UNIVERSIDAD CARLOS III DE MADRID

DOCTORAL THESIS

Risk theory and optimal control of Lévy driven processes.

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Resumen

Esta tesis contiene tres artículos de investigación con aportes originales. El primer artículo, que coincide con el Capítulo 2, ha sido publicado (Diko and Usábel (17)) en Insurance: Mathematics and Economics, una revista de reconocimiento internacional incluída en JCR. En el citado capítulo se propone un método numérico que permite evaluar la función de utilidad en un marco de proceso de Poisson compuesto con cambio de régimen. Esto supone que los parámetros del modelo de Poisson compuesto pueden variar en el tiempo, gobernados por un proceso de Markov subyacente. Este modelo es una generalización de los procesos que se analizan en la literatura relevante hasta el momento, por tanto el aporte de este capítulo consiste tanto en el desarrollo de un modelo nuevo, capaz de reflejar un entorno económico variable, como en el método de cálculo de cuantías de interés relacionadas con éste. Éstas incluyen entre otras la probabilidad de la ruina, supervivencia o el déficit medio al producirse la ruina.

El Capítulo 3 expone el tratamiento genérico de un problema de control estocástico en el marco de procesos generales de difusión de Lévy. Este tipo de problemas es conocido por su dificultad a la hora de obtener soluciones concretas, ya que las equaciones diferenciales o integro-diferenciales que caracterizan la solución no admiten tratamiento analítico exacto. Habitualmente se aplican métodos numéricos de discretización de tiempo. En esta tesis, se desarrolla un método de solución alternativo que consiste en Erlangizar (dividir en intervalos aleatorios exponenciales) el horizonte temporal establecido con lo que se consigue simplificar la complejidad de las equaciones diferenciales involucradas. Esta transformación lleva a una metodología de aproximación iterativa aplicable a un gran abanico de problemas del área de finanzas y seguros. Los resultados

de este capítulo están en el proceso de revisión en Mathematical Finance, una de las revistas de finanzas estocásticas más importantes en el mundo.

Por último, el Capítulo 4 ofrece una aplicación de la metodología presentada anteriormente en el marco de solvencia de una compañía de seguros. En este contexto se plantea un problema de decisión sobre la composición de la cartera de inversión óptima con el fin de maximizar la utilidad esperada de una cartera sometida a un proceso de riesgo. Aplicando el algoritmo iterativo del Capítulo 3 se calculan las cuantías de interés y se demuestra la rápida convergencia y buenas propiedades del método propuesto. El contenido de este capítulo también representa un aporte original y está actualmente bajo revisión en la revista ASTIN Bulletin, referente principal en el campo de investigación actuarial.

En conlusión, los tres aportes de investigación original presentados en esta tesis permiten una aplicación de métodos numéricos para obtener resultados concretos en situaciones que hasta ahora no han sido tratadas en la literatura.

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1

Introduction

An approximate answer to the right problem

is worth a good deal more

than an exact answer to an approximate problem.

– John Tukey –

This thesis contains three original research articles related to the area of risk theory and stochastic control. The exposition of the work begins with the introductory chapter that contains basic notions about Lévy processes, stochastic control and risk theory as developed in standard textbooks on the topics and are necessary to present the original contributions of the subsequent chapters. In Chapter 2 we present our original article (Diko and Usábel (17)) that has been published in Insurance: Mathematics and Economics, an international peer reviewed journal. In this article we developed a numerical method that allows to evaluate a general penalty-reward function for Markov-modulated compound Poisson processes. First, the function of interest is characterised as a solution of an integro-differential system of equations and then its solution is approximated by Chebyshev polynomial expansion. Chapter 3 presents our original contribution that treats the problem of optimal control of general Lévy diffusion processes including Markov-modulation of the parameters. Through Erlangisation approach, the value function is characterised as a solution to fairly simple integro-differential equation. This gives rise to an iterative scheme that yields the solution of a wide class of stochastic control problems. The general nature of the results of this work permits its application to variety of problems from finance and insurance related areas. The work developed in this chapter is currently submitted and in a review process of a top peer reviewed stochastic finance journal. Finally, Chapter 4 contains an application of the results from the previous chapters to the environment of risk control of an insurance company with respect to the optimal portfolio selection. It illustrates the modelisation possibilities of the framework presented in Chapter 3 and the relevance of the results by obtaining numerical solutions for a problem that has not been previously solved in the literature. The results of this chapter are submitted for the revision process to ASTIN Bulletin, a leading international peer-reviewed journal in actuarial science.

1.1 Description of the context

Risk is inherent to the insurance business and so is the necessity to quantify it. Insurance companies operating in a branch of insurance business need to cover the claims resulting from a portfolio of the contracts. Since the amounts and the timing of these claims is unknown in advance, the company needs to determine some regular patterns in the uncertain quantities to accrue appropriate funds to cover its liabilities. The decision on the volume of these funds, often called reserves, is a trade-off between the solvency and the efficiency of the capital management. Insufficient reserves will lead a company to bankruptcy, while excessive reserves mean a waste of capital resources and loss of competitiveness in the market. The problem of solvency is not only important for the insurance companies themselves but also for the regulator of the insurance market. The requirement of a minimum obligatory reserves that have to be withhold in a company operating in the insurance industry is indeed one of the three pillars sustaining the new set of the regulatory requirements being prepared for the European Union insurance market: Solvency II. The reliable quantitative tools to assess the adequacy of the monetary requirements are not only interesting on their own sake as a theoretical challenge but are also essential for practical purposes, especially in view of recent international financial crisis that, far from being exclusive to banking sector, may affect the insurance industry as well.

The sources of funds necessary to cover the liabilities from a portfolio of contracts in an insurance company is composed of accrued premiums, share capital, retained earnings and actuarial reserves assigned to this portfolio. These funds are also known by the general term *reserves*. At the beginning of the insurance activity, the reserves are formed exclusively by the share capital, called the *initial reserves*. Subsequently, the reserves increase by the premiums collected from the insured and the capital gains earned and decreases by the claim amounts when claims occur. The critical event, referred to as *ruin*, is when reserves are insufficient to cover the claims: the company becomes insolvent. This happens when the successive claims reduce the available financial resources of a company so much that, at some moment, it is impossible to face the claims.

1. INTRODUCTION

Ruin theory, a field of applied probability, studies the evolution of the financial reserves in these settings. The risk associated with a particular portfolio is quantified using several criteria. The most common one, is the probability that ruin happens within a given horizon. Though being the principal quantity of interest, other important aspects related to the reserve process have drawn continuous attention, most notably the deficit at ruin and recovery time. One can argue that the bankruptcy does not really occur until the deficit is important enough so that the company cannot recover through a short term loan. The deficit at ruin is the amount of money the company lacks to cover the claims when ruin occurs. It indicates the severity of the financial insufficiency. Closely related to this is the concept of recovery time that represents how long it takes until the company recovers to positive reserves, or even to required minimum reserves levels. Other quantities that appear in ruin theory literature is the distribution of the surplus just before the ruin, the deficit just after the ruin and the time at which the ruin occurred. The joint distribution of these three events gives detailed insight into how dangerous is a particular financial situation at a given moment.

A general framework to analyse these indicators have been developed in a series of papers by Gerber and Shiu (Gerber and Shiu (24, 25, 26)). A general utility function dependent on time and severity of ruin or remaining surplus in case of the survival of a given horizon is introduced and denoted as the *penalty-reward* function. Its expected discounted present value is studied and the above mentioned quantities (ruin and survival probabilities, time of ruin, etc.) are shown to be special cases. In this work we will focus our attention on characterisation and calculation of the expected penalty-reward function in various scenarios.

The model that will be used in this work do describe the evolution of the reserves in time is a **Lévy diffusion** process. It is general enough to accommodate the realistic behaviour of the underlying phenomenon, as was argued by Morales (38) yet tractable, at least numerically, to obtain results relevant to possible practical application.

Based on the model and the criteria indicators mentioned above, the insurance company or the regulator body can adjust the controllable variables such as initial reserves, premiums collected, and the investment decisions on the funds kept as reserves, to assure sufficient financial resources to cover the liabilities corresponding to a portfolio of its business. This dissertation sets up a theoretical framework to analyse this decision process and provides quantitative tools to evaluate the impact of possible decisions.

1.2 Process definition: Lévy diffusion

In this chapter we present the theoretical framework around which this work is developed. Its principal pieces are: the model describing the evolution of the reserves in time, the objective that is pursued by the modelisation process, and the analytic vehicle that is used to obtain the conclusions.

The cornerstone of the modelisation effort is the process describing the evolution of the reserves in time. Traditionally, in risk theory literature the compound Poisson process was used for this purpose (Asmussen (3)). However, the limitations of this model led quickly to its generalisations (Dufresne and Gerber (18), Li and Garrido (35), Sarkar and Sen (47) and Morales (38)) even at cost of tractability – the more complex the model gets the more difficult its analysis and the derivation of conclusions becomes. A considerably general family of stochastic processes encompassing most of the models analysed in recent research work on the subject are the Lévy diffusion processes. This is the family that will be studied in this work and in order to introduce it properly we will spend some space with expositions of basic notions to keep this work self-contained. The definitions and theorems exposed in this chapter are standard and can be found in principal textbook references on the field such as Kushner and Dupuis (32), Bertoin (11) and Øksendal and Sulem (40).

Throughout this thesis we will assume that the random variables involved are all defined on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$ even if is not mentioned explicitly. A **stochastic process** X on $(\Omega, \mathcal{F}, \mathrm{P})$ is a collection of $\mathbb{R}^d$-valued random variables $\{X_t : t \in [0;\infty)\}$. For any $\omega \in \Omega$ fixed, $X_t(\omega)$ as a function of t is called a **trajectory** of the process.

Let $\mathbf{F} = \{\mathcal{F}_t : t \in [0;\infty)\}$ be a filtration of σ-algebras on Ω (that is, for each $s \leq t$ also $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$) then process X_t is said to be **adapted** to filtration $\mathbf{F}$ if for every $t \in [0;\infty)$, X_t is $\mathcal{F}_t$ measurable.

Definition 1.1 (Lévy process)**.** *A $\mathbb{R}^d$-valued stochastic process $\{X_t : t \in [0;\infty)\}$ is called a Lévy process if it satisfies the following properties:*

start at zero

$X_0 = 0$ *a.s.*

independent increments

for any $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, random variables X_{t_0}, $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, ..., $X_{t_n} - X_{t_{n-1}}$ are independent

stationary increments

for any $t \geq 0$ the distribution of $X_{t+s} - X_t$ does not depend on s

continuous in probability

for every $t \geq 0$ and $\epsilon > 0$, $\lim_{s \to t} \mathrm{P}\left[|X_s - X_t| > \epsilon\right] = 0$

cádlág

almost all trajectories of the process are right-continuous and have left limits

The definition of the Lévy process may look abstract at the first sight. For better understanding of the behaviour of these processes we cite the decomposition theorem which unveils that every Lévy process is an addition of a Wiener process with a drift and the collection of jumps $\Delta X_t = X_t - X_{t^-}$.

Let us denote for all $t \in \mathbb{R}_+$ and Borel sets $U \subset \mathbb{R}$ the number of jumps of size $\Delta X_t \in U$ as $N(t, U)$. $N(t, U)$ is called the **Poisson random measure** of $\{X_t\}$ and the set function $\nu(U) = \mathrm{E}\left[N(1, U)\right]$ the **Lévy measure** of X_t.

Theorem 1.1 (Lévy decomposition). *Let $\{X_t\}$ be a Lévy process, then for some constants $\alpha, \beta \in \mathbb{R}$ it holds that*

$$X_t = \alpha t + \beta W_t + \int\limits_{\mathbb{R}} z\overline{N}(t, \mathrm{d}z),$$

with W_t being a Wiener process and

$$\overline{N}(\mathrm{d}t, \mathrm{d}z) = \begin{cases} N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t & \text{if} |z| < 1 \\ N(\mathrm{d}t, \mathrm{d}z) & \text{if} |z| \geq 1 \end{cases}$$

the compensated Poisson measure of $\{X_t\}$

Proof. See e.g. (31) □

From the theorem above one can see that Wiener process, which is often used as limiting process describing the evolution of reserves (Asmussen (3)), is a also of Lévy process. Also the compound Poisson process mentioned earlier, an important special case treated in insurance context belongs to the class of Lévy processes (Sato (48)).

1.2.1 Lévy driven stochastic differential equations

The class of stochastic processes that are studied in this thesis are Lévy diffusions. They are the solutions of the stochastic differential equations driven by Lévy processes. Let us consider functions $\alpha : [0,T] \times \mathbb{R}^n \to \mathrm{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$, n–dimensional Wiener process W_t then the solution to the stochastic differential equation (SDE)

$$X_t = \alpha(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t + \int\limits_{\mathbb{R}^n} \gamma(t, X_{t^-}, z)\overline{N}(\mathrm{d}t, \mathrm{d}z) \tag{1.1}$$

with the initial condition $X_0 = x_0 \in \mathbb{R}^n$ or its equivalent integral form

$$X_t = X_0 + \int\limits_0^t \alpha(s, X_s)\mathrm{d}s + \int\limits_0^t \sigma(s, X_s)\mathrm{d}W_s + \int\limits_0^t \int\limits_{\mathbb{R}^n} \gamma(s, X_{s^-}, z)\overline{N}(\mathrm{d}s, \mathrm{d}z). \tag{1.2}$$

is called a **Lévy diffusion**.

The next theorem gives sufficient conditions for the existence of a unique solution.

Theorem 1.2. *Consider the stochastic differential equation* (1.1)*. If the functions* α*,* σ *and* γ *satisfy the following conditions:*

at most linear growth

There exists a constant $C_1 < \infty$ *such that*

$$\|\sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int\limits_{\mathbb{R}} \sum_{k=1}^{l} |\gamma_k(t,x,z)|^2 \nu_k(\mathrm{d}z_k) \le C_1(1 + |x|^2)$$

for all $x \in \mathbb{R}^n$*.*

Lipschitz continuity

There exists a constant $C_2 < \infty$ such that

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + |\alpha(t,x) - \alpha(t,y)|^2 + \\ + \int_{\mathbb{R}} \sum_{k=1}^{l} |\gamma_k(t,x,z) - \gamma_k(t,y,z)|^2 \nu_k(\mathrm{d}z_k) \leq C_2 |x-y|^2$$

for all $x, y \in \mathbb{R}^n$

The Lévy diffusions that are time-homogeneous (that is, functions α, σ and γ do not depend on t) are called **jump diffusions**. Jump diffusions are strong Markov processes, therefore we can define their infinitesimal generator as follows

Definition 1.2 (Infinitesimal Generator)**.** *Let $\{X_t\}$ be a $\mathbb{R}^n$ valued jump diffusion. Then its infinitesimal generator $\mathcal{A}$ is defined on functions $f : \mathbb{R}^n \to \mathbb{R}$ as*

$$\mathcal{A}f(x) = \lim_{t \to 0^+} \frac{1}{t} \left\{ \mathrm{E}\left[f(X_t) \right] - f(x) \right\} \tag{1.3}$$

where $X_0 = x$.

In the case of jump diffusion processes the infinitesimal generator of a twice continuously differentiable functions can be expressed in terms of functions α, σ and γ.

Theorem 1.3. *If $f \in C_0^2(\mathbb{R}^n)$ then $\mathcal{A}f(x)$ exists and is given by*

$$\mathcal{A}f(x) = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \\ + \int_{\mathbb{R}} \sum_{k=1}^{l} \left\{ f(x + \gamma_k(x,z)) - f(x) - \nabla f(x) \cdot \gamma_k(x,z) \right\} \nu_k(\mathrm{d}z_k). \tag{1.4}$$

The infinitesimal generator collects the relevant information about the process in question. In subsequent sections we will present the characteristics related to the process we want to study and we show how they can be identified using the infinitesimal generator of the underlying process.

1.3 Penalty-reward function: a unifying approach

The ultimate goal of all model development is to answer questions about certain problems. In the context of risk theory the questions are related to downward barrier-crossing. Gerber and Shiu in a series of papers: Gerber and Shiu (24), Gerber and Shiu (25), Gerber and Shiu (26), introduced a general framework comprising the most relevant quantities of interest studied in risk theory literature. In the context of financial applications a penalty-reward function has been used by Avram et al. (6) to obtain the valuation formula for an American put option. The so-called penalty-reward function

$$\phi_t(x) = \mathrm{E}[L\left(X_\tau\right)\mathbb{I}\left(\tau \le t\right) + P\left(X_t\right)\mathbb{I}\left(\tau > t\right) \mid X_0 = x] \tag{1.5}$$

is studied in various contexts. Quantity τ is the first downcrossing of the process X_t below level 0, that is $\tau = \inf\limits_{s>0}\{X_s < 0\}$ and is commonly referred to as **ruin**. The function L represents the loss realised upon downcrossing of the level 0 and the function P the reward upon arrival to moment $t$and P. Both are assumed to be continuous. Similar scenarios occur in the context of financial investment optimisation where usually only the reward function is considered. It typically represents the utility function of an investment realised upon reaching certain horizon.

Example 1.1 (Ruin probability)**.** *Let $P \equiv 0$ and $L \equiv 1$ then $\phi_t(x)$ is called the* ***ruin probability*** *of the process X_t in finite horizon t. If we let $t \to \infty$ then $\lim\limits_{t\to\infty} \phi_t(x) \equiv \phi(x)$*

*is called the **ultimate ruin probability** of the process X_t.*

Example 1.2 (Expected shortfall). *Let $P \equiv 0$ and $L(y) \equiv y$ then $\phi_t(x)$ is the expected shortfall of the process X_t at the ruin.*

Example 1.3 (Expected utility). *Let $P(y) \equiv U(y)$ and $L \equiv 0$ where $U(y)$ is an arbitrary utility function then $\phi_t(x)$ is the expected utility of the risky investment subject to bankruptcy (τ).*

The penalty-reward function will be the main focus of attention in this work. It comprises all the sensible answers we pretend to unveil concerning the problems that will be studied.

1.3.1 Feynman-Kac formula

The direct calculation of the penalty-reward function $\phi_t(x)$ presented in the previous section is unfeasible in all but few special cases. The analytical approach yields results for very specific distributions involved in the underlying process X_t or requires additional simplifying assumptions. Most explicit results are related to compound Poisson models with exponential or phase-type claim size distributions and unrealistic assumptions about the economic environment.

If one is to move the frontier of available results to practical dimensions one powerful tool to analyse the desired quantities is the Feynman-Kac formula (see e.g. Pham (44)). If $\phi_t(x)$ is as defined by (1.5) and the underlying process is a jump-diffusion with infinitesimal generator $\mathcal{A}f(x)$ then the penalty-reward function is the solution of

$$\mathcal{A}\phi_t(x) - \frac{\partial \phi_t(x)}{\partial t} = 0 \tag{1.6}$$

with its corresponding boundary conditions. Since we are dealing with a partial differential equation, the boundary conditions are essential to identify the solution uniquely. Here we state the characterisation in a general way, below we will identify the boundary conditions for particular cases. Often this is the most difficult part of finding the solution to a particular problem.

2

Chebyshev approximation in risk processes

2.1 Introduction

In this chapter we present our original work (Diko and Usábel (17)) published in Insurance: Mathematics and Economics journal. The classical compound Poisson risk process perturbed by a diffusion is enriched by introducing the Markov-modulation of the drift term. This results in an increased modelisation versatility that allows, for example, the incorporation of variable interest rate in the model. The approximation method proposed, based on the Chebyshev polynomials, provides the numerical evaluation of the penalty-reward function in this context. This results move forward the frontier of the available models applicable to practical situations.

The risk process presented by Gerber (22) extends the classical model of risk theory introducing a Brownian diffusion. The total claims follow a compound Poisson process $\{X_t, t \geq 0\}$ with Lévy measure $\lambda f(x)\,\mathrm{d}x$, λ being the intensity of arrivals and f the density of jumps. The collection of premiums is driven by a Wiener process W_t^c independent of X_t with drift c and volatility σ, thus the perturbed risk process with initial surplus u is given by

$$\mathrm{d}R_t = c\mathrm{d}t + \sigma \mathrm{d}W_t^c - \mathrm{d}X_t, \qquad R_0 = u. \tag{2.1}$$

This process has been considered by Dufresne and Gerber (18) where a defective renewal equation was derived for the probability of ruin $\psi(u) = \Pr(\tau < \infty)$ where $\tau = \inf\{t \geq 0 : R_t < 0\}$. A review of the research on this type of processes can be found in Asmussen and Albrecher (4), Chapter 11. Generalisation of the model are treated in Li and Garrido (35), Sarkar and Sen (47), and Morales (38), whereas Ren (45) gives explicit formulae to calculate the ruin probability and related quantities for phase-type distributed claims.

Let us now allow the insurer to invest the reserves U_t into an asset with time-dependent Markov modulated return rate (drift) Δ_t and volatility $\kappa(U_t)$, that possibly depends on the amount invested U_t, driven by a Wiener process W_t^I independent of the risk process R_t

$$\mathrm{d}U_t = \left(\Delta_t \mathrm{d}t + \kappa(U_t)\,\mathrm{d}W_t^I\right) U_t + dR_t, \qquad U_0 = R_0 = u \tag{2.2}$$

The drift parameter Δ_t is governed by a finite state homogeneous Markov process with state space $\{\delta_1, \ldots, \delta_n\}$, intensity matrix $Q = (q_{ij})_{n\times n}$ and initial state δ_i. For example, Δ_t can be used to model the risk free rate announced by a central bank that evolves according to the Markov process by, for instance, 25 basis points jumps. The state space would be in this case e.g.,

$$1.00\%, 1.25\%, 1.50\%, 1.75\%, 2.00\%, \ldots, 9.00\%.$$

This environment offers considerable versatility in capturing the evolution of interest rates since any diffusion model to forecast the yield curve can be approximated arbitrarily well by continuous time Markov chains, see Kushner and Dupuis (33). Variation of the volatility according to the size of the funds invested is justified, for example, by Berk and Green (10) as an implication of their study of the performance of mutual funds and resulting rational capital flows. A particular shape of κ suggested in the cited paper, $\kappa(u) = \frac{\sigma_r}{\sqrt{u}}$, yields a surplus process in the form of an affine diffusion that was studied by Avram and Usábel (7) in this context. Many practical ideas support a fund-dependent volatility, for instance the possibility to obtain more efficient portfolios, due to transaction costs, when more money is available. Model (2.2) is a generalisation of the process considered most frequently in the literature where the return rate and the volatility are constant in time, $\Delta_t = \delta$, $\kappa(\cdot) = \sigma_r$, like in Paulsen (42), Paulsen and Gjessing (43), Wang (51), Ma and Sun (37), Gaier and Grandits (21), Grandits (27), Cai and Yang (13), Wang and Wu (52).

The stochastic differential equation (2.2) can be arranged into

$$\mathrm{d}U_t = (c + \Delta_t U_t)\,\mathrm{d}t + \sqrt{\sigma^2 + \kappa^2(U_t)\,U_t^2}\,\mathrm{d}W_t - \mathrm{d}X_t \tag{2.3}$$

with initial condition $(U_0, \Delta_0) = (u, \delta_i)$. The expected penalty-reward function, see

Gerber and Landry (23), is introduced

$$\phi_t^i(u) = \mathrm{E}\left[\pi(U_\tau)\,\mathbb{I}(\tau \le t) + P(U_t)\,\mathbb{I}(\tau > t) \mid U_0 = u, \Delta_0 = \delta_i\right] \tag{2.4}$$

where $\tau = \inf\{s \ge 0 : U_s < 0\}$. If ruin occurs before the time horizon t, the penalty $\pi(U_\tau)$ applies to the overshoot U_τ at the ruin. Otherwise, the reward function $P(U_t)$ applies to the reserves at time t. The concept of the expected penalty-reward function presented in Gerber and Shiu (24) and Gerber and Shiu (25) is a quite general framework comprising several quantities of interest as a special case, such as the time to ruin, the amount at and immediately prior to ruin or survival probabilities.

For further analysis the smoothed version of the function $\phi_t^i(u)$ will be considered, namely its Laplace-Carson transform in time defined as

$$\Upsilon_\alpha^i(u) = \int_0^\infty \alpha \mathrm{e}^{-\alpha t}\phi_t^i(u)\,\mathrm{d}t.$$

Further, letting H_α be an exponentially distributed random variable with parameter α, the former expression may be viewed as a penalty-reward function with an exponentially killed time horizon, see expression (6) in Avram and Usábel (7),

$$\begin{aligned} \Upsilon_\alpha^i(u) &= \int_0^\infty \alpha \mathrm{e}^{-\alpha t}\phi_t^i(u)\,\mathrm{d}t = \mathrm{E}\left(\phi_{H_\alpha}^i(u)\right) \\ &= \mathrm{E}\left(\pi(U_\tau)\,\mathbb{I}(\tau \le H_\alpha) + P(U_{H_\alpha})\,\mathbb{I}(\tau > H_\alpha) \mid U_0 = u, \Delta_0 = \delta_i\right) \end{aligned} \tag{2.5}$$

where the last equality comes from substituting the definition of $\phi_t^i(u)$, in (2.4).

The function $\Upsilon_\alpha^i(u)$ is analytically more tractable than the original function while, at the same time, retains a probabilistic interpretation as a penalty-reward function considering an exponential random time horizon H_α.

The results in this chapter are organised as follows: in Section (2.2) an integro-differential system that characterises the function of interest $\Upsilon_\alpha^i(u)$ is derived and the existence of the solution discussed. In Section (2.3) a numerical method to approximate the solution of the system via Chebyshev polynomials is considered and Section (2.4) offers some numerical illustrations.

2.2 Integro-differential System

This section presents further treatment of the transformed expected penalty-reward function defined by (2.5). The function $\Upsilon_\alpha^i(u)$ is dependent on the initial reserves $U_0 = u$ and the starting return rate $\Delta_0 = \delta_i$. Since the process driving the return rate Δ_t has a finite state space, the number of initial conditions is also finite. Therefore, one can consider the set of functions $\Upsilon_\alpha(u) = \left(\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \dots \Upsilon_\alpha^n(u)\right)$, each corresponding to different starting return rate from the state space $\{\delta_1, \dots, \delta_n\}$. Below, a Volterra integro-differential system of equations for the functions $\Upsilon_\alpha^1(u), \Upsilon_\alpha^2(u), \dots \Upsilon_\alpha^n(u)$ is derived and, applying the result of Le and Pascali (34), sufficient conditions for the existence of the solution are established.

Theorem 2.1. *For all $\alpha \geq 0$, functions $\Upsilon_\alpha^i : [0, \infty) \to \mathbb{R}$ defined in (2.5) satisfy the* *following system of integro-differential equations*

$$\begin{aligned} \textit{For } i &= 1,\dots,n \\ & \frac{1}{2}\left(\sigma^2+u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Upsilon_\alpha^i(u)+(c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\Upsilon_\alpha^i(u)+ \\ & +\sum_{j=1}^n q_{ij}\Upsilon_\alpha^j(u)-(\alpha+\lambda)\Upsilon_\alpha^i(u)+\lambda\int_0^u \Upsilon_\alpha^i(u-x)\ f(x)\ \mathrm{d}x+ \\ & +\alpha P(u)+\lambda\int_u^\infty \pi(u-x)\ f(x)\ \mathrm{d}x=0. \end{aligned} \tag{2.6}$$

Given that $\lim_{u\to\infty} P(u)$ *exists,* $\sigma > 0$ *and assuming positive security loading for the reserve process (2.2), the boundary conditions of the system are*

$$\begin{aligned} \Upsilon_\alpha^i(0) &= \pi(0-) \\ \lim_{u\to\infty}\Upsilon_\alpha^i(u) &= \lim_{u\to\infty}P(u)\equiv P(\infty) \end{aligned} \tag{2.7}$$

Moreover, if $f\in C^2[0,\infty)$*,* $P(u)$ *and* $\kappa(u)$ *are continuous for* $u\geq 0$ *and* $\pi(u)$ *integrable, then the system of equations (2.6) has a solution* $\Upsilon_\alpha^i\in C^2[0,\infty)$*,* $i=1,\dots,n$*.*

Proof. First, a straightforward application of Ito's lemma yields the infinitesimal generator of the process U_t, which applied to the functions $\phi_t^i(u)$, $i=1,...,n$ defined by (2.4), yields

$$\begin{aligned} \mathcal{A}\phi_t^i(u) &= \frac{1}{2}\left(\sigma^2+u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u)+(c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u)+\sum_{j=1}^n q_{ij}\phi_t^j(u)+ \\ & +\lambda\int_0^\infty\left(\phi_t^i(u-x)\ -\phi_t^i(u)\right)f(x)\ \mathrm{d}x. \end{aligned}$$

Functions $\phi_t^i(u)$ satisfy the Fokker-Planck equation, see e.g. Risken (46),

$$\mathcal{A}\phi_t^i(u) - \frac{\partial \phi_t^i(u)}{\partial t} = 0 \tag{2.8}$$

with boundary conditions

$$\phi_0^i(u) = P(u) \qquad u > 0 \tag{2.9a}$$

$$\phi_t^i(u) = \pi(u) \qquad u < 0 \text{ and } t \geq 0 \tag{2.9b}$$

for each $i = 1, 2 \ldots, n$. Using (2.9b) the following holds

$$\int_0^\infty \phi_t^i(u-x) f(x)\,\mathrm{d}x = \int_0^u \phi_t^i(u-x) f(x)\,\mathrm{d}x + \int_u^\infty \pi(u-x)\ f(x)\,\mathrm{d}x. \tag{2.10}$$

Substituting the infinitesimal generator and (2.10) into the Fokker-Planck equation yields

$$\begin{aligned}\frac{1}{2} \quad & \left(\sigma^2 + u^2\kappa^2(u)\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}\phi_t^i(u) + (c+\delta_i u)\frac{\mathrm{d}}{\mathrm{d}u}\phi_t^i(u) + \sum_{j=1}^n q_{ij}\phi_t^j(u) - \lambda\phi_t^i(u) + \\ & +\lambda\int_0^u \phi_t^i(u-x) f(x)\,\mathrm{d}x + \lambda\int_u^\infty \pi(u-x)\ f(x)\,\mathrm{d}x - \frac{\partial\phi_t^i(u)}{\partial t} = 0.\end{aligned}$$

The system (2.6) is obtained taking the Laplace-Carson transform with respect to t on both sides and expanding the last term integrating by parts

$$\int_0^\infty \alpha \mathrm{e}^{-\alpha t}\frac{\partial\phi_t^i(u)}{\partial t}\mathrm{d}t = -\alpha P(u) + \alpha\int_0^\infty \alpha\mathrm{e}^{-\alpha t}\phi_t^i(u)\,\mathrm{d}t = -\alpha P(u) + \alpha\Upsilon_\alpha^i(u)$$

where the first boundary condition (2.9a) of the Fokker-Planck equation was used.

Concerning the boundary conditions of the integro-differential system, when the initial reserves are 0 and $\sigma > 0$, the presence of the Wiener fluctuation in premiums causes immediate crossing of 0 level – see for example the proof of Theorem 2.1 in Paulsen and Gjessing (43). The second condition is the asymptotic case $u \to \infty$ when under the assumption of positive security loading $\lim_{u\to\infty} \Upsilon_\alpha^i(u) = \lim_{u\to\infty} P(u) < \infty$.

To prove the existence of the solution, an equivalent system will be considered. A change of variable is now introduced in the System (2.6), $h(v) = u$, where $h : [0,1] \to [0,\infty)$ is an arbitrary strictly monotone, twice continuously differentiable function. The system can now be written in terms of the functions $\Gamma_\alpha^i(v) = \Upsilon_\alpha^i(h(v))$.

$$\begin{aligned}\text{For } i &= 1,\ldots,n\\ &A(v)\frac{\mathrm{d}^2}{\mathrm{d}v^2}\Gamma_\alpha^i(v) + B_i(v)\frac{\mathrm{d}}{\mathrm{d}v}\Gamma_\alpha^i(v) + \sum_{j=1}^n q_{ij}\Gamma_\alpha^j(v) -\\ &-(\alpha+\lambda)\Gamma_\alpha^i(v) + \lambda\int_0^v \Gamma_\alpha^i(y) f(h(v)-h(y)) h'(y)\,\mathrm{d}y +\\ &+\lambda S(v) + \alpha P(h(v)) = 0 \qquad (2.11)\end{aligned}$$

where

$$\begin{aligned}A(v) &= \frac{\sigma^2 + h^2(v)\kappa^2(h(v))}{2[h'(v)]^2}\\ B_i(v) &= \frac{c+\delta_i h(v)}{h'(v)} - \frac{\left[\sigma^2 + h^2(v)\kappa^2(h(v))\right]h''(v)}{2[h'(v)]^3}\\ S(v) &= \int_v^1 \pi(h(v)-h(y)) f(h(y)) h'(y)\,\mathrm{d}y\end{aligned}$$

with boundary conditions

$$\begin{aligned} \Gamma_\alpha^i(0) &= \pi(0-) \\ \Gamma_\alpha^i(1) &= \lim_{u\to\infty} P(u). \end{aligned} \tag{2.12}$$

Here h' and h'' denote the first and the second derivative of function h. Finally, by integration

$$\begin{aligned} \Gamma_\alpha^i(s) &= \Gamma_\alpha^i(0) + \int_0^s \frac{h'(v)}{B_i(v)} \left[H(v) - \lambda \int_0^v f(h(v) - h(y)) \frac{h'(y)}{h'(v)} \Gamma_\alpha^i(y)\,\mathrm{d}y \right] \mathrm{d}v \\ H(v) &= \frac{-1}{h'(v)} [A(v) \frac{\mathrm{d}^2}{\mathrm{d}v^2} \Gamma_\alpha^i(v) + \sum_{j=1}^n q_{ij} \Gamma_\alpha^i(v) - \\ &\quad - (\alpha + \lambda) \Gamma_\alpha^i(v) + \alpha P(h(v)) + \lambda S(v)]. \end{aligned}$$

The existence of the solution $\Gamma_\alpha^i \in C^2[0,1]$ is guaranteed by Theorem 2 in Le and Pascali (34), as $H(v)$ is a continuous function and $f(h(v) - h(y)) \frac{h'(y)}{h'(v)}$ is integrable. The integrability is immediate as f is a density function and $\frac{h'(y)}{h'(v)}$ is a bounded function of y on $[0,v]$ for all v. This implies that $\Upsilon_\alpha^i(u) = \Gamma_\alpha^i(h^{-1}(u))$, a solution to (2.6), exists and $\Upsilon_\alpha^i \in C^2[0,\infty)$. □ □

2.3 Numerical Solution

The second order system of integro-differential equations (2.6) that characterises the Laplace-Carson transform of the expected penalty-reward function (2.5) does not have an explicit solution. In Akyuz-Dascioglu and Sezer (2) and Akyuz-Dascioglu (1) a numerical method was proposed for fairly general families of Fredholm-Volterra integro-differential systems of higher order which include the system treated in this chapter as a special case. The authors approximate the solution to the system by shifted Chebyshev polynomials on the interval $[0,1]$. A collocation method is used to fit the Chebyshev expansion of the solution. In order to adapt the procedure to system (2.6), we need to transform the domain of the unknown functions Υ_α^i, as was done in the proof of Theorem 2.1, from the interval $[0,\infty)$ to $[0,1]$. First, the solution Γ_α^i of the transformed system is found and then, applying the inverse transform, the functions of interest Υ_α^i are recovered. The convergence of the method is treated in the original article along with the illustrative examples that compare the approximation and the exact solutions showing outstanding performance. The following section describes the method adapted to the setting of this chapter to keep it self-contained. The presentation follows the development in Akyuz-Dascioglu and Sezer (2).

2.3.1 Approximation by Chebyshev Polynomials

In matrix notation the transformed system is given by

$$\begin{aligned}\mathbf{P}_2(v)\frac{\mathrm{d}^2}{\mathrm{d}v^2}\mathbf{\Gamma}_\alpha(v) \quad &+\mathbf{P}_1(v)\frac{\mathrm{d}}{\mathrm{d}v}\mathbf{\Gamma}_\alpha(v)+\mathbf{P}_0(v)\mathbf{\Gamma}_\alpha(v)= \\ &=\mathbf{g}(v)+\int_0^v \mathbf{K}(v,y)\,\mathbf{\Gamma}_\alpha(y)\,\mathrm{d}y\end{aligned} \tag{2.13}$$

where $\mathbf{\Gamma}_\alpha(v)$ is the column vector of unknown functions
$\mathbf{\Gamma}_\alpha(v) = \left(\Gamma_\alpha^1(v), \Gamma_\alpha^2(v), \ldots, \Gamma_\alpha^n(v)\right)^\top$. Coefficient matrices are as follows

$$\begin{aligned}
\mathbf{P}_2(v) &= \frac{A(v)}{h'(v)} \cdot \mathrm{I}_n \\
\mathbf{P}_1(v) &= h'(v)^{-1}\, diag\,(B_i(v)) \\
\mathbf{P}_0(v) &= h'(v)^{-1}\left[Q - (\alpha+\lambda)\cdot \mathrm{I}_n\right] \\
\mathbf{K}(v,y) &= -\lambda f(h(v) - h(y))\,\frac{h'(y)}{h'(v)} \cdot \mathrm{I}_n \\
\mathbf{g}(v) &= -h'(v)^{-1}\left[\alpha P(h(v)) + \lambda S(v)\right]\cdot 1_n \\
S(v) &= \int_v^1 \pi(h(v) - h(y))\, f(h(y))\, h'(y)\,\mathrm{d}y,
\end{aligned}$$

where I_n is the identity matrix of order $n \times n$ and 1_n is the column vector of ones of order $n \times 1$. The transform is performed with an arbitrary strictly monotone, twice continuously differentiable function $h : [0,1] \to [0,\infty)$.

The aim of the method is to approximate the solution by a truncated Chebyshev expansion

$$\Gamma_\alpha^i(v) = \sum_{r=0}^{N} a_{ir}^* T_r^*(v) \qquad i = 1, \ldots, n$$

on the interval $[0,1]$, where $T_r^*(v)$ are shifted Chebyshev polynomials of the first kind (see, for example, Boyd (12)) and a_{ir}^* are the unknown coefficients to be determined. In matrix notation

$$\Gamma_\alpha^i(v) = T^*(v)\, A_i^*,$$

where $T^*(v) = (T_0^*(v), T_1^*(v), \ldots, T_N^*(v))$ is a row vector of shifted Chebyshev polynomials up to degree N and $A_i^* = (a_{i0}^*, a_{i1}^*, \ldots a_{iN}^*)^\top$ is a column vector of the corresponding coefficients. Similarly, the $n-th$ derivative of $\Gamma_\alpha^i(v)$ can be expanded into

$$\frac{\mathrm{d}^n}{\mathrm{d}v^n}\Gamma_\alpha^i(v) = T^*(v)\,A_i^{*(n)}. \tag{2.14}$$

The link between coefficients $A_i^{*(n)}$ and A_i^* from Sezer and Kaynak (49) is

$$A_i^{*(n)} = 4^n M^n A_i^*, \tag{2.15}$$

where

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & \frac{N}{2} \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(N+1)\times(N+1)} \quad \text{for odd } N$$

$$M = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \end{pmatrix}_{(N+1)\times(N+1)} \quad \text{for even } N$$

yields the expansion of the $n-th$ derivative $\frac{\mathrm{d}^n}{\mathrm{d}v^n}\Gamma_\alpha^i(v)$ in terms of Chebyshev coefficients A_i^*.

On the other hand, functions $K_{ij}(v,y)$ can be expanded in variable y into a Chebyshev series

$$K_{ij}\left(v,y\right)=\sum_{r=0}^{N}k_r^{*ij}\left(v\right)T_r^*\left(y\right)$$

where the Chebyshev coefficients k_r^{*ij} are functions of v. Using matrix notation for convenience

$$K_{ij}\left(v,y\right)=k^{*ij}\left(v\right)T^*\left(y\right)^{\top}, \tag{2.16}$$

where k^{*ij} is the row vector of coefficients determined by Clenshaw-Curtis quadrature, see Clenshaw and Curtis (16).

Substituting (2.14), (2.15) and (2.16), the $i-th$ equation ($i=1,\ldots,n$) of the system (2.13) is finally obtained:

$$\begin{aligned}
&h'\left(v\right)^{-1}A\left(v\right)16M^2T^*\left(v\right)A_i^*+h'\left(v\right)^{-1}B_i\left(v\right)4MT^*\left(v\right)A_i^*+\\
&+h'\left(v\right)^{-1}\left[\sum_{j=1}^{n}q_{ij}-\left(\alpha+\lambda\right)\right]T^*\left(v\right)A_i^*=g_i\left(v\right)-\int_0^v k^{*ij}\left(v\right)T^*\left(y\right)^{\top}T^*\left(y\right)A_i^*\mathrm{d}y.
\end{aligned}$$

The matrix of the inner product of Chebyshev polynomials

$$\begin{aligned}
Z^*\left(v\right)=\left(z_{ij}^*\left(v\right)\right) &\equiv \int_0^v T^*\left(y\right)^{\top}T^*\left(y\right)\mathrm{d}y=\\
&= \frac{1}{2}\int_{-1}^{2v-1}T\left(x\right)^{\top}T\left(x\right)\mathrm{d}x=\frac{1}{2}\left(z_{ij}\left(2v-1\right)\right)=\frac{1}{2}Z\left(2v-1\right)
\end{aligned}$$

can be computed as shown in Akyuz-Dascioglu (1), where

$$z_{ij}(v) = \frac{1}{4}\begin{cases} 2v^2 - 2 & \text{for } i+j=1 \\ \frac{T_{i+j+1}(v)}{i+j+1} - \frac{T_{i+j-1}(v)}{i+j-1} - \frac{1}{i+j+1} + \frac{1}{i+j-1} + v^2 - 1 & \text{for } |i-j|=1 \\ \frac{T_{i+j+1}(v)}{i+j+1} + \frac{T_{1-i-j}(v)}{1-i-j} + \frac{T_{1+i-j}(v)}{1+i-j} + \frac{T_{1-i+j}(v)}{1-i+j} + 2\left(\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right) & \text{for even } i+j \\ \frac{T_{i+j+1}(v)}{i+j+1} + \frac{T_{1-i-j}(v)}{1-i-j} + \frac{T_{1+i-j}(v)}{1+i-j} + \frac{T_{1-i+j}(v)}{1-i+j} - 2\left(\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right) & \text{for odd } i+j \end{cases},$$

which yields the system

$$(2.17)\quad h'(v)^{-1} A(v)\, 8M^2 T^*(v)\, A_i^* + h'(v)^{-1} B_i(v)\, 4MT^*(v)\, A_i^* + \\ +h'(v)^{-1}\left[\sum_{j=1}^{n} q_{ij} - (\alpha+\lambda)\right] T^*(v)\, A_i^* = g_i(v) - k^{*ij}(v)\, Z^*(v)\, A_i^*,$$

for all $i = 1, \ldots, n$. The only unknown values are Chebyshev expansion coefficients A_i^*. The collocation method proposed by the authors fits the solution through the collocation points

$$x_s = \frac{1}{2}\left(1 + \cos\left(\frac{s}{N}\pi\right)\right), \qquad s = 1, 2, \ldots, (N-1).$$

Each of the $N-1$ collocation points x_s is substituted into the system (2.17) and yields n linear equations of unknown variable A_i^*, whence $n(N-1)$ equations are obtained. The boundary conditions (2.12) for $i = 1, \ldots, n$,

$$\begin{aligned} T^*(0)\, A_i^* &= \pi(0-) \\ T^*(1)\, A_i^* &= P(\infty), \end{aligned}$$

yield another $2n$ equations. A linear system of $n(N+1)$ equations is constructed and solved for the Chebyshev coefficients A_i^*. Once the approximation $\widetilde{\Gamma}_\alpha^i(v) = \sum_{r=0}^{N} a_{ir}^* T_r^*(v)$ is obtained, the relationship between the solution of the transformed and the original system from the Theorem 2.1 yields the approximation of the expected penalty-reward function $\widetilde{\Upsilon}_\alpha^i(u) = \widetilde{\Gamma}_\alpha^i\left(h^{-1}(u)\right)$.

2.4 Numerical Examples

As mentioned before, $\Upsilon_t^i(u)$ is the Laplace-Carson transform in time of the expected penalty-reward function in a jump-diffusion process. This function has a probabilistic interpretation as the penalty-reward function in an exponentially killed time horizon H_α. The ultimate case is also unveiled by a straightforward application of the Tauberian theorem

$$\begin{aligned}
(2.18)\quad \lim_{\alpha\to 0}\Upsilon_\alpha^i = \lim_{\alpha\to 0}\int_0^\infty \alpha e^{-\alpha t}\phi_t^i(u)\,dt &= -\phi_0^i(u) + \lim_{\alpha\to 0}\int_0^\infty e^{-\alpha t}\frac{d}{dt}\phi_t^i(u)\,dt \\
&= -\phi_0^i(u) + \int_0^\infty \frac{d}{dt}\phi_t^i(u)\,dt = \phi_\infty^i(u)\,.
\end{aligned}$$

For the more challenging finite time horizon penalty-reward, a numerical inversion of the Laplace transform recovers the original function $\phi_\alpha^i(u)$, see Usábel (50). The relationship $C(s) = sL(s)$ between the Laplace transform $L(s)$ and the Laplace-Carson transform $C(s)$ applies.

2.4.1 Ultimate Survival Probability

The survival probability is a special case of the function $\Upsilon^i_\alpha(u)$. For $\pi(x) \equiv 0$ and $P(x) \equiv 1$

$$\phi^i_\infty(u) = \mathrm{E}\left[\mathbb{I}(\tau = \infty) \mid U_0 = u, \Delta_0 = \delta_i\right].$$

The premium collection rate is $c = 11$, the volatility of premium accruals $\sigma^2 = 0.04$, the intensity of claim arrivals $\lambda = 4$, and claims follow a Gamma distribution $Gamma\,(5; 2)$. The interest rate is assumed to be fixed at 3% with no volatility ($\sigma_r^2 = 0$). The ultimate survival probability $\phi^i_\infty(u)$ is considered in this context and thus $\alpha = 0$ as motivated by (2.18). For the change of variables, the function $h(v) = -\ln(1-v)$ was used. The following table shows the approximations for various starting reserves and precision levels (order of Chebyshev polynomials).

		N – precision level					
		200	250	300	350	400	450
u	1	0.318081594	0.318079845	0.318079373	0.318079219	0.318079161	0.318079137
	2	0.435631392	0.43562899	0.435628343	0.435628132	0.435628053	0.43562802
	5	0.753759689	0.753755453	0.753754322	0.753753953	0.753753813	0.753753756
	10	0.987580029	0.987573342	0.987571486	0.98757086	0.987570616	0.987570511
	15	0.999982762	0.999973643	0.99997087	0.999969864	0.999969447	0.999969256

2.4.2 Markov-modulated Interest Rate Structure

The second example presents an interest rate structure driven by a Markov process and a reserve dependent volatility. Let us assume two regimes (high interest rate and low interest rate) comprising several interest rate levels. The intensity matrix Q, characterising the Markov process, governs the evolution of the interest rate:

$$
Q = \begin{array}{c} \delta_i \\ 1\% \\ 2\% \\ 7\% \\ 8\% \\ 9\% \end{array}
\begin{array}{c} \begin{array}{ccccc} 1\% & 2\% & 7\% & 8\% & 9\% \end{array} \\
\begin{pmatrix} -2 & 2 & 0 & 0 & 0 \\ 1.9 & -2 & 0.1 & 0 & 0 \\ 0 & 0.1 & -3 & 2.9 & 0 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix} \end{array}.
$$

The low interest rate regime embeds two levels 1% and 2% while the high interest rate regime considers three levels 7%, 8%, and 9%. Let the premium collection rate be 1 with the volatility of premium accruals 0.25, the intensity of claims arrival $\frac{1}{3}$ (one claim every three time periods on average), the distribution of claim size lognormal $\ln \mathcal{N}\left(0.5; 1\right)$. The volatility of the return on investment, dependent on the reserves level, is $\kappa^2(u) = \frac{\sigma_r^2}{u}$, as motivated in the introduction, with $\sigma_r^2 = 0.81$. The probability of survival of a random horizon of 20 years on average is approximated ($\alpha = 0.05$, $\pi\left(x\right) \equiv 0$ and $P\left(x\right) \equiv 1$). Regarding the change of variables, the function $h\left(v\right) = -\ln\left(1 - v\right)$ was used again. In the following table the survival probabilities conditional on various initial interest rates and starting reserve levels are presented.

		$N-$ number of polynomials used for the approximation				
u	δ_i	250	300	350	400	450
1	1%	0.144815222	0.144815829	0.144815469	0.144814893	0.144814330
	2%	0.146306443	0.146307016	0.146306644	0.146306063	0.146305499
	7%	0.188906830	0.188906028	0.188905174	0.188904470	0.188903928
	8%	0.190954404	0.190953560	0.190952690	0.190951981	0.190951439
	9%	0.191794388	0.191793534	0.191792659	0.191791949	0.191791406
10	1%	0.676382452	0.676390197	0.676389970	0.676387433	0.676384493
	2%	0.689328522	0.689335954	0.689335662	0.689333147	0.689330254
	7%	0.855051985	0.855051380	0.855048719	0.855045981	0.855043652
	8%	0.865060563	0.865059778	0.865057124	0.865054446	0.865052186
	9%	0.870653629	0.870652847	0.870650244	0.870647620	0.870645408
15	1%	0.845057051	0.845074744	0.845078953	0.845078338	0.845076134
	2%	0.864897819	0.864914116	0.864918001	0.864917442	0.864915420
	7%	0.977203365	0.977208141	0.977208683	0.977207860	0.977206693
	8%	0.981995614	0.981999965	0.982000462	0.981999717	0.981998658
	9%	0.984633935	0.984638092	0.984638607	0.984637937	0.984636962
20	1%	0.949938439	0.949967119	0.949978339	0.949982157	0.949982771
	2%	0.967826609	0.967849612	0.967858765	0.967861995	0.967862632
	7%	0.999402042	0.999408967	0.999411851	0.999412960	0.999413267
	8%	0.999715443	0.999721564	0.999724130	0.999725128	0.999725416
	9%	0.999823318	0.999828873	0.999831216	0.999832138	0, 999832412
25	1%	0.993079369	0.993110103	0.993125249	0.993132811	0.993136514
	2%	0.998186486	0.998205547	0.998215150	0.998220073	0.998222575
	7%	0.999986074	0.999991170	0.999993799	0.999995183	0.999995911
	8%	0.999988832	0.999993290	0.999995592	0.999996805	0.999997445
	9%	0.999990114	0.999994091	0.999996147	0.999997232	0.999997804

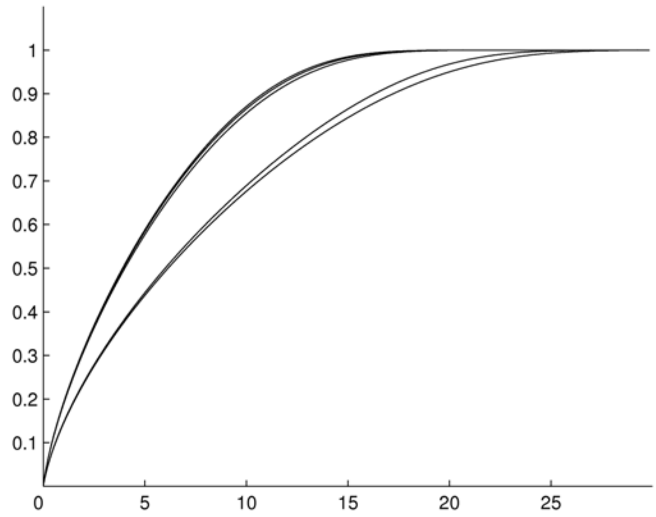

Figure 2.1: Survival probability curves as a function of initial reserves. Each curve represents different initial interest level, the lowest curve corresponding to 1%, the uppermost to 9%.

Figure 2.1 unveils the impact of the initial conditions on the survival probability. Each curve represents different initial interest rate, the lowest curve corresponds to $\Delta_0 = 1\%$ and the uppermost to $\Delta_0 = 9\%$. The horizontal axis shows the initial reserves level U_0, the vertical axis the survival probability $\Upsilon_{\alpha}^{i}(u)$.

2.5 Conclusions

A general model for the risk process of an insurance company was presented in this chapter allowing arbitrary distributions of the claim sizes, a Wiener fluctuation in premium collection and investment in a, possibly, risky asset. The evolution of the return rate is modulated by Markov process implementing a non-constant interest rates in a risk process. In particular, we suggest the possibility of interpretation as interest rates announced by a central bank that in practice move by a quarter percentile jumps. A method is developed to calculate the Gerber-Shiu expected penalty-reward function in this framework that comprises several interesting particular cases such as the calculation of ruin probabilities or moments of the deficit at ruin. The method is based on Chebyshev polynomials approximations and shows an outstanding convergence rate.

3

Optimal control of Lévy diffusions

3.1 Introduction

In this chapter we present a theoretical framework for simplifying the characterisation of the value function of the stochastic control problems. The Erlangisation approach turns an analytically unsolvable problem into a series of simple differential equations. Since the setting considered is very general it admits applications to wide range of problems. In the next chapter an application to risk theory is presented. The original results presented here were submitted to a Mathematical Finance journal and are currently under revision process.

Stochastic control problems have been studied in several applied contexts ranging from engineering, physics, biology to finance and actuarial science. A thorough treatment of the general theory can be found in (20). The most recurring tool to solve this type of problems is the dynamic programming approach originated by (9) that focuses on the value function of the optimisation problem. The value function is characterized as a solution of the Hamilton-Jacobi-Bellman (HJB) equation. The success of this approach depends heavily on the availability of methods to solve the corresponding HJB equation. Explicit solutions are seldom available, most real problems need numerical treatment. The basis of existent numerical methods is the discretisation of the continuous-time process. As the process treated by these methods is typically Markovian, the approximating process is a Markov chain and the problem is usually solved in discrete time and state space. An overview of numerical methods is presented in (32).

In this chapter an alternative approach that provides a semi-analytic treatment is presented. First, the stochastic control problem in finite horizon is treated by approximating the horizon by a partition composed of exponential horizons. Exponential horizon, viewed as an integral transform of the fix horizon problem, eliminates the dependence of the control on time. Second, the optimal control is approximated by a piecewise constant control with respect to this partition. The assumption of constant control with respect to the level of the controlled process reduces the problem of dynamic programming to a series of regular optimisation problems. These two steps result in a considerable simplification of the corresponding HJB equation and open a way to obtain solutions through standard procedures, numerical or analytical where available, for differential equations.

Section 3.2 presents the class of processes that will be treated and the optimisation framework. Consequently, in Section 3.3, the randomisation of the horizon is explained and further development is motivated. Principal results of the chapter are collected in Section 3.4 with implications to practical application of the method. Finally, in Section 3.5 an example of the application of the method is shown; Section 3.6 concludes.

3.2 Problem Formulation

The class of stochastic processes considered in this chapter are Lévy diffusions, a subclass of semi-martingale family. Let $U_t, t \geq 0$ be a real-valued stochastic process that satisfies the stochastic differential equation

$$
\begin{aligned}
\mathrm{d}U_t &= a(U_t, Y_t, \sigma_t)\mathrm{d}t + b(U_t, Y_t, \sigma_t)\mathrm{d}W_t + \\
&\quad + \int_{\mathbb{R}^k} \gamma(U_{t^-}, Y_{t^-}, \sigma_{t^-}, z) N(\mathrm{d}t, \mathrm{d}z) \\
U_0 &= u
\end{aligned}
\tag{3.1}
$$

where a, b and γ are known real valued functions, W_t is a standard Wiener process and $N(\mathrm{d}t, \mathrm{d}z)$ is a compensated Poisson random measure. See (11) for more details on Lévy processes. The process σ_t is the control process assumed to be adapted and cádlág. Process Y_t is a Markov process with finite state space $\{\delta_1, \ldots, \delta_m\}$ and intensity matrix $Q = \{q_{ij}\}$ that modulates the coefficients of (3.1). Markov-modulated models are widely used to model phenomena where abrupt changes in otherwise stable behaviour of the system occur. Markov modulation provides a set of model parameters for each behaviour state and governs the switching between them. For example in finance, see e.g. (19), Markov modulating is shown to perform better in explaining the behaviour of financial assets than usual Gaussian models.

The stochastic control is considered in a fixed horizon T or until the process U_t

exits a region $S \subseteq \mathbb{R}$. The performance criterion v to be maximised is

$$v^\sigma(T, u, \delta_i) = \mathrm{E}\left[P(U_T, Y_T) \cdot \mathbb{I}_{\{\tau \geq T\}} + \right. \\ \left. + L(U_\tau, Y_\tau) \cdot \mathbb{I}_{\{\tau < T\}} | U_0 = u, Y_0 = \delta_i\right] \tag{3.2}$$

where $\tau = \inf\{t : U_t \notin S\}$ is the exit time of the process U_t from the region S. Functions P and L are arbitrary but continuous and represent the utility realised upon termination of the horizon and upon exit of the controlled process from the region S. Let us denote $J(T, u, \delta_i)$ the optimal value of the maximisation problem

$$J(T, u, \delta_i) \equiv \max_{\sigma \in \Pi} v^\sigma(T, u, \delta_i). \tag{3.3}$$

The set Π contains all the admissible controls, that is such σ_t for which a strong solution to the equation (3.1) exists and is unique. Moreover, the attention will be restricted to controls of the form $\sigma_t = \sigma(U_{t-}, Y_{t-})$ also called Markov controls. Øksendal (39, Th. 11.2.3.) gives fairly week sufficient conditions under which the optimal value of the problem restricted to Markov controls equals the optimal value of the problem with arbitrary adapted control. Therefore narrowing the control space Π to Markov controls is not too restrictive.

Solving the Problem (3.3) directly is not feasible since an explicit expression for $v^\sigma(T, u, \delta_i)$ is not available in most general case. However, following the dynamic programming approach, one can write the Hamilton-Jacobi-Bellman equation that characterises the value function $J(t, u, \delta_i)$

$$\sup_{\sigma \in \Pi} \left\{ \mathcal{A}^\sigma J(t, u, \delta_i) - \frac{\partial J(t, u, \delta_i)}{\partial t} \right\} = 0 \tag{3.4}$$

where $\mathcal{A}^\sigma$ is the infinitesimal generator of the controlled process U_t. $\mathcal{A}^\sigma$ can be expressed, see Øksendal and Sulem (40, pg. 40), for each $i = 1, \ldots, m$ as

$$
\begin{aligned}
(3.5)\quad \mathcal{A}^\sigma J(t,u,\delta_i) = {} & \frac{1}{2} b(u,\delta_i,\sigma(u,\delta_i)) \frac{\partial^2}{\partial u^2} J(t,u,\delta_i) + \\
& + a(u,\delta_i,\sigma(u,\delta_i)) \frac{\partial}{\partial u} J(t,u,\delta_i) + \sum_{j=1}^{m} q_{ij} J_u(t,u,\delta_j) + \\
& + \int_{\mathbb{R}} \Big\{ J(t, u + \gamma(u,\delta_i,\sigma(u,\delta_i),z),\delta_i) \\
& - J(t,u,\delta_i) - \gamma(u,\delta_i,\sigma(u,\delta_i),z) \frac{\partial}{\partial u} J(t,u,\delta_i) \Big\} \nu(\mathrm{d}z)
\end{aligned}
$$

where $\nu(\mathrm{d}z) = \mathrm{E}\left[N(1,\mathrm{d}z)\right]$ is the Lévy measure of the process U_t. Moreover, if σ^* is such that

$$
\mathcal{A}^{\sigma^*} J(t,u,\delta_i) - \frac{\partial J(t,u,\delta_i)}{\partial t} = 0 \tag{3.6}
$$

then σ^* is the optimal control for the Problem (3.3) and $J(T,u,\delta_i) = v^{\sigma^*}(T,u,\delta_i)$. In this case, the optimal control can be found as the maximiser of the expression under the supremum in (3.4) whence the following must hold

$$
\frac{\partial}{\partial \sigma} \mathcal{A}^{\sigma^*} J(t,u,\delta_i) = 0. \tag{3.7}
$$

The HJB equation (3.4) together with optimality condition (3.7), in the view of (3.5), form a system of non-linear second order partial integro-differential equations. The solution of such system is usually a very difficult task even using numerical procedures. Analytic solutions are available only in very few particular cases. For example (8) solve this problem in cases when no jumps are present and the dependence of the coefficients on the control is linear. On the other hand, (41) study this type of models in a pricing problem scenario but no explicit solutions are available due to complexity of the setting.

3.3 Randomised horizon

A method based on approximation of the fixed horizon by a random horizon with Erlang distribution was presented by (14) in an article on the optimal execution of a put option time. Similar principle has later been used to find explicit expressions for the default risk problem with underlying fluid flow process by (5). This approach is often called randomisation of the horizon.

The Erlang distribution $\mathrm{Er}\,(\alpha, k)$ is equivalent to the distribution of a sum of k independent variables with identical exponential distribution of parameter α. Its density function is given by

$$p\,(x) = \frac{\alpha^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\alpha x}.$$

Its mean is $\frac{k}{\alpha}$ and variance $\frac{k}{\alpha^2}$.

Let H^n_α be a random variable with distribution $\mathrm{Er}(\alpha, n)$. Let us consider a series of random variables

$$H^n_{\frac{n}{T}} \sim \mathrm{Er}(\frac{n}{T}, n). \tag{3.8}$$

One can observe that $\mathrm{E}(H^n_{\frac{n}{T}}) = T$ and

$$\mathrm{E}(H^n_{\frac{n}{T}} - T)^2 = \frac{T^2}{n} \to 0 \text{ as } n \to \infty$$

that is $H^n_{\frac{n}{T}}$ indeed converges to T in L_2 and therefore in probability.

The complication of treating a series of convergent random horizons is compensated by the advantage of the memoryless property of the individual exponential horizons. This simplifies the dynamic programming problem since the dependence on time within each horizon is eliminated. If the problem can be treated in exponential horizon, appending such horizons the Erlangian horizon is reproduced and the convergence argument applied to approximate the solution of a fixed horizon problem. Here this idea is exploited further not only to convergence of the horizon but also to the convergence of optimal control.

3.3.1 Exponential horizon

As a first step, the maximisation problem (3.3) will be considered in a hypothetical exponential random horizon H_α instead of the fixed horizon T. Let H_α be an exponentially distributed random variable with parameter α. If one lets $\alpha = \frac{1}{T}$ then $\mathrm{E}\left(H_\alpha\right) = T$, that is, in expected terms, the random horizon H_α and the fixed horizon T coincide.

Let us denote Υ the performance criterion to be maximised in a random horizon H_α, that is

$$\Upsilon^\sigma(\alpha, u, \delta_i) \equiv \mathrm{E}\left[v^\sigma(H_\alpha, u, \delta_i)\right] = \mathrm{E}\big[P(U_{H_\alpha}, Y_{H_\alpha}) \cdot \mathbb{I}_{\{\tau \geq H_\alpha\}} + \\ + L(U_\tau, Y_\tau) \cdot \mathbb{I}_{\{\tau < H_\alpha\}} | U_0 = u, Y_0 = \delta_i\big] .$$

Notice that the second expectation is taken with respect to the random horizon H_α and the stochastic process (U_t, Y_t). The optimisation problem is similar to (3.3), J_1

will represent the optimal value,

$$J_1(\alpha, u, \delta_i) \equiv \max_{\sigma \in \Pi} \Upsilon^{\sigma}(\alpha, u, \delta_i). \tag{3.9}$$

The HJB equation in this case simplifies to

$$\sup_{\sigma \in \Pi} \{\mathcal{A}^{\sigma} J_1(\alpha, u, \delta_i)\} = 0 \tag{3.10}$$

where, compared to the fixed horizon equation (3.4), partial derivative with respect to time vanishes due to memoryless property of the exponential distribution. This guarantees the independence of the optimal control of the time horizon. Similarly to (3.6), if σ_1^* is such that

$$\mathcal{A}^{\sigma_1^*} J_1(\alpha, u, \delta_i) = 0, \tag{3.11}$$

then σ_1^* is the optimal control for Problem (3.9) and $J_1(\alpha, u, \delta_i) = \Upsilon^{\sigma_1^*}(\alpha, u, \delta_i)$. The optimal control σ_1^* must satisfy, in line with (3.7), the following equation

$$\frac{\partial}{\partial \sigma} \mathcal{A}^{\sigma_1^*} J(\alpha, u, \delta_i) = 0. \tag{3.12}$$

Altogether, equations (3.11) and (3.12) form a more tractable system of equations than (3.6) and (3.7) of the original problem. Since partial derivative with respect to time vanishes the problem turns from partial to ordinary system of differential equations. One can verify, see e.g. (7), that the exponential horizon has a mathematical representation as a Laplace-Carson transform in time of the objective function v defined in

(3.2), indeed

$$\Upsilon^{\sigma}(\alpha, u, \delta_i) = \mathrm{E}\left[v^{\sigma}(H_{\alpha}, u, \delta_i)\right] = \int_0^{\infty} v^{\sigma}(t, u, \delta_i)\, \alpha \mathrm{e}^{-\alpha t} \mathrm{d}t. \tag{3.13}$$

Laplace-Carson transform $C(s)$ of an integrable function is closely related to its Laplace transform $L(s)$ through the relationship $C(s) = sL(s)$. This fact can be exploited to obtain the solution of the problem (3.9) since the Laplace transform of the function v is more easily obtained than its original form in certain scenarios, see for example (6).

3.3.2 Erlangian horizon

As was shown above, introducing the exponential horizon the analytical treatment of the problem is simplified. The solution of the fixed horizon problem (3.3) can be recovered concatenating exponential horizons into Erlangian horizon. To formalise this step let us state the optimisation problem in a horizon that has Erlang distribution H_{α}^{n}. The performance criterion to be maximised in the horizon H_{α}^{n} is

$$\Upsilon_n^{\sigma}(\alpha, u, \delta_i) \equiv \mathrm{E}\left[v^{\sigma}(H_{\alpha}^{n}, u, \delta_i)\right] = \mathrm{E}\big[P(U_{H_{\alpha}^{n}}, Y_{H_{\alpha}^{n}}) \cdot \mathbb{I}\{\tau \geq H_{\alpha}^{n}\} + \\ + L(U_{\tau}, Y_{\tau}) \cdot \mathbb{I}_{\{\tau < H_{\alpha}^{n}\}} | U_0 = u, Y_0 = \delta_i\big].$$

The optimisation problem is

$$J_n(\alpha, u, \delta_i) \equiv \max_{\sigma \in \Pi} \Upsilon_n^{\sigma}(\alpha, u, \delta_i) \tag{3.14}$$

where J_n represents the value function. The next theorem shows the convergence of the value function of the Erlangian horizon problem to the value function of the fixed horizon problem.

Theorem 3.1. *Let* $J(T, u, y)$ *be the value function of the Problem* (3.3) *and* $J_n(\alpha, u, y)$ *be the value function of the Problem* (3.14) *then*

$$\lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right) = J(T, u, y).$$

Proof. (Following the proof of Theorem 2 in (36).) Let σ be any feasible control in Π such that $v^\sigma(t, u, y)$ is continuous in t on $[0, \infty)$. Then

$$\mathrm{E}\left[v^\sigma\left(H_\alpha^n, u, y\right)\right] = \Upsilon_n^\sigma(\alpha, u, y) \le J_n(\alpha, u, y),$$

in particular for $\alpha = \frac{n}{T}$, taking limit $n \to \infty$

$$v^\sigma(T, u, y) = \lim_{n\to\infty} \mathrm{E}\left[v^\sigma\left(H^n_{\frac{n}{T}}, u, y\right)\right] \le \lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right) \tag{3.15}$$

where the first equality comes from a variant of Helly-Bray Theorem (see Chow and Teicher (15, corollary 8.1.6)). Taking maximum over all admissible controls on the left side of (3.15) yields

$$J(T, u, y) \le \lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right). \tag{3.16}$$

On the other hand, notice that

$$\begin{aligned}
J_n(\alpha,u,y) \equiv \max_{\sigma\in\Pi} \Upsilon_n^{\sigma}(\alpha,u,y) &= \max_{\sigma\in\Pi} \int_0^{\infty} v^{\sigma}(t,u,y)\,\frac{\alpha^n}{(n-1)!} t^{n-1} \mathrm{e}^{-\alpha t} \mathrm{d}t \\
&\leq \int_0^{\infty} \max_{\sigma\in\Pi} v^{\sigma}(t,u,y)\,\frac{\alpha^n}{(n-1)!} t^{n-1} \mathrm{e}^{-\alpha t} \mathrm{d}t \\
&= \int_0^{\infty} J(t,u,y)\,\frac{\alpha^n}{(n-1)!} t^{n-1} \mathrm{e}^{-\alpha t} \mathrm{d}t.
\end{aligned}$$

The inequality comes from the fact that the optimal control σ^* that maximises the whole integral in the first line is feasible in the maximisation problem under the integral in the second line for each t. Letting $\alpha = \frac{n}{T}$, taking the limit on both sides of the inequality and applying the Helly-Bray Theorem again the complementary inequality to (3.16) follows

$$\lim_{n\to\infty} J_n\left(\frac{n}{T},u,y\right) \leq J(T,u,y)$$

thus completing the proof. □

Once the convergence of the value function is guaranteed, the next theorem presents the basis of the iterative procedure to actually evaluate the value function in the Erlangian horizon. In order to state the argument formally we need to introduce the following notation

$$J_n(\alpha,u,\delta_i,P) \equiv J_n(\alpha,u,\delta_i)$$

$$\Upsilon_n^{\sigma}(\alpha,u,\delta_i,P) \equiv \Upsilon_n^{\sigma}(\alpha,u,\delta_i)$$

whenever the utility function P needs to be specified explicitly.

Theorem 3.2. *Let P be a utility function, u the initial condition, $\alpha > 0$ a real parameter. For every natural $k \geq 2$ we have*

$$J_k(\alpha, u, \delta_i, P) = J_1(\alpha, u, \delta_i, P_{k-1})$$

where $P_{k-1}(w, y) \equiv J_{k-1}(\alpha, w, y, P)$.

Proof. It is assumed that the termination of the exponential horizons composing H_α^n is observable. For that purpose let us introduce a Poisson process V_t^α, independent of (U_t, Y_t), with jump intensity α. Let σ_t^* be the optimal control for $J_k(\alpha, u, \delta_i, P)$. Conditioning $v^\sigma(H_\alpha^n, u, \delta_i)$ on the instant of the first jump of the process V_t^α, denoted T_1, the value of (U_t, Y_t) at T_1 and occurrence of the ruin one can write

$$\begin{aligned} J_k(\alpha, u, \delta_i, P) &= \Upsilon_n^{\sigma^*}(\alpha, u, \delta_i, P) = \mathrm{E}\left[v^{\sigma^*}(H_\alpha^n, u, \delta_i, P)\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H_\alpha^n, u, \delta_i) \mid T_1, U_{T_1}, Y_{T_1}, \tau\right]\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H_\alpha^{n-1}, U_{T_1}, Y_{T_1})\right] \mathbb{I}_{\{\tau \geq T_1\}}\right] \\ &\quad + \mathrm{E}\left[L(U_\tau, Y_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = \delta_i\right] \end{aligned} \tag{3.17}$$

where the first term of (3.17) comes from the Markovian nature of the process U_t. Given that the ruin did not occur before T_1 the future of the process is independent of the past conditional on the current state U_{T_1}. Moreover, the horizon $H_\alpha^n \sim \mathrm{Er}(\alpha, n)$ is reduced by $T_1 \sim \mathrm{Exp}(\alpha)$ what yields a new horizon $H_\alpha^{n-1} \sim \mathrm{Er}(\alpha, n-1)$. In the second term, given that the ruin occurred before T_1, the expected utility $L(U_\tau, Y_\tau)$ is incurred given the initial state of the process. Developing the first term yields

$$\mathrm{E}\left[v^{\sigma^*}(H_\alpha^{n-1}, U_{T_1}, Y_{T_1})\right] = \Upsilon_{n-1}^{\sigma^*}(\alpha, U_{T_1}, Y_{T_1}) = J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P) \tag{3.18}$$

where for the last equality remember that the optimisation is made on Markov controls, that is σ^*. As a conclusion σ^* is the optimal control for Υ_{n-1}^{σ}. Substituting (3.18) into (3.17) one gets

$$\begin{aligned} J_k(\alpha, u, \delta_i, P) &= \mathrm{E}\left[J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)\mathbb{I}_{\{\tau \geq T_1\}}\right. \\ &\qquad \left. + L(U_\tau, T_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = \delta_i\right] \\ &= \Upsilon_1^{\sigma^*}(\alpha, u, \delta_i, P_{k-1}) \leq J_1(\alpha, u, \delta_i, P_{k-1}). \end{aligned} \tag{3.19}$$

Notice that in the first line term $J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)$ can be included in the conditioning since it is independent of U_0.

For the inverse inequality assume that σ_1 is the optimal control for J_1. Let us consider a control

$$\sigma_1^* = \begin{cases} \sigma_1 & \text{if } V_t^\alpha = 0 \\ \sigma^* & \text{if } V_t^\alpha > 0 \end{cases}.$$

Since σ_1^* is admissible for Υ_n, applying (3.19) one can write

$$\begin{aligned} J_1(\alpha, u, \delta_i, P_{k-1}) &= \Upsilon_1^{\sigma_1^*}(\alpha, u, \delta_i, P_{k-1}) \\ &= \Upsilon_n^{\sigma_1^*}(\alpha, u, \delta_i, P) \leq J_k(\alpha, u, \delta_i, P) \end{aligned}$$

what completes the proof. □

Theorem 3.2 shows that once the stochastic control problem can be treated in exponential horizon, the Erlangian horizon (as an approximation to the fixed horizon) can be recovered by iteratively updating the utility function P_k involved in the performance criterion. This procedure offers an alternative approach to direct numerical approximation of the initial problem and, at the same time, avoids the mathematical complexity of the analytical approach.

Although the approximation procedure is centred on the value function J, the optimal control σ^* is of the same interest. Let us denote σ_n^* the optimal control that yields the approximated value function J_n. The next proposition treats the convergence and calculation of σ_n^*.

Proposition 3.1. *Let J be the value function of the Problem* (3.3) *with σ^* the corresponding optimal control. Let J_n be the approximation of J as shown in Theorem 3.2 and let σ_n^* be the solution to*

$$\frac{\partial}{\partial \sigma}\mathcal{A}^\sigma J_n(\alpha, u, \delta_i) = 0.$$

then

$$\sigma_n^* \to \sigma^* \quad \textit{for} \quad n \to \infty$$

Proof. From (3.7) one have that σ^* is the solution to

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{A}^\sigma J(t, u, \delta_i) = 0.$$

Realising that $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{A}^\sigma$ is a continuous operator, the convergence of J_n to J guarantees the convergence of optimal controls. □

3.4 Erlangian approximation

The Laplace-Carson transform of the horizon introduced in previous section simplified the equations that characterise the value function and the optimal control to a system of integro-differential equations (3.11), (3.12). Usually, the optimal control σ_1 is obtained as the solution to (3.12) expressed as a function of unknown J_1 and then substituted to (3.11) in order to solve for the value function. This second equation is typically non-linear and poses important obstacles to analytical treatment.

In this section the approximation of the value function will be taken further with the objective to reduce the complexity of the differential equations involved. As before, the fixed horizon T is substituted by a random horizon $H^n_{\frac{n}{T}}$ with distribution $\mathrm{Er}(\frac{n}{T}, n)$ that converges to T with increasing n. This results in elimination of the partial derivative with respect to time from the HJB equation. Additionally, the control that in principle is an adapted process evolving in time, will be restricted to piecewise constant process (constant on each exponential horizon composing $H^n_{\frac{n}{T}}$). By intuition, since the length of each exponential interval is infinitesimal with probability 1 as n increases, the optimisation on restricted set of controls will converge to the unrestricted one, therefore the convergence of the procedure outlined earlier is not compromised. Theorem 3.3 in this section proves this idea formally.

We will introduce some necessary notation. Let $\overline{J}_1$ be the value function of the problem in exponential horizon restricted to piecewise constant control. Since on a particular exponential horizon the control is constant, the space of admissible controls is $\mathbb{R}$, hence

$$\overline{J}_1(\alpha, u, \delta_i) \equiv \max_{\sigma \in \mathbb{R}} \Upsilon^{\sigma}(\alpha, u, \delta_i)\,, \tag{3.20}$$

Notice that the optimal (constant) control will still depend on the values U_{T_1}, Y_{T_1} at the beginning of the exponential horizon and on the intensity α. In a strict sense one should write $\sigma_{\alpha,U_{T_1},Y_{T_1}}$, however, we will omit the subscripts in favour of the simplicity of the notation.

Assuming that controls are restricted to be constant on each exponential interval composing the Erlangian horizon H^n_α, the value function of the restricted problem is denoted $\overline{J}_n$. The set of all piecewise constant controls is denoted $\overline{\Pi}$ therefore

$$\overline{J}_n(\alpha, u, \delta_i) \equiv \max_{\sigma\in\overline{\Pi}} \Upsilon^\sigma_n(\alpha, u, \delta_i). \tag{3.21}$$

Theorem 3.3. *Let $J(T, u, y)$ be the value function of the Problem* (3.2) *and $\overline{J}_n(\frac{n}{T}, u, \delta_i)$ the value function of the Problem* (3.21) *then*

$$\lim_{n\to\infty} \overline{J}_n(\frac{n}{T}, u, \delta_i) = J(T, u, \delta_i)$$

Proof. Let $V_t^{\frac{n}{T}}$ be a Poisson process, independent of (U_t, Y_t) with intensity $\dfrac{n}{T}$, let $T^n_1, \ldots, T^n_n$ be the first n jump times of $V_t^{\frac{n}{T}}$ and define process $(\overline{U}^n_t, \overline{Y}^n_t) = (U_{T^n_k}, Y_{T^n_k})$ where $k = \max\{i : T^n_i < t\}$. That is $(\overline{U}^n_t, \overline{Y}^n_t)$ is a process that remains constant on exponential horizon intervals $\left[T^n_i, T^n_{i-1}\right)$ and equal to the value of the process (U_t, Y_t) at the beginning of each interval. Let us consider the stochastic differential equation

$$\begin{aligned} \mathrm{d}U^n_t &= a(U^n_t, Y_t, \sigma_t(\overline{U}^n_t, \overline{Y}^n_t))\mathrm{d}t + b(U^n_t, Y_t, \sigma_t(\overline{U}^n_t, \overline{Y}^n_t))\mathrm{d}W_t \\ &\quad + \int_{\mathbb{R}^k} \gamma(U^n_{t^-}, Y_{t^-}, \sigma_{t^-}(\overline{U}^n_t, \overline{Y}^n_t), z)N(\mathrm{d}t, \mathrm{d}z) \\ U^n_0 &= u. \end{aligned} \tag{3.22}$$

Applying the Theorem 6.9 from Jacod and Shiryaev (31, pg. 578) yields $U_t^n \to U_t$ in law. To verify the assumptions of the theorem one needs to check that $\overline{U}_t^n \to U_t$ and $\overline{Y}_t^n \to Y_t$ in law. For that purpose notice that the process $\overline{U}_t^n$ can be written as $\overline{U}_t^n = U_0 + (U_{t-}^n - \overline{U}_{t-}^n) \cdot V_t^n$. Since $\frac{V_t^n}{n}$ converges in law to $\frac{t}{T}$ and the equation

$$\overline{U}_t^n = U_0 + \frac{n}{T}\int_0^t (U_{t-}^n - \overline{U}_{t-}^n)\mathrm{d}t$$

has the solution

$$\overline{U}_t^n = \mathrm{e}^{\frac{nt}{T}} U_t^n - \mathrm{e}^{-\frac{nt}{T}} \int_0^t \mathrm{e}^{\frac{nt}{T}} U_t^n \mathrm{d}t \to U_t^n \quad \text{as} \quad n \to \infty.$$

To complete the proof, observe that $\overline{J}_n$ and J are continuous functionals of U_t^n and U_t respectively, what yields the convergence (see i.e. Proposition 3.8 Jacod and Shiryaev (31, pg.348)). The same reasoning yields the convergence of $\overline{Y}_t^n$. □

The next corollary to Theorem 3.2 provides an iterative scheme to actually evaluate the value function $\overline{J}_n$ based on $\overline{J}_1$.

Corollary 3.1. *Let P be a reward function, u the initial condition, $\alpha > 0$ a real parameter. For every natural $k \geq 2$ we have*

$$\overline{J}_k(\alpha, u, \delta_i, P) = \overline{J}_1(\alpha, u, \delta_i, P_{k-1})$$

where $P_{k-1}(w, y) \equiv J_{k-1}(\alpha, w, y, P)$.

Proof. Notice that the optimisation problem for $\overline{J}_k$ is the same as for J_k only with respect to different set of admissible controls. Since the set of controls is not relevant in the proof of Theorem 3.2 the result follows. □

Corollary 3.1, indicates that in order to evaluate the value function $\overline{J}_n$ of the optimization problem in Erlangian horizon it is sufficient to be able to evaluate the value function of the problem in exponential horizon $\overline{J}_1$ restricted to constant control. This calculation is iterated n times while in each step the value function of the previous step becomes the reward function of the next step.

The value function $\overline{J}_1$ is the basic building block of the method. Since in each exponential horizon the control is constant it can be treated as a parameter and $\Upsilon^\sigma(\alpha, u, \delta_i)$ can be obtained from the Laplace-Carson transform of the usual Fokker-Planck equation (see i.e. (46))

$$\mathcal{A}^\sigma \Upsilon^\sigma(\alpha, u, \delta_i) - \alpha \Upsilon^\sigma(\alpha, u, \delta_i) + \alpha P(u) = 0. \tag{3.23}$$

Posterior maximisation with respect to σ yields $\overline{J}_1$. Notice that the maximisation is a standard optimisation problem on real numbers.

The simplicity of the equation (3.23) to be solved, compared to system (3.6), (3.7), is the main advantage of the method that together with Corollary 3.1 provides a semi-analytic treatment of the stochastic control problem presented.

The optimal control can be recovered in a similar way as above. Let us denote $\overline{\sigma}_n^*$ the optimal control that yields the approximated value function $\overline{J}_n$. The next proposition treats the convergence and calculation of $\overline{\sigma}_n^*$.

Proposition 3.2. *Let J be the value function of the Problem* (3.3) *with σ^* the corresponding optimal control. Let $\overline{J}_n$ be the approximation of J as shown in Corollary 3.1 and let $\overline{\sigma}_n^*$ be the solution to*

$$\frac{\partial}{\partial \sigma} \mathcal{A}^\sigma \overline{J}_n(\alpha, u, \delta_i) = 0.$$

then

$$\overline{\sigma}_n^* \to \sigma^* \quad \textit{for} \quad n \to \infty$$

Proof. The proof is equivalent to the proof of the Proposition 3.1. □

3.5 Example

Theorems 3.3 and 3.2 provide a framework to approximate the solution of the dynamic programming problems in the form (3.3) by partition of the time in a series of exponential intervals with discretised control. In this section an application of the Erlangian approximation presented above is revisited in context of the classical Merton portfolio problem with one risky and one riskless asset.

3.5.1 Portfolio selection problem

Consider a market composed of two assets, a risk-free bond with constant yield r that follows $\mathrm{d}B_t = rB_t\mathrm{d}t$ and a stock with mean return μ and volatility ρ that follows a geometric Brownian motion $\mathrm{d}S_t = \mu S_t\mathrm{d}t + \rho S_t\mathrm{d}W_t$. An agent invests at time t a proportion σ_t into the stock and a proportion $1 - \sigma_t$ into the bond. The evolution of the total wealth then follows

$$\mathrm{d}U_t = (r + \sigma_t(\mu - r))U_t\mathrm{d}t + \sigma_t\rho U_t\mathrm{d}W_t, \qquad U_0 = u$$

where u is the initial wealth. The objective is to determine the optimal control σ_t such that a utility function P is maximised at fixed horizon T, that is

$$J(T, u) = \max_{\sigma_t \in \Pi} \mathrm{E}[U_T]$$

where $J(T,u)$ is the value function of the optimisation problem and Π the set of all admissible controls.

It was shown in (8) that for the utility function $P(u) = \log(u)$ the optimal control $\sigma_t = \dfrac{\mu - r}{\rho^2}$ is constant in time and the value function is

$$J(T,u) = \log(u) + T\left[r + \frac{1}{2}\left(\frac{\mu - r}{\rho}\right)^2\right].$$

The solution to this problem for arbitrary utility function is, however, not available. Here, the approximation procedure presented in this chapter will be used to obtain the value function and the optimal control for an arbitrary continuous utility function.

Let P be a continuous utility function, the infinitesimal generator of the process, in view of (3.5), is

$$\mathcal{A}^{\sigma} J(T,u) = \frac{1}{2}u^2\sigma_t^2\rho^2\frac{\partial^2}{\partial u^2}J(T,u) + u(r + \sigma_t(\mu - r))\frac{\partial}{\partial u}J(T,u).$$

From (3.7) the optimal control has to satisfy

$$\sigma^* = -\frac{\mu - r}{\rho}\frac{\frac{\partial}{\partial u}J(T,u)}{u\frac{\partial^2}{\partial u^2}J(T,u)}.$$

Substituting the optimal control into (3.6) the value function $J(T,u)$ is characterised as the solution to

$$\left(\frac{\mu - r}{\rho}\right)^2\left(\frac{1}{2} - \rho u\right)\frac{\frac{\partial}{\partial u}^2 J(T,u)}{u\frac{\partial^2}{\partial u^2}J(T,u)} + ur\frac{\partial}{\partial u}J(T,u) - \frac{\partial}{\partial T}J(T,u) = 0,$$

what is a non-linear partial differential equation characterising the value function $J(T,u)$.

In an exponential time, as developed in Section 3.3 the optimal control, as a solution to (3.12), is

$$\sigma_1^* = -\frac{\mu - r}{\rho} \frac{\frac{\partial}{\partial u} J(\alpha, u)}{u \frac{\partial^2}{\partial u^2} J(\alpha, u)}$$

with corresponding HJB equation characterising the value function $J(\alpha, u)$

$$\left(\frac{\mu - r}{\rho}\right)^2 \left(\frac{1}{2} - \rho u\right) \frac{\frac{\partial}{\partial u}^2 J(\alpha, u)}{u \frac{\partial^2}{\partial u^2} J(\alpha, u)} + ur \frac{\partial}{\partial u} J(\alpha, u) = 0.$$

The differential equation obtained is still non-linear but the partial derivative with respect to time vanished.

If one further assumes piecewise constant control, then over an exponential horizon the performance criterion $\Upsilon^\sigma(\alpha, u)$ satisfies, following (3.23),

$$\frac{1}{2} u^2 \sigma^2 \rho^2 \frac{\partial^2}{\partial u^2} \Upsilon^\sigma(\alpha, u) + u(r + \sigma(\mu - r)) \frac{\partial}{\partial u} \Upsilon^\sigma(\alpha, u) + \\ + \alpha \Upsilon^\sigma(\alpha, u) - \alpha P(u) = 0.$$

This is a second order linear ordinary differential equation of Euler-Cauchy type that can be treated analytically using the substitution $u = \mathrm{e}^w$. The solution represents the performance criterion $\Upsilon^\sigma(\alpha, u)$ as a function of constant control σ. Consequent maximisation yields

$$\overline{J}_1(\alpha, u, P) = \max_{\sigma \in \mathbb{R}} \Upsilon^\sigma(\alpha, u, P)$$

in line with notation introduced in Theorem 3.2. The approximation of the value function $\overline{J}_n$ is obtained iterating the value function as explained in Corollary 3.1

$$\overline{J}_k(\alpha, u, P) = \overline{J}_1(\alpha, u, P_{k-1})$$

and

$$P_{k-1}(u) = \overline{J}_{k-1}(\alpha, u, P).$$

Moreover, the approximation of the optimal control is recovered as

$$\overline{\sigma}_n^* = -\frac{\mu - r}{\rho} \frac{\frac{\partial}{\partial u}\overline{J}_n(\alpha, u)}{u\frac{\partial^2}{\partial u^2}\overline{J}_n(\alpha, u)}.$$

3.6 Conclusions

The stochastic control problem in finite horizon with terminal and exit utility function has been treated in a context of Markov-modulated Lévy diffusion processes. An alternative numerical procedure to usual discretisation approach was presented in this chapter. Approximating the fixed horizon by a sequence of exponential horizons with piecewise constant control the stochastic programming problem has been transformed to a sequence of standard optimization problems. Moreover, the complexity of the Hamilton-Jacobi-Bellman equations involved has been reduced opening ways for analytic treatment. The convergence of both, the value function and the optimal control is guaranteed.

4

Risk theory and optimal investment

4.1 Introduction

In this chapter, a general risk process with investment into a portfolio of risky assets is analysed. The previous chapter introduced a theoretical framework for dealing with this type of problems, here we present a non-trivial application into a Markov-modulated environment with compound Poisson reserve process. The contents of this chapter forms an original research article that has been submitted to ASTIN Bulletin, the journal of International Actuarial Association.

The principal goal is to determine optimal investment strategies for an insurance company in order to maximise an objective penalty-reward function. The methodology presented in previous chapters will be adapted an applied to obtain numerical procedures that will be illustrated by particular examples.

4.1.1 The Model

The risk process driving the reserves of an insurance company is assumed to evolve according to the following stochastic differential equation

$$\mathrm{d}R_t = c\mathrm{d}t + \rho\mathrm{d}W_t^{(1)} - X_t, \qquad R_0 = u.$$

This represents premiums collected at constant rate c perturbed by a diffusion with volatility ρ and claims payment that follows a compound Poisson X_t process with intensity λ and jump density function f. $W_t^{(1)}$ is a standard Wiener process independent of X_t.

It is assumed that reserves are invested into a portfolio of assets with expected return μ and volatility σ. The insurance company selects a combination of return and volatility amongst its investment possibilities. In this chapter, the investment possibilities for an insurer are modelled subject to two factors: the general state of economic environment, and the level of funds available for investment. The economic environment is represented by a homogeneous Markov process Y_t with finite state space $\{\delta_1, \ldots, \delta_n\}$ and intensity matrix $Q = \{q_{ij}\}$ and summarises macroeconomic factors that determine investment options such as risk-free rate, inflation rate or economic cycle. The level of funds available for investment U_t conditions the investment options due to transaction costs, divisibility constraints or as a consequence of rational behaviour of market agents as argued by Berk and Green (10).

Assuming rational behaviour of the investor only Pareto optimal pairs of (μ, σ) will be considered. That is, for a given level of expected return the company would choose the smallest possible level of volatility and, similarly, for a given value of volatility the highest possible level of expected return. Therefore, we can assume the existence of a function reflecting the efficient frontier of the investment possibilities that relates the parameters μ and σ on one-to-one basis. The natural point of view of an insurer is to control the level of risk. For that reason, the volatility is assumed to be chosen based on the parameters of the model and the economic environment Y_t. The value of the corresponding expected return μ is a function of the chosen volatility and the investment opportunities of the insurer (Y_t, U_t).

The stochastic differential equation representing the total reserves process U_t including investment is expressed as

$$\text{(4.1)} \qquad \mathrm{d}U_t = [c + \mu_t(\sigma_t, Y_t, U_t)U_t]\,\mathrm{d}t + \sqrt{\rho^2 + U_t^2\sigma_t^2(Y_t)}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0 = u.$$

In this work it is assumed that μ_t is a continuous function of time.

4.1.2 Stochastic Control

Once the insurer selects the desired level of volatility $\sigma_t(Y_t)$ the corresponding expected return $\mu_t(\sigma_t, Y_t, U_t)$ is implied. This way the control variable of the optimisation problem has been reduced to selection of σ_t. The objective function v to be maximised is expressed as an expected penalty-reward function

$$\text{(4.2)} \qquad v^\sigma(T, u, \delta_i) = \mathrm{E}\left[P(U_T, Y_T)\cdot \mathbb{I}_{\{\tau\geq T\}} + L(U_\tau, Y_\tau)\cdot \mathbb{I}_{\{\tau<T\}}|U_0 = u, Y_0 = \delta_i\right]$$

where $\tau = \inf\{t : U_t \notin S\}$ is the exit time of the process U_t form the solvency region S (typically $S = [0, \infty)$)

Let us denote $J(T,u,y)$ the optimal value of the maximisation problem

$$J(T,u,y) \equiv \max_{\sigma\in\Pi} v^{\sigma}(T,u,y). \tag{4.3}$$

The set Π contains all admissible strategies σ_t, that is the strategies for which a solution of (4.1) exists. We will focus only on Markov strategies, that is, σ_t depends on the process $\{U_s\}_{0\le s\le\infty}$ only through U_t. Suppose that σ^* is the maximising value for $v^{\sigma}(T,u,y)$ then $J(T,u,y) = v^{\sigma^*}(T,u,y)$.

Solving the problem (4.3) directly is not feasible since an explicit expression for $v^{\sigma}(T,u,y)$ is not available in most general case. However, following the dynamic programming approach, one can write the Hamilton-Jacobi-Bellman equation that the value function $J(t,u,y)$ satisfies under some regularity conditions, similar to the Fokker-Planck equation

$$\begin{aligned} \sup_{\sigma\in\Pi}\Bigg\{ & -\frac{\partial J}{\partial t} + \frac{1}{2}(\rho^2+\sigma^2(\delta_i)u^2)\frac{\partial^2}{\partial u^2}J + (c+\mu(\sigma(\delta_i),\delta_i,u)u)\frac{\partial}{\partial u}J \\ & + \sum_{j=1}^{n} q_{ij} J\left(t,u,\delta_i\right) + \lambda\int_0^{\infty}\left(J\left(t,u-x,\delta_i\right) - J\left(t,u,\delta_i\right)\right) f\left(x\right)\mathrm{d}x \Bigg\} = 0. \end{aligned} \tag{4.4}$$

In this chapter we present a numerical method to approximate the value function $J(T,x,y)$. The idea of Carr (1998) to approximate a fixed horizon T by a series of consecutive exponential intervals (random horizon with Erlang-n distribution) will be applied assuming that the strategy σ_n^* is constant on each interval. It will be shown that this solution converges to the optimal solution as the number of intervals n approaches infinity.

4.2 Randomised horizon

4.2.1 Exponential horizon

As a first step, the maximisation problem (4.3) will be considered in a hypothetical exponential random horizon H_α instead of a fixed horizon T. Let H_α be an exponentially distributed random variable with parameter α. If one lets $\alpha = \frac{1}{T}$ then $\mathrm{E}\left(H_\alpha\right) = T$, that is, in expected terms, the random horizon H_α and the fixed horizon T coincide.

Let us denote Υ the objective expected penalty-reward function that is to be maximised in a random horizon H_α, that is

$$\begin{aligned}\Upsilon^\sigma\left(\alpha,u,y\right) &\equiv \mathrm{E}\left[v^\sigma\left(H_\alpha,u,y\right)\right] = \\ &= \mathrm{E}\left[P(U_{H_\alpha},Y_{H_\alpha})\cdot\mathbb{I}_{\{\tau\geq H_\alpha\}} + L(U_\tau,Y_\tau)\cdot\mathbb{I}_{\{\tau<H_\alpha\}}|U_0=u,Y_0=y\right].\end{aligned}$$

Notice that the second expectation is taken with respect to random horizon H_α and the stochastic process U_t. The optimisation problem is similar to (4.3), J_1 will represent the optimal value

$$J_1\left(\alpha,u,y\right) \equiv \max_{\sigma\in\Pi}\Upsilon^\sigma\left(\alpha,u,y\right). \tag{4.5}$$

The objective function Υ can be seen as a Laplace-Carson transform in time of v defined in (4.2), indeed

$$\Upsilon^\sigma\left(\alpha,u,y\right) = \mathrm{E}\left[v^\sigma\left(H_\alpha,u,y\right)\right] = \int_0^\infty v^\sigma\left(t,u,y\right)\alpha \mathrm{e}^{-\alpha t}\mathrm{d}t. \tag{4.6}$$

Laplace-Carson transform $C(s)$ of an integrable function is closely related to its Laplace transform $L(s)$ by the relationship $C(s) = sL(s)$. This fact can be exploited to obtain the solution of the problem (4.5) since the Laplace transform of the function v is more easily obtained in many scenarios. In an analogous way to (4.4) one can obtain the Hamilton-Jacobi-Bellman Equation for $J_1(\alpha, u, y)$ as developed in Diko and Usábel (17).

$$\begin{aligned}\sup_{\sigma\in\Pi}\Bigg\{ &\frac{1}{2}(\rho^2+\sigma^2(\delta_i)u^2)\frac{\partial^2}{\partial u^2}J_1 + (c+\mu(\sigma(\delta_i),\delta_i,u)u)\frac{\partial}{\partial u}J_1 \\ &+\sum_{j=1}^{n} q_{ij}J_1 - (\lambda+\alpha)J_1 + \lambda\int_0^u J(\alpha, u-x,\delta_i)f(x)\mathrm{d}x + \\ &+\alpha P(u,\delta_i) + \lambda\int_u^\infty L(u-x,\delta_i)f(x)\mathrm{d}x\Bigg\} = 0. \end{aligned} \tag{4.7}$$

Since the time dependence is eliminated, the optimal strategy $\sigma^*(\delta_i)$ is found for each δ_i differentiating (4.7) with respect to $\sigma(\delta_i)$ as a solution to

$$\sigma(\delta_i)u^2J_1^{(uu)} + \frac{\mathrm{d}}{\mathrm{d}\sigma}\mu(\sigma,\delta_i,u)\,uJ_1^{(u)} = 0,$$

where $J_1^{(u)}$ and $J_1^{(uu)}$ is the first and the second derivative with respect to u. In case of a linear relationship between volatility σ and expected return μ this reduces to

$$\sigma(\delta_i)u^2J_1^{(uu)} + \mu(\delta_i,u)\,uJ_1^{(u)} = 0 \tag{4.8}$$

whence

$$\sigma^*(\delta_i) = -\mu(\delta_i,u)\,\frac{J_1^{(u)}}{uJ_1^{(uu)}}.$$

In particular, assuming that the investment possibilities do not depend on available capital one gets

$$\sigma^*(\delta_i) = -\mu(\delta_i)\frac{J_1^{(u)}}{uJ_1^{(uu)}}$$

a solution found in Hipp and Plum (28) and Bauerle and Rieder (8). The latter authors realise that the optimal strategy is constant for CRRA utility functions and linear specification of underlying risk process. In general, however, the solution is hard to find. Irgens and Paulsen (30) study the optimal investment (and other solvency variables) under exponential utility, Yang and Zhang (53) give an explicit solution for exponential utility under simplified market model.

4.2.2 Erlangian horizon

The next step is to approximate the fixed horizon T in the problem (4.3) by a series of consecutive exponential horizons. The distribution of a sum of k independent variables with identical exponential distribution of parameter α is the Erlang(α, k) distribution. Its density function is given by

$$p(x) = \frac{\alpha^k}{(k-1)!}x^{k-1}\mathrm{e}^{-\alpha x}.$$

Its mean is $\frac{k}{\lambda}$ and variance $\frac{k}{\lambda^2}$.

Let H_α^n be a random variable with Erlang(α, n) distribution. Let us consider a series of random variables

$$H_{\frac{n}{T}}^n \sim \mathrm{Er}(\frac{n}{T}, n). \tag{4.9}$$

One can observe that $\mathrm{E}(H^n_{\frac{n}{T}}) = T$ and

$$\mathrm{E}(H^n_{\frac{n}{T}} - T)^2 = \frac{T^2}{n} \to 0 \text{ as } n \to \infty$$

that is $H^n_{\frac{n}{T}}$ indeed converges to T in L_2 and therefore in probability.

Let us state the optimisation problem similar to (4.5) with a horizon that has Erlang distribution H^n_α. It is assumed that the termination of the exponential horizons composing H^n_α is observable. For that purpose a Poisson process H_t, independent of (U_t, Y_t), with jump intensity α is introduced. Then $H^n_\alpha = \inf\{t : H_t \geq n\}$ is a stopping time. The expected penalty-reward function to be maximised in the horizon H^n_α is

$$\begin{aligned}\Upsilon^\sigma_n(\alpha, u, y) &\equiv \mathrm{E}\left[v^\sigma\left(H^n_\alpha, u, y\right)\right] = \\ &= \mathrm{E}\left[P(U_{H^n_\alpha}, Y_{H^n_\alpha}) \cdot \mathbb{I}_{\{\tau \geq H^n_\alpha\}} + L(U_\tau, Y_\tau) \cdot \mathbb{I}_{\{\tau < H^n_\alpha\}} | U_0 = u, Y_0 = y\right].\end{aligned}$$

Taking into account the density function of Erlang distribution, one can write

$$\begin{aligned}\Upsilon^\sigma_n(\alpha, u, y) &= \mathrm{E}\left[v^\sigma\left(H^n_\alpha, u, y\right)\right] = \\ &= \int_0^\infty v^\sigma(t, u, y) \frac{\alpha^n}{(n-1)!} t^{n-1} \mathrm{e}^{-\alpha t} \mathrm{d}t\end{aligned}$$

The optimisation problem is

$$J_n(\alpha, u, y) \equiv \max_{\sigma \in \Pi} \Upsilon^\sigma_n(\alpha, u, y) \tag{4.10}$$

where J_n represents the optimal value. Only markovian strategies with respect to

(U_t, Y_t, H_t) are considered. The next theorem establishes the relationship between the value function $J_n(\alpha, u, y)$ and the value function $J(T, u, y)$ defined by (4.3).

Theorem 4.1. *Let $J(T, u, y)$ be the value function of the problem (4.3) and $J_n(\alpha, u, y)$ the value function of the problem (4.10) then*

$$\lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right) = J(T, u, y).$$

Proof. (Following the proof of Theorem 2 in (36).) Let σ be any feasible strategy in Π such that $v^{\sigma}(t, u, y)$ is continuous in t on $[0, \infty)$. Then

$$\mathrm{E}\left[v^{\sigma}\left(H_{\alpha}^{n}, u, y\right)\right] = \Upsilon_n^{\sigma}(\alpha, u, y) \leq J_n(\alpha, u, y),$$

in particular for $\alpha = \frac{n}{T}$, taking limit $n \to \infty$

$$v^{\sigma}(T, u, y) = \lim_{n\to\infty} \mathrm{E}\left[v^{\sigma}\left(H_{\frac{n}{T}}^{n}, u, y\right)\right] \leq \lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right) \tag{4.11}$$

where the first equality comes from a variant of Helly-Bray Theorem (see Chow and Teicher (15, Corolary 8.1.6)). Taking maximum over all admissible strategies on the left side of (4.11) yields

$$J(T, u, y) \leq \lim_{n\to\infty} J_n\left(\frac{n}{T}, u, y\right). \tag{4.12}$$

On the other hand, notice that

$$\begin{aligned}
J_n(\alpha,u,y) &\equiv \max_{\sigma\in\Pi}\Upsilon_n^{\sigma}(\alpha,u,y) = \max_{\sigma\in\Pi}\int_0^{\infty} v^{\sigma}(t,u,y)\,\frac{\alpha^n}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t \\
&\leq \int_0^{\infty}\max_{\sigma\in\Pi} v^{\sigma}(t,u,y)\,\frac{\alpha^n}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t \\
&= \int_0^{\infty} J(t,u,y)\,\frac{\alpha^n}{(n-1)!}t^{n-1}\mathrm{e}^{-\alpha t}\mathrm{d}t.
\end{aligned}$$

The inequality comes from the fact that the optimal σ^* that maximises the whole integral in the first line is a feasible strategy in the maximisation problem under the integral in the second line for each t. Letting $\alpha = \frac{n}{T}$, taking the limit on both sides of the inequality, and applying the Helly-Bray Theorem again the complementary inequality to (4.12) follows

$$\lim_{n\to\infty} J_n\left(\frac{n}{T},u,y\right) \leq J(T,u,y)$$

thus completing the proof. □

Theorem 4.1 provides a tool to approximate the fixed horizon by a series of consecutive exponential horizons. As will be shown in the next Theorem, this translates the original problem of maximisation in a fixed horizon T into a series of optimisation problems in exponential horizon. If $H^n_{\frac{n}{T}}$ is an Erlangian random horizon as defined in (4.9) it can be expressed as

$$H^n_{\frac{n}{T}} = \sum_{i=1}^{n} T_i^n \tag{4.13}$$

where $T_1^n, T_2^n, \ldots, T_n^n$ are independent random variables with common exponential distribution with parameter $\frac{n}{T}$. Variables T_i^n can be interpreted as consecutive random exponential horizons that compose the Erlangian horizon $H^n_{\frac{n}{T}}$. This in limit converges to the fixed horizon T. In order to state the next theorem formally we need to introduce the following notation

$$J_n(\alpha, u, y, P) \equiv J_n(\alpha, u, y)$$
$$\Upsilon^\sigma_n(\alpha, u, y, P) \equiv \Upsilon^\sigma_n(\alpha, u, y)$$

when the reward function P needs to be specified explicitly.

Theorem 4.2. *Let P be a reward function, u and y the initial conditions, $\alpha > 0$ a real parameter. For every natural $k \geq 2$ we have*

$$J_k(\alpha, u, y, P) = J_1(\alpha, u, y, P_{k-1}) \tag{4.14}$$

where $P_{k-1}(w, z) \equiv J_{k-1}(\alpha, w, z, P)$.

Proof. Let σ^*_t be the optimal strategy for $J_k(\alpha, u, y, P)$. Conditioning $v^\sigma(H^n_\alpha, u, y)$ on the instant of the first jump of the process H_t, which will be denoted T_1, the value of (U_t, Y_t) at T_1 and occurrence of the ruin one can write

$$J_k(\alpha, u, y, P) = \Upsilon^{\sigma^*}_n(\alpha, u, y) = \mathrm{E}\left[v^{\sigma^*}(H^n_\alpha, u, y, P)\right] = \tag{4.15}$$

$$= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H^n_\alpha, u, y) \mid T_1, (U_{T_1}, Y_{T_1}), \tau\right]\right] \tag{4.16}$$

$$= \mathrm{E}\left[\mathrm{E}\left[v^{\sigma^*}(H^{n-1}_\alpha, U_{T_1}, Y_{T_1})\right] \mathbb{I}_{\{\tau \geq T_1\}}\right]$$

$$+ \mathrm{E}\left[L(U_\tau, Y_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = y\right] \tag{4.17}$$

where the first term of (4.17) comes from the Markovian nature of the process (U_t, Y_t). Given that the ruin did not occur before T_1 the future is independent of the past of the process conditional on the current state U_{T_1}, Y_{T_1}. Moreover, the horizon $H_\alpha^n \sim \mathrm{Er}(\alpha, n)$ is reduced by $T_1 \sim \mathrm{Exp}(\alpha)$ what yields a new horizon $H_\alpha^{n-1} \sim \mathrm{Er}(\alpha, n-1)$. In the second term, given that the ruin occurred before T_1, the expected loss $L(U_\tau, Y_\tau)$ is incurred given the initial state of the process. Developing the first term yields

$$\mathrm{E}\left[v^{\sigma^*}(H_\alpha^{n-1}, U_{T_1}, Y_{T_1})\right] = \Upsilon_{n-1}^{\sigma^*}(\alpha, U_{T_1}, Y_{T_1}) = J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P) \tag{4.18}$$

where for the last equality remember that the optimisation is made on Markovian strategies, that is σ^* depends on process (U_s, Y_s, H_s) only through (U_t, Y_t, H_t). Therefore σ^* is the optimal the strategy for Υ_{n-1}^{σ}. Substituting into (4.17) one gets

$$\begin{aligned} J_k(\alpha, u, y, P) &= \mathrm{E}\left[J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)\mathbb{I}_{\{\tau \geq T_1\}}\right. \\ &\qquad \left. + L(U_\tau, Y_\tau)\mathbb{I}_{\{\tau < T_1\}} \mid U_0 = u, Y_0 = y\right] \\ &= \Upsilon_1^{\sigma^*}(\alpha, u, y, P_{k-1}) \leq J_1(\alpha, u, y, P_{k-1}). \end{aligned} \tag{4.19}$$

Notice that in the first line term $J_{k-1}(\alpha, U_{T_1}, Y_{T_1}, P)$ can be included in the conditioning since it is independent of U_0, Y_0.

On the other hand, assume that σ_1 is the optimal strategy for J_1. Let us consider a strategy

$$\sigma_1^* = \begin{cases} \sigma_1 & \text{if } H_t = 0 \\ \sigma^* & \text{if } H_t > 0 \end{cases}. \tag{4.20}$$

Since σ_1^* is admissible for Υ_n, applying (4.19) one can write

$$\begin{aligned} J_1(\alpha, u, y, P_{k-1}) &= \Upsilon_1^{\sigma_1^*}(\alpha, u, y, P_{k-1}) \\ &= \Upsilon_n^{\sigma_1^*}(\alpha, u, y, P) \leq J_k(\alpha, u, y, P) \end{aligned}$$

what completes the proof. □

Previous theorems provide an approximation method for cases when the solution to the stochastic control problem in exponential time is available. Theorem 4.2 presents a recursive procedure to approximate the value function in Erlangian time by iterating through n exponential horizons. The value function J_k is updated in each step until the final J_n is calculated. Theorem 4.1 guarantees the convergence of value function J_n to its fixed horizon counterpart J as n goes to infinity.

4.3 Value function approximation

Since the exponential horizon can be seen as a Laplace transform of time, as illustrated by (4.6), the solution of the stochastic control in that case tends to be more tractable, since the dependence on time is eliminated (Avram et al. (6)). Nevertheless, for complex models, in particular for the model defined by (4.1) that is treated in this chapter, the explicit solution to the stochastic control problem is not available even in exponential horizon. In the next section we present a tool to treat this cases by approximating not only the fixed horizon by a convergent series of Erlangian distributions but also approximating admissible optimal controls σ by a class of controls that are piecewise constant.

For an Erlangian horizon H_α^n determined as hitting time of $\{n\}$ by a Poisson process H_t and a control σ we define piecewise constant control σ_n that changes only with jumps of H_t. In theorem 4.3 it will be shown that the value function of the stochastic control problem in exponential horizon constrained to the class of controls σ_n converges to the value function of the unconstrained problem. Let us denote $\overline{J_n}(\alpha, u, y)$ the solution to the problem (4.10) constrained to the piecewise constant strategies defined as admissible strategies in Π that remain constant unless a jump of the process H_t^n occurs.

Theorem 4.3. *Let $J(T, u, y)$ be the solution to the problem* (4.2) *then*

$$\lim_{n\to\infty} \overline{J_n}(\alpha, u, y) = J(T, u, y) \tag{4.21}$$

Before we prove Theorem 4.3 some notation is introduced. Let us consider the Erlangian horizon H_α^n as a sum of n independent exponential distributions with parameter $\alpha = \frac{n}{T}$. Let V_t^n be a Poisson process independent of (W_t, X_t) with intensity $\frac{n}{T}$, let $T_1^n, \ldots, T_n^n$ be the first n jump times of V_t^n and define process $G_t^n = Y_{T_k^n}$ where $k = \max\{i : T_i^n < t\}$. That is G_t^n is a process that remains constant on exponential horizon intervals $\left[T_i^n, T_{i-1}^n\right)$. If we consider the stochastic differential equation

$$\mathrm{d}U_t^n = (\mu_t U_t^n + c)\,\mathrm{d}t + \sqrt{(U_t^n)^2\sigma_t^2(G_t^n, Y_t) + \rho^2}\mathrm{d}W_t - \mathrm{d}X_t, \qquad U_0^n = u \tag{4.22}$$

then the optimisation problem (4.3) of the expected penalty reward function (4.2) under the process U_t^n in an Erlangian horizon yields the value function $\lim_{n\to\infty} \overline{J_n}(\alpha, u, y)$.

Let us state the following Lemma

Lemma 4.1. *Let U_t^n be the solution to* (4.22) *and U_t the solution to* (4.1) *then U_t^n converges in Law to U_t.*

Proof of Lemma 4.1. Process G_t^n can be written as $G_t^n = U_0 + (U_{t-}^n - G_{t-}^n) \cdot V_t^n$. Since $\frac{V_t^n}{n}$ converges in Law to $\frac{t}{T}$ and the equation

$$G_t^n = U_0 + \frac{n}{T}\int_0^t (U_{t-}^n - G_{t-}^n)\mathrm{d}t \tag{4.23}$$

has the solution

$$G_t^n = \mathrm{e}^{\frac{nt}{T}} U_t^n - \mathrm{e}^{-\frac{nt}{T}} \int_0^t \mathrm{e}^{\frac{nt}{T}} U_t^n \mathrm{d}t \to U_t^n \text{ as } n \to \infty \tag{4.24}$$

applying the Theorem 6.9 from Jacod and Shiryaev (31, pg. 578) the result follows. □

Proof of Theorem 4.3. Since P and L are continuous functions of U_t^n or U_t, by Proposition 3.8 Jacod and Shiryaev (31, pg. 348) yields the convergence of the expectation of continuous functionals $\overline{J_n}$ to J. □

4.4 Example

In this section we will illustrate the application of the theorems proved above. The risk process considered follows

$$\mathrm{d}R_t = c\mathrm{d}t + \rho \mathrm{d}W_t - \mathrm{d}X_t, \qquad R_0 = u \tag{4.25}$$

where X_t is a compound Poisson process with intensity $\lambda = \frac{1}{3}$ and lognormal claim size distribution $\mathcal{LN}(1,2)$. This process represents claims collected at a constant rate $c = 3$ perturbed by a diffusion with volatility $\rho^2 = 0.25$ that can be interpreted as aggregate small claims and claims collection accruals. The Poisson process then represents catastrophic claims (with average occurrence once every 3 periods) with lognormal (heavy-tail) severity distribution. The investment opportunities will be represented by a riskless asset $\mathrm{d}S_t^{(1)} = r\mathrm{d}t$ and a risky asset $\mathrm{d}S_t^{(2)} = \nu\mathrm{d}t + \xi\mathrm{d}W_t$. The proportion invested into a risky asset will be denoted as π. No short-selling is allowed, therefore $\pi \in [0,1]$. Altogether, the reserve process, including investment, can be written as

$$\mathrm{d}U_t = [c + (r + \pi(\nu - r))U_t]\,\mathrm{d}t + \sqrt{\rho^2 + \pi^2\xi^2U_t^2}\mathrm{d}W_t - \mathrm{d}X_t \qquad U_0 = u \tag{4.26}$$

In the view of the equation (4.1) this implies the following linear relationship between the volatility $\sigma = \pi\xi$ and expected return on investment $\mu\sigma$

$$\mu(\sigma) = r + \pi(\nu - r) = r + \sigma\frac{\nu - r}{\xi}. \tag{4.27}$$

Notice that no Markov modulation is considered in this example (what is equivalent to taking Y_t = constant) as no further insight would be added besides more complex notation. The penalty-reward function considered is $P(u) = 1$, $L(u) = 0$

$$v^{\sigma}(T,u) = \mathrm{E}\left[\mathbb{I}_{\{\tau \geq T\}} | U_0 = u\right] = \mathrm{P}\left[\tau \geq T | U_0 = u\right] \equiv \varphi(u,T) \tag{4.28}$$

what represents the survival probability. The optimization problem (4.3) then turns to maximisation of survival probability in a fixed horizon T. Similar problems have been treated in Hipp and Plum (29) and others but no closed form solution exist. Following the development presented above, in order to be able to apply the iterative scheme from Theorem 4.2, the fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$

$$\varphi^*(u,\alpha) = \int_0^\infty \varphi(u,T)\alpha e^{-\alpha T} dt. \tag{4.29}$$

The fixed horizon T will be approximated by a series of n exponential horizons with parameter $\frac{n}{T}$, whereas in each of the horizons the problem to be solved is

$$J_i(\frac{n}{T}, u) \equiv \max_{\pi\in\Pi} \Upsilon^\sigma(\alpha, u, J_{i-1}). \tag{4.30}$$

with $J_0 = P = 1$. As proved in the theorem 4.3, to achieve convergence, it is sufficient to consider strategies π constant on each exponential interval. The function Υ for a constant π satisfies satisfies the following integro-differential equation

$$\begin{aligned} \frac{1}{2}(\rho^2 + \pi^2\xi^2u^2)\frac{\partial^2}{\partial u^2}\Upsilon + (c + (r + \pi(\nu - r))u)\frac{\partial}{\partial u}\Upsilon \\ - (\lambda + \alpha)\Upsilon + \lambda\int_0^u \Upsilon(\alpha, u - x)f(x)dx + \alpha J_{i-1}(u) = 0. \end{aligned} \tag{4.31}$$

as derived in Diko and Usábel (17). The cited paper also proposes an approximation method by chebyshev polynomials to calculate the solution to this problem. Since feasible strategies are bounded it is possible to evaluate Υ for a grid of possible values of $\pi \in [0,1]$ and take the maximum value as an approximation to the solution of (4.30). In this example we took equidistant grid of granularity 0.1. The table 1 shows the results of approximated value function $J(u,T)$ for various values of initial reserves u

and number of exponential intervals n that approximate the fixed horizon $T = 10$. The convergence is achieved to up to 3 decimal places for as few as 100 intervals.

n – number of intervals

u	1	2	5	10	20	50	100
0.1	0.354370	0.306438	0.287529	0.288241	0.291537	0.295415	0.297638
0.5	0.413865	0.361477	0.341409	0.341409	0.346224	0.349630	0.350752
1	0.427487	0.377567	0.359446	0.361308	0.365365	0.369168	0.370406
2	0.456898	0.412100	0.397678	0.400614	0.405446	0.410055	0.411589
5	0.570571	0.542250	0.536840	0.541189	0.547234	0.554050	0.556663
10	0.886121	0.882775	0.883860	0.885951	0.888517	0.891156	0.891655
15	0.999024	0.998997	0.999008	0.999027	0.999049	0.999072	0.999074

Figure 4.1 depicts $J(u, T)$ (maximal survival probability in horizon $T = 10$) as a function of u for $n = 1, 2, 5, 10, 20, 50, 100$. The optimal strategy that leads to the value function can be recovered using the relationship between the value function and the optimal strategy given by (4.8).

4.5 Conclusions

An application of the theoretical framework developed in Chapter 3 has been developed. The application of Chebyshev polynomials provided a numerical method to obtain the solution of an optimal investment problem in a Markov-modulated framework for a compound Poisson process. The results of this type are new in the risk theory framework.

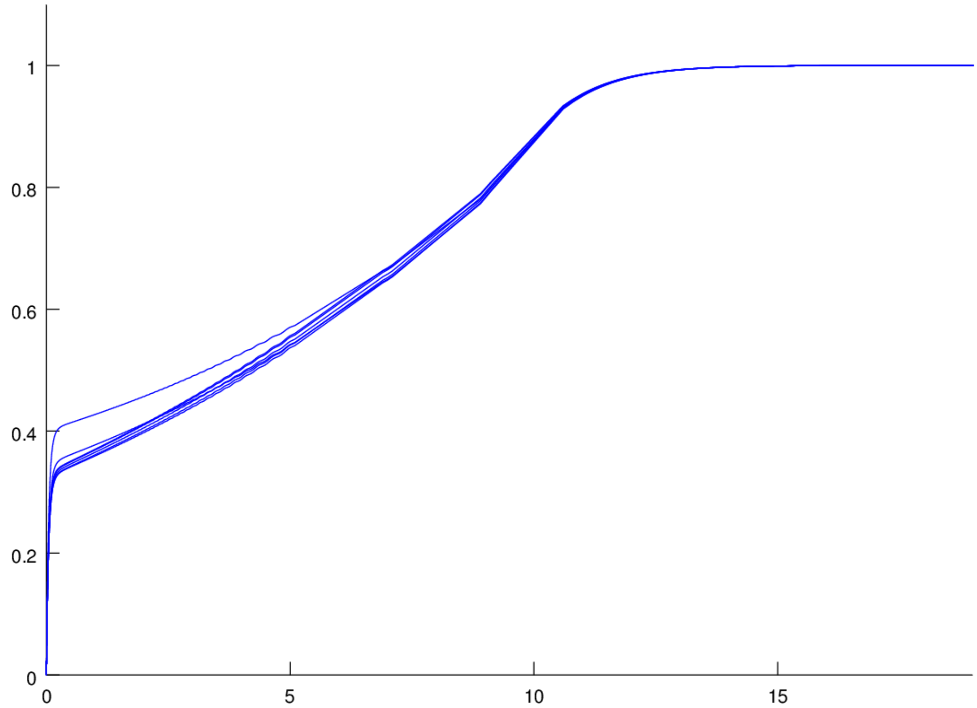

Figure 4.1: Convergence of the maximal survival probability in the horizon $T = 10$ as a function of the initial reserve.

5

Conclusions

In this work, we have treated the problems related to risk theory and stochastic control in the context of Lévy diffusion processes. It was argued that the Lévy diffusions provide a fairly general framework covering varied modelisation paradigms from finance and insurance areas. In particular a compound Poisson process with Markov-modulated parameters has been analysed, nevertheless, general results in Chapter 3 include wider spectrum of stochastic models.

Altogether, the work included in this dissertation forms three research articles. One has already been published in an international peer reviewed journal and the other two are under the revision process.

In Chapter 2 we have presented a new approximation procedure for the calculation of the penalty-reward function in a risk theory context. The importance of the contribution is underlined by the fact that previously no solution was available in the general setting that has been proposed in this work.

Chapter 3 is purely theoretical framework for simplifying a generic stochastic control of a Lévy diffusions into series of treatable standardised problems. The reduction of the complexity, achieved through the Erlangisation of the horizon, comes at a cost of iterative evaluation of the solution. Nevertheless, as have been shown, for certain cases the problem simplifies enough so that the analytical procedures can be used to obtain the solution.

Chapter 4 demonstrates the power of the approaches presented earlier by solving a stochastic control problem of optimal investment of an insurer facing risk management decisions in a context that has not been treated previously. An example shows how maximum survival probability curve can be obtained for different levels of initial conditions.

Besides the theoretical interest of the presented results we believe that these could become relevant analytical tools in practical applications, in both regulatory bodies and internal control processes within insurance companies. Sensible models that are able to explain the behaviour and quantify the answers posed about solvency, profitability or other nature of insurance business are needed in decision making processes.

Possible extensions of the results of this work include generalisation of the penalty-reward function. In the context presented here the value of utility depends solely on the value of the process at the moment of ruin or at the end of the established horizon. However, it is often interesting to include the whole path of the process into the objective function of the optimisation problem. The application of this generalisation includes the possibility of discounting the penalty-reward function at a given interest rate or the valuation of path-dependent financial assets. This extension, however, presents a non-trivial challenge to the methods proposed.

Additionally, it has to be pointed out that the decisions of the control problems have to be made taking into account exogenous variables that describe the environment of the particular branch of the insurance business. Typically, the exogenous variables are severity and frequency of claims but can include other quantities such as financial market evolution (interest rates in particular) or macroeconomic variables (such as inflation). These quantities are usually out of control of any insurance company or regulatory body and have to be estimated from the available data, a process that presents further challenge. These unexplored questions are, however, beyond the scope of this work.

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