_n(\boldsymbol{\lambda}, \mathbf{c}, r; \cdot)$, with $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_r)$ and $\mathbf{c} = (c_0, c_1, \ldots, c_r)$, orthogonal with respect to the discrete Sobolev inner product of the form

$$\langle p, q \rangle_{\mathcal{D}_r} = \int_I p(x)q(x)d\mu(x) + \sum_{i=0}^{r} \lambda_i p^{(i)}(c_i)q^{(i)}(c_i), \quad c_i \in \mathbb{R}, \quad \lambda_i \geq 0, \quad r \geq 0, \tag{7.1}$$

where μ is a positive Borel measure supported on $I \subseteq \mathbb{R}$. In the Appendix A we consider a particular case of (7.1).

We now formulate some questions: fix λ_i, $i = 0, 1, \ldots, k-1, k+1, \ldots, r$, in (7.1), and consider the zeros of S_n as functions of λ_k. Then, how many zeros of S_n lie in the interior of the convex hull of I? Moreover, are these zeros monotonic functions with respect to the parameter λ_k? Do they converge when λ_k goes to infinity? If so, what is the speed of convergence? For some partial results towards these problems, for specific measures and vectors $\boldsymbol{\lambda}$, we refer to Alfaro et al. [1992, 2010]; Meijer [1993]; Pérez and Piñar [1993], Rafaeli [2010] and the references therein. The most general result in this direction was obtained in Alfaro et al. [1996]. There it is stated that every $S_n(\boldsymbol{\lambda}, \mathbf{c}, r; \cdot)$ possesses at least $n - r$ zeros in the interior of the convex hull of I, when $c_0 = c_1 = \ldots = c_r$.

Infinity discrete Sobolev inner products on a rectifiable Jordan curve or arc

The sequences of orthogonal polynomials associated with an orthogonality measure with finitely many point masses outside a curve or arc have also been studied by Kaliaguine and Kononova Kaliaguine [1993, 1995]; Kaliaguine and Kononova [2010]. The case of an infinity number of mass point for measures of the Szegő supported on the real line is considered by Peherstorfer and Yuditskii in Peherstorfer and Yuditskii [2001]. Let $f(z)$ be a function of one variable and let

$$Z = (\underbrace{z_1,\ldots,z_1}_{l_1},\underbrace{z_2,\ldots,z_2}_{l_2},\ldots,\underbrace{z_m,\ldots z_m}_{l_m},\ldots)$$

be an infinite vector. Denote

$$\boldsymbol{f}(Z) = (f(z_1),\ldots,f^{(l_1)}(z_1),\ldots,f(z_m),\ldots f^{(l_m)}(z_m),\ldots).$$

The infinite discrete Sobolev inner product in the complex plane is defined by

$$\langle f,g\rangle_{S_I} = \int_C f(z)\overline{g(z)}\omega(z)|dz| + \mathbf{f}(Z)\ \mathbf{A}\ \mathbf{g}(Z)^H, \tag{7.2}$$

where C is a rectifiable Jordan curve or arc in the complex plane; $\mathbf{A}_M$, the principal $M\times M$ hermitian submatrix of the infinity matrix $\mathbf{A}$, is quasi-definite; $z_i \in \Omega$, $i = 1,2,\ldots$, and Ω denotes the connect component of $\mathbb{C}\backslash C$ such that $\infty \in \Omega$. It is thus fundamental to ask: What can be said about the asymptotic behavior of discrete Sobolev orthogonal polynomial with respect to the inner product (7.2)?

Toda lattices

The study of integrable systems on the unit circle is not so performant as in the real line case, and its development started at the end of the previous century Ammar and Gragg [1994], Faybusovich and Gekhtman [1999], when the system of non-linear differential-difference equations (1.6) was studied. The main focus was based on the spectral theory of the GGT matrix $\mathbf{H}$. It was found that the resulting GGT matrices satisfy a Lax equation

$$\mathbf{H}^{'} = [(\mathbf{H}+\mathbf{H}^{-1})_+,\mathbf{H}],.$$

More recently, some developments in this direction have been done; see Chapter 1. In Golinskii [2007], the author considers the equations (1.6) with a similar approach, but using an alternative matrix representation of the multiplication operator for Laurent orthogonal polynomials, the CMV matrix. The corresponding CMV matrices satisfy a Lax equation

$$\mathbf{C}^{'} = [\mathbf{B},\mathbf{C}],$$

where $\mathbf{B}$ is an upper triangular matrix with two non-zero diagonals above the main diagonal, whose entries are also expressed in terms of the Verblunsky coefficients. In this situation, the non-linear differential-difference equation of their associated sequence of monic orthogonal polynomials satisfy

$$\Phi_n^{'}(z,t) = \Phi_{n+1}(z,t) - (z + \Phi_{n+1}(0,t)\overline{\Phi_n(0,t)})\Phi_n(z,t) - (1 - |\Phi_n(0,t)|^2)\Phi_{n-1}(z,t),$$

and the orthogonality measure associated with the Verblunsky coefficients is given by (1.7).

An interesting open problem is the study of Schur flows associated with the canonical linear spectral transformation of C-functions obtained in Chapter 6. Another interesting question is to analyze the analog of the Lax equations for the corresponding GGT and CMV matrices associated with such perturbations.

Appendix A

Discrete Sobolev orthogonal polynomials on the real line

Let $\{Q_n(\lambda, c, j; \cdot)\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the discrete Sobolev inner product

$$\langle p, q\rangle_{\mathcal{D}_1} = \int_a^b p(x)q(x)d\mu(x) + \lambda p^{(j)}(c)q^{(j)}(c), \quad c \notin (a,b), \quad \lambda \in \mathbb{R}_+, \quad j \geq 0, \tag{A.1}$$

where μ is a positive Borel measure supported in the interval (a,b) (either a or b can be infinity) and p, q are polynomials with real coefficients. In what follows we assume all orthogonal polynomials, both those with respect to μ and the Sobolev ones, to be monic.

In this appendix we prove that the zeros of discrete Sobolev orthogonal polynomials are monotonic functions of the parameter λ and establish their asymptotics when either λ converges to zero or to infinity. The precise location of the extreme zeros is also analyzed.

A.1 Monotonicity and asymptotics of zeros

Let $x_{n,k}(\lambda, c, j)$, $k = 1, \ldots, n$, be the zeros of $Q_n(\lambda, c, j; \cdot)$. For $\lambda = 0$ or $n \leq j$, the polynomials $P_n = Q_n(\lambda, c, j; \cdot)$ are orthogonal with respect to the inner product

$$\langle p, q\rangle_\mu = \int_a^b p(x)q(x)d\mu(x).$$

When $\lambda > 0$ and $n > j$ some natural questions arise. Are the zeros of $Q_n(\lambda, c, j; \cdot)$ all real and do they belong to (a,b)? If so, do the zeros of $Q_n(c, \lambda, j; \cdot)$ interlace with the zeros of P_n? Moreover, are the zeros $x_{n,k}(\lambda, c, j)$ monotonic functions with respect to the parameter λ? Do the zeros $x_{n,k}(\lambda, c, j)$ converge when λ goes to infinity? If so, what is the speed of convergence?

The answer of the first two questions was given in Meijer [1993]. He proved that the polynomial $Q_n(\lambda, c, j; \cdot)$ possesses n real simple zeros and at most one of them is located outside the interval (a,b).

In addition, he showed the following interlacing property. If $c \leq a$, then

$$x_{n,1}(\lambda, c, j) < x_{n,1} < x_{n,2}(\lambda, c, j) < x_{n,2} < \cdots < x_{n,n}(\lambda, c, j) < x_{n,n}. \tag{A.2}$$

If $c \geq b$, then

$$x_{n,1} < x_{n,1}(\lambda, c, j) < x_{n,2} < x_{n,2}(\lambda, c, j) < \cdots < x_{n,n} < x_{n,n}(\lambda, c, j). \tag{A.3}$$

In this appendix our main contribution deals with the remaining questions posed above. Our result is general in three aspects: the measure involved is any positive Borel measure, the point c is any value outside (a, b), and j is any positive integer. We obtain a new interlacing property, the monotonicity of $x_{n,k}(\lambda, c, j)$ with respect to λ, as well as their convergence – when λ tends to infinity – to the zeros of some polynomial with a speed of convergence of order $1/\lambda$. For this propose we define the polynomial $G_n(c, j; \cdot)$ by

$$G_n(c, j; x) = P_n(x) - \frac{P_n^{(j)}(c)}{K_{n-1}^{(j,j)}(c, c)} K_{n-1}^{(j,0)}(c, x), \tag{A.4}$$

where $K_{n-1}^{(r,s)}$ denotes the generalized Kernel polynomial of degree $n-1$,

$$K_{n-1}^{(r,s)}(x, y) = \sum_{k=0}^{n-1} \frac{P_k^{(r)}(x) P_k^{(s)}(y)}{\langle P_k, P_k \rangle_\mu}.$$

Notice that when $r = s = 0$, $K_{n-1} = K_{n-1}^{(0,0)}$ is the usual Kernel polynomial.

Theorem A.1.1. *Let $\lambda > 0$ and $y_{n,1}(c, j), \ldots, y_{n,n}(c, j)$ be the zeros of the polynomial $G_n(c, j; \cdot)$ defined by (A.4). For every $n > j$ and $c \leq a$, then*

$$y_{n,1}(c, j) < x_{n,1}(\lambda, c, j) < x_{n,1} < \cdots < y_{n,n}(c, j) < x_{n,n}(\lambda, c, j) < x_{n,n}.$$

Moreover, each $x_{n,k}(\lambda, c, j)$ is a decreasing function of λ, for each $k = 1, \ldots, n$.

On the other hand, if $c \geq b$, then

$$x_{n,1} < x_{n,1}(\lambda, c, j) < y_{n,1}(c, j) < \cdots < x_{n,n} < x_{n,n}(\lambda, c, j) < y_{n,n}(c, j). \tag{A.5}$$

In addition, each $x_{n,k}(\lambda, c, j)$ is an increasing function of λ, for $1 \leq k \leq n$.

In both cases, when λ goes to infinity,

$$\lim_{\lambda \to \infty} x_{n,k}(\lambda, c, j) = y_{n,k}(c, j)$$

and

$$\lim_{\lambda \to \infty} \lambda (y_{n,k}(c, j) - x_{n,k}(\lambda, c, j)) = \frac{P_n(y_{n,k}(c, j))}{K_{n-1}^{(j,j)}(c, c) G_n'(c, j; y_{n,k}(c, j))}. \tag{A.6}$$

Proof. Let us consider the following expression for the discrete Sobolev orthogonal polynomial Marcellán and Ronveaux [1990],

$$Q_n(\lambda, c, j; x) = P_n(x) - \frac{\lambda P_n^{(j)}(c)}{1 + \lambda K_{n-1}^{(j,j)}(c,c)} K_{n-1}^{(j,0)}(c,x). \tag{A.7}$$

Considering the normalization

$$\widetilde{Q}_n(\lambda, c, j; x) = \left(1 + \lambda K_{n-1}^{(j,j)}(c,c)\right) Q_n(\lambda, c, j; x),$$

we derive a simple representation for the n-th discrete Sobolev orthogonal polynomial

$$\widetilde{Q}_n(\lambda, c, j; x) = P_n(x) + \lambda K_{n-1}^{(j,j)}(c,c) G_n(c, j; x).$$

Observe that $G_n(c, j; x) = \lim_{\lambda \to \infty} Q_n(\lambda, c, j; x)$ is independent of λ . Then, evaluating the above expression in the zeros of P_n and $Q_n(\lambda, c, j; \cdot)$, we conclude that

$$\operatorname{sgn}\left(\widetilde{Q}_n(\lambda, c, j; x_{n,k})\right) = \operatorname{sgn}\left(G_n(c, j; x_{n,k})\right), \quad \operatorname{sgn}\left(P_n(x_{n,k}(\lambda, c, j)\right) = -\operatorname{sgn}\left(G_n(c, j; x_{n,k}(\lambda, c, j))\right),$$

for every $k = 1, \ldots, n-1$. Therefore, using (A.2) and (A.3), $G_n(c, j; \cdot)$ changes sign $n-1$ times at the zeros of $Q_n(\lambda, c, j; \cdot)$ and P_n. In other words, each interval $(x_{n,k}, x_{n,k+1})$ and $(x_{n,k}(\lambda, c, j), x_{n,k+1}(\lambda, c, j))$, $1 \leq k \leq n-1$, contains one zero of $G_n(c, j; \cdot)$. It remains to find the location of one zero of $G_n(c, j; \cdot)$. Taking into account that the polynomial $G_n(c, j; \cdot)$ has a positive leader coefficient and the location of the point c with respect to the interval (a, b), we obtain the inequalities stated.

To prove the monotonicity of the zeros $x_{n,k}(\lambda, c, j)$ with respect to λ, for every non-negative ε, we consider the polynomial

$$\widetilde{Q}_n(\lambda + \varepsilon, c, j; x) = P_n(x) + (\lambda + \varepsilon) K_{n-1}^{(j,j)}(c,c) G_n(c, j; x) = \widetilde{Q}_n(\lambda, c, j; x) + \varepsilon K_{n-1}^{(j,j)}(c,c) G_n(c, j; x).$$

Thus, for $\varepsilon > 0$,

$$\widetilde{Q}_n(\lambda + \varepsilon, c, j; x_{n,k}(\lambda, c, j)) = \varepsilon K_{n-1}^{(j,j)}(c,c) G_n(c, j; x_{n,k}(\lambda, c, j)),$$

and

$$\widetilde{Q}_n(\lambda + \varepsilon, c, j; x_{n,k+1}(\lambda, c, j)) = \varepsilon K_{n-1}^{(j,j)}(c,c) G_n(c, j; x_{n,k+1}(\lambda, c, j))$$

have opposite sign. Therefore, each interval $(x_{n,k}(\lambda, c, j), x_{n,k+1}(\lambda, c, j))$ contains at least one zero of $\widetilde{Q}_n(\lambda + \varepsilon, c, j; x)$. In addition, if $c \leq a$, then

$$\operatorname{sgn}\left(\widetilde{Q}_n(\lambda + \varepsilon, c, j; x_{n,1}(\lambda, c, j))\right) = (-1)^{n+1}, \quad \lim_{x \to -\infty} \widetilde{Q}_n(\lambda + \varepsilon, c, j; x) = \begin{cases} +\infty, & n \quad \text{even}; \\ -\infty, & n \quad \text{odd}. \end{cases}$$

Hence,

$$x_{n,1}(\lambda + \varepsilon, c, j) < x_{n,1}(\lambda, c, j) < \cdots < x_{n,n}(\lambda + \varepsilon, c, j) < x_{n,n}(\lambda, c, j).$$

In the other situation, if $c \geq b$, then

$$\operatorname{sgn}\left(\widetilde{Q}_n(\lambda+\varepsilon, c, j; x_{n,n}(\lambda, c, j))\right) = -1, \qquad \lim_{x\to+\infty} \widetilde{Q}_n(\lambda+\varepsilon, c, j; x) = +\infty.$$

Hence,

$$x_{n,1}(\lambda, c, j) < x_{n,1}(\lambda+\varepsilon, c, j) < \cdots < x_{n,n}(\lambda, c, j) < x_{n,n}(\lambda+\varepsilon, c, j).$$

It remains to prove the limit relations stated in the theorem. For this purpose, we define the polynomial $\widehat{Q}_n(\lambda, c, j; \cdot)$ by

$$\widehat{Q}_n(\lambda, c, j; x) = \frac{1}{\lambda} P_n(x) + K_{n-1}^{(j,j)}(c,c) G_n(c, j; x).$$

Notice that the zeros of $Q_n(\lambda, c, j; \cdot)$ and $\widehat{Q}_n(\lambda, c, j; \cdot)$ are the same for each $\lambda > 0$. Since

$$\lim_{\lambda\to\infty} \widehat{Q}_n(\lambda, c, j; x) = G_n(c, j; x),$$

by Hurwitz's Theorem Szegő [1975], the zeros $x_{n,k}(\lambda, c, j)$ of $Q_n(\lambda, c, j; \cdot)$ converge to the zeros $y_{n,k}(c, j)$ of $G_n(c, j; \cdot)$ when λ tends to infinity, that is,

$$\lim_{\lambda\to\infty} x_{n,k}(\lambda, c, j) = y_{n,k}(c, j), \quad k = 1, \ldots, n.$$

On the other hand, by the Mean Value Theorem, there exist real numbers $\theta_{n,k}$ between $y_{n,k}(c, j)$ and $x_{n,k}(\lambda, c, j)$, $k = 1, \ldots, n$, such that

$$\frac{\lambda G_n(c, j; y_{n,k}(c, j)) - \lambda G_n(c, j; x_{n,k}(\lambda, c, j))}{y_{n,k}(c, j) - x_{n,k}(\lambda, c, j)} = \lambda G'_n(c, j; \theta_{n,k}),$$

or, equivalently,

$$\lambda\left(y_{n,k}(c, j) - x_{n,k}(\lambda, c, j)\right) = \frac{P_n(x_{n,k}(\lambda, c, j))}{K_{n-1}^{(j,j)}(c,c) G'_n(c, j; \theta_{n,k})}.$$

Since $\lim_{\lambda\to\infty} x_{n,k}(\lambda, c, j) = y_{n,k}(c, j)$ and $\theta_{n,k}$ is located between $y_{n,k}(c, j)$ and $x_{n,k}(\lambda, c, j)$ we also have $\lim_{\lambda\to\infty} \theta_{n,k} = y_{n,k}(c, j)$. Thus

$$\lim_{\lambda\to\infty} \lambda\left(y_{n,k}(c, j) - x_{n,k}(\lambda, c, j)\right) = \lim_{\lambda\to\infty} \frac{P_n(x_{n,k}(\lambda, c, j))}{K_{n-1}^{(j,j)}(c,c) G'_n(c, j; \theta_{n,k})} = \frac{P_n(y_{n,k}(c, j))}{K_{n-1}^{(j,j)}(c,c) G'_n(c, j; y_{n,k}(c, j))},$$

and our statements hold. □

We used for the proof of Theorem A.1.1 the technique developed in Rafaeli [2010] and the references therein concerning the zeros of a linear combination of two polynomials of the seam degree with interlacing zeros. We emphasize an interesting consequence of this theorem. It says that, when λ goes from zero to infinity, each zero $x_{n,k}(\lambda, c, j)$ runs monotonically over the entire interval $(y_{n,k}(c, j), x_{n,k})$ or

$(x_{n,k}, y_{n,k}(c,j))$. In other words, if $c \le a$, then

$$x_{n,k}(\lambda,c,j) \in (y_{n,k}(c,j) = \sup_{\lambda}\{x_{n,k}(\lambda,c,j)\}, x_{n,k} = \inf_{\lambda}\{x_{n,k}(\lambda,c,j)\})$$

and, if $c \ge a$, then

$$x_{n,k}(\lambda,c,j) \in (x_{n,k} = \inf_{\lambda}\{x_{n,k}(\lambda,c,j)\}, y_{n,k}(c,j) = \sup_{\lambda}\{x_{n,k}(\lambda,c,j)\}).$$

Meijer Meijer [1993] obtained the inequalities (A.2) and (A.3) for $j = 1$ using another technique. Furthermore, for $n \ge 3$ he showed that for some choice of c in (a,b), $Q_n(\lambda,c,1;\cdot)$ has two complex zeros, if λ is sufficiently large. Earlier Marcellán, Pérez, and Piñar Marcellán et al. [1992] had shown the interlacing properties (A.2) and (A.3) when $j = 1$. Moreover, the monotonicity of $x_{n,k}(\lambda,c,1)$ with respect to λ, and the convergence of one of the extreme zeros to c when n goes to $+\infty$ were established in Marcellán et al. [1992].

In the sequel we analyze the location of the smallest (resp. greatest) zero of $Q_n(\lambda,c,j;\cdot)$ with respect to the point a (resp. b).

Corollary A.1.1. *Let $n > j$ and $\lambda > 0$.*

i) If $c \le a$ and $y_{n,1}(c,j) < a$, then the smallest zero $x_{n,1}(\lambda,c,j)$ satisfies

$$\begin{aligned} x_{n,1}(\lambda,c,j) &> a, \quad \lambda < \lambda_0, \\ x_{n,1}(\lambda,c,j) &= a, \quad \lambda = \lambda_0, \\ x_{n,1}(\lambda,c,j) &< a, \quad \lambda > \lambda_0, \end{aligned}$$

where

$$\lambda_0 = \lambda_0(n,a,c,j) = \frac{P_n(a)}{K_{n-1}^{(j,0)}(c,a)P_n^{(j)}(c) - K_{n-1}^{(j,j)}(c,c)P_n(a)}.$$

ii) If $c \ge b$ and $y_{n.n}(c,j) > b$, then the largest zero $x_{n,n}(\lambda,c,j)$ satisfies

$$\begin{aligned} x_{n,n}(\lambda,c,j) &< b, \quad \lambda < \lambda_0, \\ x_{n,n}(\lambda,c,j) &= b, \quad \lambda = \lambda_0, \\ x_{n,n}(\lambda,c,j) &> b, \quad \lambda > \lambda_0, \end{aligned}$$

where $\lambda_0 = \lambda_0(n,b,c,j)$.

The proofs are an immediate consequence of determining the value of the polynomial $Q_n(\lambda,c,j;\cdot)$ via (A.7), together with the fact that $G_n(c,j;a)P_n(a) < 0$ if $y_{n,1}(c,j) < a$ or $G_n(c,j;b)P_n(b) < 0$ if $y_{n,n}(c,j) > b$. Observe that for $c \notin [a,b]$ we derive explicitly the value λ_0 of the mass, such that for $\lambda > \lambda_0$ one of the zeros is located outside (a,b).

A similar result about the mutual location of c and the extreme zeros of $Q_n(\lambda,c,j;\cdot)$ is the following corollary.

Corollary A.1.2. *Let $n > j$ and $\lambda > 0$.*

i) If $c \le a$ and $y_{n,1}(c,j) < c$, then the smallest zero $x_{n,1}(\lambda,c,j)$ satisfies

$$\begin{aligned} x_{n,1}(\lambda,c,j) &> c, \quad \lambda < \lambda_1,\\ x_{n,1}(\lambda,c,j) &= c, \quad \lambda = \lambda_1,\\ x_{n,1}(\lambda,c,j) &< c, \quad \lambda > \lambda_1, \end{aligned}$$

where $\lambda_1 = \lambda_1(n,c,j) = \dfrac{P_n(c)}{K_{n-1}^{(j,0)}(c,c)P_n^{(j)}(c) - K_{n-1}^{(j,j)}(c,c)P_n(c)}.$

ii) If $c \ge b$ and $y_{n,n}(c,j) > c$, then the largest zero $x_{n,n}(\lambda,c,j)$ satisfies

$$\begin{aligned} x_{n,n}(\lambda,c,j) &< c, \quad \lambda < \lambda_1,\\ x_{n,n}(\lambda,c,j) &= c, \quad \lambda = \lambda_1,\\ x_{n,n}(\lambda,c,j) &> c, \quad \lambda > \lambda_1, \end{aligned}$$

where $\lambda_1 = \lambda_1(n,c,j)$.

It means that, depending on the value of the parameter λ, one zero can be located outside the interval $(\min\{a,c\},\max\{b,c\})$.

A.1.1 Jacobi polynomials

Let $\{Q_n^{(\alpha,\beta)}(\lambda,c,j;\cdot)\}_{n\geqslant 0}$ be the polynomials which are orthogonal with respect to the discrete Sobolev inner product (A.1) with $d\mu = d\mu(\alpha,\beta;\cdot)$; see Chapter 1. Let us denote by $x_{n,k}(\alpha,\beta)$ and $x_{n,k}(\lambda,c,j;\alpha,\beta)$ the zeros of $P_n^{(\alpha,\beta)}$ and $Q_n^{(\alpha,\beta)}(\lambda,c,j;\cdot)$, respectively.

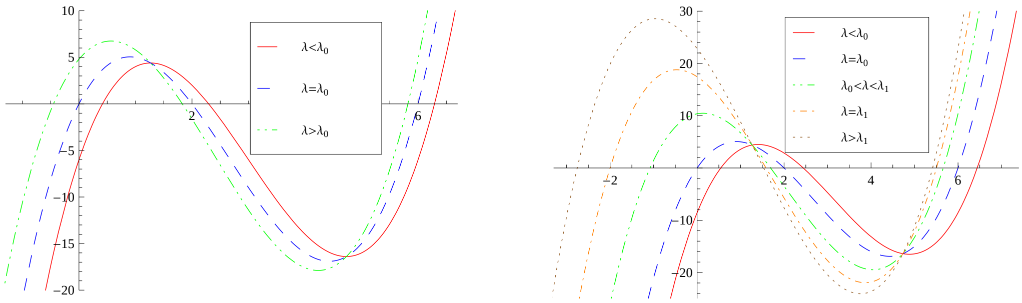
To illustrate the results obtained in Corollary A.1.1 and A.1.2 we consider two figures. In the Figure A.1(a) we consider $n = 3$, $\alpha = -1/2, \beta = 1/2$, $j = 2$, and $c = 1$ for some values of λ. In the Figure A.1(b) we take the same values of n, α, β, and j but we choose $c = 2$, that is, now c is not an endpoint of $(-1,1)$ and we vary the parameter λ. We see that in both figures at least one zero of

$$Q_3^{(-1/2,1/2)}(\lambda,c,2;x) = x^3 - \frac{384c\lambda+\pi}{2(64\lambda+\pi)}x^2 + \frac{192c\lambda-96\lambda-\pi}{2(64\lambda+\pi)}x + \frac{384c\lambda+\pi}{8(64\lambda+\pi)}$$

is outside of the support $(-1,1)$. Moreover, one zero coincides with $b = 1$ when $\lambda = \lambda_0 = \pi/(128(3c-1))$ and with c when $\lambda = \lambda_1 = \pi(1-4c-4c^2+8c^3)/(256c^2(4c-3))$. In Figure A.1(a) observe that since $b = c = 1$ then $\lambda_0 = \lambda_1$.

Table A.1: Zeros of $Q_4^{(-1/2,1/2)}(\lambda,1,2;x) = x^4 - (1/2+2400\lambda/(\pi+1664\lambda))x^3 + [-3/4+960\lambda/(\pi+1664\lambda)]x^2 + [1/4+1320\lambda/(\pi+1664\lambda)]x + 1/16 - 240\lambda/(\pi+1664\lambda)$ for some values of λ

λ	$x_{4,1}(\lambda,1,2;-1/2,1/2)$	$x_{4,2}(\lambda,1,2;-1/2,1/2)$
$1/3000$	-0.74365	-0.09607
$1/2000$	-0.73703	-0.07079
$\lambda_0 = \lambda_1 = \pi/4096$	-0.72955	-0.04152
1	-0.69680	0.08019
10	-0.69676	0.08032
100	-0.69675	0.08034
λ	$x_{4,3}(\lambda,1,2;-1/2,1/2)$	$x_{4,4}(\lambda,1,2;-1/2,1/2)$
$1/3000$	0.59496	0.96120
$1/2000$	0.63516	0.97465
$\lambda_0 = \lambda_1 = \pi/4096$	0.68774	1
1	0.85899	1.69721
10	0.85907	1.69940
100	0.85908	1.69962



(a) $Q_3^{(-1/2,1/2)}(\lambda,1,2;x)$ (b) $Q_3^{(-1/2,1/2)}(\lambda,2,2;x)$

Figure A.1: Location of the smallest and greatest zero of $Q_n(\lambda,c,j;x)$

We also provide two tables illustrating the monotonicity of the zeros of $Q_n^{(\alpha,\beta)}(\lambda,c,j;\cdot)$ as functions of λ, and the convergence of these zeros to the zeros of $G_n^{(\alpha,\beta)}(c,j;\cdot)$, which are the polynomials defined in (A.4). In Table A.1 we can observe the behavior of the zeros of $Q_4^{(\alpha,\beta)}(\lambda,c,j;\cdot)$ when $\alpha=-1/2$, $\beta=1/2$, $j=2$ and $c=1$, for some values of λ. It is quite clear that the zeros are increasing functions of λ, and they converge to the zeros of $G_4^{(-1/2,1/2)}(1,2;\cdot)$, which are -0.696751, 0.0803371, 0.859082, and 1.69964.

In Table A.2 we present the zeros of $Q_4^{(\alpha,\beta)}(\lambda,c,j;\cdot)$ when $\alpha=-1/2, \beta=1/2$, $j=2$, $c=2$ for several choices of λ. In this table, since the zeros of $G_4^{(-1/2,1/2)}(2,2;\cdot)$ are -0.659576, 0.160050, 0.886989, and 3.76418, we observe that the zeros have the same monotonic and asymptotic behavior when the point c does not coincide with an endpoint of the interval $(-1,1)$.

Table A.2: Zeros of $Q_4^{(-1/2,1/2)}(\lambda,2,2;x) = x^4-(1/2+28512\lambda/(\pi+7808\lambda))x^3+(-3/4+12960\lambda/(\pi+7808\lambda))x^2+(1/4+14904\lambda/(\pi+7808\lambda))x+1/16-3240\lambda/(\pi+7808\lambda)$ for some values of λ

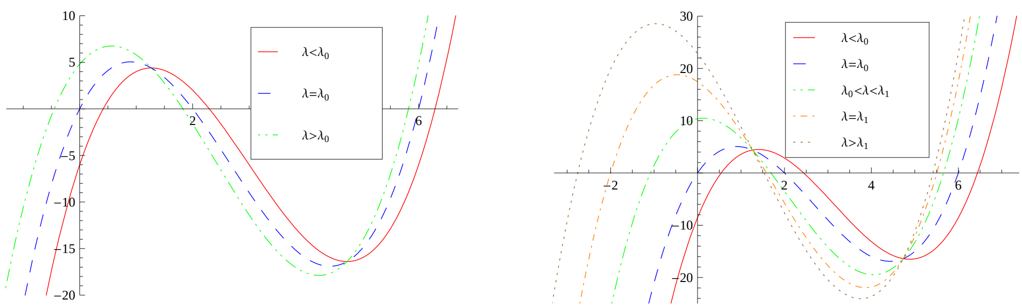
λ	$x_{4,1}(\lambda,2,2;-1/2,1/2)$	$x_{4,2}(\lambda,2,2;-1/2,1/2)$
$\lambda_0=\pi/54400$	-0.71792	-0.02081
$1/3000$	-0.67724	0.11537
$\lambda_1=17\pi/133376$	-0.67465	0.12258
1	-0.65958	0.16003
10	-0.65958	0.16005
100	-0.65958	0.16005

λ	$x_{4,3}(\lambda,2,2;-1/2,1/2)$	$x_{4,4}(\lambda,2,2;-1/2,1/2)$
$\lambda_0=\pi/54400$	0.69706	1
$1/3000$	0.87010	1.84630
$\lambda_1=17\pi/133376$	0.87350	2
1	0.88699	3.76274
10	0.88699	3.76403
100	0.88699	3.76416

A.1.2 Laguerre polynomials

Let $x_{n,k}(\lambda,c,j;\alpha)$ be the zeros of the polynomials $\{Q_n^{(\alpha)}(\lambda,c,j;\cdot)\}_{n\geqslant 0}$ which are orthogonal with respect to the discrete Sobolev inner product (A.1) with $d\mu=d\mu(\alpha;\cdot)$; see Chapter 1.

To illustrate the behavior of $x_{n,k}(\lambda,c,j;\alpha)$ we show two figures and two tables. We plot $Q_3^{(\alpha)}(\lambda,c,j;x)$ for α, c, and j fixed and vary λ, to show that, depending on the value of λ, the smallest zero can be less than or equal to c. In Figure A.2(a) we choose $\alpha=1$, $j=2$, and $c=0$, that is, c is an endpoint of the orthogonality interval. In Figure A.2(b) we choose $\alpha=1$, $j=2$, and $c=-2$.



(a) $Q_3^{(1)}(\lambda,0,2;x)$ (b) $Q_3^{(1)}(\lambda,-2,2;x)$

Figure A.2: Location of the smallest and greatest zero of $Q_n(\lambda,c,j;x)$

In order to illustrate the behavior of the zeros of the polynomials $Q_4^{(\alpha)}(\lambda,c,j;\cdot)$ presented in Theorem A.1.1, we present some numerical computations in Tables A.3 and A.4. In the first one we show the zeros $x_{4,k}(\lambda,c,j;\alpha)$, with $\alpha=1$, $c=0$ and $j=2$ for several choices of λ. Observe that, since the point

c is on the left side of the interval of orthogonality, all the zeros of these polynomials are decreasing functions of λ, and they converge to the zeros of $G_4^{(1)}(0,2;\cdot)$, defined in (A.4), which are -3.63913, 1.16543, 4.00543, and 9.23750.

Table A.3: Zeros of $Q_4^{(1)}(\lambda,0,2;x) = x^4-(20-120\lambda/(3+13\lambda))x^3+(360/(3+13\lambda))x^2-(240-5040\lambda/(3+13\lambda))x+120-3600\lambda/(3+13\lambda)$ for some values of λ ($\lambda_0=\lambda_1=3/17$)

λ	$x_{4,1}(\lambda,0,2;1)$	$x_{4,2}(\lambda,0,2;1)$	$x_{4,3}(\lambda,0,2;1)$	$x_{4,4}(\lambda,0,2;1)$
$1/100$	0.71843	2.49658	5.60038	10.8012
$1/10$	0.39195	1.89442	4.85101	10.0719
$\lambda_0=\lambda_1$	0	1.60262	4.57292	9.82446
1	−2.21826	1.22608	4.11868	9.37350
10	−3.46229	1.17104	4.01682	9.25187
100	−3.62101	1.16599	4.00657	9.23895
1000	−3.63731	1.16549	4.00554	9.23765

In Table A.4 we choose the same values of α and j, but in this case $c=-2$. The zeros $x_{4,k}(\lambda,c,j;\alpha)$ are also monotonic functions and converge to the zeros of $G_4^{(1)}(-2,2;\cdot)$, which are -7.76110, 1.07579, 3.76689, and 8.77556.

Table A.4: Zeros of $Q_4^{(1)}(\lambda,-2,2;x) = x^4-(20-396\lambda/(3+28\lambda))x^3+(120-5016\lambda/(3+28\lambda))x^2-(240-15840\lambda/(3+28\lambda))x+120-11088\lambda/(3+28\lambda)$ for some values of λ ($\lambda_0=15/322$ and $\lambda_1=471/3854$)

λ	$x_{4,1}(\lambda,0,2;1)$	$x_{4,2}(\lambda,0,2;1)$	$x_{4,3}(\lambda,0,2;1)$	$x_{4,4}(\lambda,0,2;1)$
$1/100$	0.660855	2.33685	5.33702	10.4580
λ_0	0	1.58118	4.50358	9.62953
$1/10$	−1.47545	1.26126	4.13306	9.25354
λ_1	−2	1.219141	4.06667	9.17819
10	−7.62158	1.07713	3.77041	8.78110
100	−7.74700	1.07593	3.76724	8.77611
1000	−7.75969	1.07580	3.76693	8.77561

Appendix B

Uvarov spectral transformation

In this appendix we deal with a transformation of a quasi-definite functional by the addition of Dirac's linear functionals supported on r different points located either on the unit circle or on its complement. Consider the quasi-definite linear functional $\mathcal{L}$ introduced in (2.22), and let $\mathcal{L}_U$ be the linear functional such that its associated bilinear functional satisfies

$$\langle f,g\rangle_{\mathcal{L}_U}=\langle f,g\rangle_{\mathcal{L}}+\sum_{i=1}^{r}m_i f(\alpha_i)\overline{g(\alpha_i)},\quad m_i\in\mathbb{R}\backslash\{0\},\quad |\alpha_i|=1.$$

Necessary and sufficient conditions for the regularity of the perturbed linear functional $\mathcal{L}_U$ are deduced. We also obtain the corresponding linear C-functions.

B.1 Mass points on the unit circle

Using an analog method to the one used in Daruis et al. [2007], we can show

Theorem B.1.1. *The following statements are equivalent:*

i) $\mathcal{L}_U$ *is a quasi-definite linear functional.*

ii) *The matrix* $\mathbf{D}_r^{-1}+\mathbb{K}_{n-1}$ *is non-singular, and*

$$\mathbf{k}_n+\mathbf{\Phi}_n^H\left(\mathbf{D}_r^{-1}+\mathbb{K}_{n-1}\right)^{-1}\mathbf{\Phi}_n\neq 0,\quad n\geqslant 1.$$

Moreover, the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_U$ *is given by*

$$\Psi_n(z)=\Phi_n(z)-\mathbf{K}_{n-1}(z)\left(\mathbf{D}_r^{-1}+\mathbb{K}_{n-1}\right)^{-1}\mathbf{\Phi}_n,\quad n\geqslant 1,\tag{B.1}$$

with

$$\mathbf{K}_{n-1}(z)=[K_{n-1}(z,\alpha_1),K_{n-1}(z,\alpha_2),\dots K_{n-1}(z,\alpha_r)],\quad \mathbf{D}_r=diag[\boldsymbol{m}_1,\boldsymbol{m}_2,\dots,\boldsymbol{m}_r],$$
$$\boldsymbol{\Phi}_n=[\Phi_n(\alpha_1),\Phi_n(\alpha_2),\dots,\Phi_n(\alpha_r)]^T,$$

and

$$\mathbb{K}_{n-1}=\begin{bmatrix} K_{n-1}(\alpha_1,\alpha_1) & K_{n-1}(\alpha_1,\alpha_2) & \cdots & K_{n-1}(\alpha_1,\alpha_r)\\ K_{n-1}(\alpha_2,\alpha_1) & K_{n-1}(\alpha_2,\alpha_2) & \cdots & K_{n-1}(\alpha_2,\alpha_r)\\ \vdots & \vdots & \ddots & \vdots\\ K_{n-1}(\alpha_r,\alpha_1) & K_{n-1}(\alpha_r,\alpha_2) & \cdots & K_{n-1}(\alpha_r,\alpha_r)\end{bmatrix}.$$

Proof. First, let us assume that $\mathcal{L}_U$ is a quasi-definite linear functional and denote by $\{\Psi_n\}_{n\geqslant 0}$ its corresponding sequence of monic orthogonal polynomials. Thus,

$$\Psi_n(z)=\Phi_n(z)+\sum_{k=0}^{n-1}\lambda_{n,k}\Phi_k(z),\quad \lambda_{n,k}=-\frac{1}{\mathbf{k}_k}\sum_{i=1}^{r}m_i\Psi_n(\alpha_i)\overline{\Phi_k(\alpha_i)},\quad n\geq 1.$$

Then, we have

$$\Psi_n(z)=\Phi_n(z)-\sum_{i=1}^{r}m_i\Psi_n(\alpha_i)K_{n-1}(z,\alpha_i). \tag{B.2}$$

In particular, for $j=1,\dots,r$, we have the following system of r linear equations and r unknowns $\Psi_n(\alpha_j)$, $j=1,2,...,r$,

$$\Psi_n(\alpha_j)=\Phi_n(\alpha_j)-\sum_{i=1}^{r}m_i\Psi_n(\alpha_i)K_{n-1}(\alpha_j,\alpha_i).$$

Therefore,

$$\begin{bmatrix} 1+m_1K_{n-1}(\alpha_1,\alpha_1) & m_2K_{n-1}(\alpha_1,\alpha_2) & \cdots & m_rK_{n-1}(\alpha_1,\alpha_r)\\ m_1K_{n-1}(\alpha_2,\alpha_1) & 1+m_2K_{n-1}(\alpha_2,\alpha_2) & \cdots & m_rK_{n-1}(\alpha_2,\alpha_r)\\ \vdots & \vdots & \ddots & \vdots\\ m_1K_{n-1}(\alpha_r,\alpha_1) & m_2K_{n-1}(\alpha_r,\alpha_2) & \cdots & 1+m_rK_{n-1}(\alpha_r,\alpha_r)\end{bmatrix}\boldsymbol{\Psi}_n=\boldsymbol{\Phi}_n,$$

where $\boldsymbol{\Psi}_n=[\Psi_n(\alpha_1),\Psi_n(\alpha_2),\dots,\Psi_n(\alpha_r)]$. In other words,

$$(\mathbb{K}_{n-1}\mathbf{D}_r+\mathbf{I}_r)\boldsymbol{\Psi}_n=\boldsymbol{\Phi}_n.$$

Since $\mathcal{L}_U$ is assumed to be quasi-definite, the matrix $\mathbb{K}_{n-1}\mathbf{D}_r+\mathbf{I}_r$ is non-singular and, therefore, (B.1) follows from (B.2).

On the other hand, assume *ii*) holds. For $0 \le k \le n-1$, we have

$$\begin{aligned}\langle \Psi_n, \Phi_k \rangle_{\mathcal{L}_U} &= \left\langle \Phi_n(z) - \sum_{i=1}^{r} m_i \Psi_n(\alpha_i) K_{n-1}(z,\alpha_i), \Phi_k(z) \right\rangle + \sum_{i=1}^{r} m_i \Psi_n(\alpha_i)\overline{\Phi_k(\alpha_i)} \\ &= -\sum_{i=1}^{r} m_i \Psi_n(\alpha_i) \langle K_{n-1}(z,\alpha_i), \Phi_k(z)\rangle + \sum_{i=1}^{r} m_i \Psi_n(\alpha_i)\overline{\Phi_k(\alpha_i)} = 0,\end{aligned}$$

using the reproducing kernel property in the last expression. Furthermore,

$$\langle \Psi_n, \Phi_n \rangle_{\mathcal{L}_U} = \mathbf{k}_n + \sum_{i=1}^{r} m_i \Psi_n(\alpha_i)\overline{\Phi_n(\alpha_i)} = \mathbf{k}_n + \mathbf{\Phi}_n^H \, \mathbf{D}_r \, \mathbf{\Psi}_n = \mathbf{k}_n + \mathbf{\Phi}_n^H (\mathbb{K}_{n-1} + \mathbf{D}_r^{-1})^{-1} \mathbf{\Phi}_n \neq 0,$$

which proves that $\{\Psi_n\}_{n\geqslant 0}$, defined by (B.1), is the sequence of monic polynomials orthogonal with respect to $\mathcal{L}_U$. □

Notice that for $r = 1$, the regularity condition for $\mathcal{L}_U$ becomes $1 + m_1 K_{n-1}(\alpha_1, \alpha_1) \neq 0$, $n \geq 0$, as shown in Daruis et al. [2007].

Theorem B.1.2. *For $z \in \mathbb{D}$, the C-function associated with $\mathcal{L}_U$ is*

$$F_U(z) = F(z) + \sum_{i=1}^{N} m_i \left(\frac{\alpha_i + z}{\alpha_i - z} \right).$$

Proof. Denoting $\widetilde{c}_{-k} = \langle \mathcal{L}_U, z^{-k} \rangle$, we have

$$F_U(z) = \widetilde{c}_0 + 2\sum_{k=1}^{\infty} \widetilde{c}_{-k} z^k = F(z) + \sum_{i=1}^{r} m_i \left(\frac{\alpha_i + z}{\alpha_i - z} \right),$$

i.e., $F_U(z)$ has simple poles at $z = \alpha_i$. □

B.2 Mass points outside the unit circle

The next step is to consider a perturbation $\mathcal{L}_D$ such that its corresponding bilinear functional satisfies

$$\langle f, g \rangle_{\mathcal{L}_D} = \langle f, g \rangle_{\mathcal{L}} + \sum_{i=1}^{r} \left(\boldsymbol{m}_i f(\alpha_i) \overline{g}(\alpha_i^{-1}) + \overline{\boldsymbol{m}}_i f(\overline{\alpha}_i^{-1}) \overline{g(\alpha_i)} \right), \quad \boldsymbol{m}_i \in \mathbf{C}\backslash\{0\}, \quad |\alpha_i| \neq 0, 1. \tag{B.3}$$

By analogy with the previous case, we can obtain the following result.

Theorem B.2.1. *The following statements are equivalent:*

i) *$\mathcal{L}_D$ is a quasi-definite linear functional.*

ii) *The matrix $\widetilde{\mathbf{D}}_{2r}^{-1} + \widetilde{\mathbb{K}}_{n-1}$ is non-singular, and*

$$\mathbf{k}_n + \widetilde{\mathbf{\Phi}}_n^H \left(\widetilde{\mathbf{D}}_{2r}^{-1} + \widetilde{\mathbb{K}}_{n-1} \right)^{-1} \widetilde{\mathbf{\Psi}}_n \neq 0, \quad n \geqslant 1. \tag{B.4}$$

Moreover, the corresponding sequence of monic polynomials orthogonal with respect to $\mathcal{L}_\Omega$ is given by

$$\widetilde{\Psi}_n(z) = \Phi_n(z) - \widetilde{\mathbf{K}}_{n-1}(z)\left(\widetilde{\mathbf{D}}_{2r}^{-1} + \widetilde{\mathbb{K}}_{n-1}\right)^{-1}\widetilde{\mathbf{\Phi}}_n, \quad n \geqslant 1, \tag{B.5}$$

with

$$\widetilde{\mathbf{\Phi}}_n(\alpha) = [\Phi_n(\alpha_1), \ldots, \Phi_n(\alpha_r), \Phi_n(\overline{\alpha}_1^{-1}), \ldots, \Phi_n(\overline{\alpha}_r^{-1})]^T, \quad \widetilde{\mathbf{D}}_{2r} = diag\{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_r, \overline{\boldsymbol{m}}_1, \ldots, \overline{\boldsymbol{m}}_r\},$$
$$\widetilde{\mathbf{K}}_{n-1} = [K_{n-1}(z,\alpha_1), \ldots K_{n-1}(z,\alpha_r), K_{n-1}(z,\overline{\alpha}_1^{-1}), \ldots, K_{n-1}(z,\overline{\alpha}_r^{-1})],$$

$$\widetilde{\mathbb{K}}_{n-1} = \left[\begin{array}{c|c} \mathbb{R}_{n-1}(\alpha,\alpha) & \mathbb{R}_{n-1}(\alpha,\overline{\alpha}^{-1}) \\ \hline \mathbb{R}_{n-1}(\overline{\alpha}^{-1},\alpha) & \mathbb{R}_{n-1}(\overline{\alpha}^{-1},\overline{\alpha}^{-1}) \end{array}\right], \; \mathbb{R}_{n-1}(\alpha,\alpha) = \begin{bmatrix} K_{n-1}(\alpha_1,\alpha_1) & \cdots & K_{n-1}(\alpha_1,\alpha_r) \\ \vdots & \ddots & \vdots \\ K_{n-1}(\alpha_r,\alpha_1) & \cdots & K_{n-1}(\alpha_r,\alpha_r) \end{bmatrix}.$$

Proceeding as in the proof of Theorem B.1.2, we obtain

Theorem B.2.2. *For $z \in \mathbb{D}$, the C-function associated with $\mathcal{L}_D$ is*

$$F_D(z) = F(z) + \sum_{i=1}^{r}\left(m_i\frac{\alpha_i + z}{\alpha_i - z} + \overline{m}_i\frac{\overline{\alpha}_i^{-1} + z}{\overline{\alpha}_i^{-1} - z}\right),$$

i.e., $F_D(z)$ has simple poles at $z = \alpha_i$ and $z = \overline{\alpha}_i^{-1}$.

Bibliography

M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.*, 16:598–603, 1975. 5

M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.*, 17:1011–1018, 1976. 5

N. I. Akhiezer. *The Classical Moment Problem and Some Related Questions in Analysis*. Hafner, New York, 1965. Russian original, 1961. 13, 26

M. Alfaro, F. Marcellán, M. L. Rezola, and A. Ronveaux. On orthogonal polynomials of Sobolev type: Algebraic properties and zeros. *SIAM J. Math. Anal.*, 23:737–757, 1992. 130

M. Alfaro, G. López, and M. L. Rezola. Some properties of zeros of Sobolev-type orthogonal polynomials. *J. Comput. Appl. Math.*, 69:171–179, 1996. 97, 130

M. Alfaro, J. J. Moreno-Balcázar, A. Peña, and M. L. Rezola. A new approach to the asymptotics for Sobolev orthogonal polynomials. *J. Approx. Theory*, 163:460–480, 2010. 130

R. Álvarez-Nodarse and J. Petronilho. On Krall-type discrete polynomials. *J. Math. Anal. Appl.*, 10: 335–347, 2004. 59

R. Álvarez-Nodarse, F. Marcellán, and J. Petronilho. WKB approximation and Krall-type orthogonal polynomials. *Acta Appl. Math.*, 54:27–58, 1998. 59

G. S. Ammar and W. B. Gragg. Schur flows for orthogonal Hessenberg matrices. Hamiltonian and gradient flows, algorithms and control. *Algorithms and Control, Fields Inst. Commun*, 3:27–34, 1994. 131

J. Arvesú, R. Álvarez-Nodarse, and F. Marcellán. Modifications of quasi-definite linear functionals via addition of delta and derivatives of delta Dirac functions. *Indag. Math. (N.S.)*, 15(1):1–20, 2004. 59

R. Askey. *Orthogonal Polynomials and Special Functions*. SIAM, Philadelphia, PA, 1975. Regional Conference Series in Applied Mathematics, Vol. 21. 1

S. Barnett. *Polynomials and Linear Control Systems*. Marcel Dekker, New York, 1983. 29

G. Baxter. Polynomials defined by difference system. *J. Math. Anal. Appl.*, 2:223–263, 1961. 105

S. Belmehdi and F. Marcellán. Orthogonal polynomials associated to some modifications of a linear functional. *Appl. Anal.*, 46:1–24, 1992. 59

C. Berg. Fibonacci numbers and orthogonal polynomials. *Arab. J. Math. Sci.*, 17:75–88, 2011. 1

F. Beukers. Legendre polynomials in irrationality proofs. *Bull. Austral. Math. Soc.*, 22:431–438, 1980. 1

S. Bochner. Über Sturm-Liouvillesche polynomsysteme. *Math. Zeit.*, 29:730–736, 1929. 16

A. Böttcher and S. M. Grudsky. *Matrix Spectral Properties of Banded Toeplitz Matrices*. SIAM, Philadelphia, PA, 2005. 79

C. F. Bracciali, J. H. McCabe, and A. Sri Ranga. On a symmetry in strong distributions. *J. Comput. Appl. Math.*, 105:187–198, 1999. 51

A. Branquinho, L. Golinskii, and F. Marcellán. Orthogonal polynomials and rational modifications of the Lebesgue measure on the unit circle. *Complex Var. Theory Appl.*, 38:137–154, 1999. 66

A. Branquinho, A. Foulquié, and F. Marcellán. Asymptotic behavior of Sobolev-type orthogonal polynomials on a rectifiable Jordan curve or arc. *Constr. Approx.*, 18:161–182, 2002. 89

C. Brezinski. *History of Continued Fractions and Padé Approximants*. Springer-Verlag, New York, 1991. Springer Series in Computational Mathematics, Vol. 12. 14

M. G. De Bruin. A tool for locating zeros of orthogonal polynomials in Sobolev inner product spaces. *J. Comput. Appl. Math.*, 49:27–35, 1993. 97

A. Bultheel and M. Van Barel. *Linear Algebra, Rational Approximation and Orthogonal Polynomials*. Elsevier Science Publ., Amsterdam, 1997. Studies in Computational Mathematics, Vol. 6. 1

A. Cachafeiro, F. Marcellán, and C. Pérez. Lebesgue perturbation of a quasi-definite Hermitian functional. The positive definite case. *Linear Algebra Appl.*, 369:235–250, 2003. 59, 66

A. Cachafeiro, F. Marcellán, and C. Pérez. Orthogonal polynomials with respect to a sum of a arbitrary measure and a Bernstein-Szegő measure. *Adv. Comput. Math.*, 26:81–104, 2007. 59, 66

M. Cámara, J. Fábrega, M. A. Fiol, and E. Garriga. Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes. *Electron. J. Combin.*, 16(1):R83, 2009. 1

M. J. Cantero. *Polinomios ortogonales sobre la circunferencia unidad. Modificaciones de los parámetros de Schur*. PhD thesis, Departamento de Matemáticas, Universidad de Zaragoza, 1997. In Spanish. 32, 105, 108, 109

M. J. Cantero, L. Moral, and L. Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Linear Algebra Appl.*, 362:29–56, 2003. 5, 28

M. J. Cantero, L. Moral, and L. Velázquez. Direct and inverse polynomial perturbations of hermitian linear functionals. *J. Approx. Theory*, 163:988–1028, 2011. 7, 32, 106, 107, 110

C. Carathéodory. Über den variabilitätsbereich der Koeffizienten von Potenzreihen die gegebene Werter nicht annehmen. *Math. Ann.*, 64:95–115, 1907. 5, 30

K. Castillo. A new approach to relative asymptotic behavior for discrete Sobolev-type orthogonal polynomials on the unit circle. *Appl. Math. Lett.*, 25:1000–1004, 2012f. 6

K. Castillo and F. Marcellán. Generators of rational spectral transformations for nontrivial $\mathcal{C}$-functions. *Math. Comp.*, 2012c. In Press. 6

K. Castillo, L. Garza, and F. Marcellán. Linear spectral transformations, Hessenberg matrices, and orthogonal polynomials. *Rend. Circ. Mat. Palermo*, 82(2):3–26, 2010a. 6

K. Castillo, L. Garza, and F. Marcellán. Laurent polynomials perturbations of linear functional. An inverse problem. *Electron. Trans. Numer. Anal.*, 36:83–98, 2010b. 6

K. Castillo, L. Garza, and F. Marcellán. Perturbations on the subdiagonals of Toeplitz matrices. *Linear Algebra Appl.*, 434:1563–1579, 2011a. 6

K. Castillo, L. Garza, and F. Marcellán. A new linear spectral transformation associated with derivatives of Dirac linear functionals. *J. Approx. Theory*, 163:1834–1853, 2011b. 6

K. Castillo, L. Garza, and F. Marcellán. Asymptotic behavior of Sobolev orthogonal polynomials on the unit circle. *Integral Transforms Spec. Funct.*, 2012a. DOI: 10.1080/10652469.2011.649751. In Press. 6

K. Castillo, L. Garza, and F. Marcellán. Zeros of discrete Sobolev orthogonal polynomials on the unit circle. 2012b. Submitted. 6

K. Castillo, R. L. Lamblém, F. R. Rafaeli, and A. Sri Ranga. Szegő polynomials and para-orthogonal polynomials on the real line: Zeros and canonical spectral transformations. *Math. Comp.*, 2012d. DOI: 10.1090/S0025-5718-2012-02593-2. In Press. 6

K. Castillo, R. L. Lamblém, and A. Sri Ranga. On a moment problem associated with Chebyshev polynomials. *Appl. Math. Comput.*, 218:9571–9574, 2012e. 6

K. Castillo, M. V. Mello, and F. R. Rafaeli. Monotonicity and asymptotics of zeros of Sobolev type orthogonal polynomials: A general case. 2012g. Submitted. 6

K. Castillo, D. K. Dimitrov, L. Garza, and F. R. Rafaeli. Perturbations on the antidiagonals of Hankel matrices. 2012h. Submitted. 6

P. L. Chebyshev. Sur les fractions continues. *J. Math. Pures Appl.*, 3(2):289–323, 1885. Translation of a Russian paper published in 1885; reprinted in Oeuvres de P. L. Tchebychef, tome 1, A. Markoff and N. Sonin (eds.), St Petersburg 203-230, 1899; reprinted Chelsea, New York, 1961. 15, 20

T. S. Chihara. *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York-London-Paris, 1978. Mathematics and Its Applications, Vol. 13. 1, 9, 17, 18, 50

E. B. Christoffel. Über die Gaussische Quadratur und eine Verallgemeinerung derselben. *J. Reine Angew. Math.*, 55:61–82, 1858. 15

A. K. Common and J. H. McCabe. The symmetric strong moment problem. *J. Comput. Appl. Math.*, 67:327–341, 1996. 51

J. B. Conway. *Functions of One Complex Variable I*. Springer-Verlag, New York, 1978. 100

C. W. Cryer. Über die Jacobischen Polynome und zwei verwandte Polynomklassen. *Math. Zeit.*, 39: 634–638, 1935. 17

C. W. Cryer. Rodrigues formula and the classical orthogonal polynomials. *Bull. Un. Mat. Ital.*, 25:1–11, 1970. 17

A. P. da Silva and A. Sri Ranga. Polynomials generated by three term recurrence relation. Bounds for complex zeros. *Linear Algebra Appl.*, 397:299–324, 2005. 25, 46

G. Darboux. Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série. *Liouville J.*, 4(3):5–56, 377–416, 1878. 15

L. Daruis, J. Hernández, and F. Marcellán. Spectral transformations for Hermitian Toeplitz matrices. *J. Comput. Appl. Math.*, 202:155–176, 2007. 66, 82, 105, 143, 145

D. Day and L. Romero. Roots of polynomials expressed in terms of orthogonal polynomials. *SIAM J. Numer. Anal.*, 43(5):1969–1987, 2005. 1

L. de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1):137–152, 1985. 1

P. A. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*. New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, Rhode Island, 1999. Courant Lecture Notes, Vol. 3. 1

P. Delsarte and Y. Genin. The split Levison algorithm. *IEEE Trans. Acoust. Speech Signal Process*, 34: 470–478, 1986. 35

P. Delsarte and Y. Genin. On the role of orthogonal polynomials on the unit circle in digital signal processing applications. *Orthogonal polynomials: Theory and Practice*, 294:115–133, 1990. P. Nevai (ed.), Kluwer Academic Publishers, Dordrecht-Boston-London, Serie C: Mathematical and Physical Sciences. 5

P. Delsarte and Y. Genin. Tridiagonal approach to the algebraic environment of Toeplitz matrices, Part I: Basic results. *SIAM J. Matrix Anal.*, 12:220–238, 1991. 35

D. K. Dimitrov, F. Marcellán, and F. R. Rafaeli. Monotonicity of zeros of Laguerre-Sobolev-type orthogonal polynomials. *J. Math. Anal. Appl.*, 368:80–89, 2010. 65, 97

D. K. Dimitrov, M. V. Mello, and F. R. Rafaeli. Monotonicity of zeros of Jacobi-Sobolev type orthogonal polynomials. *Appl. Numer. Math.*, 60:263–276, 2010b. 65, 97

S. Elhay, G. H. Golub, and J. Kautsky. Updating and downdating of orthogonal polynomials with data fitting applications. *SIAM J. Matrix Anal. Appl.*, 12:327–353, 1991. 1

J. Favard. Sur les polynômes de Tchebycheff. *C. R. Acad. Sci. Paris*, 200:2052–2055, 1935. 11

L. Faybusovich and M. Gekhtman. On Schur flows. *J. Phys. A: Math. Gen.*, 32:4671–4680, 1999. 131

L. Féjer. Über trigonometrische Polynome. *J. Reine Angew. Math.*, 146:53–82, 1915. 106

L. Féjer. Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen. *Math. Ann.*, 85:41–48, 1922. 29

E. Fermi, J. Pasta, and S. Ulam. *Studies of Nonlinear Problems*. University of Chicago Press, 1965. Collected Works of Enrico Fermi. 2

H. Flaschka. Discrete and periodic illustrations of some aspects of the inverse method. *Dynamical Systems, Theory and Applications, Lecture Notes in Phys.*, 38:441–466, 1975. 1, 2

A. Foulquié, F. Marcellán, and K. Pan. Asymptotic behavior of Sobolev-type orthogonal polynomials on the unit circle. *J. Approx. Theory*, 100:345–363, 1999. 89, 93, 94

G. Freud. *Orthogonal Polynomials*. Akadémiai Kiadó, Pergamon Press, Budapest, 1971. 5, 9, 26

L. Garza. *Transformaciones espectrales, funciones de Carathéodory y polinomios ortogonales en la circunferencia unidad*. PhD thesis, Departamento de Matemáticas, Universidad de Carlos III de Madrid, 2008. In Spanish. 32, 105, 109

W. Gautschi. *Orthogonal Polynomials: Computation and Approximation*. Oxford University Press, 2004. 1

Ya. L. Geronimus. On the polynomials orthogonal with respect to a given number sequence. *Zap. Mat. Otdel. Khar'kov. Univers. i NII Mat. i Mehan.*, 17:3–18, 1940a. 4, 21, 22

Ya. L. Geronimus. On the polynomials orthogonal with respect to a given number sequence and a theorem by W. Hahn. *Izv. Akad. Nauk. SSSR*, 4:215–228, 1940b. 4, 21, 22

Ya. L. Geronimus. On some difference equations and corresponding systems of orthogonal polynomials. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 5:203–210, 1943. In Russian. 22

Ya. L. Geronimus. On polynomials orthogonal on the unit circle, on trigonometric moment problem, and on allied Carathéodory and Schur functions. *Sb. Math.*, 15:99–130, 1944. In Russian. 5, 27

Ya. L. Geronimus. *Orthogonal Polynomials on the Unit Circle and Their Applications*. Amer. Math. Soc. Translation, 1954. no. 104. 5, 9, 66, 82, 118

Ya. L. Geronimus. *Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on An Interval*. Consultants Bureau, New York, 1961. 9

F. Gesztesy and B. Simon. m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices. *J. Anal. Math.*, 73:267–297, 1997. 4

E. Godoy and F. Marcellán. An analogue of the Christoffel formula for polynomial modification of a measure on the unit circle. *Boll. Un. Mat. Ital.*, 5(A):1–12, 1991. 66, 92, 105

E. Godoy and F. Marcellán. Orthogonal polynomials and rational modifications of measures. *Canad. J. Math.*, 45:930–943, 1993. 66, 92, 105

E. Godoy, F. Marcellán, L. Salto, and A. Zarzo. Perturbations of discrete semiclassical functionals by Dirac masses. *Integral Transforms Spec. Funct.*, 5(1-2):19–46, 1997. 59

L. B. Golinskii. Schur flows and orthogonal polynomials on the unit circle. *Sb. Math.*, 197(8):1145–1165, 2007. 1, 5, 131

A. A. Gonchar. On the convergence of Padé approximants for some classes of meromorphic functions. *Math. USSR Sbornik*, 26:555–575, 1975. 85

W. B. Gragg. Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle. *J. Comput. Appl. Math.*, 46:183–198, 1993. Russian original in Numerical Methods of Linear Algebra, pp. 16-32, Moskov. Gos. Uni., Moscow, 1982. 5, 27

U. Grenander and G. Szegő. *Toeplitz Forms and Their Applications*. 2nd edition, Chelsea, New York, 1984. 1st edition, Univ. California Press, Berkeley-Los Angeles, 1958. 5

H. Hamburger. Über eine Erweiterung des Stieltjesschen Momentproblems I-III. *Math. Ann.*, 81, 82, 82:235–319, 120–164, 168–187, 1920, 1921, 1921. 13

F. Hausdorff. Momentenprobleme für ein endliches Intervall. *Math. Z.*, 16:220–248, 1923. 13

E. Heine. *Handbuch der Kugelfunctionen, Theorie und Anwendungen*. G. Reimer, Berlin, 1878, 1881. vols 1,2. 12

G. Herglotz. Über potenzreihen mit positive, reellem teil im Einheitspreis. *Ber. Ver. Ges. Wiss. Leipzig*, 63:501–511, 1911. 30

R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990. 10, 16, 24, 27, 78

M. E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*. Cambridge University Press, Cambridge, 2009. Encyclopedia in Mathematics and its Applications, Vol. 98. 1

C. G. J. Jacobi. Über die Reduction der quadratischen Formen auf die kleinste Anzahl Glieder. *J. Reine Angew. Math.*, 39:290–292, 1848. 11

C. G. J. Jacobi. Untersuchungen über die Differentialgleichung der Hypergeometrische Reihe. *J. Reine Angew. Math.*, 56:149–165, 1859. 16

W. B. Jones and W. J. Thron. *Continued Fractions. Analytic Theory and Applications*. Addison-Wesley, Reading, Massachusetts, 1980. Encyclopedia of Mathematics and Its Applications, Vol. 11. 14

W. B. Jones, W. J. Thron, and H. Waadeland. A strong Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 261:503–528, 1980. 14, 48, 56, 57

W. B. Jones, O. Njåstad, and W. J. Thron. Orthogonal Laurent polynomials and the strong moment problem. *J. Math. Anal. Appl.*, 98:528–554, 1984. 14

W. B. Jones, O. Njåstad, and W. J. Thron. Continued fractions associated with trigonometric and other strong moment problems. *Constr. Approx.*, 2:197–211, 1986. 36

W. B. Jones, O. Njåstad, and W. J. Thron. Perron-Carathéodory continued fractions. *Lecture Notes in Math.*, 1237:188–206, 1987. J. Gilewicz, M. Pindor, and W. Siemaszko (eds.), Rational Approximation and Its Applications to Mathematics and Physics, Springer-Verlag, New York. 36

W. B. Jones, O. Njåstad, and W. J. Thron. Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle. *Bull. London Math. Soc.*, 21:113–152, 1989. 1, 36, 43, 45

T. Kailath, A. Vieira, and M. Morf. Inverse Toeplitz operators, innovations, and orthogonal polynomials. *SIAM Rev.*, 20:106–119, 1978. 1

V. A. Kaliaguine. On asymptotics of L^p extremal polynomials on a complex curve ($0 < p < \infty$). *J. Approx. Theory*, 74:226–236, 1993. 131

V. A. Kaliaguine. A note on the asymptotics of orthogonal polynomials on a complex arc: the case of a measure with a discrete part. *J. Approx. Theory*, 80:138–145, 1995. 131

V. A. Kaliaguine and A. A. Kononova. On the asymptotic of polynomials orthogonal on a system of curves with respect to a measure with discrete part. *St. Petersburg Math. J.*, 21:217–230, 2010. 131

A. Ya. Khintchine. *Continued Fractions*. Noordhoff, Groningen, 1963. 14

T. H. Kjeldsen. The early history of the moment problem. *Historia Math.*, 20:19–44, 1993. 13

D. J. Korteweg and F. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 39:422–443, 1895. 5

A. M. Krall. *Hilbert Space, Boundary Value Problems and Orthogonal Polynomials*. Birkhäuser Verlag, Basel-Boston-Berlin, 2002. Operator Theory: Advances and Applications, Vol. 133. 1

H. L. Krall. On derivatives of orthogonal polynomials. *Bull. Amer. Math. Soc.*, 42:423–428, 1936. 17

H. L. Krall. On orthogonal polynomials satisfying a certain fourth order differential equation. *Pennsylvania State College Stud.*, 6:1–24, 1940. 59, 61

R. L. Lamblém. *Polinômios typo Szegő*. PhD thesis, Department of Computer Science and Statistics, State University of São Paulo, 2011. In Portuguese. 129

R. L. Lamblém, J. H. McCabe, M. A. Piñar, and A. Sri Ranga. Szegő type polynomials and para-orthogonal polynomials. *J. Math. Anal. Appl.*, 370:30–41, 2010. 129

A. M. Legendre. Recherches sur l'attraction des sphéroïdes homogènes. *Mém. Math. Phys. Prés. à l'Acad. Roy. Sci. (Paris) par divers savants*, 10:411–434, 1785. 9

E. Levin and D. S. Lubinsky. Universality limits involving orthogonal polynomials on the unit circle. *Comput. Methods Funct. Theory*, 7:543–561, 2007. 85

X. Li and F. Marcellán. On polynomials orthogonal with respect to Sobolev inner products. *Pacific J. Math.*, 175:127–146, 1996. 93, 94

X. Li and F. Marcellán. Representation of orthogonal polynomials for modified measures. *Commun. Anal. Theory Contin. Fract.*, 7:9–22, 1999. 66, 82

G. López, F. Marcellán, and W. Van Assche. Relative asymptotics for orthogonal polynomials with respect to a discrete Sobolev inner product. *Constr. Approx.*, 11:107–137, 1995. 93

L. Lorentzen and H. Waadeland. *Continued Fractions with Applications*. Elsevier Science Publ., Amsterdam, 1992. Studies in Computational Mathematics, Vol. 3. 14

L. Lorentzen and H. Waadeland. *Continued Fractions: Convergence Theory*. Atlantis Studies in Math., Atlantis Press, 2008. Vol. 1. 14

A. K. Louis and F. Natterer. Mathematical problems of computerized tomography. *Proc. IEEE*, 71: 379–390, 1983. 1

W. Magnus and S. Winkler. *Hill's Equation*. Interscience Publishers John Wiley and Sons, New York-London-Sydney, 1966. 2

F. Marcellán and J. Hernández. Geronimus spectral transforms and measures on the complex plane. *J. Comput. Appl. Math.*, 217:441–456, 2008. 105

F. Marcellán and P. Maroni. Sur l'adjonction d'une masse de Dirac à une forme régulière et semiclassique. *Ann. Mat. Pura ed Appl.*, 162(IV):1–22, 1992. 59

F. Marcellán and L. Moral. Sobolev-type orthogonal polynomials on the unit circle. *Appl. Math. Comput.*, 128:107–137, 2002. 93, 94, 100

F. Marcellán and F. R. Rafaeli. Monotonicity and asymptotic of zeros of Laguerre-Sobolev-type orthogonal polynomials of higher derivatives. *Proc. Amer. Math. Soc.*, 139:3929–3936, 2011. 97

F. Marcellán and A. Ronveaux. On a class of polynomials orthogonal with respect to a discrete Sobolev inner product. *Indag. Math. (N. S.)*, 1:451–464, 1990. 134

F. Marcellán and A. Ronveaux. Orthogonal polynomials and Sobolev inner products. A bibliography. *Universidad Carlos III de Madrid*, 2012. 93

F. Marcellán and G. Sansigre. Orthogonal polynomials on the unit circle: Symmetrization and quadratic decomposition. *J. Approx. Theory*, 65:109–119, 1991. 123

F. Marcellán and W. Van Assche. Relative asymptotics for orthogonal polynomials with a Sobolev inner product. *J. Approx. Theory*, 72:193–209, 1993. 93

F. Marcellán, J. S. Dehesa, and A. Ronveaux. On orthogonal polynomials with perturbed recurrence relations. *J. Comput. Appl. Math.*, 30:203–212, 1990. 4

F. Marcellán, T. E. Pérez, and M. A. Piñar. On zeros of Sobolev-type polynomials. *Rend. Mat. Appl.*, 7: 455–473, 1992. 137

F. Marcellán, F. Peherstorfer, and R. Steinbauer. Orhogonality properties of linear combinations of orthogonal polynomials, I. *Adv. Comput. Math.*, 5:281–295, 1996. 66

F. Marcellán, F. Peherstorfer, and R. Steinbauer. Orhogonality properties of linear combinations of orthogonal polynomials, II. *Adv. Comput. Math.*, 7:401–428, 1997. 66

P. Maroni. Une théorie algébrique des polynômes orthogonaux: Applications aux polynômes orthogonaux semiclassiques. *MACS Ann. Comput. Appl. Math.*, 9:95–130, 1991. 61

A. Martínez-Finkelshtein and B. Simon. Asymptotics of the L^2 norm of derivatives of OPUC. *J. Approx. Theory*, 163:747–773, 2011. 28

A. Maté, P. Nevai, and V. Totik. Extensions of Szegő's theory of orthogonal polynomials II. *Constr. Approx.*, 3:51–72, 1987. 90

H. McKean and P. Van Moerbeke. The spectrum of Hill's equation. *Invent. Math.*, 30(3):217–274, 1975. 2

R. McWeeny. *Symmetry: An Introduction to Group Theory and Its Applications*. Dover, New York, 2002. 17

H. G. Meijer. Zero distribution of orthogonal polynomials in a certain discrete sobolev space. *J. Math. Anal. Appl.*, 172:520–532, 1993. 100, 130, 133, 137

G. Meurant and G. H. Golub. *Matrices, Moments and Quadrature with Applications*. Princeton University Press, Princeton, 2010. 1

P. Van Moerbeke. The spectrum of Jacobi matrices. *Invent. Math.*, 37:45–81, 1976. 2

J. Morse. Three integrable hamiltonian systems connected with isospectral deformations. *Adv. Math.*, 16:197–220, 1975a. 1, 2

J. Morse. Finitely many mass points on the line under the influence of an exponential potential - an integrable system. *Dynamical systems. Theory and Applications, Lecture Note Phys*, 38:467–497, 1975b. 1, 2

A. Mukaihira and Y. Nakamura. Schur flow for orthogonal polynomials on the unit circle and its integrable discretization. *J. Comput. Appl. Math.*, 139:75–94, 2002. 35

I. Nenciu. Lax pairs for Ablowitz-Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Not.*, 11:647–686, 2005. 5

P. Nevai. *Orthogonal Polynomials*. Amer. Math. Soc. Providence, Rhode Island, 1979. Memoirs of the Amer. Math. Soc., Vol. 213. 59, 61

A. F. Nikiforov and V. B. Uvarov. *Special Functions of Mathematical Physics*. Birkhäuser, Boston, 1988. 1

E. M. Nikishin and V. N. Sorokin. *Rational Approximations and Orthogonality*. Transl. Math. Monogr., 1991. Russian original, 1988. 1

S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov. *Theory of Solitons: The Inverse Scattering Method*. Plenum, New York, 1984. Russian original, 1980. 4

F. Peherstorfer. Finite perturbations of orthogonal polynomials. *J. Comput. Appl. Math.*, 44:275–302, 1992. 4

F. Peherstorfer. A special class of polynomials orthogonal on the unit circle including the associated polynomials. *Constr. Approx.*, 12:161–185, 1996. 8, 32

F. Peherstorfer. On Toda lattices and orthogonal polynomials. *J. Comput. Appl. Math.*, 133:519–534, 2001. 4

F. Peherstorfer and R. Steinbauer. Characterization of orthogonal polynomials with respect to a functional. *J. Comput. Appl. Math.*, 65:339–355, 1995. 31

F. Peherstorfer and P. Yuditskii. Asymptotic of orthogonal polynomials in the presence of a denumerable set of mass points. *Proc. Amer. Math. Soc.*, 129:3213–3220, 2001. 131

F. Peherstorfer, V. P. Spiridonov, and A. S. Zhedanov. Toda chain, Stieltjes function, and orthogonal polynomials. *Theoret. Math. Phys.*, 151(1):505–528, 2007. 4

T. E. Pérez and M. Piñar. Global properties of zeros for Sobolev-type orthogonal polynomials. *J. Comput. Appl. Math.*, 49:225–232, 1993. 97, 130

V. F. Pisarenko. The retrieval of harmonics from a covariance function. *Geophys. J. Royal Austral. Soc.*, 33:347–366, 1973. 66

F. R. Rafaeli. *Zeros de polinômios ortogonais na recta real*. PhD thesis, Department of Computer Science and Statistics, State University of São Paulo, 2010. In Portuguese. 130, 136

E. A. Rakhmanov. On the asymptotics of the ratio of orthogonal polynomials I, II. *Sb. Math.*, 32, 46: 199–214, 105–118, 1977, 1983. Russian original in Math. Sb., 103(145):237-252, 1977; Math. Sb., 118(160):104-117, 1982. 28

A. Sri Ranga, E. X. L. Andrade, and J. McCabe. Some consequence of symmetry in strong distributions. *J. Math. Anal. Appl.*, 193:158–168, 1995. 47, 51

R. Rasala. The Rodrigues formula and polynomial differential operators. *J. Math. Anal. Appl.*, 84: 443–482, 1981. 17

M. Raussen and C. Skau. Interview with John Milnor. *Notices Amer. Math. Soc.*, 59(3):400–408, 2012. 1

F. Riesz. Sur les opérations fonctionnelles linéaires. *C. R. Acad. Sci. Paris*, 149:974–997, 1909. 12

F. Riesz. Sur certains systèmes singuliers d'équations intégrales. *Ann. École. Norm. Sup.*, 28(3):32–62, 1911. 30

L. Riesz. Über ein Problem des Herrn Carathéodory. *J. Reine Angew. Math.*, 146:83–87, 1915. 106

A. M. Rocket and P. Szüsz. *Continued Fractions*. World Scientific, Singapore, 1992. 14

A. Ronveaux and W. Van Assche. Upward extension of the Jacobi matrix for orthogonal polynomials. *J. Approx. Theory*, 86:335–357, 1996. 23

E. J. Routh. On some properties of certain solutions of a differential equation of the second order. *Proc. London Math. Soc.*, 16:245–262, 1884. 16

W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1987. 3rd edition. 12, 27

E. B. Saff and V. Totik. What parts of a measure's support attract zeros of the corresponding orthogonal polynomials? *Proc. Amer. Math. Soc.*, 114:185–190, 1995. 81

W. Schoutens. *Stochastic Processes and Orthogonal Polynomials*. Springer-Verlag, New York, 2000. 1

I. Schur. Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, I. *J. Reine Angew. Math.*, 147:205–232, 1917. I. Gohberg (ed.), English translation in I. Schur Methods in Operator Theory and Signal Processing, pp. 31-59, Operator Theory: Advances and Applications, 18, Birkhäuser, Basel, 1986. 25

I. Schur. Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, II. *J. Reine Angew. Math.*, 148:122–145, 1918. I. Gohberg (ed.), English translation in I. Schur Methods in Operator Theory and Signal Processing, pp. 66-88, Operator Theory: Advances and Applications, 18, Birkhäuser, Basel, 1986. 25

J. A. Shohat and J. D. Tamarkin. *The Problem of Moments*. Amer. Math. Soc., New York, 1943. 13

B. Simon. The classical moment problem as a self-adjoint finite difference operator. *Adv. Math.*, 137: 82–203, 1998. 1, 13

B. Simon. Analogs of the m-function in the theory of orthogonal polynomials on the unit circle. *J. Comput. Appl. Math.*, 171:411–424, 2004. 4, 5

B. Simon. *Orthogonal Polynomials on the Unit Circle*. Amer. Math. Soc. Providence, Rhode Island, 2005. Amer. Math. Soc. Coll. Publ. Series, Vol. 54, 2 vols. 5, 9, 26, 66, 69, 70, 103, 116, 118, 124

B. Simon. CMV matrices: Five years after. *J. Comput. Appl. Math.*, 208:120–154, 2007. 29

B. Simon. *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*. Princeton University Press, Princeton, 2011. 1, 9

H. Stahl and V. Totik. *General Orthogonal Polynomials*. Cambridge University Press, Cambridge, 1992. 85

T. J. Stieltjes. Recherches sur les fractions continues. *Ann. Frac. Sci. Toulouse*, 8, 9:J1–J122, A1–A47, 1894, 1895. 13, 16, 19, 20

C. Suárez. *Polinomios ortogonales relativos a modificaciones polinomiales de funcionales regulares hermitianos*. PhD thesis, Departamento de Matemática Aplicada, Universidad de Vigo, 1993. In Spanish. 32, 105, 108

P. K. Suetin. Fundamental properties of polynomials orthogonal on a contour. *Russian Math. Surveys*, 21:35–83, 1966. 9

G. Szegő. Beiträge zur Theorie der Toeplitzschen formen. *Math. Z.*, 6:167–202, 1920. 9, 27

G. Szegő. Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören. *Math. Z.*, 9:218–270, 1921a. 9

G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen, II. *Math. Z.*, 9:167–190, 1921b. 9, 27

G. Szegő. *Orthogonal Polynomials*. Amer. Math. Soc. Providence, Rhode Island, 4th Edition, 1975. Amer. Math. Soc. Coll. Publ. Series, Vol. 24. 1, 4, 5, 9, 16, 17, 18, 20, 136

C. Tasis. *Propiedades de polinomios ortogonales relativos a la circunferencia unidad*. PhD thesis, Departamento de Matemáticas, Universidad de Cantabria, 1989. In Spanish. 82, 106

A. V. Teplyaev. The pure point spectrum of random orthogonal polynomials on the unit circle. *Soviet Math. Dokl.*, 44:407–411, 1992. Russian original in Dokl. Akad. Nauk SSSR 320:49-53, 1991. 5, 27

M. Toda. *Theory of Nonlinear Lattices*. Springer-Verlag, Berlin, 1989. Springer Series in Solid-State Sciences Vol. 20. 2

R. Usmani. Inversion of a triangular Jacobi matrix. *Linear Algebra Appl.*, 212/213:413–414, 1994. 79

S. Verblunsky. On positive harmonic functions: A contribution to the algebra of Fourier series. *Proc. London Math. Soc.*, 38(2):125–157, 1935. 25

L. Vinet and A. Zhedanov. Szegő polynomials on the real axis. *Integral Transforms Spec. Funct.*, 8: 149–164, 1999. 6, 35, 36

L. Volkovyski, G. Lunts, and I. Aramanovich. *Problems in the Theory of Functions of a Complex Variable*. Mir, Moscow, 1972. 111

H. S. Wall. *Analytic Theory of Continued Fractions*. Van Nostrand, New York, 1948. The University Series in Higher Mathematics, Vol. 1. 14

J. Wallis. *The Arithmetic of Infinitesimals*. Springer-Verlag, New York-Oxford, 1656. Translation and introduction by J. A. Stedall of Arithmetica Infinitorum, 2004. 14

D.S. Watkins. Some perspectives on the eigenvalue problem. *SIAM Rev.*, 35:430–471, 1993. 29

M. S. Webster. Orthogonal polynomials with orthogonal derivatives. *Bull. Amer. Math. Soc.*, 44:880–888, 1938. 17

B. Wendroff. On orthogonal polynomials. *Proc. Amer. Math. Soc.*, 12:554–555, 1961. 16

M. L. Wong. Generalized bounded variation and inserting point masses. *Constr. Approx.*, 30:1–15, 2009. 82

A. Zhedanov. Rational spectral transformations and orthogonal polynomials. *J. Comput. Appl. Math.*, 85:67–86, 1997. 4, 6, 20, 21, 22, 23, 105