

Nonconvex Optimization for Pricing and Hedging in Imperfect Markets



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Abstract—The paper deals with imperfect financial markets and provides new methods to overcome many inefficiencies caused by frictions. Transaction costs are quite general and far from linear or convex. The concepts of pseudoarbitrage and efficiency are introduced and deeply analyzed by means of both scalar and vector optimization problems. Their optimality conditions and solutions yield strategies to invest and hedging portfolios, as well as bid-ask spread improvements. They also point out the role of coalitions when dealing with these markets. Several sensitivity results will permit us to show that a significant transaction costs reduction is very often feasible in practice, as well as to measure its effect on the general efficiency of the market. All these findings may be especially important for many emerging and still illiquid spot or derivative markets (electricity markets, commodity markets, markets related to weather, inflation-linked or insurance-linked derivatives, etc.).

Keywords—Global optimization, Pseudoarbitrage, Spread reduction, Balance point, Sensitivity.

1. INTRODUCTION

Applications of mathematical programming in finance are becoming more and more usual in the literature. Since the seminal contribution of Markowitz, many authors started addressing portfolio choice problems by using optimization methods. As time was going back several alternative financial topics were sequentially treated, and currently many financial market linked problems are the focus of interesting optimization articles. For instance, [1] and [2] deal with pricing issues in incomplete markets and [3] or [4] focus on the risk measurement. The present paper applies optimization procedures and addresses pricing and hedging issues in imperfect markets with significant transaction costs.

Imperfect financial markets are becoming very treated too, with special attention on several arbitrage linked issues. For instance, [5–7] or [8] focus on several classical characterizations of the absence of arbitrage in perfect markets, and they extend the results. The first paper and the last one yield a martingale-like property, while the second and the third ones focus on static (or one

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The authors would like to thank Professor E. Galperin for his assistance and useful suggestions, that have led to several improvements of the paper.

Partially funded by “Comunidad Autónoma de Madrid” and Spanish Ministry of Science and Education (ref: BEC2003-09067-C04-03).

period) imperfect models and represent some arbitrage-free pricing rules by means of Choquet integrals.

Illiquid markets are a special kind of imperfect markets. Along with high transaction costs and bid-ask spreads, illiquid markets usually reflect large difficulties for trading due to the lack of available quotes. Many important emerging markets are still illiquid. For instance, electricity markets, commodity markets, some inflation-linked markets, weather-linked derivative markets, etc. In these markets it is very difficult to apply those pricing methods based on dynamic replications (see [9] or [10]).

Illiquid markets have also deserved the interest of several authors. So, for instance, [11] provides new pricing and portfolio choice models, and [12] introduces new arbitrage-linked methods for pricing and hedging catastrophe-linked derivatives that were available at the Chicago Board of Trade and reflected scarce activity.

The approach of [12] draws on real bid or ask quotes and empirically tests the possibility of outperforming some of these quotes without assuming any sort of risk. This is possible if we can hedge with an arbitrage. To prevent illiquidity effects and the impossibility of any continuous trading, static arbitrage was only tested. One year later, the same method was tested in liquid markets in [13]. Both empirical papers present clear evidence justifying the interest of this technique to increase liquidity and to reduce in practice transaction costs and bid-ask spreads.

Both papers above empirically analyzed the financial markets behavior by drawing on some scalar and linear optimization problems. Their theoretical framework, as well as the possible scope of future empirical applications, has been significantly extended in [14]. This article considers multiple combinations of securities, which lead to multiobjective optimization problems. However, it follows the approach of [5] and imperfections are represented by the bid-ask spread. Consequently, transaction costs are linear and so are all of the optimization problems proposed in the study.

Many real imperfections are nonlinear and nonconvex (commissions paid to brokers that depend on the traded volume, special prices obtained when purchasing many units of some asset, surplus to be paid when dealing with several levels of bid/ask prices, etc.). Thus, it seems interesting to focus on them and develop classic issues like arbitrage, hedging, and efficiency. This is the major object of this paper that will enlarge the empirical and practical interest of these techniques.

The article outline is as follows. Section 2 will be devoted to introduce basic assumptions, notations, and the general framework. The major concepts of arbitrage, strong-pseudoarbitrage, weak-pseudoarbitrage, and efficiency will be introduced in Section 3. Theorem 1, the most important finding in the section, provides a characterization of the arbitrage absence by means of the existence of stochastic discount factors. For nonconvex frictions this seems to be the first result of this type appearing in the literature.

Section 4 will focus on strong-pseudoarbitrage portfolios and will characterize their existence, as well as those methods that will permit us to compute these strategies in practice (if available). We will also show how to use them in order to improve (outperform) the real market quotes and reduce transaction costs.

The concept of strong-pseudoarbitrage is the theoretical representation of the empirical method proposed in [12], though we consider nonconvex imperfections as well. By means of several scalar optimization problems this concept will allow us to define the so-called “shadow bid-ask spread” and its associated hedging portfolios. Shadow spreads will be a powerful tool for practitioners who can use them in order to improve liquidity and earnings. The shadow spread major properties will be provided in Theorems 4 and 6. These results also point out the existence of critical differences with respect to the case of linear frictions proposed in [14].

Section 5 is devoted to the so-called “weak pseudoarbitrage strategies”. They involve complex combinations of securities which lead to vector optimization problems frequently containing a significant number of objectives. We will address these multiobjective problems by drawing on the “balance space approach” as introduced and developed in [15] and [16]. Once again we will

provide several results that could be interesting to both researchers and practitioners. Indeed, on the one hand the presence of weak-pseudoarbitrage will be characterized and used to improve liquidity and transaction cost levels, and several methods to measure the sensitivity of the market efficiency with respect to friction improvements will be given too. On the other hand investors and brokers will have a new method allowing them to establish coalitions and to obtain additional riskless profits. Once more we will reveal many significant differences with regard to [14]. They are caused by the nonconvex imperfections.

The last section summarizes and concludes the paper.

2. PRELIMINARIES AND NOTATIONS

This section is devoted to present some basic notations and the general framework.

Consider n arbitrary securities B_j , $j = 1, 2, \dots, n$, available in the market, and denote by $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$, $p_j \leq q_j$, $j = 1, 2, \dots, n$, the vectors of current bid and ask prices, respectively. The portfolio composed of x_j units of B_j in long position (bought assets) and y_j units of B_j in short position (sold assets), $x_j, y_j \geq 0$, $j = 1, 2, \dots, n$, will be represented by $(x, y) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \in \mathbb{R}^{2n}$.

Along with the transaction cost generated by the bid-ask spread $q - p$, agents have to pay an additional friction that depends on the portfolio $(x, y) \in \mathbb{R}^{2n}$. A discussion below will justify this assumption. Thus, we will consider a real-valued function φ such that the current price of $(x, y) \in \mathbb{R}^{2n}$ is given by¹

$$\Lambda_0(x, y) = \sum_{j=1}^n (q_j x_j - p_j y_j) + \varphi(x, y). \quad (1)$$

Suppose that T is a future date and consider the set K providing us with the states of nature that may occur at T . Assume that K is endowed with an appropriate topology and becomes a Hausdorff and compact space. If $V_j(k)$ is the payoff of B_j , $j = 1, 2, \dots, n$, at T if $k \in K$ occurs, and $W_j(k) \geq V_j(k)$ represents the liability associated with a short B_j , $j = 1, 2, \dots, n$, $k \in K$, then the real-valued function $\Lambda_T : \mathbb{R}^{2n} \times K \rightarrow \mathbb{R}$ given by

$$\Lambda_T(x, y, k) = \sum_{j=1}^n (x_j V_j(k) - y_j W_j(k)) \quad (2)$$

yields the payoff $\Lambda_T(x, y, k)$ of (x, y) at T under the state k . Expression (2) clearly shows that the function Λ_T is linear in the (x, y) variable. Moreover $V_j, W_j : K \rightarrow \mathbb{R}$, $j = 1, 2, \dots, n$, will be assumed to be continuous and, therefore, it trivially follows from (2) that $\Lambda_T(x, y, -) : K \rightarrow \mathbb{R}$ will be also continuous in the k variable.

The classical one period approach of financial economics usually considers that states of the world are given by probability spaces rather than Hausdorff and compact spaces and, therefore, future prices are given by square-integrable functions rather than continuous ones. This fact implies some advantages since compactness is not required and the space of square-integrable functions makes it possible to apply those properties only associated with Hilbert spaces (orthogonality properties, for instance). Instead, the set of states of the world must be endowed with an initial probability measure, and this is the reason why we have decided to modify the general setting. The theory allows for applications on very different types of market and it would be quite difficult in practice to provide the corresponding states of nature with realistic probabilities. On the other hand, both approaches are often quite closely related and most of the results may be easily translated from one framework to other.

Regarding the function φ , we will assume that it incorporates those costs that are not reflected by the spread $q - p$. So for instance, if investors have to pay to brokers an additional commission

¹We will assume that φ does not depend on the investor.

per traded asset, then φ should contain a term of the form $\sum_{j=1}^n (c_j x_j + c'_j y_j)$. If there is a constant fixed commission φ_0 then φ must add this parameter. If there are limit orders and the ask price of B_j increases from q_j to $q'_j > q_j$ whenever x_j overcomes the limit α_j , then φ could contain the term $(q'_j - q_j)(x_j - \alpha_j)^+$.² Obviously, φ can also reflect special discounts for significant traded volumes and many other situations.

Actually, in practice φ will be only given for isolated values of (x, y) . For instance, we often know the transaction costs whenever (x, y) is composed of integers. However, it is easy to interpolate the function φ in such a way that it verifies several properties. Thus, in what follows we will impose some assumptions that are meaningful from the economic viewpoint.

ASSUMPTION 1.

- (1a) φ is continuously differentiable on an open set containing the nonnegative cone of \mathbb{R}^{2n} .
- (1b) $\varphi(0, 0) = \frac{\partial \varphi}{\partial x_j} |_{(x,y)=(0,0)} = \frac{\partial \varphi}{\partial y_j} |_{(x,y)=(0,0)} = 0, j = 1, 2, \dots, n.$
- (1c) φ is increasing, i.e.,

$$\varphi(x, y) \leq \varphi(x', y') \quad (3)$$

whenever $0 \leq x \leq x'$ and $0 \leq y \leq y'$. ■

It might be worthwhile to point out that some assertions related to the differentiability of φ might be relaxed without significantly affecting the remainder of the paper. In such a case optimality and sensitivity results for nondifferentiable optimization problems (see [17] for a recent contribution) could be used.

3. ARBITRAGE

This section is devoted to introduce those key concepts that will often apply throughout the paper.

We will deal with arbitrage portfolios of the second type, in the sense of [9]. Arbitrage portfolios of the first type might be also considered but they are beyond the scope of this article.

DEFINITION 1. (x, y) is said to be an arbitrage portfolio if $\Lambda_0(x, y) < 0$ and $\Lambda_T(x, y, k) \geq 0$ for every $k \in K$. ■

Next we will introduce a crucial concept for this paper.

DEFINITION 2. Let $J_1, J_2,$ and J_3 be disjoint sets such that $\{1, 2, \dots, n\} = J_1 \cup J_2 \cup J_3$. Let $z = (z_j)_{j \in J_1 \cup J_2}$ be a family of real numbers such that $z_j \geq 1, j \in J_1 \cup J_2$. Portfolio $(x, y) = ((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3}))$ is said to be a pseudoarbitrage (ps-arbitrage) portfolio associated with (J_1, J_2, z) if

$$\varphi((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3})) + \sum_{j \in J_1} p_j z_j - \sum_{j \in J_2} q_j z_j + \sum_{j \in J_3} (q_j x_j - p_j y_j) < 0 \quad (4)$$

and

$$\sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) + \sum_{j \in J_3} x_j V_j(k) - \sum_{j \in J_3} y_j W_j(k) \geq 0 \quad (5)$$

for every $k \in K$. In addition, if $J_1 = \{j_0\}$ is a singleton and J_2 is empty then (x, y) will be called bid-pseudoarbitrage (bps-arbitrage) associated with (j_0, z_{j_0}) . If $J_2 = \{j_0\}$ is a singleton and J_1 is empty then (x, y) is said to be an ask-pseudoarbitrage (aps-arbitrage) associated with (j_0, z_{j_0}) . Every bps-arbitrage or aps-arbitrage will be called a strong-pseudoarbitrage portfolio (sps-arbitrage). Otherwise we will merely say weak-pseudoarbitrage portfolio (wps-arbitrage).³ ■

Let us interpret the concepts above from the economic point of view. If (x, y) is a ps-arbitrage associated with (J_1, J_2, z) , then bid prices are “too cheap” for those securities related to J_1 and

²As usual, $\beta^+ = \text{Sup}\{\beta, 0\}$ and $\beta^- = \text{Sup}\{-\beta, 0\}$ for every real number β .

³Condition $z_j \geq 1$ for $j \in J_1 \cup J_2$ could be relaxed to $z_j > 0$ for $j \in J_1 \cup J_2$, and the results would not be affected at all. Nevertheless, owing to nonlinear or convex frictions we preferred to follow the economic intuition and to impose that at least one security must be traded when implementing ps-arbitrage.

ask prices are “too expensive” for the securities of J_2 . Indeed, any trader can improve the quotes $((p_j)_{j \in J_1}, (q_j)_{j \in J_2})$ by bidding $(z_j)_{j \in J_1}$ and simultaneously asking $(z_j)_{j \in J_2}$ for a better (bigger) price than

$$\sum_{j \in J_1} p_j z_j - \sum_{j \in J_2} q_j z_j. \quad (6)$$

This trader can totally hedge her/his position if a new investor accepts her/his proposal. It is sufficient to add the strategy $((x_j)_{j \in J_3}, (y_j)_{j \in J_3})$.

Following the strategy above competition among traders trying to obtain arbitrage profits could lead to significant bid-ask spread reductions in a nonps-arbitrage free market. This could make this concept very appropriate in order to increase the level of efficiency of imperfect markets. Dealing with some concrete sps-arbitrage portfolios, the empirical papers [12] and [13] have illustrated some possibilities of these methods. Note that when some trader tries to improve real quotes by applying the previous procedure he/she will have to deal with the securities of $J_1 \cup J_2$, which suggests that sps-arbitrage portfolios, if available, will be more interesting to traders than wps-arbitrage portfolios.

Our last definition is related to the concept of “nondominated” or “efficient” strategy. Since the word “efficiency” has many nonequivalent meanings in both vector optimization and finance, it is worthwhile to indicate that we will follow the idea of the pioneering book of Pareto (see [18]).

DEFINITION 3. *Portfolio (x, y) is said to be efficient if there are no portfolios (x', y') verifying*

- (a) $\Lambda_0(x', y') \leq \Lambda_0(x, y)$.
- (b) $\Lambda_T(x', y', k) \geq \Lambda_T(x, y, k)$ for every $k \in K$.
- (c) $\Lambda_0(x', y') < \Lambda_0(x, y)$ or there exists $k \in K$ such that

$$\Lambda_T(x', y', k) > \Lambda_T(x, y, k). \quad (7) \blacksquare$$

EXAMPLE 1. Let us consider a simple example illustrating the concepts above. It will also show that the converse of the obvious implication

$$\text{ps-arbitrage absence} \implies \text{sps-arbitrage absence} \implies \text{arbitrage absence}$$

fails in general, i.e., one can find ps-arbitrage and sps-arbitrage portfolios in arbitrage free markets (see Assumption 3 below).

Suppose that $n = 4$, K is a finite set composed of three states of nature, B_1 is a bond whose final payoff is $(1, 1, 1)$, B_2 is an arbitrary stock with payoff $(0, 1, 2)$, and B_3 and B_4 are call and put options on B_2 , with strike 1 and final payoff $(0, 0, 1)$ and $(1, 0, 0)$, respectively.

Suppose that the vector of bid prices equals $p = (1, 0.7, 0.1, 0.1)$, while the vector of ask prices equals $q = (1, 0.9, 0.3, 0.5)$. There are no more transaction costs so any liability and any payoff are identical and $\varphi \equiv 0$. Theorem 1 below will show that the market is arbitrage free. On the other hand, it may be easily checked that $x = (0, 1, 0, 1)$ and $y = (1, 0, 1, 0)$ provide a sps-portfolio associated with $J_1 = \{4\}$, $z_4 = 1$, and $J_2 = \emptyset$, i.e., (x, y) is an arbitrage if B_4 may be purchased by paying its bid price.

Let us consider a new bid price $p = (1, 0.7, 0.1, 0.3)$. Then, Theorem 4 below will show that the market is sps-arbitrage free. However, it is easy to check that the portfolio above provides a wps-arbitrage associated with the partition $J_1 = \{2, 4\}$, $z_{J_1} = \{1, 1\}$, and $J_2 = \emptyset$, i.e., (x, y) is an arbitrage as long as B_2 and B_4 may be purchased by paying their bid prices.

Finally, notice that the purchase of $B_2 + B_4$ is not efficient since the current price 1.4 is bigger than the current price 1.3 of $B_1 + B_3$, although both portfolios generate the same payoff $(1, 1, 2)$. \blacksquare

Theorem 1 below will provide a characterization of the absence of arbitrage. It is in the line of other results concerning imperfect markets [5,8,14], but this seems to be the first extension of the classical theory for perfect markets ([9,10], etc.) that involves nonconvex frictions.

As usual, $\mathcal{C}(K)$ will denote the Banach space of real-valued and continuous functions on K endowed with the supremum norm. The Riesz representation theorem establishes that $\mathcal{M}(K)$ is the dual space of $\mathcal{C}(K)$, $\mathcal{M}(K)$ being the Banach space of regular σ -additive measures on the Borel σ -algebra \mathcal{B} of K endowed with the variation norm. The set of nonnegative measures

$$\mathcal{M}_+(K) = \{\mu \in \mathcal{M}(K); \mu \geq 0\} \quad (8)$$

and the set of probability measures

$$\mathcal{P}(K) = \{\mu \in \mathcal{M}(K); \mu \geq 0, \mu(K) = 1\} \quad (9)$$

are especially important subsets of $\mathcal{M}(K)$ (see [19] for further details).

Even for perfect markets, many characterizations of the absence of arbitrage are proved based on the existence of some numeraire, i.e., a security whose price and payoff are always positive. We will also impose a condition of this type.

ASSUMPTION 2. *Let us assume henceforth that $V_1(k) > 0$ for every $k \in K$.* ■

REMARK 1. Consider a continuous function $f \in \mathcal{C}(K)$. We are interested in the minimum cost portfolio whose payoff at T is at least f . This portfolio must solve the optimization problem

$$\begin{aligned} \min \Lambda_0(x, y) &= \sum_{j=1}^n (q_j x_j - p_j y_j) + \varphi(x, y), \\ \Lambda_T(x, y, k) &= \sum_{j=1}^n x_j V_j(k) - \sum_{j=1}^n y_j W_j(k) \geq f(k), \quad k \in K, \\ x_j &\geq 0, \quad y_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \quad (10)$$

It is easy to see that the Kuhn-Tucker conditions become

$$\begin{aligned} \frac{\partial \varphi}{\partial x_j} + q_j - \int_K V_j(k) d\mu(k) - \nu_j &= 0, \quad j = 1, 2, \dots, n, \\ \frac{\partial \varphi}{\partial y_j} - p_j + \int_K W_j(k) d\mu(k) - \tilde{\nu}_j &= 0, \quad j = 1, 2, \dots, n, \\ x_j \nu_j &= 0, \quad j = 1, 2, \dots, n, \\ y_j \tilde{\nu}_j &= 0, \quad j = 1, 2, \dots, n, \\ \int_K \left(\sum_{j=1}^n x_j V_j(k) - \sum_{j=1}^n y_j W_j(k) - f(k) \right) d\mu(k) &= 0, \\ x, y, \mu, \nu, \tilde{\nu} &\geq 0, \\ x, y \in \mathbb{R}^n, \quad \mu \in \mathcal{M}(K), \quad \nu, \tilde{\nu} \in \mathbb{R}^n. \end{aligned} \quad (11)$$

Recall that the Kuhn-Tucker conditions are only necessary optimality conditions if problem (10) verifies some kind of qualification, so let us prove that Assumption 2 provides the qualification we need. Indeed, according to the Fritz-John theorem (see [20]), if a portfolio (x, y) solves (10) then there exist $\tau \in \mathbb{R}$, $\nu', \tilde{\nu}' \in \mathbb{R}^n$, and a regular measure μ' on K such that $(x, y, \tau, \nu', \tilde{\nu}', \mu')$

⁴Notice that Assumption 2 is weak and easily fulfilled in practice. For instance, it trivially holds if B_1 is a riskless asset.

solves the system

$$\begin{aligned}
 \tau \frac{\partial \varphi}{\partial x_j} + \tau q_j - \int_K V_j(k) d\mu'(k) - \nu'_j &= 0, & j = 1, 2, \dots, n, \\
 \tau \frac{\partial \varphi}{\partial y_j} - \tau p_j + \int_K W_j(k) d\mu'(k) - \tilde{\nu}'_j &= 0, & j = 1, 2, \dots, n, \\
 x_j \nu'_j &= 0, & j = 1, 2, \dots, n, \\
 y_j \tilde{\nu}'_j &= 0, & j = 1, 2, \dots, n, \\
 \int_K \left(\sum_{j=1}^n x_j V_j(k) - \sum_{j=1}^n y_j W_j(k) - f(k) \right) d\mu'(k) &= 0, \\
 x, y, \tau, \mu', \nu', \tilde{\nu}' &\geq 0, \\
 (\tau, \mu', \nu', \tilde{\nu}') &\neq (0, 0, 0, 0).
 \end{aligned} \tag{12}$$

Suppose that $\tau = 0$. Then the first equation leads to

$$\int_K V_1(k) d\mu'(k) \leq 0 \tag{13}$$

and therefore $\mu' = 0$ (see Assumption 2). Hence, the first and second equation generate $\nu', \tilde{\nu}' = 0$ and we have a contradiction with the last expression of (12). Consequently $\tau \neq 0$ and (11) trivially follows if one takes $\mu = \mu'/\tau$, $\nu = \nu'/\tau$, and $\tilde{\nu} = \tilde{\nu}'/\tau$. ■

THEOREM 1. *The following assertions are equivalent.*

- (a) *The market is arbitrage free.*
- (b) *If $f \equiv 0$ then problem (10) attains a global optimal value at $(x, y) = (0, 0)$.⁵*
- (c) *There is a stochastic discount factor (also called state price, see [9]) of the market, i.e., there exists $\mu \in \mathcal{M}_+(K)$ such that*

$$p_j \leq \int_K W_j(k) d\mu(k) \quad \text{and} \quad \int_K V_j(k) d\mu(k) \leq q_j, \tag{14}$$

$$j = 1, 2, \dots, n.$$

PROOF.

(a) \implies (b) Suppose that the market is arbitrage free. Then according to Definition 1, $\Lambda_0(x, y) \geq 0$ holds for every (10)-feasible $(x, y) \in \mathbb{R}^{2n}$. Hence, (b) trivially follows from $\Lambda_0(0, 0) = 0$.

(b) \implies (c) If $(0, 0)$ solves (10) then it must be a solution of system (11) with $f = 0$. Taking $x = 0$, $y = 0$, and bearing in mind Assumption 1 and the multipliers sign we get (14).

(c) \implies (a) Assume (c) and suppose that $\Lambda_T(x, y, k) \geq 0$ for every $k \in K$. We have that

$$\Lambda_0(x, y) \geq \sum_{j=1}^n x_j q_j - \sum_{j=1}^n y_j p_j. \tag{15}$$

Since $x_j \geq 0$ and $y_j \geq 0$, $j = 1, 2, \dots, n$, (14) implies that

$$\Lambda_0(x, y) \geq \sum_{j=1}^n x_j \int_K V_j(k) d\mu(k) - \sum_{j=1}^n y_j \int_K W_j(k) d\mu(k) = \int_K \Lambda_T(x, y, k) d\mu(k) \geq 0. \tag{16a}$$

Therefore, the market is arbitrage free. ■

ASSUMPTION 3. *As usual in finance, hereafter we will assume that the market is arbitrage free.* ■

⁵Throughout this paper we will not consider local solutions of optimization problems. On the contrary, our solutions will be always global, although uniqueness will never be assumed or imposed. Accordingly, Statement (b) above means that $(0, 0)$ belongs to the set of global solutions of (10) though this set could be more than a singleton.

4. SHADOW PRICES AND STRONG-PS-ARBITRAGE

This section will yield several characterizations of the absence of sps-arbitrage, as well as those methods that will permit us to compute sps-arbitrage portfolios (if available) and to improve the real market quotes.

Let (J_1, J_2, J_3) be a partition of $\{1, 2, \dots, n\}$ and $\{z_j; j \in J_1 \cup J_2\}$ a set of positive real numbers. Take $(\pi_j)_{j \in J_1}$, $(\rho_j)_{j \in J_2}$, $(x_j)_{j \in J_3}$, and $(y_j)_{j \in J_3}$ as decision variables and consider the optimization problem

$$\begin{aligned} \max \quad & \sum_{j \in J_1} (z_j \pi_j) - \sum_{j \in J_2} (z_j \rho_j) \\ & \sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) + \sum_{j \in J_3} x_j V_j(k) - \sum_{j \in J_3} y_j W_j(k) \geq 0, \quad k \in K, \\ \varphi((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3})) + \sum_{j \in J_1} \pi_j z_j - \sum_{j \in J_2} \rho_j z_j + \sum_{j \in J_3} (q_j x_j - p_j y_j) \leq 0, \\ & x_j \geq 0, \quad y_j \geq 0, \quad j \in J_3. \end{aligned} \quad (17)$$

LEMMA 2. Let (J_1, J_2, J_3) be a partition of $\{1, 2, \dots, n\}$ and $\{z_j; j \in J_1 \cup J_2\}$ a set of strictly positive real numbers. Define

$$\Lambda_0^*(J_1, J_2, z)$$

as a maximum between the supremum of (17) and

$$\sum_{j \in J_1} z_j p_j - \sum_{j \in J_2} z_j q_j. \quad (18)$$

Then,

$$(a) \quad \sum_{j \in J_1} z_j p_j - \sum_{j \in J_2} z_j q_j \leq \Lambda_0^*(J_1, J_2, z) \leq \sum_{j \in J_1} (z_j q_j) - \sum_{j \in J_2} (z_j p_j). \quad (19)$$

(b) If $z_j \geq 1$, $j \in J_1 \cup J_2$, then there are ps-arbitrage portfolios associated with (J_1, J_2, z) if and only if

$$\Lambda_0^*(J_1, J_2, z) > \sum_{j \in J_1} (z_j p_j) - \sum_{j \in J_2} (z_j q_j). \quad (20)$$

PROOF.

(a) Suppose that $((\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2}, (x_j)_{j \in J_3}, (y_j)_{j \in J_3})$ is (17)-feasible. The first constraint of (17) leads to

$$\sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) \geq \sum_{j \in J_3} y_j W_j(k) - \sum_{j \in J_3} x_j V_j(k). \quad (21)$$

Theorem 1 and Assumption 3 guarantee the existence of μ , state price of the market. Thus, computing the integral of expression above, we get

$$\sum_{j \in J_1} z_j q_j - \sum_{j \in J_2} z_j p_j \geq \sum_{j \in J_3} y_j p_j - \sum_{j \in J_3} x_j q_j. \quad (22)$$

Besides, the second constraint of (17) and $\varphi \geq 0$ (see Assumption 1) imply that

$$\sum_{j \in J_3} y_j p_j - \sum_{j \in J_3} x_j q_j \geq \sum_{j \in J_1} \pi_j z_j - \sum_{j \in J_2} \rho_j z_j$$

and the conclusion trivially follows.

- (b) Suppose that there are ps-arbitrage portfolios. Then there exists a (17)-feasible solution $((p_j)_{j \in J_1}, (q_j)_{j \in J_2}, (x_j)_{j \in J_3}, (y_j)_{j \in J_3})$. Moreover, the second constraint holds in terms of strict inequality and therefore $(p_j)_{j \in J_1}$ (respectively, $(q_j)_{j \in J_2}$) may slightly increase (decrease) without making the restrictions fail.

Conversely, if $((\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2}, (x_j)_{j \in J_3}, (y_j)_{j \in J_3})$ is (17)-feasible and such that

$$\sum_{j \in J_1} (z_j \pi_j) - \sum_{j \in J_2} (z_j \rho_j) > \sum_{j \in J_1} (z_j p_j) - \sum_{j \in J_2} (z_j q_j) \quad (23)$$

then it is obvious that

$$\left((z_{J_1}, 0, (x_j)_{j \in J_3}) \left(0, z_{J_2}, (y_j)_{j \in J_3} \right) \right)$$

is a ps-arbitrage portfolio. ■

Problem (17) becomes easier if we deal with sps-arbitrage portfolios. So, fix $j_0 \in J$ and $z_{j_0} > 0$, take π_{j_0} (respectively, ρ_{j_0}) $(x_j)_{j \neq j_0}$ and $(y_j)_{j \neq j_0}$ as decision variables, and consider the nonnecessarily convex problems

$\max \pi_{j_0}$

$$\begin{aligned} z_{j_0} V_{j_0}(k) + \sum_{j \neq j_0} x_j V_j(k) - \sum_{j \neq j_0} y_j W_j(k) &\geq 0, & k \in K, \\ \varphi \left((z_{j_0}, (x_j)_{j \neq j_0}) \left(0, (y_j)_{j \neq j_0} \right) \right) + z_{j_0} \pi_{j_0} + \sum_{j \neq j_0} (q_j x_j - p_j y_j) &\leq 0, \\ x_j \geq 0, \quad y_j \geq 0, & \quad j \neq j_0, \end{aligned} \quad (24)$$

and

$\min \rho_{j_0}$

$$\begin{aligned} -z_{j_0} W_{j_0}(k) + \sum_{j \neq j_0} x_j V_j(k) - \sum_{j \neq j_0} y_j W_j(k) &\geq 0, & k \in K, \\ \varphi \left((0, (x_j)_{j \neq j_0}) \left(z_{j_0}, (y_j)_{j \neq j_0} \right) \right) - z_{j_0} \rho_{j_0} + \sum_{j \neq j_0} (q_j x_j - p_j y_j) &\leq 0, \\ x_j \geq 0, \quad y_j \geq 0, & \quad j \neq j_0. \end{aligned} \quad (25)$$

LEMMA 3. Fix $j_0 \in \{1, 2, \dots, n\}$. Given $z_{j_0} > 0$, define $p_{j_0}^+(z_{j_0})$ (respectively, $q_{j_0}^+(z_{j_0})$) as a maximum (respectively, minimum) between the supremum (respectively, infimum) of (24) (respectively, (25)) and p_{j_0} (respectively, q_{j_0}). Then

$$p_{j_0} \leq p_{j_0}^+(z_{j_0}) \leq q_{j_0}^+(z'_{j_0}) \leq q_{j_0} \quad (26)$$

holds for every $z_{j_0}, z'_{j_0} > 0$.

PROOF. The result trivially follows from Lemma 2 if (24) or (25) is infeasible, so let us assume that they are feasible. Lemma 2 also shows that $p_{j_0} \leq p_{j_0}^+(z_{j_0}) \leq q_{j_0}$ and $p_{j_0} \leq q_{j_0}^+(z'_{j_0}) \leq q_{j_0}$, so let us prove

$$p_{j_0}^+(z_{j_0}) \leq q_{j_0}^+(z'_{j_0}). \quad (27)$$

It is clearly sufficient to see that

$$\pi_{j_0} \leq \rho_{j_0} \quad (28)$$

whenever $(\pi_{j_0}, (x_j)_{j \neq j_0}, (y_j)_{j \neq j_0})$ is (24)-feasible and $(\rho_{j_0}, (x'_j)_{j \neq j_0}, (y'_j)_{j \neq j_0})$ is (25)-feasible.⁶ Theorem 1 and Assumption 3 point out that the set of state prices verifying (14) is nonempty. Thus, take the state price μ and computing integrals on the first constraint of (24) one has

$$z_{j_0} \int_k V_{j_0} d\mu + \sum_{j \neq j_0} q_j x_j - \sum_{j \neq j_0} p_j y_j \geq 0. \quad (29)$$

⁶Note that z_{j_0} must be substituted by z'_{j_0} in (25).

Besides, the second constraint of (24) implies that

$$-\left(\sum_{j \neq j_0} q_j x_j - \sum_{j \neq j_0} p_j y_j\right) \geq z_{j_0} \pi_{j_0}. \quad (30)$$

Therefore $\int_k V_{j_0} d\mu \geq \pi_{j_0}$. Analogously $\int_k W_{j_0} d\mu \leq \rho_{j_0}$ and the proof is complete. \blacksquare

Let us introduce another basic concept in order to study the existence of sps-arbitrage portfolios.

DEFINITION 4. *The shadow bid price of B_j is defined by*

$$p_j^+ = \sup \{p_j^+(z_j); z_j \geq 1\}, \quad (31)$$

$j = 1, 2, \dots, n$. Analogously, shadow ask prices are given by

$$q_j^+ = \inf \{q_j^+(z_j); z_j \geq 1\}, \quad (32)$$

$j = 1, 2, \dots, n$. \blacksquare

Lemma 3 points out that $p_j \leq p_j^+ \leq q_j^+ \leq q_j$ hold for $j = 1, 2, \dots, n$. Moreover if we take the ideal assumptions $\varphi \equiv 0$ and $W_j \equiv V_j$, $j = 1, 2, \dots, n$, and compute the ideal shadow prices p^* and q^* we clearly have

$$p_j \leq p_j^+ \leq p_j^* \leq q_j^* \leq q_j^+ \leq q_j \quad (33)$$

for $j = 1, 2, \dots, n$. As will be pointed out in Remark 2, inequalities above provide useful upper and lower bounds for those price improvements linked to sps-arbitrage strategies.

THEOREM 4. *The following statements are equivalent:*

- (a) *the market is bps-arbitrage free,*
- (b) *the equality $p_j = p_j^+(z_j)$ holds for $j = 1, 2, \dots, n$ and for every $z_j \geq 1$,*
- (c) *the equality $p_j = p_j^+$ holds for $j = 1, 2, \dots, n$.*

Analogously one has the equivalence amongst:

- (d) *the market is asp-arbitrage free,*
- (e) *The equality $q_j = q_j^+(z_j)$ holds for $j = 1, 2, \dots, n$ and for every $z_j \geq 1$,*
- (f) *the equality $q_j = q_j^+$ holds for $j = 1, 2, \dots, n$.*

PROOF. It trivially follows from Lemmas 2 and 3. \blacksquare

Shadow prices may be understood as those prices attainable by means of sps-arbitrage methods. A natural question arises if some investor improves a real quote from p_j (or q_j) to $\bar{p}_j \in (p_j, p_j^+)$ (or to $\bar{q}_j \in (q_j^+, q_j)$). Indeed, does the improvement modify (improve) the value of any other shadow price? The remainder of this section is devoted to provide an appropriate answer (see Theorem 6 and Remark 2 below), as well as some complementary properties related to the sps-arbitrage.

Throughout the rest of this section we will impose the following.

ASSUMPTION 4. *Inequality $V_2(k) > 0$ holds for every $k \in K$.* \blacksquare

LEMMA 5. *Suppose that $j_0 \in \{1, 2, \dots, n\}$ and $z_{j_0} > 0$. If problem (25) (respectively, (24)) is solvable then it verifies the Kuhn-Tucker conditions, and the multiplier associated with the second constraint is $\lambda = 1/z_{j_0}$.*

PROOF. Both proofs are similar so let us deal with problem (25). According to the Fritz-John theorem (see [20]) there exists a nonnull and nonnegative

$$(\tau, \lambda', \mu', \nu', \bar{\nu}') \in \mathbb{R} \times \mathbb{R} \times \mathcal{M}(K) \times \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\begin{aligned} \tau - \lambda' z_{j_0} &= 0, \\ - \int_K V_j(k) d\mu'(k) + \lambda' \frac{\partial \varphi}{\partial x_j} + \lambda' q_j - \nu'_j &= 0, \quad j \neq j_0, \\ \int_K W_j(k) d\mu'(k) + \lambda' \frac{\partial \varphi}{\partial y_j} - \lambda' p_j - \bar{\nu}'_j &= 0, \quad j \neq j_0, \end{aligned} \quad (34)$$

along with the complementary slackness conditions. If $\tau = 0$ the first equation gives $\lambda' = 0$, whence the second equation leads to $\int_K V_j(k) d\mu'(k) \leq 0$ if $j \neq j_0$. Thus, if $j_0 \neq 1$ we have $\int_K V_1(k) d\mu'(k) \leq 0$ and, otherwise, $\int_K V_2(k) d\mu'(k) \leq 0$. In both cases $\mu' = 0$. Now the system above gives $\nu' = \bar{\nu} = 0$ and we have the contradiction $(\tau, \lambda', \mu', \nu', \bar{\nu}') = 0$. Since $\tau \neq 0$ the lemma trivially follows if one takes $(\lambda, \mu, \nu, \bar{\nu}) = (1/\tau)(\lambda', \mu', \nu', \bar{\nu}')$. \blacksquare

THEOREM 6. *Let $I_1, I_2 \subset J$ be such that $p_j < p_j^+$ if $j \in I_1$ and $q_j > q_j^+$ if $j \in I_2$. Consider $\bar{p}_j \in [p_j, p_j^+)$ whenever $j \in I_1$ and $\bar{q}_j \in (q_j^+, q_j]$ whenever $j \in I_2$. Fix $j_0 \notin I_1 \cup I_2$, $z_{j_0} \geq 1$, and suppose that (25) (respectively, (24)) achieves its optimal value at $(Q, (x_j)_{j \neq j_0}, (y_j)_{j \neq j_0})$ (respectively, $(P, (x'_j)_{j \neq j_0}, (y'_j)_{j \neq j_0})$) when the bid price of B_j is \bar{p}_j for $j \in I_1$ and its ask price is \bar{q}_j for $j \in I_2$. Then,*

$$\frac{\partial Q}{\partial \bar{q}_j} = \frac{x_j}{z_{j_0}}, \quad \frac{\partial P}{\partial \bar{q}_j} = -\frac{x'_j}{z_{j_0}} \quad (35)$$

if $j \in I_2$, and

$$\frac{\partial Q}{\partial \bar{p}_j} = -\frac{y_j}{z_{j_0}}, \quad \frac{\partial P}{\partial \bar{p}_j} = \frac{y'_j}{z_{j_0}} \quad (36)$$

if $j \in I_1$.

PROOF. Both P and Q are similar so let us analyze the partial derivatives of Q . The latter lemma guarantees that the envelope theorem applies (see for instance [21] or [22], where the result is also called ‘‘envelopent theorem’’). Accordingly, $\frac{\partial Q}{\partial \bar{p}_j} = \frac{\partial \mathcal{L}_2}{\partial \bar{p}_j}$, $j \in I_1$, and $\frac{\partial Q}{\partial \bar{q}_j} = \frac{\partial \mathcal{L}_2}{\partial \bar{q}_j}$, $j \in I_2$, where

$$\begin{aligned} & \mathcal{L}_2 \left(\rho_{j_0}, (x_j)_{j \neq j_0}, (y_j)_{j \neq j_0}, \mu, \lambda, (\nu_j)_{j \neq j_0}, (\bar{\nu}_j)_{j \neq j_0} \right) \\ &= \rho_{j_0} + \int_K W_{j_0}(k) d\mu(k) - \sum_{j \neq j_0} x_j \int_K V_j(k) d\mu(k) + \sum_{j \neq j_0} y_j \int_K W_j(k) d\mu(k) \\ & \quad + \lambda \left(\varphi - z_{j_0} \rho_{j_0} + \sum_{j \neq j_0} (\bar{q}_j x_j - \bar{p}_j y_j) \right) - \sum_{j \neq j_0} \nu_j x_j - \sum_{j \neq j_0} \bar{\nu}_j y_j \end{aligned}$$

is the Lagrangian function of (25). Hence the latter lemma implies that $\frac{\partial Q}{\partial \bar{q}_j} = \lambda x_j = x_j/z_{j_0}$ for $j \in I_2$ and $\frac{\partial Q}{\partial \bar{p}_j} = -\lambda y_j = -y_j/z_{j_0}$ for $j \in I_1$. \blacksquare

REMARK 2. Theorem 6 reflects that the partial derivatives of P and Q have the adequate sign, in the sense that improvements on market quotes will never have a negative influence on the rest of shadow prices. The importance of this effect increases as so does the absolute value of the derivatives. In particular, if they do not vanish then improvements on market quotes will also imply improvements on shadow prices. This is a very important difference with respect to those situations for which frictions are linear. Indeed, a result in [14] proves that improvements on real quotes do not modify the remainder shadow prices if $\varphi \equiv 0$ and the final transaction costs also vanish ($W_j - V_j \equiv 0$, $j = 1, 2, \dots, n$).

Ideas of the paragraph above provoke a new question. In fact, shadow prices provide those quotes that may be achieved by sps-arbitrage methods in the first step but, once a given quote has been improved, which are the new shadow prices? Do they remain constant? Are they significantly better? Of course the value of the derivatives of Theorem 6 yields an important

answer, though it is not complete. Anyway, we can obtain additional and useful (incomplete) information by computing the initial shadow prices p^* and q^* of the market under the ideal assumptions $\varphi \equiv 0$ and $W_j \equiv V_j$, $j = 1, 2, \dots, n$. According to [14] p^* and q^* remain constant after improvements and therefore they provide bounds for those shadow prices that may be reached after several steps and correspond to the real φ and W_j , $j = 1, 2, \dots, n$ (see (33)). ■

5. EFFICIENCY, WEAK-PS-ARBITRAGE AND COALITIONS

Weak-ps-arbitrage strategies may be also an appropriate tool in order to outperform real market quotes, though in this case several assets are simultaneously involved in the new offer. So, the purpose of this section is to analyze the absence or presence of wps-arbitrage portfolios and those methods permitting us to compute them in practice.

First of all it is worth pointing out the close relationship between the existence of inefficient portfolios (see Definition 3) and the existence of ps-arbitrage. Furthermore we can provide a necessary condition (see Statement (d) below) for a portfolio to be efficient.

THEOREM 7. *Let (J_1, J_2, J_3) be a partition of $\{1, 2, \dots, n\}$ and $z = (z_j)_{j \in J_1 \cup J_2}$ a family of strictly positive real numbers. Consider the portfolio $(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2n}$ given by*

$$\begin{aligned}\tilde{x}_{J_1} &= \tilde{x}_{J_3} = 0, \\ \tilde{x}_{J_2} &= z_{J_2}, \\ \tilde{y}_{J_1} &= z_{J_1}, \\ \tilde{y}_{J_2} &= \tilde{y}_{J_3} = 0.\end{aligned}$$

Then, statements below verify the implications (a) \implies (b) and (a) \implies (c) \implies (d).

- (a) Portfolio (\tilde{x}, \tilde{y}) is efficient.
- (b) If $z_j \geq 1$, $j \in J_1 \cup J_2$, then there are no ps-arbitrage portfolios (x, y) associated with (J_1, J_2, z) .
- (c) Portfolio (\tilde{x}, \tilde{y}) solves problem (10) for $f = \sum_{j \in J_2} z_j V_j - \sum_{j \in J_1} z_j W_j$.⁸
- (d) There exists $\mu \in \mathcal{M}_+(K)$ such that (\tilde{x}, \tilde{y}) solves the system

$$\begin{aligned}\frac{\partial \varphi}{\partial x_j} + q_j &= \int_K V_j(k) d\mu(k), & j \in J_2, \\ \frac{\partial \varphi}{\partial x_j} + q_j &\geq \int_K V_j(k) d\mu(k), & j \in J_1 \cup J_3, \\ -\frac{\partial \varphi}{\partial y_j} + p_j &= \int_K W_j(k) d\mu(k), & j \in J_1, \\ -\frac{\partial \varphi}{\partial x_j} + p_j &\leq \int_K W_j(k) d\mu(k), & j \in J_2 \cup J_3.\end{aligned}\tag{37}$$

PROOF.

- (a) \implies (b) Suppose that (x, y) exists. One has that

$$\varphi(x, y) + \sum_{j \in J_1} p_j z_j - \sum_{j \in J_2} q_j z_j + \sum_{j \in J_3} (q_j x_j - p_j y_j) < 0,\tag{38}$$

from where

$$\begin{aligned}\Lambda_0(\tilde{x}, \tilde{y}) &\geq \sum_{j \in J_2} q_j z_j - \sum_{j \in J_1} p_j z_j \\ &> \varphi(x, y) + \sum_{j \in J_3} (q_j x_j - p_j y_j) \\ &\geq \varphi(x', y') + \sum_{j \in J_3} (q_j x_j - p_j y_j) = \Lambda_0(x', y'),\end{aligned}\tag{39}$$

⁸For example, (\tilde{x}, \tilde{y}) is a global solution of the problem, though uniqueness is not guaranteed and therefore more global solutions might exist.

(x', y') being the portfolio $x'_{J_1} = x'_{J_2} = y'_{J_1} = y'_{J_2} = 0$, $x'_{J_3} = x_{J_3}$ and $y'_{J_3} = y_{J_3}$. Besides,

$$\sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) + \sum_{j \in J_3} (x_j V_j(k) - y_j W_j(k)) \geq 0, \quad (40)$$

for every $k \in K$, from where

$$\begin{aligned} \Lambda_T(\tilde{x}, \tilde{y}, k) &= \sum_{j \in J_2} z_j V_j(k) - \sum_{j \in J_1} z_j W_j(k) \\ &\leq \sum_{j \in J_2} z_j W_j(k) - \sum_{j \in J_1} z_j V_j(k) \\ &\leq \sum_{j \in J_3} (x_j V_j(k) - y_j W_j(k)) = \Lambda_T(x', y', k) \end{aligned} \quad (41)$$

for every $k \in K$. Expressions (39) and (41) clearly contradict (a).

(a) \implies (c) It trivially follows from the definition of efficient portfolio.

(c) \implies (d) System (11) becomes

$$\begin{aligned} \frac{\partial \varphi}{\partial x_j} + q_j - \int_K V_j(k) d\mu(k) - \nu_j &= 0, \quad j = 1, 2, \dots, n, \\ \frac{\partial \varphi}{\partial y_j} - p_j + \int_K W_j(k) d\mu(k) - \tilde{\nu}_j &= 0, \quad j = 1, 2, \dots, n, \\ \nu_j = 0, \quad j \in J_2, \quad \tilde{\nu}_j = 0, \quad j \in J_1, \quad x, y, \tau, \mu, \nu, \tilde{\nu} &\geq 0, \end{aligned} \quad (42)$$

and we immediately have (d). ■

The interest of Theorem 7 above seems to be clear. As shown in Example 1 there may exist inefficient (dominated) portfolios even for arbitrage-free and sps-arbitrage-free markets, so it is important to have practical rules detecting that. In this sense, system (37) gives a necessary condition that may be useful to traders. Besides, when testing in practice the existence of ps-arbitrage we can also check (37) due to the implication (a) \implies (b).

Despite the comments above we also need new direct criteria indicating the presence of ps-arbitrage, in the line of those ones reached in Section 4 for sps-arbitrage. Thus, let us use the same notations as in (17), suppose that $z_j \geq 1$, $j \in J_1 \cup J_2$, and consider the vector problem

$$\begin{aligned} \max (\pi_j)_{j \in J_1}, \quad \min (\rho_j)_{j \in J_2}, \\ \sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) + \sum_{j \in J_3} x_j V_j(k) - \sum_{j \in J_3} y_j W_j(k) &\geq 0, \quad k \in K, \\ \varphi((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3})) + \sum_{j \in J_1} \pi_j z_j - \sum_{j \in J_2} \rho_j z_j + \sum_{j \in J_3} (q_j x_j - p_j y_j) &\leq 0, \\ \pi_j \leq q_j, & \quad j \in J_1, \\ \rho_j \geq p_j, & \quad j \in J_2, \\ x_j \geq 0, \quad y_j \geq 0, & \quad j \in J_3. \end{aligned} \quad (43)$$

In order to guarantee that problem (43) is feasible it is sufficient to show that so is problem (17). It may be ensured if there are ps-arbitrage strategies associated with (J_1, J_2, z) , which, according to Lemma 2b, will hold if henceforth we impose the following.

ASSUMPTION 5. $\Lambda_0^*(J_1, J_2, z) > \sum_{j \in J_1} (z_j p_j) - \sum_{j \in J_2} (z_j q_j)$. ■

Problem (43) will play the role played by (24) and (25) when dealing with sps-arbitrage strategies. In particular, every optimal value $((\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2})$ of (43) will be considered an attainable market quote. This is the major reason why we have added the third and fourth constraints since

they cannot be derived and their failure would imply that we would not be outperforming the whole set of market prices $((q_j)_{j \in J_1}, (p_j)_{j \in J_2})$ (see Lemma 2a). Hence, if $\pi_j > q_j$ for some $j \in J_1$ (respectively, $\rho_j < p_j$ for some $j \in J_2$) then agents trying to implement ps-arbitrage would not need to bid asset B_j (respectively, ask asset B_j) because they can reach a better price in the market. Consequently they would reduce J_1 (respectively, J_2) making it easier to apply ps-arbitrage in practice.

We will address (43) by drawing on the “balance space approach” [15] and, consequently, we have to compute the vector of partial optimal values

$$\left((p_j^+(z))_{j \in J_1}, (q_j^+(z))_{j \in J_2} \right) \quad (44)$$

by maximizing π_j for each $j \in J_1$ (respectively, minimizing ρ_j for each $j \in J_2$) under the constraints of (43). If (as usual) (44) is not an attainable objective value (i.e., if problem (43) is not balanced) then we choose and fix a vector of preferential deviations $d = (d_j)_{j \in J_1 \cup J_2}$ composed of strictly positive real numbers. Vector d indicates the ratio $d_j/d_{j'}$ of “losses” in the j^{th} -objective per unit lost in the j'^{th} -one (see [15] for a further discussion). We will yield the closest to (44) Pareto solution of (43) proportional to d . The mentioned Pareto solution will take the form

$$\left((\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2} \right) = \left((p_j^+(z))_{j \in J_1} - \gamma_0 (d_j)_{j \in J_1}, (q_j^+(z))_{j \in J_2} + \gamma_0 (d_j)_{j \in J_2} \right), \quad (45)$$

for some $\gamma_0 > 0$. In order to ensure that the Pareto solution (45) exists we will impose the following.

ASSUMPTION 6. *There exists a balance point of (43) in the direction of d .* ■

A possible way to detect the balance point in the direction of preferential deviations d (or the Pareto solution (45)) is to solve (see [21] or [22] for further details)

$$\begin{aligned} \min \gamma \\ & \sum_{j \in J_1} z_j V_j(k) - \sum_{j \in J_2} z_j W_j(k) + \sum_{j \in J_3} x_j V_j(k) - \sum_{j \in J_3} y_j W_j(k) \geq 0, & k \in K, \\ & \varphi((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3})) + \sum_{j \in J_1} \pi_j z_j - \sum_{j \in J_2} \rho_j z_j + \sum_{j \in J_3} (q_j x_j - p_j y_j) \leq 0, & (46) \\ & \pi_j + \gamma d_j \geq p_j^+(z), & j \in J_1, \\ & \rho_j - \gamma d_j \leq q_j^+(z), \quad j \in J_2, \quad \gamma \geq 0, \quad x_j \geq 0, \quad y_j \geq 0, \quad j \in J_3, \end{aligned}$$

where $\gamma \in \mathbb{R}$, $(x_j)_{j \in J_3}$, $(y_j)_{j \in J_3}$, $(\pi_j)_{j \in J_1}$, and $(\rho_j)_{j \in J_2}$ are the decision variables. Indeed, the solution of (46) provides the solution of (43) (i.e., the ps-arbitrage portfolio and the “optimal market quotes”) and the value γ_0 of (45).

Notice that $\pi_j \leq q_j$, $j \in J_1$ and $\rho_j \geq p_j$, $j \in J_2$ do not have to be imposed in (46) since they will hold for any optimal solution. Indeed, they trivially follow from (45), $\gamma_0 \geq 0$, $p_j^+(z) \leq q_j$ for $j \in J_1$ and $q_j^+(z) \geq p_j$ for $j \in J_2$.

Ideas above may be summarized as follows.

THEOREM 8. *Under Assumptions 5 and 6 there are ps-arbitrage strategies associated with (J_1, J_2, z) . Furthermore, if $(\gamma_0, (\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2}, x_{J_3}, y_{J_3})$ solves (46) then the following assertions are fulfilled.*

- (a) Portfolio $(x, y) = ((z_{J_1}, 0, x_{J_3})(0, z_{J_2}, y_{J_3}))$ is a ps-arbitrage strategy associated with (J_1, J_2, z) and $((\pi_j)_{j \in J_1}, (\rho_j)_{j \in J_2})$ is an attainable market quote.
- (b) Expression (45) holds. ■

REMARK 3. Theorems 7 and 8 point out several interesting and practical properties.

First of all let us remark that Example 1 illustrates how inefficient portfolios may be generated by adding efficient ones. Indeed, the example has presented a wps-arbitrage strategy in an sps-arbitrage free model. As shown in [14], under linear frictions the existence of ps-arbitrage is equivalent to the existence of inefficiencies (in other words, (b) \Rightarrow (a) holds in Theorem 7) so the sps-arbitrage absence permits us to conclude that portfolios with a single security are efficient, while portfolio $B_2 + B_4$ is inefficient.

The example above suggests the interest of possible coalitions among traders. So for instance, two investors interested in B_2 and B_4 , respectively, could accept a coalition and buy the efficient portfolio $B_1 + B_3$. They would pay 1.3 dollars (less than 1.4, price of $B_2 + B_4$) and would achieve the similar payoff (1, 1, 2). Consequently, the coalition could assist them to outperform the market ask quotes 0.9 and 0.5 of B_2 and B_4 .

The example also illustrates how brokers can use orders of their clients to provide them with better prices. So, a broker with two orders to purchase B_2 and B_4 , respectively, could better buy $B_1 + B_3$ and make her/his clients improve the performance.

In general, brokers or coalitions of traders can outperform market quotes by drawing on the ideas of Theorems 7 and 8. They can add several partial portfolios and use Theorem 7 or system (37) to check the efficiency of the whole strategy. If it were not efficient they could replace it by a better performing one. Moreover, if Theorem 7 reveals inefficiencies then brokers can use Theorem 8 with the appropriate value of z . Theorem 8 and problem (46) will show if ps-arbitrage strategies are available. If so, the broker can obtain additional riskless arbitrage earnings and simultaneously provide her/his clients with better quotes.

Finally, let us remark that once again the introduction of nonlinear or convex frictions implies important differences with regard to the linear case of [14]. For instance, the failure of (b) \Rightarrow (a) in Theorem 7 for nonconvex costs may merit special attention. ■

REMARK 4. As we did when dealing with sps-arbitrage strategies it will be useful to analyze the effects of ps-arbitrage-linked price improvements on the remainder ps-arbitrage-linked achievable prices. It may be done by drawing on a procedure quite parallel to that used when proving Theorem 6. In fact, these effects may be analyzed by sensitivities that may be measured by applying a general envelope (or envolvent) theorem for balance points that was established in [22]. As in the proofs of Remark 1 and Lemma 5, one has to provide conditions guaranteeing that both problem (46) and those scalar problems leading to the vector of partial optima satisfy the Kuhn-Tucker theorem. Then, those sensitivities we are interested in will be easily measured and obtained by straightforward applications of the results in the above-mentioned paper. ■

6. CONCLUSIONS

The paper has focused on imperfect markets with quite general nonconvex transaction costs and has provided new methods to overcome many inefficiencies caused by frictions. By using scalar optimization methods we have characterized the absence of arbitrage and sps-arbitrage and have yielded necessary conditions for a portfolio to be efficient. By using multiobjective optimization methods, with special focus on the balance space approach, we have studied the existence and properties of wps-arbitrage strategies.

The presence of general frictions significantly broadens the set of possible applications. Furthermore, important differences have been pointed out when compared with the more restricted case of linear or convex imperfections.

We have illustrated how to use ps-arbitrage in order to outperform the revealed market quotes. The improvement of prices and spreads also increases the level of the market efficiency and reduces illiquidity. Furthermore, the analysis could be also very interesting to practitioners since the proposed optimization techniques may allow them to compose hedged or efficient portfolios and to obtain riskless additional earnings.

All these findings may be especially useful for many emerging and still illiquid spot or derivative markets (electricity markets, commodity markets, markets related to weather, inflation-linked or insurance-linked derivatives, etc.).

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