UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 03-67
Statistics and Econometrics Series 16
November 2003

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A POWERFUL TEST FOR CONDITIONAL HETEROSCEDASTICITY FOR FINANCIAL TIME SERIES WITH HIGHLY PERSISTENT VOLATILITIES.

Julio Rodríguez and Esther Ruiz*

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Keywords: Autocorrelations of non-linear transformations; GARCH; Long-memory; McLeod-Li statistic; Stochastic volatility.

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A powerful test for conditional heteroscedasticity for financial time series with highly persistent volatilities

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Traditional tests for conditional heteroscedasticity are based on testing for significant autocorrelations of squared or absolute observations. In the context of high frequency time series of financial returns, these autocorrelations are often positive and very persistent, although their magnitude is usually very small. Moreover, the sample autocorrelations are severely biased towards zero, specially if the volatility is highly persistent. Consequently, the power of the traditional tests is often very low. In this paper, we propose a new test that takes into account not only the magnitude of the sample autocorrelations but also possible patterns among them. This aditional information makes the test more powerful in situations of empirical interest. The asymptotic distribution of the new statistic is derived and its finite sample properties are analized by means of Monte Carlo experiments. The performance of the new test is compared with other alternative tests. Finally, we illustrate the results analysing several real time series of financial returns.

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JEL classification: C22

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1. INTRODUCTION

It is well known that high frequency time series of returns are characterized by evolving conditional variances. As a consequence, some non-linear transformations of returns, such as squares or absolute values, are autocorrelated. The corresponding autocorrelations are often small, positive and decay very slowly towards zero. This last characteristic has been usually related with long-memory in volatility; see, for example, Ding *et al.* (1993). Two of the most popular models to represent the dynamic evolution of volatilities are the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model of Engle (1982) and Bollerslev (1986) and the Autoregressive Stochastic Volatility (ARSV) model proposed by Taylor (1986). Both models generate series with autocorrelated squares. However, Carnero *et al.* (2001) show that, unless the kurtosis is heavily restricted, GARCH models are not able to represent autocorrelations as small as the ones often observed in practice. On the other hand, ARSV models are more flexible to represent the empirical characteristics often observed in high frequency returns. Furthermore, little is known about the autocorrelations of squared or absolute returns generated by the most popular long-memory GARCH model, the Fractionally Integrated GARH (FIGARCH) model proposed by Baillie *et al.* (1996) while the autocorrelations of powers of absolute observations of Long Memory Stochastic Volatility (LMSV) models have been derived by Harvey (1998). Consequently, in this paper, we focus on testing for conditional homoscedasticity in the context of Stochastic Volatility (SV) models.

For the reasons previously explained, the identification of conditional heteroscedasticity is often based on testing whether squared or absolute returns are autocorrelated; see, for example, Andersen and Bollerslev (1997) and Bollerslev and Mikkelsen (1999) among many others. Testing for uncorrelatedness of a particular transformation of returns, $f(y_t)$, can be carried out using the portmanteau statistic suggested by Box and Pierce (1970) given by

$$Q(M) = T \sum_{k=1}^{T} r^2(k) \tag{1}$$

where $r(k) = \frac{\sum_{t=k+1}^{T} (f(y_t)-\overline{f})(f(y_{t-k})-\overline{f})}{\sum_{t=1}^{T} (f(y_t)-\overline{f})^2}$, $\overline{f} = \frac{\sum_{t=1}^{T} f(y_t)}{T}$ and T is the sample size. In this paper, we consider two popular transformations, namely $f(y_t) = y_t^2$ or $f(y_t) = |y_t|$.

The $Q(M)$ statistic applied to squared observations was proposed by McLeod and Li (1983) who show that, if the eighth order moment of y_t exists, its asymptotic distribution can be approximated by a $\chi^2_{(M)}$ distribution. From now on, the statistic in (1) is denoted as McLeod-Li statistic even if it is applied to absolute returns. Notice that the $\chi^2_{(M)}$ asymptotic distribution of the Box-Pierce statistic requires that the series to be tested for uncorrelation is an independent sequence with finite fourth order moment; see Hannan (1970). Therefore, when the $Q(M)$ statistic is implemented to absolute values, only the fourth order moment of returns should be finite for the asymptotic distribution to hold.

Alternatively, Peña and Rodriguez (2002) have proposed a portmanteau test based on the M^{th} root of the determinant of the autocorrelation matrix of order M. The proposed statistic, denoted by D_M, is given by

$$D_M = T\left[1 - |\mathbf{R}_M|^{1/M}\right] \tag{2}$$

where

$$\mathbf{R}_M = \begin{bmatrix} 1 & \widetilde{r}(1) & ... & \widetilde{r}(M) \\ \widetilde{r}(1) & 1 & ... & \widetilde{r}(M-1) \\ ... & ... & ... & ... \\ \widetilde{r}(M) & \widetilde{r}(M-1) & ... & 1 \end{bmatrix}$$

and $\widetilde{r}(j)$ is the standardized sample autocorrelation of order j given by $\widetilde{r}(j) = \sqrt{\frac{T+2}{T-j}}r(j)$[1].

If the eighth order moment of y_t exists, the asymptotic distribution of the D_M statistic applied to squared observations can be approximated by a Gamma distribution, $\mathcal{G}(\theta, \tau)$ with $\theta = 3M(M+1)/4(2M+1)$ and $\tau = 3M/2(2M+1)$. The same result holds for absolute values if the fourth order moment of returns is finite.

Although Peña and Rodriguez (2002) show that for squared returns, the D_M test is more powerful than the McLeod-Li test, both tests have rather low power specially when the volatility is very persistent; see, Pérez and Ruiz (2003) for exhaustive Monte Carlo experiments in the context of LMSV models. The low power could be attributed to substantial finite sample negative biases of the sample autocorrelations and to the very small magnitude of the population autocorrelations. Therefore, these tests may fail to reject homoscedasticity when the returns are conditionally heteroscedastic.

However, notice that asymptotically the sample autocorrelations of independent series with finite fourth order moment are not only identically distributed normal variables with zero mean and variance $1/T$ but also mutually independent; see Hannan (1970). Therefore, the estimated autocorrelations are not expected to have any distinct pattern in large samples. However, the McLeod-Li and Peña-Rodriguez tests have only focus on the first implication of the null hypothesis, namely that the sample autocorrelations should have zero mean. These tests ignore the information on the patterns of successive estimated autocorrelations and, consequently, cannot distinguish between the correlogram of an uncorrelated variable that has all the autocorrelation coefficients small and randomly distributed around zero and the correlogram of a variable that has relatively small autocorrelations with a distinct pattern for very long lags. In this paper, we propose a new statistic to test for uncorrelatedness in non-linear transformations of returns that considers the information about possible patterns in successive correlations. This test is based on ideas developed by Koch and Yang (1986) in the context of testing for zero cross-correlations between series of multivariate dynamic systems.

Finally, given that, as mentioned before, we are analyzing the performance of tests for conditional homoscedasticity in the context of SV models, we also consider the test

[1] Peña and Rodriguez (2003a) has proposed a modified version of the D_M statistic based on the logarithm of the determinant. This new statistic has better size properties for large M and better power properties.

proposed by Harvey and Streibel (1998) who focus on the ARSV(1) model given by

$$\begin{aligned} y_t &= \sigma_* \varepsilon_t \sigma_t, \ t = 1, ..., T \\ \log(\sigma_t^2) &= \phi \log(\sigma_{t-1}^2) + \eta_t \end{aligned} \tag{3}$$

where σ_* is a scale parameter, σ_t is the volatility and ε_t and η_t are mutually independent Gaussian white noise processes with zero mean and variances one and σ_η^2 respectively. The model is stationary if $|\phi| < 1$. The same condition guarantees the existence of the fourth order moment; see Ghysels *et al.* (1996) for a detailed description of the statistical properties of SV models. The variance of the log-volatility process is given by $\sigma_h^2 = \sigma_\eta^2/(1-\phi^2)$ and it is assumed to be finite and fixed. Therefore, the variance of η_t can be written as a function of the persistence parameter as follows, $\sigma_\eta^2 = (1-\phi^2)\sigma_h^2$. Observe that if, as it is often observed in real time series of high frequency returns, the persistence parameter ϕ is close to one, then σ_η^2 should be close to zero for a given value of the variance of $\log(\sigma_t^2)$. In this case, the volatility evolves very smoothly through time. In the limit, if $\phi = 1$ then $\sigma_\eta^2 = 0$ and y_t is conditionally homoscedastic. Harvey and Streibel (1998) proposes to test the null of conditional homoscedasticity, i.e. $H_0 : \sigma_\eta^2 = 0$, using the following statistic:

$$NM = -T^{-1} \sum_{k=1}^{T-1} r(k)k. \tag{4}$$

They show that, if the second order moment of $f(y_t)$ is finite, the NM statistic has asymptotically the Crámer-von Mises distribution for which the 5% critical value is 0.461. Furthermore, the corresponding test is the Locally Best Invariant (LBI) test for the presence of a random walk[2]. They implement the test to squared and absolute observations and show that the finite sample power is higher when the latter transformation is used.

The rest of the paper is organized as follows. Section 2 describes the new statistic and derives its asymptotic distribution. In section 3, we carry out Monte Carlo experiments to assess the finite sample size and power of the new statistic against short memory ARSV models. These finite sample properties are compared with the properties of the McLeod-Li, Peña-Rodriguez and Harvey-Streibel tests. In section 4, the test is implemented to test for uncorrelatedness in squared and absolute of daily returns of several financial prices. Finally, section 5 concludes the paper.

2. A NEW TEST FOR CONDITIONAL HOMOSCEDASTICITY

As we mentioned before, conditionally heteroscedastic processes generate time series with autocorrelated squares and absolute observations. Consequently, we propose to test for conditional homoscedasticity using the information contained in the sample autocorrelations of these non-linear transformations, $f(y_t)$. The new test for uncorrelatedness of $f(y_t)$ takes into account that, under the null hypothesis, if the fourth order moment

[2] See Ferguson (1967) for the definition of the Locally Best test.

of $f(y_t)$ exists, the sample autocorrelations of $f(y_t)$ are asymptotically independent and identically distributed Normal variables with zero mean and variance $1/T$. Therefore, this statistic not only tests whether the sample autocorrelations are significantly different from zero but also incorporates information about possible patterns among successive autocorrelation coefficients, $r(k)$. We propose the following statistic

$$Q_i^*(M) = T \sum_{k=1}^{M-i} \left[\sum_{l=0}^{i} r(k+l) \right]^2 ; i = 0, 1, 2, \ldots, M-1. \tag{5}$$

Notice that for each value of the number of autocorrelations considered, M, we have a collection of statistics, choosing different values of i. Each of these statistics has different information on the possible pattern of the sample autocorrelations. For example, when $i = 0$, the McLeod-Li statistic in (1) is obtained as a particular case. In this case, the statistic is obtained adding up the squared estimated autocorrelations. If all of these autocorrelations are small, the statistic will be small and the null hypothesis is not rejected. However, when $i = 1$, the statistic incorporates information about the correlation between sample autocorrelations one lag apart. In this case, if they are strongly correlated, the null hypothesis can be rejected even if the coefficients $r(j)$ are very small. When $i = 2$, the correlations between coefficients two lags apart is also considered and so on.

The statistic $Q_i^*(M)$ is a quadratic form in $T^{1/2}\mathbf{r}_{(M)}$, where $\mathbf{r}_{(M)} = (r(1), \ldots, r(M))$ given by

$$Q_i^*(M) = T\mathbf{r}'_{(M)}\mathbf{A}_i\mathbf{r}_{(M)},$$

where $\mathbf{A}_i = \mathbf{C}_i'\mathbf{C}_i$ is a symmetric matrix of dimension M. In general $\mathbf{C}_i'$ is a matrix of dimension $M \times (M-i)$, where each column is composed by the first $i+1$ values equal to ones and the rest equal to zeroes. Given that, under the null hypothesis, the asymptotic distribution of $T^{1/2}\mathbf{r}_{(M)}$ is $N(\mathbf{0}, \mathbf{I}_M)$, the statistic $Q_i^*(M)$ has asymptotically the same distribution as the random variable,

$$Q(W) = \sum_{j=1}^{M} \lambda_j W_j^2,$$

where W_j are independent standard normal variables and λ_j are the eigenvalues of $\mathbf{A}_j$; see Box (1954). Therefore, the asymptotic distribution of $Q_i^*(M)$ depend on the eigenvalues of $\mathbf{A}_i$, and consequently on M and i. Peña and Rodriguez (2002) propose to use a computationally simpler approximation of the asymptotic distribution due to Satterthwaite (1941, 1946) and Box (1954). In particular, the distribution of $Q_i^*(M)$ can be approximated by a gamma distribution, $\mathcal{G}(\theta, \tau)$ with parameters $\theta = \frac{a^2}{2b}$ and $\tau = \frac{a}{2b}$ where $a = (i+1)(M-i)$ and $b = (M-2i)(i+1)^2 + 2\sum_{j=1}^{i} j^2(M-1+i-3(j-1))$. Notice that a is the trace of the matrix $\mathbf{A}_i$ given by $tr(\mathbf{A}_i) = \sum_{j=1}^{M} \lambda_j$ and b the trace of $\mathbf{A}_i\mathbf{A}_i$, given by $tr(\mathbf{A}_i\mathbf{A}_i) = \sum_{j=1}^{M} \lambda_j^2$. For example, if $i = 0$, then $a = M$ and $b = M$ and, therefore, the usual $\chi^2_{(M)}$ asymptotic distribution is obtained. On the other hand, if for instance, $i = 1$, then the parameters of the Gamma distribution are given by $\theta = \frac{(M-1)^2}{3M-4}$ and $\tau = \frac{M-1}{3M-4}$. For reasons that will be clearer later, another interesting case is $i = M/3 - 1$. In this case, the corresponding parameters are $\theta = \frac{54M+72M^2+24M^3}{2(45M+12M^2+7M^3+54)}$ and $\tau = \frac{108M+162}{45M+12M^2+7M^3+54}$. Finally, consider the case $i = M-1$, with $\theta = \frac{-6M}{2(M^3-4M^2-M-2)}$ and $\tau = \frac{-6}{M^3-4M^2-M-2}$. In this case, the asymptotic distribution of the statistic $Q^*_{M-1}(M)/M$ can be approximated by a $\chi^2_{(1)}$.

Finally, the asymptotic distribution of $Q^*_i(M)$ can be further simplified using the power transformation proposed by Chen and Deo (2001) to improve the normality of test statistics in finite samples. In particular,

$$\sigma^{-1}\left(Q^*_i(M)^{1/\beta} - \mu\right) \sim N(0,1)$$

where $\beta = \frac{3}{2}\left(\sum_{j=1}^{M}\lambda_j^2\right)^2\left[\frac{3}{2}\left(\sum_{j=1}^{M}\lambda_j^2\right)^2 - \left(\sum_{j=1}^{M}\lambda_j\right)\left(\sum_{j=1}^{M}\lambda_j^3\right)\right]^{-1}$, $\mu = a^{\frac{1}{\beta}} - \frac{1}{2}\frac{\beta-1}{\beta^2}a^{\frac{1}{\beta}-2}2b$ and $\sigma = \frac{(2b)^{1/2}}{\beta a^{1-\frac{1}{\beta}}}$. To facilitate the use of this approximation, Table 1 reports the values of the constants μ, σ and β for some particular cases that can be useful in the empirical analysis of real time series.

3. SIZE AND POWER IN FINITE SAMPLES

In this section, we analyze the finite sample performance of the $Q^*_i(M)$ statistic by means of Monte Carlo experiments. The main objectives are to analyze whether the asymptotic distribution is an adequate approximation to the finite sample distribution under the null hypothesis and to compare the size and power of the new statistic with the McLeod-Li, Peña-Rodriguez and Harvey-Streibel statistics. Furthermore, we give some guidelines as to which values of M and i are more adequate for the cases of interest from an empirical point of view.

3.1 Size

To analyze the finite sample size of the test, we have generated series by white noise processes with three different distributions, Normal and Student-t with $\nu = 5$ and 9 degrees of freedom. The Student-t distributions have been chosen because it has been often observed in empirical applications that the marginal distribution of financial returns is leptokurtic and we want to analyze the performance of the tests in the presence of leptokurtic although homoscedastic time series. Moreover, the degrees of freedom are selected in such a way that the eight order moment of returns exists when $\nu = 9$ and does not exit when $\nu = 5$. All the results are based on 20000 replicates.

Table 2 reports the empirical sizes of the D_M, $Q(M)$, NM and $Q^*_i(M)$ tests. We consider $M = 12, 24$ and 36 and $i = 1$, $M/3-1$ and $M-1$. The nominal size is 5% and the sample sizes are $T = 500$ and 2000. The critical values have been obtained using both the Gamma and Normal approximations described in Section 2 with similar results. Therefore, Table 2 only reports the empirical sizes obtained using the Gamma distribution. All the statistics are implemented for both squared and absolute observations. The empirical sizes obtained for the Student-9 distribution are always between the sizes reported in Table 2 for the Gaussian and Student-5 cases, and are not reported here to save space.

Looking first at the results when the series are generated by a Student-5 white noise, it is possible to observe that all the tests considered can suffer of important size distortions when they are applied to squared observations. In this case, the empirical size is larger than the nominal for the D_M, $Q(M)$ and $Q_i^*(M)$ tests with small values of i. Consequently, the tests would reject the hypothesis of homoscedasticity more often than expected when homoscedastic series are generated by a leptokurtic Student-t distribution with less than 8 degrees of freedom. Furthermore, the size distortions are not reduced when the sample size increases. For example, the sizes of D_{24}, $Q(24)$ and $Q_1^*(24)$ are 6.4%, 6.4% and 5.5% respectively when $T = 500$ and 7.8%, 8.6% and 7.3% when $T = 2000$. Therefore, it is evident that, as expected given that the eight order moment is not defined, in this case, the asymptotic distribution is a bad approximation to the finite sample distribution of the statistics considered. The size distortions of $Q_i^*(M)$ are smaller than for the $Q(M)$ and D_M tests but are still big enough as to be taken into account. Figure 1, that plots the differences between empirical and nominal sizes of $Q_i^*(M)$, $i = 0, ..., M-1$ and $M = 12$, 24 and 36, illustrates these size distortions. On the other hand, notice that the asymptotic distribution of the NM test only requires the second moment of y_t to be finite and consequently, as reflected in Table 2 its size is close to the nominal. Finally, Table 2 shows that when the tests are applied to absolute observations, the nominal and empirical sizes of all the tests considered are very close; see also Figure 2 that shows that the size of $Q_i^*(M)$ is rather close to the nominal for all M and i even for moderate sample sizes.

Focusing now on the results reported in Tables 2 and 3 for absolute observations generated by homoscedastic Student-5 and Gaussian white noises respectively, it is possible to observe that given T and M, the empirical sizes of $Q_i^*(M)$ decreases with i. Figures 1 and 2, that plots the corresponding differences between the empirical and nominal sizes, show that for $M = 12, 24$ and 36, the smallest differences are obtained when i is approximately 3, 5 and 11 respectively. Therefore, it seems that the nominal size is closer to the nominal if the following *ad hoc* rule is applied $i = [M/3] - 1$. In any case, it is important to point out that for all values of M and i considered, the size of $Q_i^*(M)$ is remarkably close to the nominal specially for the larger sample sizes.

Summarizing, the Monte Carlo experiments reported in this section show that, if $i = M/3 - 1$ and the fourth order moment of y_t exists, the asymptotic distribution provides an adequate approximation to the sample distribution of $Q_i^*(M)$ when it is applied to absolute returns. If is implemented to squared observations, the eighth order moment should be finite. Consequently, we recommend to test for conditional heteroscedasticity using absolute returns; see also Harvey and Streibel (1998) and Pérez and Ruiz (2003). From now on, in this paper, all the results are based on implementing the alternative statistics considered to absolute observations.

3.2 Power short memory models

To analyze the finite sample power of $Q_i^*(M)$, we have generated artificial series by the ARSV(1) in (3) for different coefficients of variation given by squared $C.V. = \exp(\sigma_h^2) - 1 = 0.22$, 0.82 and 1.72. These values have been chosen to resemble the parameter values often estimated when the ARSV model is fitted to real time series of financial returns; see Jacquier *et al.* (1994). The sample sizes considered are $T = 100$, 512 and 1024.

Figure 3 plots the percentage of rejections of the $Q(M)$, D_M, NM and $Q_i^*(M)$ tests as a function of the persistence parameter, ϕ, for $M = 24$ and $T = 100$ and 512. Remember that when $\phi = 1$, $\sigma_\eta^2 = 0$ and, consequently, given that the series are homoscedastic the percentage of rejections is 5%. First of all, this figure shows that the power of the NM test is highest when ϕ is close to the boundary and, consequently, the series is close to the null of heteroscedasticity and the sample size is very small. When the persistence of volatility decreases or the sample size is moderately large, the NM has important losses of power relative to its competitors. Notice that for the sample sizes usually encountered in the empirical analysis if financial time series, the powers of the $Q(M)$, D_M and $Q_i^*(M)$ tests are larger than the power of NM. On the other hand, comparing now the powers of $Q(M)$, D_M and $Q_i^*(M)$, we can observe that if the sample size is $T = 100$, the power of $Q_i^*(M)$ is the largest for all the values of the parameters considered in Figure 3, i.e. $\phi \geq 0.8$. Finally, if $T = 512$, the power of $Q_i^*(M)$ is larger when $C.V. = 0.22$ and very similar for the other two $C.V.$ considered. The powers of the three tests are similar and close to one for larger sample sizes.

To illustrate how the proposed test has higher power than its competitors, even in the more persistent cases, in which the Mc-Leod-Li and Peña-Rodriguez tests are well known to have difficulties to identify the presence of conditional heterocedasticity, Table 3 reports the powers of these tests implemented to absolute observations for $M = 12$, 24 and 36 and $T = 512$ and 1024 when the parameters are $\phi = 0.98$ and $\sigma_\eta^2 = 0.1$. The power of the $Q_{M/3-1}^*(M)$ test is the higest for all M and T. For example, if $T = 512$ and $M = 12$, then the power is 67.6% if the D_M test is implemented for absolute returns, 74% if the McLeod-Li is used and 83.3% if the new test with $i = M/3 - 1$ is implemented. Therefore, the new test has higher power and better size properties without increasing much the computational burden.

Notice that in Figure 3, the power of $Q_i^*(M)$ has been considered as a function of ϕ for fixed $C.V.$ Therefore, in this figure both parameters ϕ and σ_η^2 are moving together. However, it could be of interest to analyze how the power depends on the two parameters of the ARSV(1) model separately. It is expected that the power increases separately with both parameters, ϕ and σ_η^2. To analyze this point, Figure 4 plots the powers of $Q_i^*(24)$ for $T = 500$ as a function of ϕ and σ_η^2. This figure illustrates clearly that, as expected, the power is an increasing function of both parameters, σ_η^2 and ϕ. It also shows that the power depends more heavily on the persistence parameter ϕ than on the variance σ_η^2. When the persistence parameter is relatively low, the power is low even if σ_η^2 is large.

However, if ϕ is large, the power is large even if σ_η^2 is small.

4. Power long memory models

As we have mentioned before, another stylized fact often observed in the sample autocorrelations of squared and absolute returns is their slow decay towards zero, suggesting that volatility may have long memory. In this section, we analyze the power of $Q_i^*(M)$ in the presence of long memory. Furthermore, we also explore how the information contained in the $Q_i^*(M)$ statistic for successive values of i can be used to obtain an indication of the possible presence of long memory.

In the context of SV models, Breidt *et al.* (1998) and Harvey (1998) have proposed independently the LMSV model where the log-volatility follows an ARFIMA(p, d, q) process. The corresponding LMSV(1,d,0) model is given by

$$\begin{aligned} y_t &= \varepsilon_t \sigma_t, \ t = 1, ..., T \\ (1-L)^d (1-\phi L) \log(\sigma_t^2) &= \mu + \eta_t \end{aligned} \tag{6}$$

where $0 \leq d < 1$ is the long memory parameter and L is the lag operator such that $L^j x_t = x_{t-j}$. All the parameters and noises are defined as in the short memory ARSV(1) model in (3) except the variance of η_t that, in model (6), is given by $\sigma_\eta^2 = \frac{[\Gamma(1-d)]^2(1+\phi)}{\Gamma(1-2d)F(1;1+d;1-d;\phi)}\sigma_h^2$ where $\Gamma(\cdot)$ and $F(\cdot;\cdot;\cdot;\cdot)$ are the Gamma and Hypergeometric functions respectively.

To analyze whether the $Q_i^*(M)$ statistic is also more powerful than its competitors in the presence of long memory, 5000 time series have been generated by model (6) with the same $C.V.$ as in Section 3 and two values of the long memory parameter $d = \{0.2, 0.4\}$. These values have been chosen because the asymptotic properties of the sample autocorrelations of squared and absolute observations are only known when $d < 0.25$, see Perez and Ruiz (2003). Figures 5 and 6 plot the powers of the $Q(M)$, D_M, NM and $Q_i^*(M)$ tests for $d = 0.2$ and $d = 0.4$ respectively. These figures show that the conclusions about the relative performance in terms of the power of the alternative tests are the same as in the short memory case. The NM is more powerful only when the series are very close to be homoscedastic and the sample size is very small. On the other hand, the $Q_i^*(M)$ test clearly overperforms the $Q(M)$ and D_M tests when the sample size or the $C.V.$ are small. Finally, notice that for large sample sizes and $C.V.$ the power of the three tests are very similar and close to one. In any case, it is important to point out that, comparing Figures 4, 5 and 6, it is evident that the power of all tests decreases dramatically with the long memory parameter d. To illustrate this loss of power, Figure 7 plots the powers of the $Q_0^*(12)$ and $Q_4^*(12)$ as a function of d for fixed $\{\phi = 0.9, \sigma_\eta^2 = 0.01\}$ and $\{\phi = 0, \sigma_\eta^2 = 0.1\}$. The power of both statistics seem to depend heavily on the parameter ϕ. When $\phi = 0$, even for moderate samples as $T = 1024$, very large values of d (over 0.35) are needed for the power to be over 20%. However, when $\phi = 0.9$, the power is bigger than 20% for all values of the long-memory parameter d. Furthermore, Figure 7 also illustrate the gains of power of the $Q_i^*(M)$ test with respect to the McLeod-Li test.

Finally, to illustrate the power gains of the $Q_i^*(M)$ test in the context of LMSV models, Table 3 reports the results of the Monte Carlo experiments for some selected designs which are characterized by generating series where it is hard to detect the conditional heteroscedasticity. In particular, we consider $\{\phi = 0.9, d = 0.2, \sigma_\eta^2 = 0.01\}$ and $\{\phi = 0, d = 0.4, \sigma_\eta^2 = 0.1\}$. In this table, it is possible to observe that in the presence of long-memory, the gains in power of $Q_i^*(M)$ with respect to the $Q(M)$, D_M and NM statistics can be very important. For example, when $\phi = 0.9$, $d = 0.2$ and $\sigma_\eta^2 = 0.01$ and $T = 512$, the powers of the $Q(12)$, D_{12} and NM tests are 45%, 39.2% and 37.6% respectively, while the power of $Q_3^*(12)$ is 59.3%. Therefore, in this case, the power of $Q_i^*(M)$ is 31.78%, 51.28% and 57.71% larger than the powers of $Q(12)$, D_{12} and NM . There is a substantial increase in power. Even for relatively large sample sizes as $T = 1024$, the powers of the $Q(12)$, D_{12} and NM are 71.5%, 69% and 46.6% respectively, while the power of $Q_3^*(M)$ is 85.7%. Consequently, the gains in power of the new test proposed in this paper compared with the alternative tests, could be very important specially in the presence of long-memory in the volatility process.

5. EMPIRICAL APPLICATION

In this section, we implement the $Q(M)$, D_M, NM and $Q_i^*(M)$ statistics to test for conditional homoscedasticity of returns of exchange rates of the Canadian Dollar, Euro and Swiss Franc against the US Dollar, observed daily from 1st April 2000 until 21st of May 2003 with $T = 848$[3]. To avoid the influence of large outliers on the properties of the homoscedasticity tests, the series of returns have been filtered by the observations larger than 5 sample standard deviations; see Carnero et al. (2003) for the influence of outliers on tests for conditional homoscedasticity in the context of GARCH models. The three series of returns have been plotted in Figure 8 together with the corresponding autocorrelations of absolute returns which are rather small, always under 0.1 in absolute value. However, observe that, with the exception of the Swiss Franc, the autocorrelations are mainly positive which is imcompatible with independent observations.

Table 4 reports, for each exchange rate, the ratio between the value of the statistic and the corresponding 5% critical value for the NM, D_M, $Q(M)$ and $Q_{M/3-1}^*(M)$ statistics for $M = 10$, 20, 30 and 50 when implemented to absolute returns. Looking at the results for the Canadian Dollar, the McLeod-Li and Peña-Rodriguez tests do not reject the null hypothesis of conditional homoscedasticity for any value of M. The Harvey-Streibel test rejects the null although the statistic is relatively close to the critical value. Finally, the $Q_{M/3-1}^*(M)$ statistic rejects clearly the null, specially for large values of M as, for example, $M = 30$ or 50. Therefore, the conclusions on whether the Canadian Dollar returns are homoscedastic are contradictory depending on the statistics used to test the null.

A similar result is obtained for the returns of the exchange rates of the Euro against the

[3]The data are freely available from the web page of Professor Werner Antweiler, University of British Columbia, Vancouver BC, Canada. The exchange rates have been transformed into returns as usual by taking first differences of logarithms and multiplying by 100, i.e., $y_t = 100(\log(p_t) - \log(p_{t-1}))$.

Dollar. In this case, the McLeod-Li and Peña-Rodriguez tests are just on the boundary of the non-rejection region when the size is 5% and the number of correlations considered in the statistic is 20, 30 or 50. Furthermore, if $M = 10$, these tests suggest that this series is conditionally homoscedastic. However, the Harvey-Streibel and the $Q^*_{M/3-1}(M)$ statistics clearly reject the null hypothesis.

Finally, looking at the results for the Swiss-Franc exchange rates, the situation is somehow reverse. As in previous examples, the $Q(M)$ and $D(M)$ tests are close to the boundary of the rejection region when the size is 5%. However, the other two tests are more conclusive and they do not reject the null.

Therefore, in the three examples considered in this section, it seems that taking into account not only the magnitude but also the pattern of the sample autocorrelations of absolute returns help to obtain a clearer answer on whether the corresponding returns are homoscedastic or heteroscedastic. In the three cases, the statistics that only accounts for the magnitude of the autocorrelations are rather inconclusive while our proposed test gives a clearer answer. In these empirical examples, the answer of the test proposed by Harvey and Streibel (1998) is in concordance with the $Q^*_{M/3-1}(M)$ statistic.

6. CONCLUSIONS

In this paper, we propose a new test for conditional heteroscedasticity that takes into account that the autocorrelations of squared and absolute returns are usually small but always positive. Incorporating this additional information, the new test has larger power than several alternative tests previously proposed in the literature.

We derive the asymptotic distribution of the statistic and show, by means of Monte Carlo experiments, that it is an adequate approximation to the finite sample distribution, at least for the sample sizes usually encountered in financial time series. The results of these experiments also show that, in general, the size of the test is closer to the nominal when the test is implemented on absolute observations rather than on squares.

With respect to the power, we show that, in the context of SV models, the new statistic has larger power than the McLeod-Li and Peña-Rodriguez tests specially if the volatility is highly persistent or has long memory. Therefore, the new test is more powerful without loosing its size properties. When compared with the Harvey and Streibel (1998) test, we show that the latter test has larger power in a very narrow region close to the homoscedasticity if the sample size is very small. However, the power of the test seriously deteriorates when the volatility is not highly persistent as can be the case, for example, when analyzing environmental series; see Tol (1996) and Peña and Rodriguez (2003b). Furthermore, even if the volatility is highly persistent if the sample size is moderate or large, as often encountered in the empirical analysis of financial time series of returns, the power properties of our proposed test are clearly better than the Harvey-Streibel test.

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$M\backslash i$	1	$[M/3]-1$	$M-1$
10	$(3.93, 1.55, 1.46)$	$(3.93, 1.54, 1.45)$	$(3.35, 1.19, 1.57)$
20	$(3.89, 1.28, 1.75)$	$(5.62, 2.33, 1.45)$	$(3.16, 0.87, 2.02)$
30	$(3.88, 1.14, 1.94)$	$(6.22, 2.55, 1.49)$	$(3.10, 0.73, 2.33)$
50	$(3.87, 1.00, 2.22)$	$(6.41, 2.46, 1.60)$	$(3.06, 0.60, 2.80)$

Table 1. Values of the constants $(\beta, \mu, \sigma^{-1})$ for some particular cases in the normal approximation to the asymptotic distribution of Q(M).

		Gaussian white noise				Student-5 white noise			
		y_t^2		$\|y_t\|$		y_t^2		$\|y_t\|$	
$M\backslash T$		500	2000	500	2000	500	2000	500	2000
12	D_M	0.048	0.050	0.051	0.050	0.061	0.069	0.047	0.051
	$Q(M)$	0.048	0.050	0.053	0.048	0.065	0.074	0.051	0.054
	$Q_1^*(M)$	0.048	0.048	0.052	0.053	0.052	0.061	0.049	0.053
	$Q_3^*(M)$	0.045	0.049	0.049	0.051	0.039	0.048	0.047	0.051
	$Q_{11}^*(M)$	0.040	0.046	0.044	0.047	0.028	0.037	0.041	0.048
24	D_M	0.045	0.048	0.048	0.049	0.064	0.078	0.044	0.049
	$Q(M)$	0.054	0.051	0.058	0.050	0.064	0.086	0.054	0.055
	$Q_1^*(M)$	0.056	0.052	0.059	0.054	0.055	0.073	0.054	0.054
	$Q_7^*(M)$	0.046	0.049	0.051	0.049	0.032	0.045	0.048	0.049
	$Q_{23}^*(M)$	0.034	0.046	0.039	0.045	0.023	0.036	0.036	0.046
36	D_M	0.041	0.048	0.050	0.049	0.062	0.082	0.039	0.048
	$Q(M)$	0.059	0.051	0.062	0.052	0.063	0.089	0.056	0.053
	$Q_1^*(M)$	0.059	0.056	0.064	0.054	0.053	0.079	0.059	0.056
	$Q_{11}^*(M)$	0.049	0.050	0.052	0.051	0.030	0.044	0.050	0.048
	$Q_{35}^*(M)$	0.028	0.043	0.033	0.045	0.020	0.034	0.031	0.043
	NM	0.050	0.047	0.053	0.051	0.044	0.046	0.048	0.049

Table 2. Empirical sizes of Q(M), D_M, NM and $Q_i^*(M)$ tests, $i = 1, M/3-1$ and $M-1$ for squared and absolute observations of homoscedastic Gaussian and Student-5 white noise series of sizes $T = 500$ and 2000.

		$T = 512$			$T = 1024$		
Parameters		$M = 12$	$M = 24$	$M = 36$	$M = 12$	$M = 24$	$M = 36$
$\{\phi, d, \sigma_\eta^2\}$							
$\{0.98, 0, 0.01\}$	D_M	.676	.674	.647	.939	.946	.938
	$Q(M)$	.727	.725	.713	.965	.968	.964
	$Q^*_{M/3-1}$	.828	.855	.860	.986	.994	.990
	NM	.665	.665	.665	.775	.775	.775
$\{0.9, 0.2, 0.01\}$	D_M	.392	.370	.326	.706	.690	.646
	Q^*_0	.450	.405	.365	.746	.715	.664
	$Q^*_{M/3-1}$	.593	.573	.563	.857	.850	.830
	NM	.376	.376	.376	.466	.466	.466
$\{0, 0.45, 0.1\}$	D_M	.229	.209	.189	.493	.476	.453
	Q^*_0	.253	.251	.220	.504	.516	.514
	$Q^*_{M/3-1}$	.363	.380	.392	.661	.683	.693
	NM	.382	.382	.382	.564	.564	.564

Table 3. Empirical powers of the D_M, $Q(M)$, $Q^*_{M/3-1}$ and NM tests for absolute observations of $LMSV(1, d, 0)$ processes.

	a) Canadian dollar				b) EUROS				c) Swiss franc			
M	10	20	30	50	10	20	30	50	10	20	30	50
$Q(M)$	0.70	0.75	0.68	0.61	0.94	1.30	1.31	1.17	0.59	0.91	1.01	0.91
$Q^*_i(M)$	1.05	1.72	2.29	1.97	1.77	2.60	3.66	3.48	0.56	0.31	0.50	0.37
D_M	0.51	0.65	0.64	0.62	0.84	1.04	1.11	1.07	0.56	0.83	0.98	0.95
NM	1.28	1.28	1.28	1.28	3.34	3.34	3.34	3.34	0.69	0.69	0.69	0.69

Table 4. Ratio between the value of the statistic and the corresponding 5% critical value for the NM, D_M, $Q(M)$ and $Q^*_{M/3-1}(M)$ statistics for $M = 10, 20, 30$ and 50 when implemented to absolute returns..

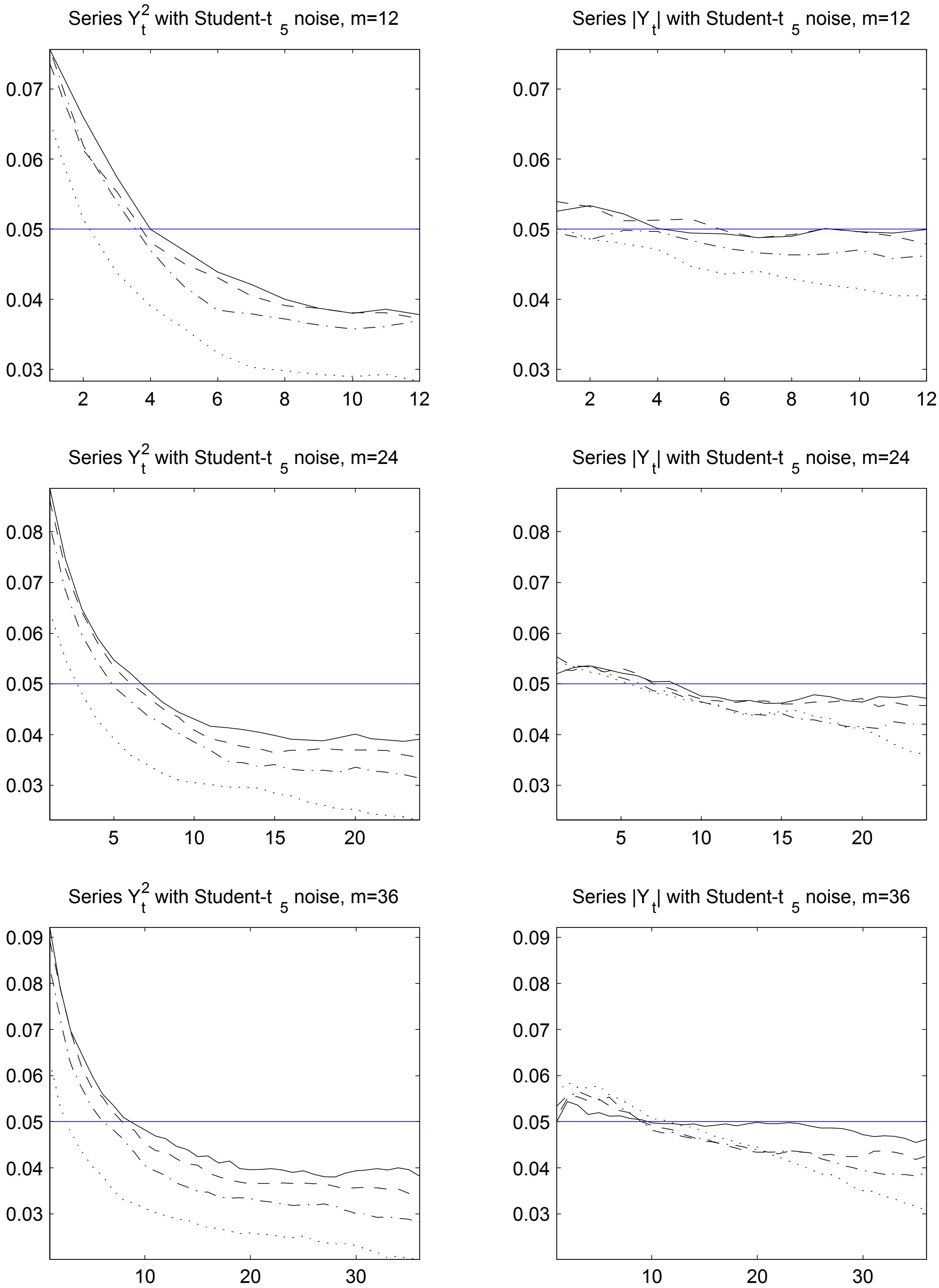

FIG. 1. Differences between empirical and nominal rejection probabilities, $\alpha = 0.05$, of the $Q_i^*(M)$ test, $i = 0, \ldots, M-1$, for non-linear transformations of Student-t_5 noises with T=500 ($\cdots$), T=1000 ($-\cdot$), T=2000 ($--$) and T=4000 (—).

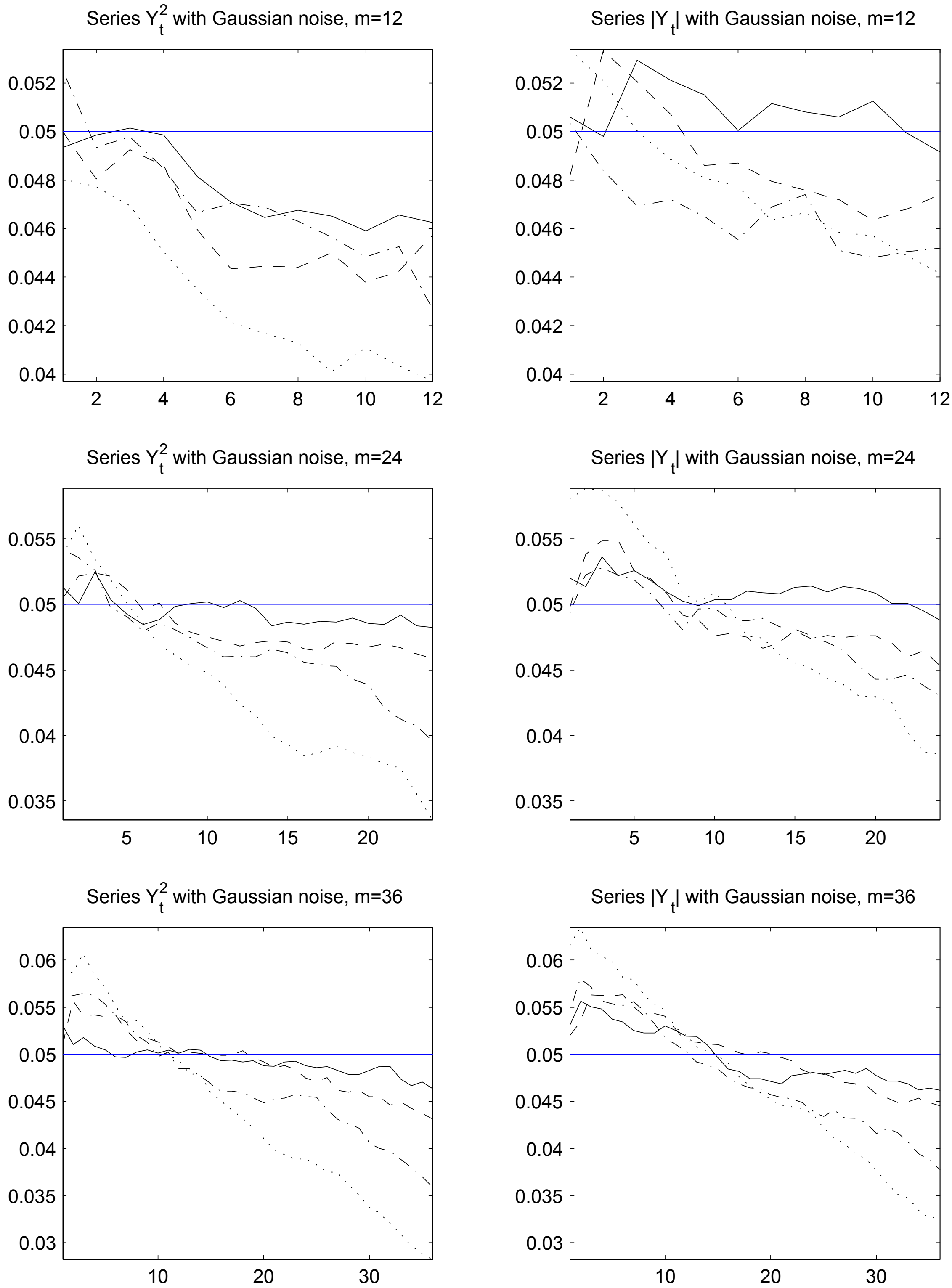

FIG. 2. Differences between empirical and nominal rejection probabilities, $\alpha = 0.05$, of the $Q_i^*(M)$ test, $i = 0, \ldots, M-1$, for non-linear transformations of Gaussian noises with T=500 ($\cdots$), T=1000 ($-\cdot$), T=2000 ($--$) and T=4000 (—).

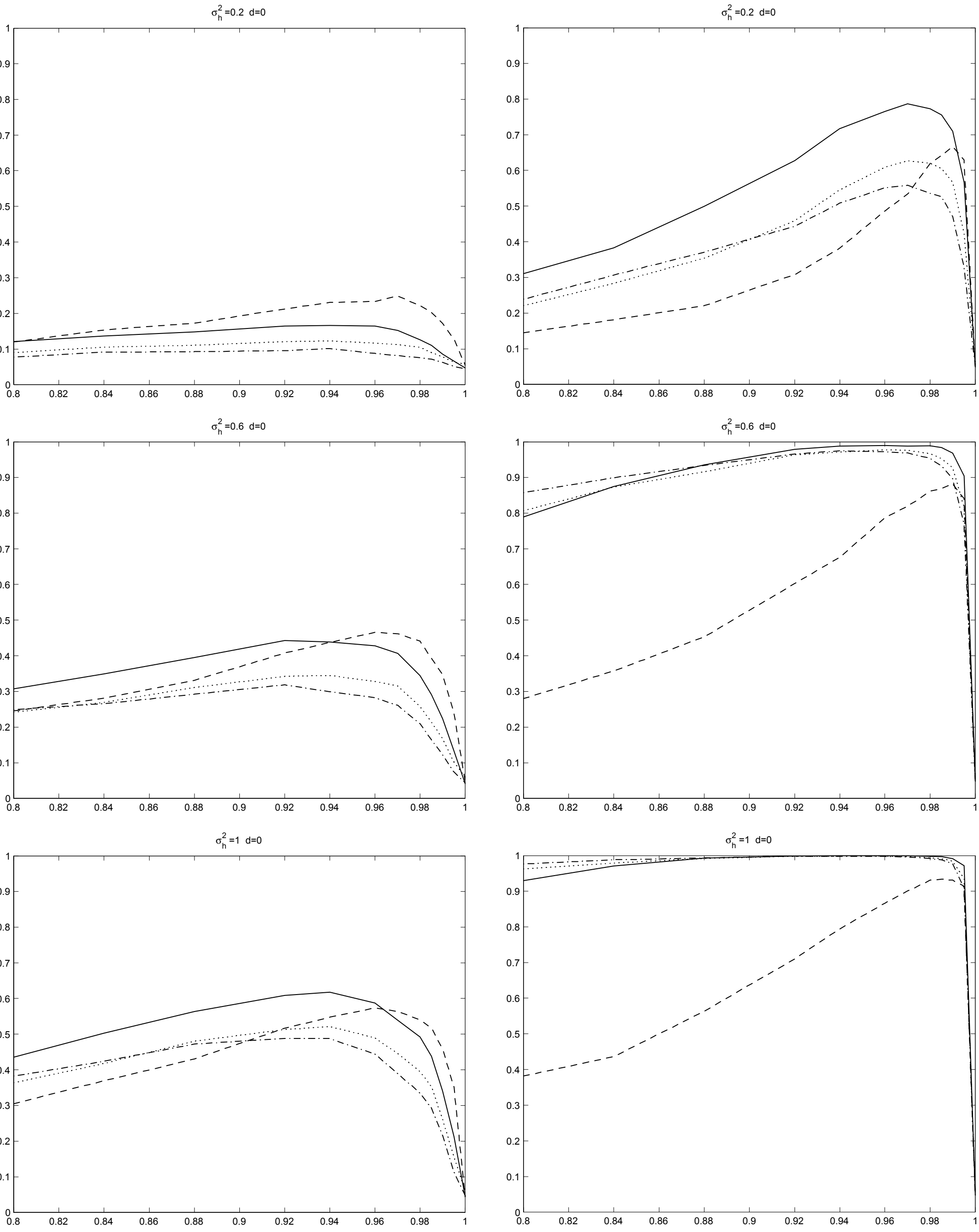

FIG. 3. Empirical powers of the Q(24) ($\cdots$), D_{24} ($-\cdot$), NM ($--$) and $Q_7^*(24)$ (—) tests for absolute observations of $ARSV(1)$ processes with T=100 (left panels) and T=512 (right panels) and C.V.=0.22 (first row), 0.82 (second row) and 1.72 (third row).

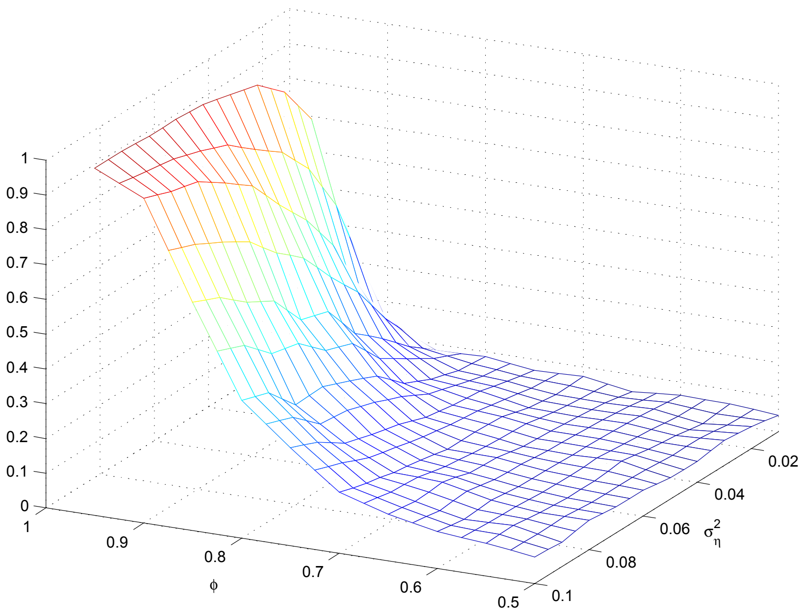

FIG. 4. Powers of $Q_i^*(24)$ test, in ARSV(1), as a function of the persistence parameter, ϕ,and the variance of volatility,σ_η^2, when $T = 500$.

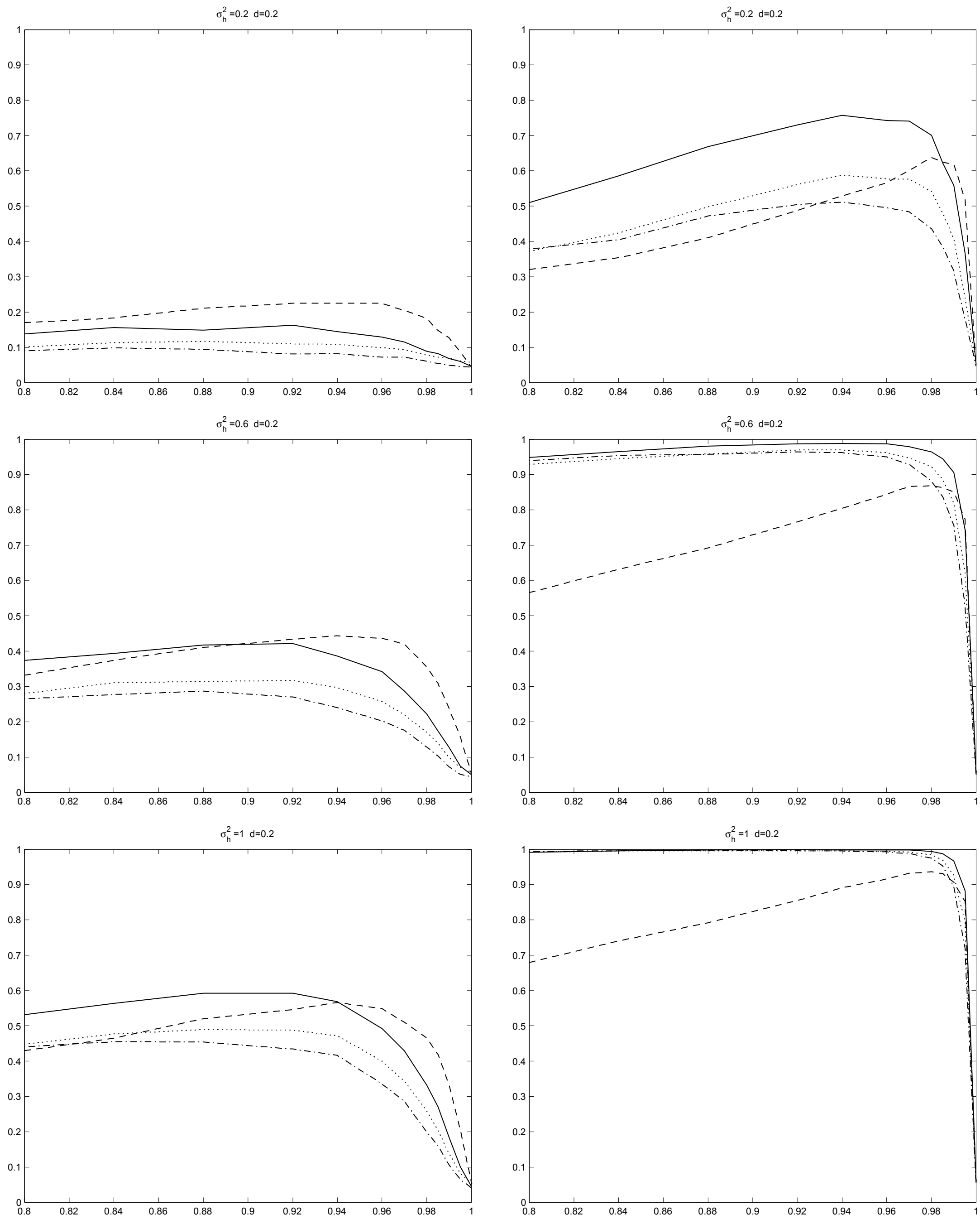

FIG. 5. Empirical powers of the Q(24) ($\cdots$), D_{24} ($-\cdot$), NM ($--$) and $Q_7^*(24)$ (—) tests for absolute observations of $LMSV$ processes with long memory parameter d=0.2 and sample sizes T=100 (left panels) and T=512 (right panels) and C.V.=0.22 (first row), 0.82 (second row) and 1.72 (third row).

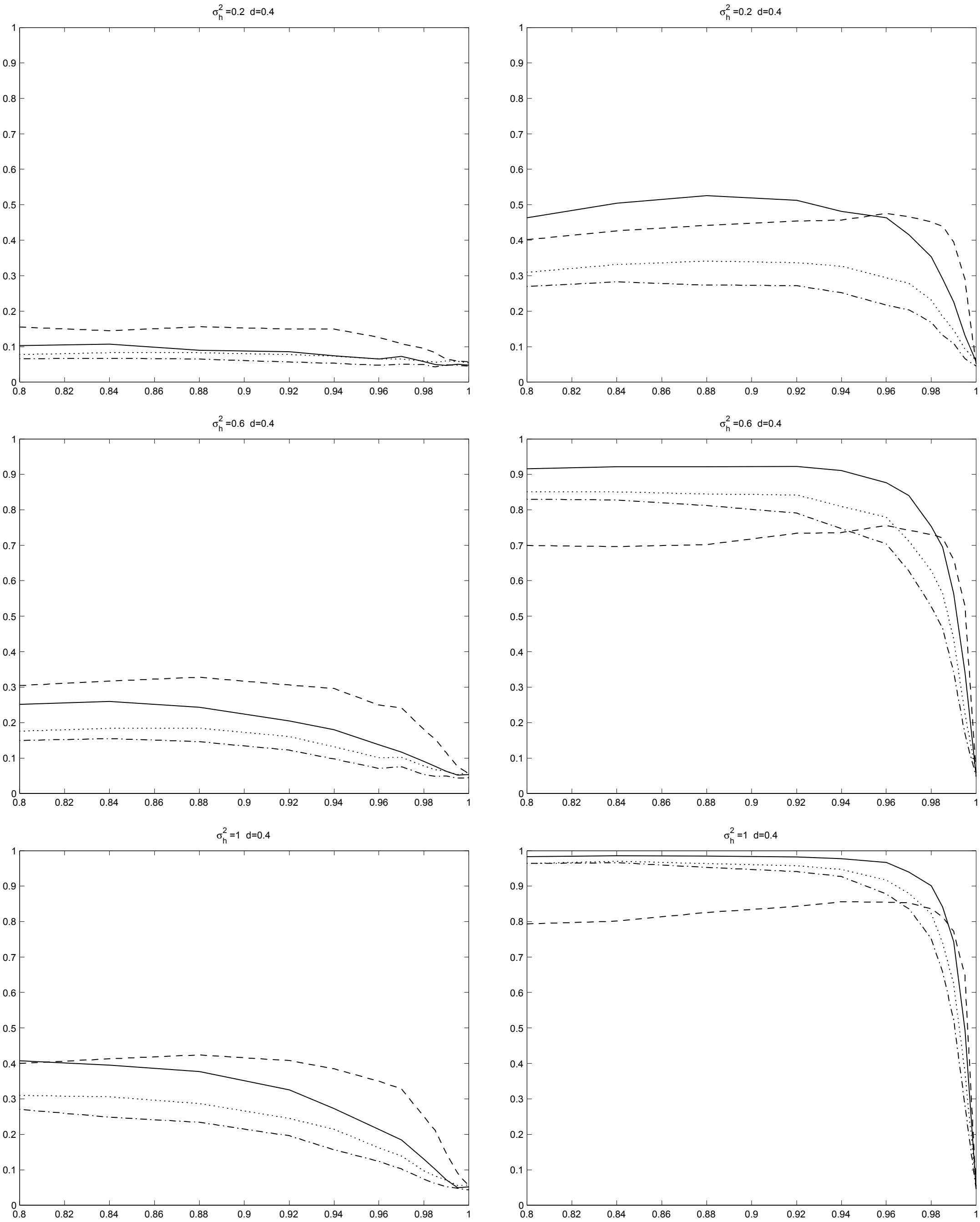

FIG. 6. Empirical powers of the Q(24) $(\cdots)$, D_{24} $(-\cdot)$, NM $(--)$ and $Q_7^*(24)$ (—) tests for absolute observations of $LMSV$ processes with long memory parameter d=0.4 and sample sizes T=100 (left panels) and T=512 (rigt panels) and $C.V. = 0.22$ (first row), 0.82 (second row) and 1.72 (third row).

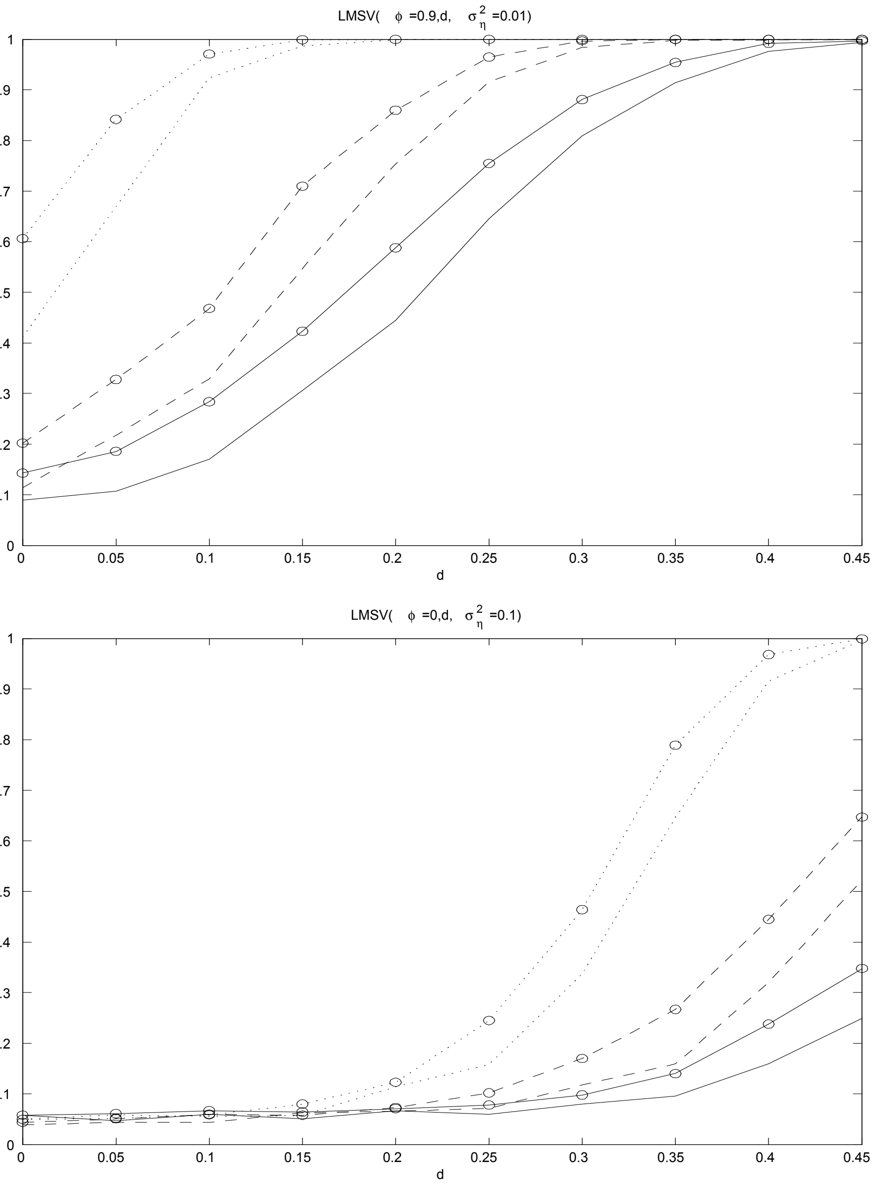

FIG. 7. Empirical powers of the nominal 5% $Q_{ML}(12)$ (lines) and $Q_4^*(12)$ (lines with circles) tests for the absolute transformation of $LMSV(1, d, 0)$ processes with T=512 (—), T=1024 (−−) and T=4096 $(\cdots)$.

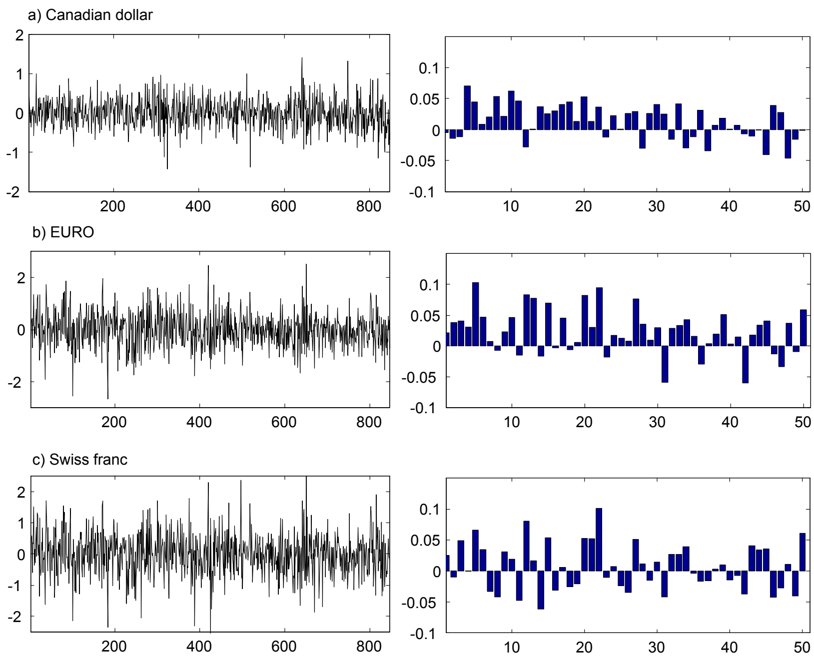

FIG. 8. Series of returns with the corresponding autocorrelations of absolute returns.