

SINE-GORDON WOBBLER THROUGH BÄCKLUND  
TRANSFORMATIONS

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ABSTRACT. In this work we construct the wobble exact solution of sine-Gordon equation by means of Bäcklund Transformations. We find the parameters of the transformations corresponding to the Bianchi diagram for the wobble as a particular 3-soliton solutions. We show that this solution agrees with the wobbles obtained by Kälbermann and Segur by means of the Inverse Scattering Transform, and by Ferreira et al. using the Hirota method. The new formulation introduced allows to identify easily the parameters that define the building blocks of this solution – a kink and a breather, and can be used in further studies of this solution in the perturbed sine-Gordon equation.

1. **Introduction.** Klein-Gordon type models bear topological solutions called kink (or antikink) solitons, which are solitary waves that connect two minima of the potential. These solutions evolve in time with constant velocity without changing their shape. There are also oscillatory, non-topological solutions called breathers, which are bound states of a kink and an antikink. Yet another solution is the “wobble”, which can be interpreted as a nonlinear superposition of a kink and a breather [36, 5, 18].

Among these models, the sine-Gordon (sG) equation,

$$\phi_{tt} - \phi_{xx} + \sin \phi = 0, \quad (1)$$

differs from the others in being an integrable system. An infinite number of conservation laws [22, 35, 37], elastic interaction among solitons [38, 8, 14], existence of

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2000 *Mathematics Subject Classification.* 37K35, 37K40, 70K75.

*Key words and phrases.* Soliton theory, Bäcklund transformations, nonlinear dynamics.

the Lax pairs [23], or integrability by Inverse Scattering Transform (IST) [17, 39, 1] belong to the set of properties of any integrable system. Due to the sG integrability, its breather and its wobble are exact solutions. In contrast, for the non-integrable  $\phi^4$  model, “breather” and wobbling kink are not exact solutions. They decay slowly because they lose energy through the radiation [5, 9]. Another important, intrinsic property of integrable models is that they do not have internal modes (see e.g. Ref. [10], where the relation between internal modes and non-integrability is studied). Internal modes correspond to the discrete spectrum of the eigenvalue problem arising from a linear stability analysis [26, 25, 9] and arise as an oscillation of the kink shape, this being the reason why they also receive the name of shape modes. Internal modes can change the kink dynamics in so far as they can temporarily store energy from the kink translational motion and then restore it back. This mechanism has been shown to lead to non-trivial resonance phenomena in solitons driven by external perturbations [21, 28, 24], and to explain [11, 12] the phenomenon of length scale competition induced by an external, periodic potential constant in time [34, 33].

The knowledge summarized above about internal modes was not clarified until early this century. Indeed, during the decade of the 1990s, there was quite some excitement about the possibility of internal modes arising in the sG equation. This conjecture was raised in a work by Boesch and Willis in Ref. [7], where they numerically observed internal oscillations of the sG kink with a frequency above the lower phonon band, which they termed “quasimode”. In the subsequent years, a number of researchers tried to reproduce their results without success, and the discussion lingered on. Finally, in 2000, Quintero et al. [29] (see also [30]) showed that the quasimode did not exist by means of numerical studies of the nonlinear response of the sG kink to external periodic forces. In addition, they also showed that Boesch and Willis were in fact interpreting incorrectly their simulations, as they used solitons with nonzero velocity but ignored the corresponding Doppler shift in measuring frequencies: When corrected by the Doppler effect, it becomes clear that their quasimode is nothing but the first usual phonon mode, which lies within the continuum spectrum with frequency above 1, in full agreement with the linear stability analysis of the sG equation [32, 16]. Interestingly, it can be shown that in the sG equation there are resonance phenomena similar to those originated by the internal mode, but once again they can be traced back to the influence of some phonon modes [27]. Therefore, the question is fully settled by now and it is clear and generally accepted that internal modes do not exist in the sG equation; note, however, that this is only true of the continuum sG equation, and when discretized in space, internal modes can appear in the system due to the discretization (as well as to other perturbations, see [20]), which can only be observed for very large spatial steps and not for those used in the typical discretizations [7, 29, 30].

In this context, in 2004 a study was carried out by Kälbermann [19] to obtain the expression of a wobble solution of the sG equation. While this was done correctly, Kälbermann tried to make a connection with Willis’s quasimode. However, such a connection is simply not possible: Kälbermann finds a continuum of such possible quasimodes, which is not surprising because his wobble solution is a combination of a kink and a breather, but he never finds a specific, unique quasimode as Willis claims, and Kälbermann’s only numerical example has a frequency below the phonon band, very different from Willis’s quasimode. In fact, Kälbermann shows that the frequency of the oscillations ranges from non-stationary, decreasing values above 1

to steady values below 1 [19]; and so it has been suggested that a quasimode could be an intermediate stage between distorted kinks [kinks as initial conditions of (1) with parameters such that they do not verify the equation, for instance, a static kink too wide] and wobbles. Here we show that Kälbermann expression is simply the well known sG wobble and, in doing that, we prove that his results have nothing to do with the long ago resolved problem of the sG quasimode, which does not exist.

As we said, the expression of the wobble has been known for long. It was first obtained by Segur [36]; after that, Barashenkov et al. [4] used the dressing method to obtain the  $N$ -soliton solution and, more recently, Ferreira et al. [15] constructed the wobble as a 3-soliton solution using the Hirota method. As far as we know, Bäcklund Transformations (BTs) have been applied to obtain explicitly the expressions for 1-soliton and 2-soliton solutions [31, 13]. To obtain the 3-soliton solutions, in some references [31, 2] the Bianchi diagram associated to the transformations is shown, but the parameters of the transformations are not given. Indeed, there is not a direct way to find these parameters. The aim of this work is to use the BTs to obtain the wobble and make the connection to Kälbermann’s expression transparent. In order to show this, in Section 2, we start from the BT that yields to the kink and the Bianchi diagram that generates the breather. Afterwards, we construct the Bianchi diagram associated to the wobble and find out the parameters related with the transformations. Finally in Section 3 we briefly summarize our findings.

**2. Bäcklund transformations.** BTs were developed in the late of 19th century to study the pseudospherical surfaces (surfaces with constant negative curvature) in differential geometry. In the framework of the well known sG equation (1) (with Gaussian curvature  $-1$ ), or equivalently in terms of characteristic coordinates  $\xi = (x + t)/2$  and  $\tau = (x - t)/2$

$$\phi_{\xi\tau} = \sin \phi, \tag{2}$$

Bäcklund discovered a transformation allowing to calculate new surfaces of constant negative curvature from known ones [3, 31, 13]. The BT is given by a differential relation between two solutions  $\phi$  and  $\psi$  of (2)

$$\psi_{\xi} - \phi_{\xi} = 2a \sin \left( \frac{\psi + \phi}{2} \right), \tag{3}$$

$$\psi_{\tau} + \phi_{\tau} = \frac{2}{a} \sin \left( \frac{\psi - \phi}{2} \right), \tag{4}$$

where  $a$  represents a parameter of the transformation. The generic procedure is usually denoted by  $\phi = \mathbb{B}_a(\psi)$ , i.e. the new solution  $\phi$  can be obtained by means of BTs with a parameter  $a$  using the known solution  $\psi$ . Starting with a trivial solution  $\psi = 0$  in (3)-(4), one can find the 1-soliton solution by integration. Applying this procedure many times, the  $N$ -soliton solution,  $\phi$ , can be found from the  $N - 1$  soliton solution,  $\psi$ . However, from BTs and Bianchi’s theorem of permutability [6] (schematically represented as a Bianchi diagram 1) it is possible to find  $N$ -soliton solutions by a purely algebraic method. For the sG equation, after some straightforward calculations (see [31] for details), the following *nonlinear superposition formula* is obtained

$$\tan \left( \frac{\phi_{12} - \phi_0}{4} \right) = \frac{a_2 + a_1}{a_2 - a_1} \tan \left( \frac{\phi_2 - \phi_1}{4} \right), \tag{5}$$

where  $\phi_1 = \mathbb{B}_{a_1}(\phi_0)$  and  $\phi_2 = \mathbb{B}_{a_2}(\phi_0)$  are obtained via solving differential equation (3). Then, from (5) the new solution  $\phi_{12}$  (which fulfils  $\phi_{12} = \phi_{21} = \mathbb{B}_{a_2}(\phi_1) = \mathbb{B}_{a_1}(\phi_2)$ , see Fig. 1) can be obtained.

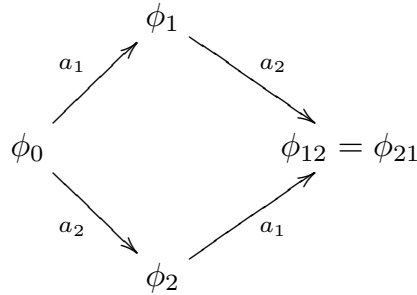


FIGURE 1. A Bianchi diagram.

2.1. 1-soliton solution of sG equation via BTs.

**Lemma 2.1.** *Let  $a = 1$ , and  $\phi(x, t) = \mathbb{B}_a(\phi_0)$  with  $\phi_0 = 0$ . Then,*

$$\tan\left(\frac{\phi}{4}\right) = e^{z_0}, \tag{6}$$

where  $z_0 = x - x_k$ ,  $x_k$  being a constant.

*Proof.* Setting  $\psi = \phi_0 = 0$  in (3)-(4), then multiplying the reduced equations by  $d\xi$  and  $d\tau$ , respectively; and finally adding both expressions we obtain

$$\phi_\xi d\xi + \phi_\tau d\tau = 2 \sin \frac{\phi}{2} \left( a d\xi + \frac{d\tau}{a} \right), \tag{7}$$

which can be integrated in  $\phi$  since  $d\phi = \phi_\xi d\xi + \phi_\tau d\tau$ . This procedure yields

$$\phi(\xi, \tau) = 4 \tan^{-1} \left[ \exp \left( a\xi + \frac{\tau}{a} + \delta \right) \right], \tag{8}$$

where  $\delta$  is a constant of integration. Returning to the original sG equation (1),

$$\phi(x, t) = 4 \tan^{-1} \left( \exp \left[ \left( a + \frac{1}{a} \right) \frac{x}{2} + \left( a - \frac{1}{a} \right) \frac{t}{2} + \delta \right] \right). \tag{9}$$

The expression (6) follows from (9), taking  $a = 1$  and  $\delta = -x_k$ . □

**Remark 1.** The expression (9) with real parameter  $a$  represents 1-soliton solution (kink or antikink) of sG equation (1). It can be also written in the form

$$\phi(x, t) = 4 \tan^{-1} \left[ \exp \left( \pm \frac{x - vt - x_k}{\sqrt{1 - v^2}} \right) \right], \tag{10}$$

where the sign  $\pm$  refers to the kink and antikink solutions, respectively, the center of the soliton at  $t = 0$  is represented by  $x_k = -\delta\sqrt{1 - v^2}$  and the constant velocity by  $v = (1 - a^2)/(1 + a^2)$ . In this case, the parameter  $a$  of the BT fixes the soliton velocity. For instance, if  $|a| < 1$ , then  $v > 0$  and if  $|a| > 1$ ,  $v < 0$ . Taking  $a = 1$ , i.e.  $v = 0$  and the  $+$  sign in (10), we obtain (6). Then,  $\phi(x, t)$  from (6) represents the static kink solution for the sG equation.

**2.2. 2-soliton solution of sG equation.** Now, instead of using the BT (3)-(4), we can apply the formula (5) to obtain the 2-soliton solution.

**Lemma 2.2.** *Let assume that  $\phi_0 = 0$ , and  $\phi_1(x, t)$  and  $\phi_2(x, t)$  are given by 1-soliton solution (10), both centered at  $x_k = 0$ , and with velocities  $v_1$  and  $v_2$ , respectively. Then, taking the kink solutions for both  $\phi_1$  and  $\phi_2$  with  $v_1 = -v_2 = v$ , such that  $a_2 = 1/a_1 = \sqrt{(1+v)/(1-v)}$ , the solution for a kink-antikink pair can be obtained in the form*

$$\phi(x, t) = 4 \tan^{-1} \left[ \frac{\sinh \left( \frac{vt}{\sqrt{1-v^2}} \right)}{v \cosh \left( \frac{x}{\sqrt{1-v^2}} \right)} \right]. \tag{11}$$

*Proof.* The expression (11) follows from (5). For more details see [31, 13]. □

**Remark 2.** The evolution of the kink-antikink pair looks like a linear superposition of a kink connecting the minimum 0 and  $2\pi$  and an antikink going from  $2\pi$  to 0. However, it has been obtained from a nonlinear superposition of two kinks with opposite velocities.

**Lemma 2.3.** *Let*

$$v = -\frac{i\alpha}{\beta}, \quad \alpha^2 + \beta^2 = 1, \tag{12}$$

*in (11). Then, the oscillatory breather static solution with frequency  $|\omega| < 1$  reads*

$$\phi(x, t) = 4 \tan^{-1} \left[ \frac{\sqrt{1-\omega^2}}{\omega} \frac{\sin(\omega(t-t_b))}{\cosh(\sqrt{1-\omega^2}(x-x_b))} \right], \tag{13}$$

*where  $\alpha = \omega$  and  $\beta = \sqrt{1-\omega^2}$  with the parameters  $t_b$  and  $x_b$ .*

*Proof.* The proof is trivial. The two added parameters  $t_b$  and the center of the breather,  $x_b$ , can be obtained by considering that the sG equation is invariant under any translation in space and time. □

**Lemma 2.4.** *Let  $\phi_0 = 0$ ,  $\phi_1 = \mathbb{B}_{a_1}(\phi_0)$  and  $\phi_2 = \mathbb{B}_{a_2}(\phi_0)$  with  $a_1 = \beta + i\alpha$  and  $a_2 = a_1^*$  such that  $\alpha^2 + \beta^2 = 1$ . Then, the breather solution (13) can be obtained from (5).*

*Proof.* Noticing that the parameters  $a_1$  and  $a_2$  of lemma 2.2 become in  $a_1 = \beta + i\alpha$  and  $a_2 = a_1^*$  by using (12), the proof follows from lemmas 2.2 and 2.3. □

**Remark 3.** In this way, the intermediate solutions  $\phi_1 = \mathbb{B}_{a_1}(\phi_0)$  and  $\phi_2 = \mathbb{B}_{a_2}(\phi_0)$  are imaginary, so they are not of physical interest, but they are essential for the construction not only of the breather, but also of the wobble.

**Remark 4.** Furthermore, since (1) is invariant under Lorentz transformations  $t' = (x - v_b t) / \sqrt{1 - v_b^2}$  and  $x' = (t - v_b x) / \sqrt{1 - v_b^2}$  from the stationary breather solution (13) we can always obtain a moving breather with a velocity  $v_b$ .

**Lemma 2.5.** *Let  $a_1 = \beta + i\alpha$  and  $a_2 = a_1^*$ , such that  $\alpha^2 + \beta^2 = 1$ . Let  $\phi_1 = \mathbb{B}_{a_1}(\phi_0)$  and  $\phi_2 = \mathbb{B}_{a_2}(\phi_0)$  with  $\phi_0 = 0$ . Then,*

$$\tan \left( \frac{\phi_1}{4} \right) = e^{z_1 + iz_2}, \tag{14}$$

$$\tan \left( \frac{\phi_2}{4} \right) = e^{z_1 - iz_2}, \tag{15}$$

where  $z_1 = \beta(x - x_b)$  and  $z_2 = \alpha(t - t_b)$ .

*Proof.* Substituting  $a = a_1$  and  $\delta = -\beta x_b + i\alpha t_b$  in (9), we obtain (14). Setting  $a = a_2$  and  $\delta = -\beta x_b - i\alpha t_b$  in (9), we obtain (15).  $\square$

**2.3. The wobble as a 3-soliton solution.** To obtain the 3-soliton solution it is necessary to add one parameter in the Bianchi diagram 1. The new diagram for constructing the 3-soliton-like solutions, and in particular the wobble, is represented in Fig. 2.

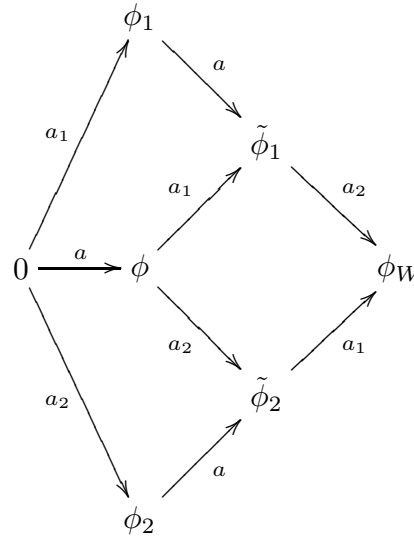


FIGURE 2. A Bianchi diagram for 3-soliton solutions.

**Theorem 2.6.** Let  $a = 1$ ,  $a_1 = \beta + i\alpha$  and  $a_2 = a_1^*$ . Let  $\phi = \mathbb{B}_a(\phi_0)$ ,  $\phi_1 = \mathbb{B}_{a_1}(\phi_0)$  and  $\phi_2 = \mathbb{B}_{a_2}(\phi_0)$  with  $\phi_0 = 0$ . Furthermore, let  $\tilde{\phi}_1 = \mathbb{B}_a(\phi_1)$  and  $\tilde{\phi}_2 = \mathbb{B}_a(\phi_2)$ . Then, the wobble solution is given by  $\phi_W = \mathbb{B}_{a_2}(\tilde{\phi}_1)$  and reads

$$\phi_W = 4 \tan^{-1} \left[ \frac{V_s}{U_s} \right], \tag{16}$$

$$V_s(x, t) = \frac{1 + \beta}{1 - \beta} e^{z_0} + e^{z_0 + 2z_1} - \frac{2\beta}{1 - \beta} e^{z_1} \cos z_2, \tag{17}$$

$$U_s(x, t) = 1 + \frac{1 + \beta}{1 - \beta} e^{2z_1} - \frac{2\beta}{1 - \beta} e^{z_0 + z_1} \cos z_2. \tag{18}$$

where  $z_0 = x - x_k$ ,  $z_1 = \beta(x - x_b)$  and  $z_2 = \alpha(t - t_b)$ ,  $x_k$ ,  $x_b$  and  $t_b$  being constants.

*Proof.* At a first step in the Bianchi diagram 2 we notice that  $\phi(x, t) = \mathbb{B}_a(\phi_0)$  with  $a = 1$  and  $\phi_0 = 0$  is the static kink solution given in (6). Furthermore, the parameters  $a_1$  and  $a_2$  are the same that we use to obtain the breather stationary solution, however, the Bianchi diagram to calculate the breather solution is different from the one to derive  $\phi_1$  or  $\phi_2$ . Therefore, the imaginary intermediate solutions  $\phi_1$  and  $\phi_2$  are not breathers and follow from (14) and (15), respectively.

The next step is to calculate the solutions  $\tilde{\phi}_1$  and  $\tilde{\phi}_2$ . Using the formula (5) and with the help of the Bianchi diagram 2, we obtain

$$\tilde{\phi}_n(x, t) = 4 \tan^{-1} \left[ \frac{a_n + a}{a - a_n} \tan \left( \frac{\phi - \phi_n}{4} \right) \right], \quad n = 1, 2. \tag{19}$$

Notice that if  $\phi_j(x, t) = 4 \tan^{-1}(e^{\theta_j(x,t)})$ , the following relation holds

$$\tan \left( \frac{\phi_j - \phi_k}{4} \right) = \frac{\sinh \left( \frac{\theta_j - \theta_k}{2} \right)}{\cosh \left( \frac{\theta_j + \theta_k}{2} \right)}. \tag{20}$$

According to (19) and considering (20) we obtain

$$\tilde{\phi}_1(x, t) = \frac{i\alpha}{\beta} \frac{\sinh \left( \frac{z_0 - z_1 - iz_2}{2} \right)}{\cosh \left( \frac{z_0 + z_1 + iz_2}{2} \right)} = \frac{i\alpha}{1 - \beta} \frac{e^{z_0} - e^{z_1 + iz_2}}{1 + e^{z_0 + z_1 + iz_2}}, \tag{21}$$

and  $\tilde{\phi}_2 = \tilde{\phi}_1^*$ . Now, from (5) and following the last step in the Bianchi diagram, we obtain the wobble solution

$$\phi_W(x, t) = \phi(x, t) + 4 \tan^{-1} \left[ \frac{a_1 + a_2}{a_2 - a_1} \tan \left( \frac{\tilde{\phi}_2 - \tilde{\phi}_1}{4} \right) \right]. \tag{22}$$

Noticing that  $\tilde{\phi}_1 - \tilde{\phi}_2 = 2\mathbb{I}(\tilde{\phi}_1)$ , after some straightforward calculations we obtain

$$\frac{\phi_W - \phi}{4} = \tan^{-1} \left[ \frac{N}{D} \right] \tag{23}$$

where the functions  $N$  and  $D$  are given by

$$N(x, t) = 2\beta (e^{z_0} - e^{z_0 + 2z_1} + e^{2z_0 + z_1} \cos z_2 - e^{z_1} \cos z_2), \tag{24}$$

$$D(x, t) = (1 - \beta) (1 + e^{2z_0 + 2z_1} + 2e^{z_0 + z_1} \cos z_2) + (1 + \beta) (e^{2z_0} + e^{2z_1} - 2e^{z_0 + z_1} \cos z_2), \tag{25}$$

respectively. Taking into account that

$$\tan^{-1} \left( \frac{Y}{X} \right) = \arg(X + iY), \quad X > 0, \tag{26}$$

the function  $\phi(x, t)$  given by (6) can be rewritten in the form

$$\phi(x, t) = 4 \arg(1 + ie^{z_0}), \tag{27}$$

and the r.h.s. of (23) is  $\tan^{-1}[N/D] = \arg(D + iN)$ . Therefore, we obtain

$$\begin{aligned} \frac{\phi_W}{4} = \arg \{ & (1 + e^{2z_0})[(1 - \beta) + (1 + \beta)e^{2z_1} - 2\beta e^{z_0 + z_1} \cos z_2 \\ & + i(1 + \beta)e^{z_0} + i(1 - \beta)e^{z_0 + 2z_1} - i2\beta e^{z_1} \cos z_2] \}. \end{aligned} \tag{28}$$

Now using the property  $\arg[Z(X + iY)] = \arg(X + iY)$ , for  $Z > 0$  and taking  $(1 - \beta)(1 + e^{2z_0})$  as a common factor in the r.h.s. of (28), we obtain

$$\frac{\phi_W}{4} = \arg(U_s + iV_s), \tag{29}$$

where  $V_s$  and  $U_s$  are given by (17) and (18), respectively. It is clear that the wobble solution (16)-(18) follows from (26) and (29).  $\square$

**Remark 5.** The expression for the wobble (16) contains explicitly the parameters related with the center of the kink, and the center, phase and frequency of the breather. It can be useful for the construction of the approximated wobble solutions (an Ansatz) in the perturbed sG equations through the evolution of a few degrees of freedom, namely the collective coordinates [11]. The wobble (16) agrees with the ones obtained in [19, 15]. However, these parameters do not appear explicitly in the previous obtained wobbles in [19, 15].

**Lemma 2.7.** *The wobble, obtained by Kälbermann (equations (1)-(5) of [19]) and represented by*

$$\frac{\phi_W}{4} = \arg(U + iV), \tag{30}$$

with

$$U = 1 + \frac{|m_2|^2 \left(\frac{1}{4} - \omega_k^2\right)}{\omega_k^2} e^{4\omega_k x} - \frac{2m_1|m_2| \left(\frac{1}{2} - \omega_k\right)}{\frac{1}{2} + \omega_k} e^{x+2\omega_k x} \cos(2\sigma(t + t_0)), \tag{31}$$

$$V = -\frac{m_1|m_2|^2 \left(\frac{1}{2} - \omega_k\right)^3}{\omega_k^2 \left(\frac{1}{2} + \omega_k\right)} e^{x+4\omega_k x} - m_1 e^x + 2|m_2| e^{2\omega_k x} \cos(2\sigma(t + t_0)), \tag{32}$$

where  $\omega_k^2 + \sigma^2 = 1/4$ ,  $m_1 \in \mathbb{R}$ ,  $m_2 \in \mathbb{C}$ ,

$$\tan(2\sigma t_0) = \frac{\text{Im}(m_2)\omega_k + \text{Re}(m_2)\sigma}{\text{Im}(m_2)\sigma - \text{Re}(m_2)\omega_k} \tag{33}$$

agrees with the wobble given by (16)-(18).

*Proof.* A comparison between  $U$  (31) and  $U_s$  (18) and between  $V$  (32) and  $V_s$  (17) yields that both expressions of the wobble agree each other if

$$\begin{aligned} m_1 &= -\frac{1 + \beta}{1 - \beta} e^{-x_k}, & 2\sigma t_0 &= -\alpha t_b + \pi, \\ |m_2| &= \frac{\beta}{1 - \beta} e^{-\beta x_b}, & 2\sigma &= \alpha, \\ & & 2\omega_k &= \beta. \end{aligned} \tag{34}$$

□

**Remark 6.** Since in [15] has been shown that the wobble obtained by Hirota method agrees with the one obtained by Kälbermann, the three expressions for the wobble, obtained by means of IST, Hirota method and BT, agree.

**Remark 7.** To obtain the wobble solution via IST one uses the scattering data of the kink and the breather, contained in the  $3 \times 3$  matrix  $M$  [19]. The elements of this matrix are related with the aforementioned parameters  $m_1$  and  $m_2$  and with  $\zeta_1 = i/2$ ,  $\zeta_2 = \sigma + i\omega_k$  and  $\zeta_3 = -\zeta_2^*$ . Notice that  $m_1$  and  $m_2$  are related with the frequency of the breather  $\omega$  since  $\beta = \sqrt{1 - \omega^2}$ , and with the centers of the kink and the breather (see (34)). Interestingly, rescaling the parameters  $\tilde{\zeta}_j = -2i\zeta_j$  with  $j = 1, 2, 3$  we obtain that  $\tilde{\zeta}_1 = a$ ,  $\tilde{\zeta}_2 = a_1$  and  $\tilde{\zeta}_3 = a_2$ , i.e. these parameters of the IST agree with the parameters of the BT and with the parameters related with the Hirota method [15] for the wobble.



**3. Conclusions.** In this work we have used the Bäcklund Transformation as an alternative method to construct the wobble solution for the sG equation (1). This wobble solution is represented by the relations (16)-(18). This solution looks like a breather sitting in the top of the kink, so it is a topological soliton with internal oscillations with a fixed frequency less than 1. In the construction of this solution we have used the Bianchi diagram for 3-soliton solution and we have found out the parameters of the transformations to obtain the wobble. We have shown that this wobble agrees with the ones obtained by Kälbermann by the Inverse Scattering Transform thus confirming that they do not have anything to do with the notorious, non-existent sG quasimode. Furthermore, we have shown that some parameters related with the elements of the matrix containing the scattering data of the kink and the breather agree with the parameters of BT for the wobble. The advantage of our expression for the wobble, obtained from a nonlinear superposition principle of a sG breather and kink, is that the parameters related with the center of the kink and with the center, frequency, and phase of the breather appear explicitly in (16)-(18). Therefore, our solution can be useful in the problems related with the dynamic of the wobble in the perturbed sG equation.

**Acknowledgments.** We acknowledge financial support from MICINN through grants MOSAICO (SC, AS) and FIS2008-02380/FIS (NRQ), and grants FQM207 and P06-FQM-01735 (NRQ) (from Junta de Andalucía, Spain), and SIMUMAT-CM (AS) (from Comunidad de Madrid, Spain).

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