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Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

NETWORKS AND COLLECTIVE ACTION

Ramón Flores^a, Maurice Koster^b, Ines Lindner^c and Elisenda Molina^a

Abstract

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Keywords: Collective action, Social networks, Influence and Diffusion models

^a Departamento de Estadística, Universidad Carlos III de Madrid, Spain. e-mail addresses: ramonjesus.flores@uc3m.es and elisenda.molina@uc3m.es

^b Department of Quantitative Economics, University of Amsterdam, The Netherlands. e-mail address: M.A.L.Koster@uva.nl

^c Department of Econometrics & OR, Free University, The Netherlands. e-mail address: ilindner@feweb.vu.nl

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Ramón Flores

Department of Statistics, Universidad Carlos III de Madrid, Spain

e-mail: ramonjesus.flores@uc3m.es

Maurice Koster

Department of Quantitative Economics, University of Amsterdam, The Netherlands

e-mail: M.A.L.Koster@uva.nl

Ines Lindner

Department of Econometrics & OR, Free University , The Netherlands

e-mail: ilindner@feweb.vu.nl

Elisenda Molina

Department of Statistics, University Carlos III de Madrid, Spain

e-mail: elisenda.molina@uc3m.es

Abstract

Given a social network, we are interested in the problem of measuring the influence of a group of agents to lead the society to adopt their behavior. Motivated by the description of terrorist movements, we provide a markovian dynamical model for non-symmetric societies, which takes into account two special features: the hard core terrorist group cannot be influenced, and the remaining agents may change from active to non-active and vice versa during the process. In this setting, we interpret the absorption time of the model, which measures how quickly the terrorist movement achieve the support of all society, as a group measure of power. In some sense, our model generalizes the classical approach of DeGroot to consensus formation.

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1 Introduction

The purpose of this paper is to provide a model to determine the influence of a group of individuals to prompt a given society to adopt their position. Which individuals are able to prompt the adoption of a particular action alternative?, What kinds of ties are most important for collective action and what features are especially relevant?, How quickly the new *collective behavior* arises?, Can we measure the extent to which any individual or group of individuals is able to become a *opinion leaders group* that controls the collective outcome? This kind of questions arise in many different areas, such as, sociology (e.g. [22], [23], [30]), epidemiology (e.g. [16], [17]), economics (e.g. [2], [7], [21], [31], [34], [39],[41], [44], [45]), social choice (e.g. [9],[10],

[43]), computer sciences (e.g. [18], [27], [28], [33]) and systems reliability (e.g. [1]) covering the analysis of riot behavior, innovation and rumor diffusion, propaganda, strikes, consumption, network externalities, spread of fashions, migration, runs on banks, voting power, cascading failures in power systems, etc. With this information, a social planner, a political party, or a firm's management may be able to actively address the most powerful individuals and thus help to prevent or to stimulate action in a given context¹.

In order to answer those questions it is crucial to understand the role that the social structure plays in the sharing of information and the formation of opinions. Therefore, first of all, we need a model to describe how beliefs and behaviors of a society evolve over time, taking into account that their members are connected through a social network which is the primary conduit of information, opinions, and behaviors. Various models have been proposed to describe that evolution:

- *Decreasing cascade models*, proposed by Kempe, Kleinberg and Tardos [27, 2003], [28, 2005], in which a behavior spreads in a cascading fashion according to a probabilistic rule, beginning with a set of *initially active* nodes. These models generalize the economic models of diffusion based on *thresholds*, where individuals are assumed to have different *thresholds* that determine whether they will adopt as a function of the number (or proportion) of others in the population who have already adopted². In this context, Kempe *et al.* study which they called the *target set selection* problem (to choose a set of individuals to target such that the cascade is as large as possible in expectation), and show that a natural greedy algorithm is a good approximation in this case.
- *Interactive Markov chains (symmetric) models*, in which the next state of an individual depend on his current state and on the current frequency distribution of the population among the states modeling the different positions. The dynamics of these models were first considered by Conlisk [11, 1976]³, where the concept of an *interactive Markov chain* as a framework for stochastic flows when the effects on the decisions of individuals of imitation, fashion, popularity, contagion, and so on cannot be ignored was introduced. The work of Stadje [38, 1997] is a deep generalization of the Conlisk model. However, although the symmetric case provides insight into broad patterns of social behavior, it does

¹See, for instance, Rogers [35, 1995], where the important role that opinion leaders play in the dissemination of information and their influence on opinions and decisions in marketing, social programs, education, campaigning, and more diffusion properties is studied.

²The dynamics of these models were first studied by Schelling [36, 1971], [37, 1978], Granovetter [23, 1978] and Granovetter and Soong [24, 1988]. For more recent work, see Valente [39, 1995], [40, 1996], [41, 2005], Macy [30, 1991], Watts [42, 2002], Dodds and Watts [16, 17, 2004, 2005], Lopez-Pintado and Watts [29, 2008], and Young [45, 2009].

³He also studies his original model in [12, 1978], [13, 1982], and [14, 1992].

not incorporate the micro-details of who interact with whom. To incorporate networked interactions, a richer non-symmetric structure is needed.

- *Interactive Markov chains (non-symmetric) models*, in which an individual's choice of actions may depend on arbitrary neighborhoods of others. This is precisely, the kind of dynamics we have considered. To be specific, we have adopted the *voter model*, also called the *invasion process*, which has been first introduced along with their dual process called *coalescing random walks* in Holley and Liggett [25, 1975], and also independently in Clifford and Sudbury [8, 1973]. Originally developed in the context of interacting stochastic processes, the voter model has been also considered as a model of social interaction, as for instance by Asavathiratham [1, 2000] and Even-Dar and Shaphira [18, 2007]. These authors are also interested in the target set selection problem, and they assume that the set of *initially active* nodes can be deactivated during the process, since they are also influenced by the remaining agents. What all the referred papers based on the voter model have in common is that they are able to answer their questions by means of analyzing a much tractable process, the Markov chain obtained when the sociomatrix describing the social structure is interpreted as a transition matrix. Ni, Xie and Liu [33, 2010] also consider an interactive Markov (non-symmetric) chain based on a monotonic modification of the voter model, which they called the *incremental chance model*.

We are interested here in the problem of target selection in a non-symmetric social dynamics where the individuals' decisions are influenced by those of their neighbors. To be specific, we are interested on analyzing the social dynamics when there is an *active hard core* set of individuals, which are not under the influence of the others. However, contrary to the usual cascade models and most of the innovation literature (see Young's comment [44, 2003] on page 5 about this fact), we do *not* assume monotonicity, so the the rest of individuals may change from active to non-active, and vice versa. Under these assumptions, we formulate a Markov model whose transition matrix is derived (but not the same as) the sociomatrix of the process, and which in this way relates the voter model with the seminal network interaction model of information transmission, opinion formation, and consensus formation of DeGroot [15, 1974], and therefore, also with eigenvector-based centrality measures (Katz [26, 1953], Bonacich [4, 1972], [5, 1987] and Brin and Page [6, 1998]). The reader is referred to section 3 for details. The target selection will be then undertaken by means of a power measure, which is defined as an absorption time in the Markov chain.

In this study, we were motivated by the analysis of a terrorist movement inside a society, but applies to a wide range of social networks.

The remainder of the paper is organized as follows. Section 2 is devoted to a general presentation of the dynamic model we have adopted to describe how beliefs and behaviors of a

society evolve over time. In Section 3 we introduce the measure (based on that model) that we propose to evaluate the extent to which any individual or group of individuals is able to control the collective outcome. In Section 4 monotonicity properties of the proposed measure (which we will refer to as the *power to initiate full action* of a group of agents) are analyzed.

2 The model: Influence as a Finite Markov Chain

As we have pointed out, our goal is to define an appropriate measure of the power that a subgroup of agents in a society has to initiate a certain action (such as to adopt an innovation, to join a revolutionary movement, to smoke, etc.). We base our study on a model of network influence in which the social structure of a society is described by a *sociomatrix*, which is the primary conduit of information. That is, individuals outweigh actions and opinions of others depending on the strength of a tie (Granovetter [22, 1973]) as indicated by the corresponding weight in the sociomatrix. We will interpret this weight as the probability that an actor is dominantly influenced by it. This assumption - in the spirit of a priori power - provides insight into the power implications of the network structure itself.

The Society: agents and their relations

The focus will be on a finite and fixed set $N = \{1, 2, \dots, n\}$ of agents who interact according to a social network. Relations between agents are exogenously given by a stochastic⁴ $n \times n$ matrix $\mathbf{W} = (w_{ij})_i^j$, the elements of which are understood as *influence parameters*; w_{ij} is the weight that agent i assigns to agent j . The matrix \mathbf{W} may be asymmetric, and the influences can be one-sided, so that $w_{ij} > 0$ while $w_{ji} = 0$. We refer to \mathbf{W} as the *sociomatrix*. In our setting, where agents are updating their probabilities of taking action by calculating weighted averages, w_{ij} is the weight that agent i assigns to the current action of agent j in forming his or her belief for the next period.

The Social Dynamics: updating process

Agents have two options, to take or not to take action. At each date, agents communicate with their neighbors in the social network and update their beliefs about taking or not taking action. The updating process is simple: an agent's new belief (which determines his/her probability of taking action) is the weighted average of his/her neighbors' actions from the previous period. In that setting, at each date, agents are of two kinds, passive or active, characteristics indicated by 0 or 1, respectively, and determined according to their beliefs at this date. Formally, consider the following definitions.

⁴A $n \times n$ matrix $\mathbf{Q} = (q_{ij})_i^j$ is *stochastic* if $q_{ij} \geq 0$ for all $i, j = 1, 2, \dots, n$, and $\sum_{j=1}^n q_{ij} = 1$ for all $i = 1, 2, \dots, n$.

A state of the society (N, \mathbf{W}) is a tuple $x^N \in \sigma(N) := \{0, 1\}^n$, $x_i^N \in \{0, 1\}$ being the state of agent i , $i = 1, \dots, n$, where n is the cardinality of N . Special states are $\mathbf{z}^N, \mathbf{e}^N \in \sigma(N)$, according to which respectively none, and everyone in N takes action. We will unambiguously identify $x^N \in \sigma(N)$ with a vector in \mathbb{R}^n . For $x^N \in \sigma(N)$, let $A(x^N) := \{i \in N \mid x_i^N = 1\}$ the set of active agents in N .

Definition 1. For each society (N, \mathbf{W}) , let $p : \sigma(N) \rightarrow [0, 1]^n$ be the mapping defined by

$$p(x^N) = \mathbf{W} \cdot x^N, \quad \text{for all } x^N \in \sigma(N)$$

Then for each state x^N of society (N, \mathbf{W}) the number $p_i(x^N)$ represents the probability that i takes action upon observing state x^N of (N, \mathbf{W}) . Now, we define $m_{x^N y^N}$ as the conditional probability of state x^N turning into state y^N , given by

$$m_{x^N y^N} := P \left\{ X_{t+1} = y^N \mid X_t = x^N \right\} = \prod_{i \in A(y^N)} p_i(x^N) \prod_{j \in N \setminus A(y^N)} (1 - p_j(x^N)),$$

where $X_t^N = (X_{1t}^N, \dots, X_{nt}^N)$ is the random vector which describes the state of the society (N, \mathbf{W}) at time $t \in \{0, 1, 2, \dots\}$. Then, $\mathcal{M} = \{X_t^N\}_{t \geq 0}$ is a discrete time Markov chain with *transition matrix* \mathbf{M} given by $(M_{ij})_{ij} = (m_{s_i^N s_j^N})_{ij}$, for all $i, j = 1, 2, \dots, 2^n$, where the states are ordered according to the lexicographical order. Then $\sigma(N) = \{s_1^N, s_2^N, \dots, s_{2^n}^N\}$, and $s_1^N = \mathbf{z}^N, s_{2^n}^N = \mathbf{e}^N$.

Remark 1. As said in the Introduction, note that our model does not assume a traditional concern in the innovation literature, as we are interested in quite general processes. As H.P. Young [44, 2003] notes: "Implicit in some of this literature is the notion that innovation is essentially a one-way process: once an agent has adopted an innovation, he sticks with it". We agree with Young when he asserts that: "Yet the same feedback mechanisms that cause innovations to be adopted also cause them to be abandoned ... Thus, if we want to know how long it takes, in expectation, for a "new" behavior to replace and old one, we must analyze the balance of forces pushing the adoption process forward (to become active agents, in our setting) on the one hand, and those pushing it back (to remain passive agents) on the other".

3 Power to initiate full action: definition and examples

The resulting social dynamics follows the general description of a Markov chain, a probabilistic model to which the ideas of Coleman [10, 1990] (chapter 9) about the systemic phenomena of collective behavior directly apply. To be specific, we will be able to measure the three indexes of voting power proposed by Coleman in his widely cited article [9, 1971]: i) the *power of a collectivity to act*, ii) the *power to initiate action*, and iii) the *power to prevent action*; which will

be, respectively given by i) the ability of a group to lead society to adopt their behavior (in contrast with the case in which society is reluctant to their opinion and remain divided), and if so, ii), iii) how quickly⁵ individuals learn? To be specific, question ii) refers to how quickly individuals learn to become active when the group act as a *hard core active group* (they remain forever active), whereas question iii) refers to how quickly individuals learn to become inactive when the group act as a *hard core passive group* (they remain forever inactive). We will see that, according to our model, both questions have the same answer, since the potential of actors to ignite a chain-reaction coincides with their potential to act as a firewall.

Formally, if we assume that the society (N, \mathbf{W}) evolves according to the previously defined Markov chain, we propose to measure the *power to initiate action* of group $T \subseteq N$ by means of the expected time it takes that participation of T motivates all of the other individuals to become active, given that those remaining individuals start the process being passive agents.

Thus, as a first step, we need to analyze the scenario in which the members of a subsociety $T \subseteq N$ confine themselves to act in a certain way in every period, while the remaining agents' actions are governed by the above defined updating process. Let $s^T \in \sigma(T) := \{0, 1\}^t$, where $t = |T|$ is the cardinality of T , be the *state* of subsociety $T \subseteq N$ that describes the position of agents in T during the whole process, then the social dynamics of the subsociety $N \setminus T \subseteq N$ are described by the following *partial* Markov chain $\mathcal{M}(s^T) = \{X_t^{N \setminus T}(s^T)\}_{t \geq 0}$. Let us first adapt the previous definitions to this scenario.

Notation Each combination of $s^T \in \sigma(T)$ and $x^{N \setminus T} \in \sigma(N \setminus T)$ defines a state $[s^T, x^{N \setminus T}] \in \sigma(N)$ by

$$[s^T, x^{N \setminus T}]_i = \begin{cases} s_i^T, & \text{if } i \in T, \\ x_i^{N \setminus T}, & \text{if } i \in N \setminus T. \end{cases}$$

Definition 2. For each subsociety $T \subset N$, and for any given state $s^T \in \sigma(T)$, let $p^{s^T} : \sigma(N \setminus T) \rightarrow [0, 1]^{n-t}$ be the mapping defined by

$$p^{s^T}(x^{N \setminus T}) = \mathbf{W} \cdot [s^T, x^{N \setminus T}], \text{ for all } x^{N \setminus T} \in \sigma(N \setminus T)$$

As before, for each state $x^{N \setminus T}$ of subsociety $N \setminus T$ the number $p_i^{s^T}(x^{N \setminus T})$ represents the probability that $i \in N \setminus T$ takes action upon observing state $[s^T, x^{N \setminus T}]$ of N . Now, we define $m_{x^{N \setminus T}, y^{N \setminus T}}(s^T)$ as the conditional probability of state $x^{N \setminus T}$ turning into state $y^{N \setminus T}$, given by

$$\begin{aligned} m_{x^{N \setminus T}, y^{N \setminus T}}(s^T) &:= P \left\{ X_{t+1} = [s^T, y^{N \setminus T}] / X_t = [s^T, x^{N \setminus T}] \right\} = \\ &= \prod_{i \in A(y^{N \setminus T})} p_i^{s^T}(x^{N \setminus T}) \prod_{j \in N \setminus (T \cup A(y^{N \setminus T}))} (1 - p_j^{s^T}(x^{N \setminus T})) \end{aligned}$$

⁵As Even-Dar and Shapira [18] point out, to convince society quickly is crucial to the early stages of introducing a new technology into the market.

Then, $\mathcal{M}(s^T) = \{X_t^{N \setminus T}(s^T)\}_{t \geq 0}$ is a discrete time Markov chain with *transition matrix* $\mathbf{M}(s^T) = (m_{s_i^{N \setminus T} s_j^{N \setminus T}}(s^T))_{ij}$, for all $i, j = 1, 2, \dots, 2^{n-t}$, where the states are labeled according to the lexicographical order. Thus, $\sigma(N \setminus T) = \{s_1^{N \setminus T}, s_2^{N \setminus T}, \dots, s_{2^{n-t}}^{N \setminus T}\}$, with $s_1^{N \setminus T} = \mathbf{z}^{N \setminus T}$, $s_{2^{n-t}}^{N \setminus T} = \mathbf{e}^{N \setminus T}$.

Notice that the properties of this (partial) Markov chain will depend on the particular state s^T , and the particular sociomatrix \mathbf{W} structure. In order to propose a method to measure the power to initiate and to prevent action of T group⁶, we are interested in two special cases: when agents in T confine themselves to take action ($s^T = \mathbf{e}^T$), and the opposite one, when agents in $N \setminus T$ confine themselves to never take action ($s^T = \mathbf{z}^T$). In fact, we will only need to analyze the first case, since the positive influence of every agent to push the remaining agents to take action equals the negative influence of every agent to push the remaining agents to drop action.

Now, let us consider the Markov chain $\mathcal{M}(\mathbf{e}^T) = \{X_t^{N \setminus T}(\mathbf{e}^T)\}_{t \geq 0}$, then we are interested in the expected total time spent in some of the transient states $\{s_1^{N \setminus T}, s_2^{N \setminus T}, \dots, s_{2^{n-t-1}}^{N \setminus T}\}$ before reaching the state $s_{2^{n-t}}^{N \setminus T} = \mathbf{e}^{N \setminus T}$, given that the chain starts in state $s_1^{N \setminus T} = \mathbf{z}^{N \setminus T}$. However, before trying to quantify the extent to which a group of agent is able to push the remaining agent to adopt their behavior, we will need to know if that group is even able to achieve that goal⁷. This is very related to the notion of *closed* group of agents.

Definition 3. Let (N, \mathbf{W}) be a given society. A group of agents $C \subseteq N$ is *closed* relative to \mathbf{W} if $i \in C$ and $w_{ij} > 0$ implies that $j \in C$.

Now, let (N, \mathbf{W}) be a given society, and let $T \subseteq N$ be a given subsociety. Consider the Markov chain $\mathcal{M}(\mathbf{e}^T)$. Then, if there exists a group of agents $C \subseteq N \setminus T$ that is *closed* relative to \mathbf{W} , then the state $\mathbf{e}^{N \setminus T}$ is not reachable from $\mathbf{z}^{N \setminus T}$. In that case, group T power to initiate full action should be 0. Otherwise, the Markov chain $\mathcal{M}(\mathbf{e}^T)$ has two classes, namely, $\mathcal{T} = \{s_1^{N \setminus T}, s_2^{N \setminus T}, \dots, s_{2^{n-t-1}}^{N \setminus T}\}$ and $\mathcal{R} = \{\mathbf{e}^{N \setminus T}\}$, the first class being *transient* and the second *recurrent*. Moreover, $\mathcal{M}(\mathbf{e}^T)$ is a recurrent Markov chain with an absorbing state. That is, from every state it is possible to go to the absorbing state $\mathbf{e}^{N \setminus T}$.

Definition 4. Let (N, \mathbf{W}) be a given society, and let $T \subseteq N$ be a given subsociety. Then, if there exists a group of agents $C \subseteq N \setminus T$ that is *closed* relative to \mathbf{W} , the society remains divided in group opinions and, therefore T power to initiate full action is defined to be zero⁸. Otherwise, the T power to initiate full action is measured by means of an inverse power measure, the *society resistance to group T action*, $r(T)$, which is defined as the expected absorbing time of the recurrent Markov chain $\mathcal{M}(\mathbf{e}^T)$ with an absorbing state that models the evolution of the $N \setminus T$ remaining agents when agents in T are acting as a hard active core.

⁶Which correspond to Coleman's proposals ii) and iii), see page 5.

⁷Which will be related with Coleman's proposal i), the *power of a collectivity to act*, see page 5.

⁸In the sequel, we will assume that the *society resistance to group T action* in that case is defined as $r(T) = \infty$.

It is worth noting that the previous condition about the non existence of a group of agents $C \subseteq N \setminus T$ that is *closed* relative to \mathbf{W} , is equivalent to assure that each agent in $N \setminus T$ is directly or indirectly influenced by someone of the agents in T . Another way to put this is in terms of the *social network*. Each sociomatrix \mathbf{W} defines a (*weighted*) *directed graph* $D(\mathbf{W}) = (N, E, W)$ with agents as nodes and arcs $(i, j) \in N \times N$:

$$(i, j) \in E \Leftrightarrow w_{ij} > 0,$$

and being w_{ij} the weight of arc (i, j) . Then the above assumption is equivalent to the statement that for all $i \in N \setminus T$ there exists an agent $j \in T$ such that there exists a directed path $P[i, j]$ from i to j in $D(\mathbf{W})$ ⁹. Let us denote by $\mathcal{I}(T) \subseteq N \setminus T$ the set of agents that are directly or indirectly influenced by someone of the agents in T . That is,

$$\mathcal{I}(T) = \{i \in N \setminus T / \exists j \in T \text{ with } P[i, j] \text{ in } D(\mathbf{W})\}.$$

Then, there exists a closed set $C \subseteq N \setminus T$ if, and only if, $\mathcal{I}(T) \subsetneq N \setminus T$. Therefore, if the social structure of the society is described by a strongly connected network (i.e., for all $i, j \in N$ there exists a directed path $P[i, j]$ from i to j in $D(\mathbf{W})$), then $r(T)$ is finite for all subgroup of agents $T \subseteq N$.

Nevertheless, we should remark that we can consider the general case, when the network can consist of disconnected components, in the same framework. For instance, consider the following example.

Example 1. let (N, \mathbf{W}) be the society defined by $N = \{1, 2, 3, 4, 5\}$ and sociomatrix

$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then, $r(T) = \infty$ (and therefore its power to initiate full action is 0) for all $T \subseteq \{1, 2, 3\}$ or $T \subseteq \{4, 5\}$. Otherwise, T can influence every agent in N , and therefore $r(T)$ can be calculated. For instance: $r(\{1, 4\}) = \frac{10}{3}$, $r(\{1, 4, 5\}) = 3$ and $r(\{1, 3, 5\}) = 2$.

⁹A directed path $P[i, j]$ from i to j in $D(\mathbf{W})$ is a subgraph of (N, E) consisting of a sequence of nodes $\{i_1, \dots, i_r\}$, and arcs $(i_1, i_2), (i_2, i_3), \dots, (i_{r-1}, i_r)$, where $i_1 = i$ and $i_r = j$. Alternatively, we shall sometimes refer to a directed path as a set of (sequence of) arcs (of nodes) without any explicit mention of the nodes (without explicit mention of arcs).

Note that in this example the society is actually divided into two independent subsocieties (N_1, \mathbf{W}_1) and (N_2, \mathbf{W}_2) , where $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5\}$, and

$$\mathbf{W}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

that evolve independently. Thus, it seems reasonable to assume that agents in $T \cap N_1$ do not impose any externality over agents in $T \cap N_2$, and vice versa. Thus, we will modify slightly the previous definition in order to impose that condition when the original society is composed by some independent subsocieties.

Definition 5. Let (N, \mathbf{W}) be a given society that is fragmented into independent subsocieties. That is, the associated (weighted) directed graph $D(\mathbf{W}) = (N, E, W)$ is not connected¹⁰. Let $\mathcal{C}(D(\mathbf{W})) = \{(N_1, E_1, W_1), \dots, (N_m, E_m, W_m)\}$ be the set of connected components of $D(\mathbf{W})$, and let $T \subseteq N$ a group of agents which is able to prompt society to became active. Then, we will define the *society resistance to group T action* as the maximum resistance of each subsociety to group T members:

$$r(T) = \max\{r(T_1), \dots, r(T_m)\}, \quad (1)$$

where $r(T_k)$ is the society resistance to group $T_k = T \cap N_k$ in the corresponding subsociety (N_k, \mathbf{W}_k) , for all $k = 1, \dots, m$.

Note that $\mathcal{I}(T) = N \setminus T$ implies $\mathcal{I}(T_k) = N_k \setminus T_k$, for all $k = 1, \dots, m$, and therefore, $r(T)$ given by the maximum (1) is well defined. Otherwise, the general definition $r(T) = \infty$ applies.

As we have announced in the Introduction, the voter model can be related with the seminal network interaction model of DeGroot [15], and therefore, also with eigenvector-based centrality measures (Katz [26], Bonacich [4], [5] and Brin and Page [6]). To be specific, if we consider that the set $T \subseteq N$ of initially active nodes can be deactivated during the process, then we will always work with the general Markov chain $\mathcal{M} = \{X_t^N\}_{t \geq 0}$ which is a recurrent chain with two absorbent states, namely \mathbf{z}^N and \mathbf{e}^N , when the sociomatrix \mathbf{W} is irreducible. In that case (see Stadje [38]), the *probability of absorption in \mathbf{e}^N* , given that the initial state is $[\mathbf{e}^T, \mathbf{z}^{N \setminus T}]$ equals

$$\pi_{\mathbf{e}^N}[\mathbf{e}^T, \mathbf{z}^{N \setminus T}] = \sum_{i \in T} \tilde{\zeta}_i,$$

where $(\tilde{\zeta}_i)_{i \in N}$ is the stationary distribution of the irreducible sociomatrix \mathbf{W} . If \mathbf{W} is not irreducible, then the society (N, \mathbf{W}) is fragmented into independent subsocieties (as in the previous definition), and the probability of absorption can be calculated as the product of the corresponding absorption probabilities on each of the subsocieties.

¹⁰We will say that two nodes i and j are *connected* in $D(\mathbf{W})$ if the graph contains at least one path (not necessarily directed) from node i to node j .

If $\mathcal{M}(\mathbf{e}^T)$ is recurrent the Markov chain with an absorbent state (that is, there is no closed group of agents $C \subseteq N \setminus T$), then we can use standard Markov chain theory's results to calculate $r(T)$. The society resistance to group T action will be given by the sum of the first row of the fundamental matrix $\mathbf{D}(\mathbf{e}^T)$ of the recurrent Markov chain with an absorbent state:

$$r(T) = \sum_{j=1}^{2^{n-t}-1} d_{1j}(\mathbf{e}^T),$$

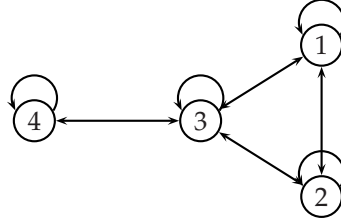
where the fundamental matrix $\mathbf{D}(\mathbf{e}^T) = (\mathbf{I}_{2^{n-t}-1} - \mathbf{Q})^{-1}$, and $\mathbf{Q}(\mathbf{e}^T)$ is obtained from the transition matrix $\mathbf{M}(\mathbf{e}^T)$ deleting its last row and its last column.

Of course, we are aware of the difficulties in tracking the full joint distribution of the Markov chain $\mathcal{M}(\mathbf{e}^T)$, which is defined by means of a $2^{n-t} \times 2^{n-t}$ transition matrix. For large societies the computational effort can be excessive. However, the power of a group can be estimated by means of simulating the behavior of the corresponding Markov chain. This is precisely the method we propose to deal with our proposal. Nonetheless, we include here the above standard result, as well as some small examples, since we think that the fundamental matrix is very interesting by itself.

Example 2. Consider a society of 4 members with socio-matrix:

$$\mathbf{W} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix},$$

which can represent the following *social network*, when each agent gives the same weight to himself and all his direct neighbors.



In that case, agent 3 should be the most powerful agent, whereas agent 4 should be the less powerful one. Agents 1 and 2, who occupy a symmetric position in the society, should have an intermediate strength.

Now, to calculate the power of agent 1 the state space of the $\mathcal{M}(\mathbf{e}_1)$ Markov chain, $\sigma(\{2, 3, 4\})$, will be given by the matrix

$$\sigma(\{2, 3, 4\}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix},$$

where $\sigma(\{2, 3, 4\})_{ij}$ represents the action of the i -th agent of $\{2, 3, 4\}$ in the j -th state of $\sigma(\{2, 3, 4\})$. The transition matrix of the Markov chain will be given by

$$M(\mathbf{e}_1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{12} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ 0 & \frac{1}{8} & 0 & 0 & \frac{3}{8} & \frac{1}{8} & 0 & \frac{3}{8} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{8} & \frac{1}{24} & \frac{1}{4} & \frac{1}{12} & \frac{1}{8} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, the fundamental matrix of the absorbing system, $D(\mathbf{e}_1) = (\mathbf{I}_7 - Q(\mathbf{e}_1))^{-1}$ will be given by

$$D(\mathbf{e}_1) \approx \begin{pmatrix} 2.779 & 1.651 & 0.889 & 0.131 & 1.681 & 0.439 & 0.201 \\ 1.010 & 2.463 & 0.708 & 0.109 & 1.819 & 0.418 & 0.173 \\ 0.674 & 0.944 & 1.492 & 0.204 & 1.223 & 0.510 & 0.296 \\ 0.907 & 0.947 & 0.645 & 1.333 & 1.167 & 0.479 & 0.448 \\ 0.294 & 0.628 & 0.226 & 0.050 & 2.186 & 0.361 & 0.096 \\ 0.459 & 0.677 & 0.422 & 0.143 & 1.109 & 1.388 & 0.310 \\ 0.203 & 0.256 & 0.165 & 0.180 & 0.376 & 0.362 & 1.452 \end{pmatrix}$$

And the expected time it takes that participation of agent 1 motivates all of the other individuals to take action, that is the *resistance* of the society against 1, will be

$$\sum_{j=1}^7 d(\mathbf{e}_1)_{1j} \approx 2.779 + 1.651 + 0.889 + 0.131 + 1.681 + 0.439 + 0.201 = 7.771.$$

The society resistance against agents 2, 3 and 4, are, 7.771, 4.17 and 15, respectively. Taking into account that agents 1 and 2 are fully symmetric, the power of all possible proper coalitions appear in the following table:

GROUP	{1}	{3}	{4}	{1,2}	{1,3}	{1,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}
RESIST.	7.77	4.17	15	5.33	2.3	3.12	3.86	2	1.33	1.5

To understand the impact of the network architecture we shall analyze the impact that the deletion of a certain link has over the individual power to initiate full action. For instance, if the link between agents 1 and 2 is deleted, then the new social network is a star centered in agent 3. In that case, the individual powers are:

$$\text{RESISTANCE} = (11.9053, 11.9053, 3.1429, 11.9053)$$

The power of agents 3 and 4 has increased and agents 1 and 2 power has decreased. Agents 3 and 4 profit from higher strength of ties on the minimal path to 1 and 2.

If we delete the link between agents 1 and 3, then the new social network is the 4 node chain 1 – 2 – 3 – 4 and

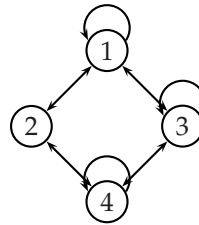
$$\text{RESISTANCE} = (15.9689, 7.6143, 7.6143, 15.9689)$$

In comparison to the first scenario, the power of agents 1 and 3 has decreased, and the power of agent 4 too. Note that, the minimal paths from 4 to the other agents were not affected when link (1, 2) is deleted, but they are affected when link (1, 3) is deleted.

Example 3. Consider the following socio-matrix:

$$W = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

In this case, the social network is the four node wheel



in which agents 1, 3 and 4 act as in the previous example. However, agent 2 gives no influence to his own past action. This fact breaks the symmetry of the wheel. Now, all agents but 2 are in a more advantageous position.

Taking into account that agents 1 and 4 are fully symmetric, the power to initiate full action in the non-symmetric wheel described by the above socio-matrix is given by:

GROUP	{1}	{2}	{3}	{1,2}	{1,3}	{1,4}	{2,3}	{1,2,3}	{1,2,4}	{1,3,4}
RESIST.	6.72	7.30	6.42	3.86	3.27	1.5	1.87	1.5	1.5	1

The power to initiate full action in a symmetric wheel is given by

GROUP	{i}	connected {i,j}	unconnected {i,j}	{i,j,k}
RESIST.	7.29	3.86	1.87	1.5

Thus, the power of all groups which 2 belongs to has not changed, whereas the power of all other groups have increased (to convince agent 2 is easier than before).

As we have commented in the introduction, our approach allows us to analyze questions related with the measurement of voting power. In that context, the objective is to quantify the extent to which a voter or a group of voters is able to control the outcome of a vote (Felsenthal and Machover [19, 1997]). Thus, the question is just how long it takes to have convinced a minimal number q of participants, instead of convincing all of them, where q plays the role of the *quota* in voting games. If that is the case, we must define the absorbent state accordingly¹¹ in order to deal with that new situation. Moreover, a general non-symmetric voting game can also be considered. For a deep analysis on this approach the reader is referred to Yeh and Koster [43, 2009], where this model is used to theoretically assess the properties of actual voting power when power is conceived as a balanced measure between constitutional voting system effects and social network effects.

4 Monotonicity

Let us consider a subsociety $T \subseteq N$ of a certain society N . In section 3 it was defined the *society resistance* to the group T action as an (inverse) measure of the power of T , given by the resistance of the individuals in $N \setminus T$ to take action, assuming that the individuals in T are confined themselves to do it. An interesting issue concerns the monotonicity of the resistance, i.e., if $T \subseteq T'$ implies that the resistance to the group T' is smaller than the resistance to the group T . Before giving a proof, we will describe an argument to check the monotonicity of a measure of this kind.

We set henceforth the framework, following [32, 2002]. Consider two Markov processes, $\mathbf{X} = \{X_t\}_{t \in \mathbb{N}}$ and $\mathbf{Y} = \{Y_t\}_{t \in \mathbb{N}}$, whose state space \mathcal{S} is discrete and partially ordered, say $(\mathcal{S}, \prec_{\mathcal{S}})$. Then the process \mathbf{X} is *stochastically smaller* than \mathbf{Y} if, for every $a \in \mathcal{S}$ and $t \in \mathbb{N}$, we have $P\{X_t \succ_{\mathcal{S}} a\} \leq P\{Y_t \succ_{\mathcal{S}} a\}$. This definition has some equivalent formulations, see [32] (chapter 1). In the sequel, we will assume that \preceq_{st} stands for the stochastic order, although a big part of the following results remains true for other orders.

It is interesting to remark that checking this condition for a certain natural number t involves the knowledge of the evolution of the process until the time t , a lot of information which is frequently not available. Then, it is useful to find characterizations of the order that could be checked, at least for homogeneous processes, only in terms of the transition matrix of the process. For this sake, denote by $\mathbb{P}_{\mathcal{S}}$ the space of the probability distributions over the state space

¹¹An absorbent state is any state with q or more active members.

\mathcal{S} , and let \mathbf{T} and \mathbf{T}' be two transition operators over $\mathbb{P}_{\mathcal{S}}$ with state space \mathcal{S} . Then \mathbf{T} is said to be smaller than \mathbf{T}' if for every $P \in \mathbb{P}_{\mathcal{S}}$ we have $\mathbf{T}P \preceq_{st} \mathbf{T}'P$ for the partial order in $\mathbb{P}_{\mathcal{S}}$. Moreover, \mathbf{T} is said *monotone* if $\mathbf{T}P \preceq_{st} \mathbf{T}P'$ for every $P, P' \in \mathbb{P}_{\mathcal{S}}$ such that $P \preceq_{st} P'$. In these conditions, we have the following fundamental result (see [32], Theorem 5.2.2):

Proposition 1. *Let \mathbf{P} and \mathbf{P}' be two homogeneous Markov processes in discrete time, \mathbf{T} and \mathbf{T}' their transition operators. Then $P_t \preceq_{st} P'_t$ for every positive t if the following three conditions hold:*

- (i) $P_0 \preceq_{st} P'_0$ for the initial distribution.
- (ii) $\mathbf{T} \preceq_{st} \mathbf{T}'$.
- (iii) There exists a monotone operator \mathbf{T}'' such that $\mathbf{T} \preceq_{st} \mathbf{T}'' \preceq_{st} \mathbf{T}'$.

The best way to apply this characterization to our particular problem is via the transition kernels.

Given a homogeneous Markov chain $\mathbf{X} = \{X_t\}_{t \in \mathbb{N}}$ and a subset $B \subseteq \mathcal{S}$ of the state space, the transition kernel $Q(x, B)$ is described as the conditional probability

$$Q(x, B) = P \{ X_{t+1} \in B / X_t = x \}, \text{ for } x \in \mathcal{S}.$$

Moreover, a set $C \subseteq \mathcal{S}$ is said *increasingly closed* with respect to a partial order $\preceq_{\mathcal{S}}$ in \mathcal{S} if for every $x, y \in \mathcal{S}$ such that $x \succeq_{\mathcal{S}} y$ and $y \in C$, then $x \in C$. We have the following criterion:

Proposition 2. *If \mathbf{T} and \mathbf{T}' are two transition operators, then $\mathbf{T} \preceq_{st} \mathbf{T}'$ if, and only if, $Q(x, C) \leq Q'(x, C)$ for every x in the state space \mathcal{S} and every increasingly closed $C \subseteq \mathcal{S}$.*

These conditions lead us to the main result we will use (see [3, 2006], Proposition 2.9):

Proposition 3. *Let $\{X_t\}_{t \in \mathbb{N}}$ and $\{Y_t\}_{t \in \mathbb{N}}$ two Markov chains over the same finite and partially ordered state space, and assume that the last state is absorbing for both chains. If $X_t \preceq_{st} Y_t$ for every t , then the absorption time in $\{Y_t\}_{t \in \mathbb{N}}$ is smaller than the absorption time in $\{X_t\}_{t \in \mathbb{N}}$ for every initial state.*

Once we have reviewed the monotonicity conditions which are useful in our context, we will precisely define the Markov processes which are involved in our framework.

Consider a society $N = \{1, \dots, n\}$ and two subsocieties T and T' , in such a way that $T = T' \cup \{j\}$, for some $j \in N \setminus T'$. Without loss of generality, we identify T' with the set $\{1, \dots, m\}$, $m < n$, and T with $\{1, \dots, m+1\}$. Our goal is to compare the society resistance to T' to the society resistance to T , and to check that the former should be, as intuition suggests, greater than the latter.

There is a difficulty one has to overthrow when facing this problem: the process $\mathcal{M}(\mathbf{e}^T) = \{X_t^{N \setminus T}(\mathbf{e}^T)\}_{t \geq 0}$ whose absorption time is the society resistance to T (see section 3) has *not* the same state space as the corresponding process for T' , $\mathcal{M}(\mathbf{e}^{T'}) = \{X_t^{N \setminus T'}(\mathbf{e}^{T'})\}_{t \geq 0}$. Indeed, the state space for the first process is $\mathcal{S}(T) := \sigma(N \setminus T)$, which has cardinal 2^{n-m-1} , while the second one, $\mathcal{S}(T') := \sigma(N \setminus T')$, has cardinal 2^{n-m} . Hence, when comparing the two processes, we should define at least another one who mimics the stochastic properties of one of them, and whose state space is the same as the one of the other.

The idea here is to redefine the process $\mathcal{M}(\mathbf{e}^T)$ as a process in the state space of T' . Recall that $\sigma(N \setminus T') = \{0, 1\}^{n-m}$ is identified in a natural way with the last 2^{n-m} components of $\sigma(N) = \{0, 1\}^n$, and $\sigma(N \setminus T) = \{0, 1\}^{n-m-1}$ with the last 2^{n-m-1} components of $\sigma(N)$. Now, we define a process \tilde{T} such that $\mathcal{S}(\tilde{T}) = \mathcal{S}(T')$ and that reproduces the stochastic features of the former. For any initial state $[1, x]$, the states $[0, y]$ will be avoided in any instant of time, and the probabilities of going from state $[1, x]$ to states $[1, y]$ and $[1, z]$ will be defined as in the process $\mathcal{M}(\mathbf{e}^T)$. To be specific, the new transition probabilities are given by the following rule (we use the notation of section 3). Here $x \in \sigma(N \setminus T)$, and $q([\cdot, x]) = p_{m+1}^{\mathbf{e}^{T'}}([\cdot, x])$ represents the probability that $m+1$ takes action upon observing state $[\cdot, x] \in \sigma(N \setminus T')$ in the Markov chain $\mathcal{M}(\mathbf{e}^{T'})$.

- The transition probability from $[0, x]$ to $[0, y]$ is zero if $q([0, x]) > 0$. Otherwise, it equals the transition probability from $[0, x]$ to $[0, y]$ in $\mathcal{M}(\mathbf{e}^{T'})$.
- The transition probability from $[0, x]$ to $[1, y]$ is zero, equal to the transition probability from $[0, x]$ to $[1, y]$ in $\mathcal{M}(\mathbf{e}^{T'})$ if $q([0, x]) = 0$.
Otherwise, it is defined by $m_{[0,x][1,y]}(\mathbf{e}^{T'}) / q([0, x])$, which is the transition probability from $[0, x]$ to $[1, y]$ in $\mathcal{M}(\mathbf{e}^{T'})$ conditioned to agent $m+1$ takes action.
- The transition probability from $[1, x]$ to $[0, y]$ is zero for every $y \in N \setminus T$.
- The transition probability from $[1, x]$ to $[1, y]$ equals the transition probability from $[1, x]$ to $[1, y]$ in $\mathcal{M}(\mathbf{e}^T)$.

Note that the transition matrix of the extended Markov chain has the following block description, where $E := \{[1, x] / x \in N \setminus T\} := [1, \mathcal{S}(T)]$ and $E^c = \{[0, x] / x \in N \setminus T\} := [0, \mathcal{S}(T)]$:

$$M(\mathbf{e}^{\tilde{T}}) = \begin{pmatrix} Q_{E^c} & Q_{E^c E} \\ \mathbf{0}_{E E^c} & M(\mathbf{e}^T) \end{pmatrix}$$

Then, if the successive visits of $\mathcal{M}(\mathbf{e}^{\tilde{T}}) = \{Y_t^{N \setminus T'}(\mathbf{e}^{T'})\}_{t \geq 0}$ to E take place at time epochs $0 < t_1 < t_2 < \dots < \dots$. Then the chain $\{Y_u^{N \setminus T'}(\mathbf{e}^{T'}) = Y_{t_u}^{N \setminus T'}(\mathbf{e}^{T'}), u = 1, 2, \dots\}$, which is called

the $\mathcal{M}(\mathbf{e}^{\tilde{T}})$ censored chain with censoring set $E = [1, \mathcal{S}(T)]$, has as transition matrix $Q_E = M(\mathbf{e}^T)$ (see [46, 1996]).

Proposition 4. *Let (N, \mathbf{W}) be a given society, and let T and T' two subsocieties defined as before. Then, if there exists a partial order \preceq_S over the state space $\mathcal{S}(\tilde{T}) = \mathcal{S}(T') = \sigma(N \setminus T')$ such that:*

(O*i*) *Given states x and y on $\mathcal{S}(\tilde{T})$, if for every active component in x , the corresponding component in y is also active, then $x \preceq_S y$,*

(O*ii*) *The transition matrix of the extended Markov chain, $M(\mathbf{e}^{\tilde{T}})$, is monotone.*

Then $r(T) \leq r(T')$.

Proof. Let $(\mathcal{S}(\tilde{T}), \preceq_S)$ a partial order satisfying conditions (O*i*) and (O*ii*). It is clear that $\mathbf{e}^{N \setminus T'}$, the absorbing state, is the unique maximal element. Then, we can invoke Proposition 3 to compare absorption times in both chains, when we start from the same initial state. Let us consider $[1, \mathbf{z}^{N \setminus T'}] \in \sigma(T')$ as the initial state. We will show that the absorption time in $\mathcal{M}(\mathbf{e}^T)$ is greater or equal than the corresponding absorption time in the extended Markov chain $\mathcal{M}(\mathbf{e}^{\tilde{T}})$. Therefore, we must prove that the process $\mathcal{M}(\mathbf{e}^T)$ is stochastically smaller than $\mathcal{M}(\mathbf{e}^{\tilde{T}})$. By Proposition 1, this amounts to prove that the transition operators for the first process are smaller than the ones for the second, and moreover that we can set a monotone matrix between the matrices of the processes. Note that this last condition is precisely what condition (O*ii*) ask for.

Let us check then the condition $M(\mathbf{e}^{T'}) \preceq_{st} M(\mathbf{e}^{\tilde{T}})$ using Proposition 2. Let us denote by $\tilde{Q}(x, B)$ and $Q'(x, B)$ the corresponding transition kernels. By Proposition 2, it is enough to prove that for every increasingly closed set C and every $x \in \mathcal{S}(T')$, $Q'(x, C) \leq \tilde{Q}(x, C)$, or equivalently,

$$P \left\{ X_{t+1}^{N \setminus T'}(\mathbf{e}^{T'}) \in C / X_t^{N \setminus T'}(\mathbf{e}^{T'}) = x \right\} \leq P \left\{ Y_{t+1}^{N \setminus T'}(\mathbf{e}^{T'}) \in C / Y_t^{N \setminus T'}(\mathbf{e}^{T'}) = x \right\}.$$

If C is an increasingly closed set, then C consists of a number of states of the form $[0, z_i]$, with $1 \leq i \leq j$, and a number of states of the form $[1, z_i]$, $1 \leq i \leq j'$, with $j \leq j'$. Observe that the condition on the ordering implies that if $[0, z_i]$ belongs to C for a certain i , then so does $[1, z_i]$. Suppose that $x = [0, z]$ for a certain $(n - m - 1)$ -uple z . If $q([0, z]) = 0$, then the probability of all transitions from x to $[1, z_i]$ is zero in both processes, whereas transitions to $[0, z_i]$ are equally probable in both processes too by definition. Thus, $\tilde{Q}(x, C) = Q'(x, C)$. Otherwise, for brevity, we will denote the transition probability from x to $[1, z_i]$ in the $\mathcal{M}(\mathbf{e}^{T'})$ process by $m_i = m_{[0, z][1, z_i]}(\mathbf{e}^{T'})$, for all $1 \leq i \leq j'$. Then, the transition kernel $Q'(x, C)$ is equal to $\frac{1}{p} \sum_{i=1}^j m_i + \sum_{i=j+1}^{j'} m_i$, while $\tilde{Q}(x, C) = \frac{1}{p} \sum_{i=1}^{j'} m_i$. As $p \leq 1$, $\tilde{Q}(x, C) \geq Q'(x, C)$. The case beginning with $x = [1, z]$ is similar.

Thus, Proposition 3 assures that $AT_{[1, \mathbf{z}^{N \setminus T}]}(\mathcal{M}(\mathbf{e}^{\tilde{T}})) \leq AT_{[1, \mathbf{z}^{N \setminus T}]}(\mathcal{M}(\mathbf{e}^{T'}))$, where $AT_{[1, \mathbf{z}^{N \setminus T}]}(\cdot)$ denotes the absorption time when the initial state is $[1, \mathbf{z}^{N \setminus T}]$.

Now, we must relate those two absorption times with resistances $r(T)$ and $r(T')$.

We will first show that $AT_{[1, \mathbf{z}^{N \setminus T}]}(\mathcal{M}(\mathbf{e}^{T'})) \leq r(T')$, that is the absorption time in $\mathcal{M}(\mathbf{e}^{T'})$ when the initial state is $[1, \mathbf{z}^{N \setminus T}] \in \mathcal{S}(T')$ is less or equal than the absorption time when the initial state is $\mathbf{z}^{N \setminus T'} = [0, \mathbf{z}^{N \setminus T}] \in \mathcal{S}(T')$. This can be easily analyzed by means of its dual processes (called *coalescing random walk* in the context of Percolation Theory, see [25, 1975]). If the Markov chain $\mathcal{M}(\mathbf{e}^{T'})$ has been absorbed in t periods when the initial state is $[0, \mathbf{z}^{N \setminus T}]$ then, for all $k \in \{m+1, \dots, n\}$ there exists a random walk¹² of length t starting at k and ending in T' . Thus, also exist random walks of length t starting at k and ending in $T = T' \cup \{m+1\}$, for all $k \in \{m+1, \dots, n\}$. Therefore, $\mathcal{M}(\mathbf{e}^{T'})$ has been absorbed in $t' \leq t$ periods when the initial state is $[1, \mathbf{z}^{N \setminus T}]$.

Finally, we will show that $r(T) \leq AT_{[1, \mathbf{z}^{N \setminus T}]}(\mathcal{M}(\mathbf{e}^{\tilde{T}}))$, that is the absorption time in $\mathcal{M}(\mathbf{e}^T)$ when the initial state is $\mathbf{z}^{N \setminus T} \in \mathcal{S}(T)$ is less or equal than the absorption time in $\mathcal{M}(\mathbf{e}^{\tilde{T}})$ when the initial state is $[1, \mathbf{z}^{N \setminus T}] \in E := [1, \mathcal{S}(T)] \subseteq \mathcal{S}(T')$. This relation follows from being $\mathcal{M}(\mathbf{e}^T)$ the $\mathcal{M}(\mathbf{e}^{\tilde{T}})$ censored chain with censoring set E , see Property 7 in [20, 2007]

□

Note that a partial order satisfying condition (Oi) obviously always exists. However, the existence of a partial order verifying condition (Oii) is much more difficult to prove in general, as it seems not easy to find a general order in the state space such that one of the operators involved is monotone. However, we have checked the existence in some random examples, which give us great hope that such an order always exists. We discuss in detail one of these examples.

Consider a society with four members and socio-matrix

$$\mathbf{W} = \begin{bmatrix} 1/5 & 0 & 4/5 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}.$$

We consider the subsocieties $T' = \{1\}$, $T = \{1, 2\}$. Then the state space of the processes

¹²Where the probability of jumping from node $k \in \{m+1, \dots, n\}$ to any other node $j \in N$ is defined to be w_{kj} (i.e., the probability that agent k adopts the current opinion of agent j in the previous step). Note that the probability of jumping from node $k \in T' = \{1, \dots, m\}$ to any other node $j \in N$ with $j \neq k$ is 0.

$\mathcal{M}(\mathbf{e}^{\tilde{T}})$ and $\mathcal{M}(\mathbf{e}^{T'})$ will be given by the columns of the matrix

$$\mathcal{S}(\{1\}) := \sigma(\{2,3,4\}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

We assume that a state x is smaller than y in the previous state space if the corresponding column for x is at the left of the column for y . Obviously, that order verifies (Oi) . Now, we will check that it also holds condition (Oii) . So we consider the matrix associated to the process $\mathcal{M}(\mathbf{e}^{\tilde{T}})$, in which we suppose that the states are ordered using the previous order:

$$M(\mathbf{e}^{\tilde{T}}) = \begin{pmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/6 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1/6 & 0 & 1/3 & 0 & 1/6 & 1/3 \\ 0 & 0 & 1/6 & 0 & 1/3 & 0 & 1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

According to Property 1 in [20, 2007], it is enough to see that for every (i, i', j) with $i, i', j \leq 8$ and $i < i'$, $\sum_{k=j}^8 a_{ik} \leq \sum_{k=j}^8 a_{i'k}$. This is easy to see by inspection, so $M(\mathbf{e}^{\tilde{T}})$ is monotone.

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