UNIVERSIDAD CARLOS III DE MADRID
working papers

Working Paper 12-24
Statistics and Econometrics Series 18
October 2012

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PYRAMIDAL VALUES

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Abstract: We propose a new type of values for cooperative TU-games, which we call pyramidal values. Assuming that the grand coalition is sequentially formed, and all orderings are equally likely, we define a pyramidal value to be any expected payoff in which the entrant player receives a salary and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. To be specific, we consider some parametric families of pyramidal values: the egalitarian pyramidal family, which coincides with the a-consensus value family introduced by Ju et al. in (2007), the proportional pyramidal family, and the weighted pyramidal family, which in turn includes the other two families as special cases. We also analyze the properties of these families, as well as their relationships with other previously defined values.

Keywords: Game Theory, TU games, piramidal values, consensus values.

Pyramidal values[1]

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Abstract

We propose a new type of values for cooperative TU-games, which we call *pyramidal values*. Assuming that the grand coalition is sequentially formed, and all orderings are equally likely, we define a pyramidal value to be any expected payoff in which the entrant player receives a salary and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. To be specific, we consider some parametric families of pyramidal values: the *egalitarian pyramidal family*, which coincides with the α-consensus value family introduced by Ju et al. (2007), the *proportional pyramidal family*, and the *weighted pyramidal family*, which in turn includes the other two families as special cases. We also analyze the properties of these families, as well as their relationships with other previously defined values.

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1 Introduction

In this paper we propose a general procedure for obtaining a broad class of solution concepts based on a pyramidal distribution of the benefits that are sequentially obtained through a dynamic process of coalition formation, in which players successively come into play and join the current coalition until the gran coalition is formed. The well-known Shapley value (Shapley, 1953) has been characterized (Weber, 1988) as the average over all permutations of a very extreme pyramidal distribution of the benefits, in which the entrant player receives all the just generated benefits (jointly created by the existing coalition of players and the entrant), when the grand coalition is sequentially formed, and all orderings are equally likely. However, such extreme shares immediately lead us to point out two questions: Why the incumbents are going to accept the deal? Why the entrant is going to stay in the coalition after receiving all his contribution?

Assuming also that all orderings are equally likely, we propose a class of values based on a more general pyramidal sharing scheme in which the entrant player receives a salary and the right

[1]This research has been supported by I+D+i research project MTM2011-27892 from the Government of Spain.

to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. In Section 2, we first introduce some standard concepts and notation on Game Theory that will be used throughout this paper, we provide a formal definition of a *pyramidal value*, and we analyze some general properties of that class of values. In Sections 3 and 4, we define parametric families of pyramidal values in which the entrant player receives as salary his own value plus a fixed proportion of the jointly created benefit less his salary, whereas the remaining benefit is distributed among the incumbent players. If the remaining benefit is equally allocated among the incumbents, then we obtain the family of *α-egalitarian pyramidal values*. On the contrary, if the remaining surplus is distributed according to each player's *contribution* to the coalition previously formed, we obtain the family of *α-proportional pyramidal values*. Finally, if the remaining surplus is distributed according to a given collection of weighting vectors $\boldsymbol{\omega}$ *exogenously* given, then we obtain the family of *α-$\boldsymbol{\omega}$-weighted pyramidal values*, which in fact includes the previous two families. In Section 3 we discuss the family of *α-egalitarian pyramidal values*, which in fact turns out to be the family of α-consensus values introduced by Ju, Borm and Ruys (2007). The other two families are introduced in Section 4, and Section 5 concludes the paper.

2 Pyramidal values

An n-person cooperative game in characteristic function form with transferable utility (TU game) is an ordered pair (N, v), where N is a finite set of n players and $v : \mathcal{P}(N) \to \mathbb{R}$ is a map assigning a real number $v(S)$, called the value of S, to each coalition $S \subseteq N$, and where $v(\emptyset) = 0$. The real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together. Let G_n be the space of all TU games with fixed player set N, where $n = |N|$, and identify $(N, v) \in G_n$ with its characteristic function v when no ambiguity appears. One of the main topics dealt with in Cooperative Game Theory is, given a game $(N, v) \in G_n$, to divide the amount $v(N)$ between players if the grand coalition N is formed. A *payoff vector*, or *allocation*, is any $\mathbf{x} \in \mathbb{R}^n$, which gives player $i \in N$ a payoff x_i. A payoff vector is said to be *efficient* if $\sum_{i \in N} x_i = v(N)$.

A *value* φ for TU games is an assignation which associates to each n-person game $(N, v) \in G_n$ a payoff vector $\varphi(N, v) \in \mathbb{R}^n$. The Shapley value (Shapley, 1953) is one of the most interesting values in Cooperative Game Theory. It can be characterized as the average of the marginal contribution vectors over all permutations (Weber, 1988). Formally, let $(N, v) \in G_n$, and let $\Pi(N)$ denotes the set of all permutations on the player set N, which we will represent as bijections $\pi : N \to N$. For a permutation $\pi \in \Pi(N)$, $\pi(i) \in N = \{1, \ldots, n\}$ represents agent i's position in order π. Define the set of all predecessors of i in π to be $P_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$, and the set of all his successors to be $S_\pi(i) = \{j \in N \mid \pi(j) > \pi(i)\}$. Moreover, the direct successor of i in the order π will be denoted by $ds_\pi(i)$, and the direct predecessor by $dp_\pi(i)$. Now, the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by

$$m_i^\pi(v) = v(P_\pi(i) \cup \{i\}) - v(P_\pi(i)), \quad i \in N,$$

which assigns to each player $i \in N$ its marginal contribution to the worth of the coalition consisting of all his predecessors in π. In that case, when player j joins coalition $P_j(\pi)$, he generates the surplus $m_j^\pi(v)$, which, according to the Shapley value, is distributed among the current coalition as follows:

- Entrant j's salary: $s_j^\pi(v) = m_j^\pi(v)$
- Incumbents $P_\pi(j)$'s shares: $a_{ij}^\pi(v) = 0$, for all $i \in P_\pi(j)$

In this setting, we define a class of values, which we call *pyramidal class*, that is based on a more general sharing scheme in which the entrant player receives a salary and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition, whereas the remaining benefit is distributed among the incumbent players. Formally:

Definition 1. Let $\mathcal{P}$ be a value for TU games. Then, $\mathcal{P}$ is called a *pyramidal value*, if for all $n \geq 1$, for all order $\pi \in \Pi(N)$, and for every n-person TU game $(N, v) \in G_n$, there exists a *pyramidal sharing scheme* $\mathcal{S} = \{(s_j^\pi(v))_{j\in N}, \{(a_{ij}^\pi(v))_{i\in P_\pi(j)} \,|\, j \in N\}\}$ such that

$$s_j^\pi(v) + \sum_{i\in P_\pi(j)} a_{ij}^\pi(v) = m_j^\pi(v), \quad \forall\, j \in N. \tag{1}$$

and verifying:

$$\mathcal{P}_i(v) = \sum_{\pi\in\Pi(N)} \frac{1}{n!} p_i^\pi(v), \quad \forall\, i \in N, \tag{2}$$

where $p_i^\pi(v) = s_i^\pi(v)$ if $\pi(i) = n$, and

$$p_i^\pi(v) = s_i^\pi(v) + \sum_{j\in S_\pi(i)} a_{ij}^\pi(v), \;\; \text{for all } i \in N \text{ with } \pi(i) < n. \tag{3}$$

In the sequel, we will refer to a pyramidal sharing scheme $\mathcal{S}$ verifying condition (1) as a $\mathcal{P}$-efficient sharing scheme. Note that negative salaries or shares are allowed in the previous definition. As usual, negative quantities must be interpreted as costs, penalties or investments in a broad sense.

The properties of the sharing scheme determine the pyramidal value properties. Then, let us formalize some interesting properties of a pyramidal sharing scheme.

Definition 2. Let $\mathcal{S} = \{(s_j^\pi(v))_{j\in N}, \{(a_{ij}^\pi(v))_{i\in P_\pi(j)} \,|\, j \in N\}\}$ be a $\mathcal{P}$-efficient sharing scheme, and let $(N, v) \in G_n$ be any n-person TU game. Then, $\mathcal{S}$ verifies,

(i) *Constant Salary*. If for all $j \in N$ there exists a real constant $k_j(v) \in \mathbb{R}$ such that $s_j^\pi(v) = k_j(v)$, for all $\pi \in \Pi(N)$.

(ii) *$\mathcal{P}$-Additivity*. If for all order $\pi \in \Pi(N)$, and for all $j \in N$ it holds $s_j^\pi(v+w) = s_j^\pi(v) + s_j^\pi(w)$, and $a_{ij}^\pi(v+w) = a_{ij}^\pi(v) + a_{ij}^\pi(w)$, for each $j \in S_\pi(i)$, for all $(N, v), (N, w) \in G_n$, where $v + w$ is given by $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

(*iii*) $\mathcal{P}$-*Dummy player*. If $s_i^\pi(v) = v(i)$, and $a_{ij}^\pi(v) = 0$, for all $j \in S_\pi(i)$, and all order $\pi \in \Pi(N)$, for all $i \in N$ being a dummy player (i.e., $v(S \cup i) = v(S) + v(i)$ for every coalition S).

(*iv*) $\mathcal{P}$-*Symmetry*. If, for all symmetric players $i, j \in N$ (i.e., $v(S \cup i) = v(S \cup j)$, for all $S \subseteq N \setminus \{i, j\}$), $s_i^\pi(v) = s_j^\pi(v)$, and it follows

$$a_{ik}^\pi(v) = a_{jk}^{\pi_{ij}}(v), \quad \text{for all } k \in S_\pi(i),$$

where $\pi_{ij}(k) = \pi(k)$, for all $k \in N \setminus \{i, j\}$, $\pi_{ij}(i) = \pi(j)$ and $\pi_{ij}(j) = \pi(i)$.

Note that constant salary property implies that the salary is an inherent attribute of each player. It can be related, for instance, to his personal training. Moreover, since $P_\pi(i) = \emptyset$ for all order π such that $\pi(i) = 1$, then each player's constant salary equals his own value $v(i)$. $\mathcal{P}$-Additivity, $\mathcal{P}$-dummy player and $\mathcal{P}$-symmetry trivially lead to the same properties for the corresponding pyramidal value. Let us recall those well-known properties of values for TU games. Formally, a *value* $\varphi : G_n \to \mathbb{R}^n$:

(*i*) is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$, for all $(N, v) \in G_n$;

(*ii*) is *additive* if $\varphi(v + w) = \varphi(v) + \varphi(w)$, for all $(N, v), (N, w) \in G_n$;

(*iii*) is *symmetric* if $\varphi_i(v) = \varphi_j(v)$, for all $(N, v) \in G_n$, and for all symmetric players $i, j \in N$;

(*iv*) is *relative invariant with respect to strategic equivalence* if $\varphi(N, w) = a\varphi(N, v) + b$, for every $(N, v) \in G_n$, $a > 0$ and $b \in \mathbb{R}^n$, where w is given by $w(S) = av(S) + \sum_{i \in S} b_i$, for all $S \subseteq N$.

(*v*) verifies the *dummy property* if $\varphi_i(v) = v(i)$, for all $(N, v) \in G_n$, and for all dummy player $i \in N$.

(*vi*) verifies the *null player property* if $\varphi_i(v) = 0$, for all $(N, v) \in G_n$, and for all *null player* $i \in N$ (i.e., $v(S \cup i) = v(S)$, for all $S \subseteq N \setminus \{i\}$).

(*vii*) is *standard for two-person games* if

$$\varphi_i(v) = v(i) + \frac{1}{2}\big(v(\{i, j\}) - v(i) - v(j)\big), \text{ for all } i \neq j, \text{ for all two-person game } (\{i, j\}, v) \in G_2.$$

Proposition 1. *Let $(N, v) \in G_n$ be a n-person TU game. Then, any efficient allocation $\varphi \in \mathbb{R}^n$ can be obtained as a pyramidal payoff.*

Proof. Let $(N, v) \in G_n$ be a n-person TU game. Since φ is efficient, it can be expressed as a linear convex combination of the extreme points $(v(N)_i, \mathbf{0}_{N \setminus i}) = (0, \ldots, 0, v(N), 0, \ldots, 0)$, $i = 1, \ldots, n$. Moreover, let $\psi_1(v), \ldots, \psi_n(v)$ be n pyramidal allocations, and let $\mathcal{S}_1(v), \ldots, \mathcal{S}_n(v)$, be their corresponding pyramidal sharing schemes. Then, the linear convex combination $\lambda_1 \mathcal{S}_1(v) + \cdots + \lambda_n \mathcal{S}_n(v)$ ($\lambda_i \geq 0$, for all $i = 1, \ldots, n$, and $\lambda_1 + \cdots + \lambda_n = 1$) is also a pyramidal sharing scheme, which gives the pyramidal allocation $\lambda_1 \psi_1(v) + \cdots + \lambda_n \psi_n(v)$.

Now, we will prove that any extreme point $(v(N)_i, \mathbf{0}_{N\setminus i})$ can be obtained by means of a pyramidal sharing scheme when negative payments are allowed. Let $\pi \in \Pi(N)$ be any given order. Let us consider the following reallocation:

- For all player $k \in P_\pi(i)$, his salary is $s_k^\pi(v) = m_k^\pi(v)$, and he distributes $a_{jk}^\pi(v) = 0$ among his predecessors $j \in P_\pi(k)$.
- For all player $k \in S_\pi(i)$, his salary is $s_k^\pi(v) = 0$, and he distributes $a_{jk}^\pi(v) = 0$ among all his predecessors $j \in P_\pi(k) \setminus \{i\}$, except for player i, who receives $a_{ik}^\pi(v) = m_k^\pi(v)$.
- When player $i \in N$ arrives, his salary is $s_i^\pi(v) = v(P_\pi(i) \cup i)$, and he distributes $a_{ji}^\pi(v) = -m_j^\pi(v)$ among his predecessors $j \in P_\pi(i)$.

Let $\mathcal{S}_i(v)$ denotes the pyramidal sharing scheme described above. Clearly, it gives the pyramidal allocation $(v(N)_i, \mathbf{0}_{N\setminus i})$, and thus the result holds. □

Proposition 2. *Any additive and efficient value φ can be obtained as a $\mathcal{P}$-additive pyramidal value.*

Proof. Let us first recall the unanimity basis for G_n, $\{(N, u_T)\}_{T\subseteq N}$, where

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

We first show that the value of any unanimity game $\varphi(u_T)$ can be obtained by means of a pyramidal sharing procedure. Let $\pi \in \Pi(N)$ be any given order. Let us consider the following redistribution, where $t_\pi \in T$ is the last member of T according to the order π.

- For all player $j \in P_\pi(t_\pi)$, his salary is $s_j^\pi(u_T) = 0$, and he distributes $a_{ij}^\pi(u_T) = 0$ among his predecessors $i \in P_\pi(j)$.
- When the last member of T arrives, he distributes 1 as follows:

$$s_{t_\pi}^\pi(u_T) = \varphi_{t_\pi}(u_T) + \sum_{j \in S_\pi(t_\pi)} \varphi_j(u_T), \tag{4}$$

$$a_{it_\pi}^\pi(u_T) = \varphi_i(u_T), \text{ for all } i \in P_\pi(t_\pi). \tag{5}$$

- For all player $j \in S_\pi(t_\pi)$, his salary is $s_j^\pi(u_T) = \varphi_j(u_T)$, which is paid by t_π. That is, $a_{ij}^\pi(u_T) = 0$, for all $i \in P_\pi(j) \setminus \{t_\pi\}$, and $a_{t_\pi j}^\pi(u_T) = -\varphi_j(u_T)$.

Clearly, the proposed sharing scheme gives $\varphi(u_T)$. Now, let $(N, v) \in G_n$ be a given TU game. Then it can be expressed as (see Shapley, 1953) $v = \sum_{T\subseteq N} \Delta(T) u_T$, where $\Delta(T)$ is the *Harsanyi dividend* of T in (N, v), given by $\Delta(T) = \sum_{S\subseteq T} (-1)^{t-s} v(S)$, s and t being the cardinalities of S and T, respectively. Thus, since $\mathcal{P}$-additivity implies $\mathcal{P}$-linearity, the $\mathcal{P}$-additive sharing scheme

$\mathcal{S}$ defined by

$$s_j^\pi(v) = \sum_{T \subseteq N} \Delta(T) s_j^\pi(u_T),$$
$$a_{ij}^\pi(v) = \sum_{T \subseteq N} \Delta(T) a_{ij}^\pi(u_T), \text{ for all } i \in P_\pi(j),$$

for all $j \in N$, and for all $\pi \in \Pi(N)$, recovers $\varphi(v)$. Note that $\mathcal{S}$ is also $\mathcal{P}$-efficient. □

Note that negative shares in the proof of Proposition 1, as well as in (5), can be interpreted as investments on human capital. With regard to (5), if the value φ verifies the null player property, then $a_{ij}^\pi(u_T) = 0$, for all $i \in P_\pi(j)$, for all $j \in S_\pi(t_\pi)$, and for all order $\pi \in \Pi(N)$.

In Flores, Molina and Tejada (2012) we show that the Shapley value can also be obtained as a non-extreme pyramidal value which is based on the second-order difference operator for a pair of players $i, j \in N$ considered by Segal (2003). Formally, the *second-order difference operator* for $i, j \in N$ is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows

$$\Delta_{ij}^2 v(S) = v(S \cup \{i,j\}) - v(S \cup \{j\}) - v(S \cup \{i\}) + v(S) = \Delta_{ji}^2 v(S), \quad \forall\, S \subseteq N \setminus \{i,j\},$$

and it is interpreted as a measure of complementarity of players i and j with respect to the players in S (see Segal, 2003). To be specific, the non-trivial pyramidal sharing scheme which defines the Shapley value is given by $s_j^\pi(v) = v(j)$, and $a_{ij}^\pi(v) = \Delta_{ij}^2(S_\pi(i,j))$, for all $i \in P_\pi(j)$, $\pi \in \Pi(N)$, and for all $(N, v) \in G_n$. That is, the Shapley value can be obtained as a constant salary pyramidal value in which the shares that player i receives when player $j \in S_\pi(i)$ arrives and joins coalition $P_\pi(j)$ depend on their complementarity with respect to the intermediate players.

In the sequel, we will consider parametric families of pyramidal values in which the entrant player $j \in N$ receives as salary his own value plus a fixed proportion of his *reduced marginal contribution*, which is precisely his added value once he has been paid accordingly to his own value $v(j)$, i.e.

$$s_j^\pi = v(j) + \alpha(m_j^\pi(v) - v(j)),$$

and the right to get part of the benefits derived from subsequent incorporations to the just formed coalition. The remaining benefit $(1-\alpha)(m_j^\pi(v) - v(j))$ is distributed among the incumbent players.

3 α-Egalitarian pyramidal values

In this section we define and analyze the family of *α-egalitarian pyramidal values*, which arises when the remaining surplus $(1-\alpha)(m_j^\pi(v) - v(j))$ generated with player j's entrance is equally allocated among the incumbents.

Definition 3. For every TU game $(N, v) \in G_n$, the *egalitarian pyramidal value* is the pyramidal value obtained by means of the following efficient sharing scheme:

(i) Entrant j's salary: $s_j^\pi(v) = v(j)$,

(ii) Incumbents $P_\pi(j)$'s shares: $a_{ij}^\pi(v) = \frac{(m_j^\pi(v) - v(j))}{|P_\pi(j)|}$,

for all $j \in N$, and for all order $\pi \in \Pi(N)$. Thus, the final payoff that player $i \in N$ receives according to the order $\pi \in \Pi(N)$ is given by:

$$ep_i^\pi(v) = v(i) + \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v) - v(j)}{|P_\pi(j)|},\ i = 1, \ldots, n.$$

Therefore, the egalitarian pyramidal value, which is the expected value under the former sharing scheme when all orders are equally likely, is given by

$$\mathcal{EP}_i(v) = v(i) + \frac{1}{n!} \sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v) - v(j)}{|P_\pi(j)|},\ i = 1, \ldots, n. \tag{6}$$

Proposition 3. *The egalitarian pyramidal value is the egalitarian CIS value defined by Driessen and Funaki (1991) as*

$$CIS_i(v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n},\ i = 1, \ldots, n.$$

Proof. We will check the following equalities:

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v)}{|P_\pi(j)|} = (n-1)!(v(N) - v(i)), \tag{7}$$

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{-v(j)}{|P_\pi(j)|} = -(n-1)! \sum_{\substack{j \in N \\ i \neq j}} v(j), \tag{8}$$

for all player $i = 1, \ldots, n$. Let $i \in N$ be a fixed player, and let $\emptyset \neq S \subsetneq N \setminus \{i\}$, then $v(S \cup i)$ appears in the sum

$$\sum_{\pi \in \Pi(N)} \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v)}{|P_\pi(j)|} \tag{9}$$

as many times as orders $\pi \in \Pi(N)$ in which $S \cup i$ members arrive in the first positions whenever i is not the last one. That is, in $((s+1)! - s!)(n-s-1)!$ orders, in which $v(S \cup i)$ should be shared among s players. On the other hand, $-v(S \cup i)$ appears as many times as orders in which $S \cup i$

arrive in the first places with independence of i's arrival. That is, in $(s+1)!(n-s-1)!$, in which $-v(S\cup i)$ should be shared among $s+1$ players. Thus, all those terms are cancelled in the above sum. Then, considering the two extreme cases $S = N\setminus\{i\}$ (in orders π such that $\pi(i) < n$) and $S=\emptyset$ (i.e., $\pi(i)=1$), the sum (9) must be equal to

$$\sum_{\substack{\pi\in\Pi(N)\\ \pi(i)<n}} \frac{v(N)}{n-1} + \sum_{\substack{\pi\in\Pi(N)\\ \pi(i)=1}} -\frac{v(i)}{1} = (n! - (n-1)!)\frac{v(N)}{n-1} - (n-1)!v(i) = (n-1)!(v(N)-v(i))$$

Now, in order to prove (8), let us consider the following arrangement, which shows $v(j)$'s contribution to the sum

$$\sum_{\pi\in\Pi(N)} \sum_{j\in S_\pi(i)} \frac{-v(j)}{|P_\pi(j)|},$$

depending on i's and j's arrivals.

ARRIVALS	$\pi(j)=2$	$\pi(j)=3$	$\cdots$	$\pi(j)=n-1$	$\pi(j)=n$	
$\pi(i)=1$	$-v(j)$	$-\frac{v(j)}{2}$	$\cdots$	$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
$\pi(i)=2$		$-\frac{v(j)}{2}$	$\cdots$	$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
			$\ddots$	$\cdots$	$\cdots$	
$\pi(i)=n-2$				$-\frac{v(j)}{n-2}$	$-\frac{v(j)}{n-1}$	
$\pi(i)=n-1$					$-\frac{v(j)}{n-1}$	
TOTAL	$-v(j)$	$-2\frac{v(j)}{2}$	$\cdots$	$-(n-2)\frac{v(j)}{n-2}$	$-(n-1)\frac{v(j)}{n-1}$	$-(n-1)v(j)$

Thus, taking into account that the number of orders $\pi\in\Pi(N)$ for which $\pi(i)=k$ and $\pi(j)=\ell$, with $\ell>k$, is $(n-2)!$, for all $\ell=k+1,\dots,n$, and for all $k=1,\dots,n-1$, and extending the previous reasoning to all $j\in N\setminus\{i\}$, equality (8) follows. □

In the egalitarian pyramidal value **all** the remaining benefit $m_j^\pi(v)-v(j)$ generated with player j's entrance is equally allocated among the incumbents. In the following definition, which generalizes the previous one, we consider the case in which **only** a fixed proportion $(1-\alpha)(m_j^\pi(v)-v(j))$, with $\alpha\in[0,1]$, is equally allocated among the incumbents.

Definition 4. For every TU game $(N,v)\in G_n$, and every $\alpha\in[0,1]$, the *α-egalitarian pyramidal value* is the pyramidal value obtained by means of the following efficient sharing scheme:

(i) Entrant j's salary: $s_j^{\pi,\alpha}(v) = v(j)+\alpha(m_j^\pi(v)-v(j))$,

(ii) Incumbents $P_\pi(j)$'s shares: $a_{ij}^{\pi,\alpha}(v) = \frac{(1-\alpha)(m_j^\pi(v)-v(j))}{|P_\pi(j)|}$,

for all $j\in N$, and for all order $\pi\in\Pi(N)$. Thus, the final payoff that player $i\in N$ receives according to the order $\pi \in \Pi(N)$ is given by:

$$ep_i^{\pi,\alpha}(v) = v(i) + \alpha(m_i^\pi(v) - v(i)) + (1-\alpha) \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v) - v(j)}{|P_\pi(j)|},\ i = 1, \ldots, n.$$

Therefore, the α-egalitarian pyramidal value, which is the expected value under the former sharing scheme when all orders are equally likely, is given by

$$\mathcal{EP}_i^\alpha(v) = v(i) + \frac{1}{n!} \sum_{\pi \in \Pi(N)} \Big(\alpha(m_i^\pi(v) - v(i)) + (1-\alpha) \sum_{j \in S_\pi(i)} \frac{m_j^\pi(v) - v(j)}{|P_\pi(j)|}\Big),\ i = 1, \ldots, n. \quad (10)$$

Corollary 1. *The family of α-egalitarian pyramidal values turns out to be the family of α-consensus values introduced by Ju, Borm and Ruys (2007).*

Proof. For every TU game $(N, v) \in G_n$, and every $\alpha \in [0, 1]$, taking into account proposition 3, it holds

$$\mathcal{EP}_i^\alpha(v) = \alpha\phi_i(v) + (1-\alpha)\mathcal{EP}_i^0(v),\ i = 1, \ldots, n,$$

where $\phi(v)$ denotes the Shapley value of the game (N, v), and $\mathcal{EP}^0(v)$, which denotes the egalitarian pyramidal value, equals the egalitarian CIS value. Therefore, the coincidence follows from result (c) in Theorem 5 in Ju *et al.* (2007). □

For every $\alpha \in [0, 1]$, the α-egalitarian pyramidal value verifies the following properties (see Ju *et al.*):

(i) Standard for two-person games.

(ii) Additivity.

(iii) Symmetry.

(iv) Relative invariance with respect to strategic equivalence.

(v) α-dummy, i.e.

$$\varphi_i(v) = \alpha v(i) + (1-\alpha)\Big(v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}\Big),$$

for all $(N, v) \in G_n$, and every dummy player $i \in N$ with respect to v.

Observe that the family of α-egalitarian pyramidal values is the standard solution for two person cooperative games for all $\alpha \in [0, 1]$, since the α-consensus family of values arises as a generalization of this solution to general n-person cooperative games. Note also that the family of α-egalitarian pyramidal values fails to verify the dummy property. On the contrary, as mentioned above, the α-pyramidal value verifies the *α-dummy property* introduced by Ju et al. (2007), which in authors' own words, "balances the tensions between the four fundamental principles of distributive justice (cf. Moulin, 2003)". To be specific, they first introduce the *neutral dummy property*, which characterizes the consensus value (when $\alpha = \frac{1}{2}$). Then, extending their arguments, they define the α-dummy property, in order to characterize the α-consensus value.

In particular, the pyramidal definition of the α-consensus values gives an alternative constructive approach to the *standardized remainder vectors* that determine the α-consensus values, which provides solid ground for it in terms of the dynamics of economic activity.

Example 1. Let us illustrate the pyramidal sharing scheme which gives rise to the consensus value by means of the same example as in Ju *et al.* (2007). Consider the following 3-person game (N, v):

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v(S)	10	0	0	18	23	0	30

The pyramidal egalitarian shares for $\alpha = \frac{1}{2}$, which is the consensus value, are depicted in the following table:

				Players' shares at each player's arrival											
	Marginal contributions			First arrival			Second arrival			Third arrival					
Order	$m_1^\pi(v)$	$m_2^\pi(v)$	$m_3^\pi(v)$	$s_1^\pi(v)$	$s_2^\pi(v)$	$s_3^\pi(v)$	$s_1^\pi(v)$ $a_{1j}^\pi(v)$	$s_2^\pi(v)$ $a_{2j}^\pi(v)$	$s_3^\pi(v)$ $a_{3j}^\pi(v)$	$s_1^\pi(v)$ $a_{1j}^\pi(v)$	$s_2^\pi(v)$ $a_{2j}^\pi(v)$	$s_3^\pi(v)$ $a_{3j}^\pi(v)$	$\frac{1}{2}$-$ep_1^\pi(v)$	$\frac{1}{2}$-$ep_2^\pi(v)$	$\frac{1}{2}$-$ep_3^\pi(v)$
(123)	10	8	12	10			4	4		3	3	6	17	7	6
(132)	10	7	13	10			6.5		6.5	1.75	3.5	1.75	18.25	3.5	8.25
(213)	18	0	12		0		14	4		3	3	6	17	7	6
(231)	30	0	0		0			0	0	20	5	5	20	5	5
(312)	23	7	0			0	16.5		6.5	1.75	3.5	1.75	18.25	3.5	8.25
(321)	30	0	0			0		0	0	20	5	5	20	5	5
Shapley	$20\frac{1}{6}$	$3\frac{2}{3}$	$6\frac{1}{6}$									Consensus value	$18\frac{5}{12}$	$5\frac{1}{6}$	$6\frac{5}{12}$

Table 1: $\frac{1}{2}$-egalitarian pyramidal Shares. Example 1

Note that the constructive approach given by the pyramidal sharing of the current benefits is more simple than the two recursions considered in Ju *et al.* (2007), and it also allows us to observe more clearly the allocation of the global final benefit, $v(N)$, in terms of the dynamics of economic activity.

Joosten (1996) introduces the family of *α-egalitarian Shapley values*, which is closely related with the family of α-consensus values. In this case, the α-egalitarian Shapley family is made by considering all convex combinations of the Shapley value and the *equal division solution*, which distributes the worth of the grand coalition equally among all players. Here, in view of Proposition 3, we can establish a pyramidal construction for the egalitarian Shapley values, in which the entrant retains a fixed fraction α of his marginal contribution, and distributes the remaining part equally among his predecessors. Formally,

Corollary 2. *For every TU game $(N, v) \in G_n$, and every $\alpha \in [0, 1]$, its α-egalitarian Shapley value $\phi^\alpha(v)$, defined by Joosten (1996) as*

$$\phi_i^\alpha(v) = \alpha\phi_i(v) + (1-\alpha)\frac{v(N)}{n}, \quad \textit{for all } i = 1, \ldots, n,$$

equals the pyramidal value obtained by means of the following $\mathcal{P}$-efficient sharing scheme:

(i) *Entrant j's salary:* $s_j^\pi(v) = \alpha m_j^\pi(v)$, *if* $\pi(j) > 1$; *and* $s_j^\pi(v) = m_j^\pi(v)$, *whenever* $\pi(j) = 1$.

(ii) *Incumbents* $P_\pi(j)$*'s shares:* $a_{ij}^\pi(v) = \frac{(1-\alpha)m_j^\pi(v)}{|P_\pi(j)|}$.

4 α-Proportional pyramidal values for monotonic games

In the α-egalitarian family, the remaining surplus, which represents the value that entrant j's participation adds to the incumbents, is shared equally among all the incumbents. In this section we consider a non egalitarian framework, in which a player's right to get part of the forthcoming benefits is determined according to his initial investment. We measure this initial investment as the value his incorporation have added to the incumbents, or in other words, by means of his marginal contribution, and define the family of *α-proportional values*. As the α-egalitarian family, the α-proportional one is also symmetric. At the end of this section, we adopt a very general point of view, and we consider non-symmetric ways of allocating the added value by means of a given collection of weighting vectors which is *exogenously* given: the family of α-$\boldsymbol{\omega}$-weighted pyramidal values.

Taking into account that a proportional allocation with respect to a given weight system in which some of the weights can be strictly negative must be carefully used, we restrict the definition of α-proportional pyramidal values to the subclass of *monotonic* TU games (i.e., $v(S) \leq v(T)$, for all $S \subseteq T$). In that case, all marginal contributions $m_j^\pi(v)$, $j \in N$, $\pi \in \Pi(N)$ are non negative.

Definition 5. For every monotonic TU game $(N, v) \in G_n$, and every $\alpha \in [0,1]$, the *α-proportional pyramidal value* is the pyramidal obtained by means of the following efficient sharing scheme:

(i) Entrant j's salary:

$$s_j^{\pi,\alpha}(v) = \begin{cases} m_j^\pi(v), & \text{if } v(P_\pi(j)) = 0, \\ v(j) + \alpha(m_j^\pi(v) - v(j)), & \text{otherwise.} \end{cases}$$

(ii) Incumbents $P_\pi(j)$'s shares:

$$a_{ij}^{\pi,\alpha}(v) = \begin{cases} 0, & \text{if } v(P_\pi(j)) = 0, \\ (1-\alpha)\frac{m_i^\pi(v)}{v(P_\pi(j))}(m_j^\pi(v) - v(j)), & \text{otherwise.} \end{cases}$$

for all $j \in N$, and for all order $\pi \in \Pi(N)$. Thus, the final payoff that player $i \in N$ receives according to the order $\pi \in \Pi(N)$ is given by:

$$pp_i^{\pi,\alpha}(v) = v(i) + \alpha(m_i^\pi(v) - v(i)) + (1-\alpha) \sum_{\substack{j \in S_\pi(i) \\ v(P_\pi(j)) \neq 0}} \frac{m_i^\pi(v)}{v(P_\pi(j))}(m_j^\pi(v) - v(j)), \tag{11}$$

if $v(P_\pi(i)) \neq 0$, and

$$pp_i^{\pi,\alpha}(v) = m_i^\pi(v) + (1-\alpha) \sum_{\substack{j \in S_\pi(i) \\ v(P_\pi(j)) \neq 0}} \frac{m_i^\pi(v)}{v(P_\pi(j))} (m_j^\pi(v) - v(j)), \tag{12}$$

if $v(P_\pi(i)) = 0$, for all $i = 1, \ldots, n$. Therefore, the α-proportional pyramidal value, which is the expected value under the former sharing scheme when all orders are equally likely, is given by

$$\mathcal{PP}_i^\alpha(v) = \frac{1}{n!}\Big(\sum_{\substack{\pi \in \Pi(N) \\ v(P_\pi(i)) \neq 0}} \big(v(i) + \alpha(m_i^\pi(v) - v(i))\big) + \sum_{\substack{\pi \in \Pi(N) \\ v(P_\pi(i)) = 0}} m_i^\pi(v)\Big) +$$

$$\frac{1-\alpha}{n!}\Big(\sum_{\pi \in \Pi(N)} \sum_{\substack{j \in S_\pi(i) \\ v(P_\pi(j)) \neq 0}} \frac{m_i^\pi(v)}{v(P_\pi(j))} (m_j^\pi(v) - v(j))\Big), \ i = 1, \ldots, n. \tag{13}$$

Proposition 4. *For every monotonic TU game $(N, v) \in G_n$, and every $\alpha \in [0, 1]$, it holds*

$$\mathcal{PP}^\alpha(v) = \alpha\phi(v) + (1-\alpha)\mathcal{PP}^0(v).$$

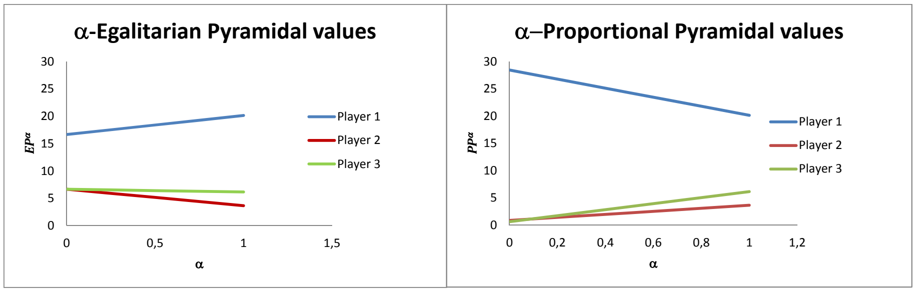
Proof. Trivially, if we express $v(i)$ and $m_i^\pi(v)$ as $\alpha v(i) + (1-\alpha)v(i)$ and $\alpha m_i^\pi(v) + (1-\alpha)m_i^\pi(v)$ in the first summand of (13), it follows that every α-proportional pyramidal value is the linear convex combination of the two extreme values for $\alpha = 0$ and $\alpha = 1$. Moreover, since the 1-proportional pyramidal value is in fact the Shapley value, then the result holds. □

Let us analyze, by means of two examples, the behavior of the extreme zero-proportional value and the α's choice effect over the final allocation of benefits. We will also look at their relation with the *core* of the game. Recall that the core of the game $(N, v) \in G_n$ is the set

$$C(v) := \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq v(S), \text{ for all } \emptyset \neq S \subseteq N \text{ and } \sum_{i \in N} x_i = v(N) \right\}.$$

If $\mathbf{x} \in C(v)$, the coalition $S \neq N$ has an incentive to split off if $\mathbf{x}$ is the proposed reward allocation in N.

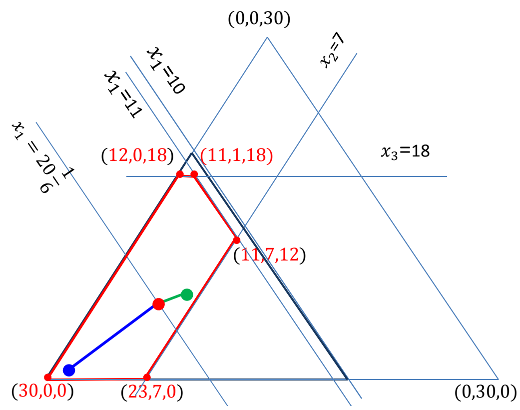
Example 2. Let us consider previous example 1. In the following graphs are represented the families of α-egalitarian and α-proportional pyramidal values, respectively.



The core of the game is non-empty and both families lie in the core, which is given by:

$$C(v) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 11, 0 \leq x_2 \leq 7, 0 \leq x_3 \leq 18, \sum_{i=1}^{3} x_i = 30 \right\}$$

In the following picture, the egalitarian family is drawn in green color, and the proportional family in blue.



It should be noted that proportional values reward a player for his contribution to the establishment of the firm as well as for his contribution to the firm's growth. The parameter α controls to what extent a player must be compensated according to his participation at the beginning of the project rather than to his contribution to its evolution. This kind of establishment compensation, which in this particular example benefits player 1, shows an extreme effect in the next one, where a non-null dummy agent receives a compensation larger than his marginal contribution.

Example 3. Consider the following 3-person game (N, v):

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v(S)	1	0	0	1	1	10	11

The families of α-egalitarian and α-proportional pyramidal values are given by:

$$\mathcal{EP}^{\alpha}(v) = \alpha(1,5,5) + (1-\alpha)(4\frac{1}{3}, 3\frac{1}{3}, 3\frac{1}{3})$$

$$\mathcal{PP}^{\alpha}(v) = \alpha(1,5,5) + (1-\alpha)(7\frac{2}{3}, 1\frac{2}{3}, 1\frac{2}{3})$$

Note that in general the proportional family fails to verify the dummy property. However, it satisfies the weaker property of *null player*, i.e. $\mathcal{PP}_i^\alpha(v) = 0$ for all $(N, v) \in G_n$ and for all *null player* $i \in N$.

In this example, the core of the game is also non empty, and the Shapley value belongs to the core. However, it is the unique value of both families which is stable, i.e., which belongs to the core. In this example, the core:

$$C(v) = \left\{(1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 + x_3 = 10 \text{ and } x_2, x_3 \geq 0\right\},$$

has an empty relative interior.[2] Otherwise, if the relative interior of the core is non-empty and the Shapley value is stable and lies in the core's relative interior, then there exist α_e and α_p in $[0, 1)$ such that $\mathcal{EP}^\alpha(v) \in C(v)$, for all $\alpha \geq \alpha_e$, and $\mathcal{PP}^\alpha(v) \in C(v)$, for all $\alpha \geq \alpha_p$.

In general, the proportional pyramidal values are not additive. For every $\alpha \in [0, 1]$, the α-proportional pyramidal value verifies the following properties:

(i) Standard for two-person games.

(ii) Symmetry.

(iii) Null player.

(iv) Null player out, defined in (Derks and Haller, 1999) as:

$$\mathcal{PP}_j^\alpha(N, v) = \mathcal{PP}_j^\alpha(N \setminus \{i\}, v|_{N\setminus\{i\}}), \text{ for all } j \neq i \in N,$$

for all $(N, v) \in G_n$ such that i is a null player in v.

The property of standard for two-person games follows because for two-person games all choices of $\alpha \in [0, 1]$ give raise to the same value, the Shapley value. The same occurs when we restrict ourselves to the class of *monotonic simple games*: $(N, u) \in G_n$ monotonic such that $u(S) \in \{0, 1\}$, for all $S \subseteq N$.

Proposition 5. *Let $(N, u) \in G_n$ be a monotonic simple game. Then $\mathcal{PP}^\alpha(u) = \phi(u)$, for all $\alpha \in [0, 1]$.*

Proof. Let $(N, u) \in G_n$ be a monotonic simple game, and let be $\pi \in \Pi(N)$ be a given order. Since there exists a unique $i_\pi \in N$ with nonzero marginal contribution, then $s_{i_\pi}^\pi(u) = m_{i_\pi}^\pi(u) = 1$, $a_{ii_\pi}^\pi(u) = 0$, and $m_j^\pi(u) = m_j^\pi(u) - u(j) = 0$, for all $j \neq i_\pi$. □

The two families of pyramidal values we have considered can be considered as special subclasses of the general class of *α-$\boldsymbol{\omega}$-weighted pyramidal values* defined as follows.

Definition 6. For every TU game $(N, v) \in G_n$, and every $\alpha \in [0, 1]$. Let $\boldsymbol{\omega} = \{(\omega_1^\pi, \ldots, \omega_n^\pi)\}_{\pi\in\Pi(N)}$ be a given collection of weighting vectors exogenously given, with $\omega_j^\pi \geq 0$, for all $j \in N$, and for all $\pi \in \Pi(N)$. Then, the *α-$\boldsymbol{\omega}$-weighted pyramidal value* is the pyramidal obtained by means of the following efficient sharing scheme:

[2] Which is defined as the interior of the core as a subset of the subspace of efficient allocations.

(*i*) Entrant *j*'s salary: $s_j^{\pi,\alpha,\omega}(v) = v(j) + \alpha(m_j^\pi(v) - v(j))$,

(*ii*) Incumbents $P_\pi(j)$'s shares: $a_{ij}^{\pi,\alpha,\omega}(v) = (1-\alpha)\frac{\omega_i^\pi}{\sum_{k\in P_\pi(j)}\omega_k^\pi}(m_j^\pi(v) - v(j))$

for all $j \in N$, and for all order $\pi \in \Pi(N)$. Thus, the final payoff that player $i \in N$ receives according to the order $\pi \in \Pi(N)$ is given by:

$$wp_i^{\pi,\alpha,\omega}(v) = v(i) + \alpha(m_i^\pi(v) - v(i)) + (1-\alpha)\sum_{j\in S_\pi(i)}\frac{\omega_i^\pi}{\sum\limits_{k\in P_\pi(j)}\omega_k^\pi}(m_j^\pi(v) - v(j)),\ i = 1,\ldots,n.$$

Therefore, the α-ω-weighted pyramidal value, which is the expected value under the former sharing scheme when all orders are equally likely, is given by

$$\mathcal{WP}_i^{\alpha,\omega}(v) = v(i) + \frac{1}{n!}\sum_{\pi\in\Pi(N)}\Big(\alpha(m_i^\pi(v) - v(i)) + (1-\alpha)\sum_{j\in S_\pi(i)}\sum_{j\in S_\pi(i)}\frac{\omega_i^\pi}{\sum\limits_{k\in P_\pi(j)}\omega_k^\pi}(m_j^\pi(v) - v(j))\Big), \tag{14}$$

for all $i = 1,\ldots,n$.

In general, the unique property that is satisfied by every α-ω-weighted pyramidal value is efficiency. It should be pointed out that the previous definition must be adapted in order to take into account that the sum $\sum_{k\in P_\pi(j)}\omega_k^\pi$ can be equal zero. In that case, the entrant *j* would receive the whole of his marginal contribution.

5 Conclusions and future research

In this paper we have proposed a general procedure for obtaining a broad class of solution concepts based on a pyramidal distribution of the benefits that are sequentially obtained through a dynamic process of coalition formation, in which players successively come into play and join the current coalition until the grand coalition is formed. To be specific, we have analyzed in detail two parametric families of pyramidal values: the α-egalitarian and the α-proportional pyramidal families, which contain the Shapley value as an extreme case. Since the α-egalitarian values coincide with the α-consensus values (Ju *et al.*, 2007), all of them are axiomatically characterized. Axiomatic characterizations for the proportional family, as well as a strategic analysis of this kind of solutions, are left for future research.

It must be pointed out that the complexity of the calculus of a pyramidal value relies crucially on the calculus of the pyramidal sharing scheme and, obviously, on the complexity of the characteristic function of the game. In the case of the two proposed families, if the marginal contributions can be calculated (or at least approximated) in polynomial time, then any pyramidal value can also be estimated in polynomial time. In fact, following Castro, Gomez and Tejada (2009), any value that can be expressed as an expectation of a polynomial function of the marginal contribution vectors over all permutations, when all orderings are equally likely, can be estimated in polynomial time, whenever the marginal contributions are computable in polinomial time.

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