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Abstract

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Key Words

Long-memory; Correlogram; financial series.

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The Correlogram of a Long Memory Process Plus a Simple Noise[1]

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Abstract

A frequent property of data, particularly in the financial area, is that the correlogram is low but remains positive for many lags. A plausible explanation for this is that the process consists of a stationary, long memory component plus a white noise component of much larger variance. The implications of such a composition are explored including the consequences for estimation of the long memory parameter.

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1. Introduction

An empirical correlogram, that is a plot of estimated autocorrelations against lag length, is sometimes found to have a very distinctive shape, with values small but positive for many lags. This is particularly found for some long financial data. In this paper, a class of models is considered which can produce such shaped empirical correlograms in which a long memory process is added to an independent white noise, or possibly a short memory process.

The paper initially considers the theoretical properties of such a sum and then the properties of estimated autocorrelations. Examples of processes having the correlogram with the shape analyzed here are discussed in the concluding section.

2. The Model and Theoretical Moments

Let $\{y_t\}_{t=-\infty}^{\infty}$ be a covariance-stationary time series with mean $E(y_t) = \mu$ and autocovariance at lag k given by $\gamma_y(k) = E(y_t - \mu)(y_{t+k} - \mu)$. Assume that $\{y_t\}_{t=-\infty}^{\infty}$ has long memory, so that its autocovariance function satisfies, as $k \to \infty$,

(1) $\qquad \gamma_y(k) \sim \lambda k^{-\alpha}$, $\lambda > 0$, $\alpha \in (0,1)$,

implying that $\sum_{k=-\infty}^{\infty} \gamma_y(k)^2 < +\infty$ but $\sum_{k=-\infty}^{\infty} |\gamma_y(k)| = +\infty$. A leading example is the fractionally integrated process, $I(d)$, for which $\alpha = 1 - 2d$, as discussed later.

Large sample properties of the sample mean, autocovariances and autocorrelations of long memory time series have been recently reported by Hosking (1996). In this paper we are interesting, instead, in the properties of the slightly modified process

(2) $\qquad x_t = y_t + \varepsilon_t$,

where $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ denotes an independently and identically distributed white noise process with zero mean and autocovariance function

(3) $$\gamma_\varepsilon(k)=\begin{cases}\sigma_\varepsilon^2 & k=0\\ 0 & k\neq 0\end{cases},$$

and where throughout this paper it is assumed that y_t and ε_τ are stochastically independent of each other for all t,τ. Moreover, it will be assumed that σ_ε^2 is large enough with respect to $\gamma_y(0)$ such that the condition

(4) $$\xi=\frac{\sigma_\varepsilon^2}{\gamma_y(0)}>1$$

will hold. Under this set-up, the theoretical first and second moments of x_t are given by

(5) $$E(x_t)=\mu,$$

(6) $$\gamma_x(0)=E(x_t-\mu)^2=\gamma_y(0)+\sigma_\varepsilon^2.$$

(7) $$\gamma_x(k)=E(x_t-\mu)(x_{t-k}-\mu)=\gamma_y(k),\quad k=1,2,\dots$$

Therefore, x_t is a second-order stationary process with mean $E(x_t)=\mu$, variance $\gamma_x(0)>\gamma_y(0)$ and autocovariance function given by $\gamma_x(k)=\gamma_y(k),\ k=1,2,\dots,$ so that $\gamma_x(k)\sim\lambda k^{-\alpha}$ for k large. Consequently, the sum of a long memory process with an independent white noise process appears to be a new long memory with greater variability but with the same degree of dependence as the original long memory process.

Consider now the shape of the autocorrelation function of x_t:

(8) $$\rho_x(k)=\frac{\gamma_x(k)}{\gamma_x(0)}=\frac{\gamma_y(k)}{\gamma_y(0)+\sigma_\varepsilon^2}=\frac{1}{1+\xi}\rho_y(k),\quad k=1,2,\dots,$$

and then, under (4), $\rho_x(k) < \rho_y(k)$, $k = 1,2,\ldots$. Moreover, since x_t is a long memory process, from (1) and (8) it follows that

(9) $$\rho_x(k) \sim \frac{\lambda}{\gamma_y(0) + \sigma_\varepsilon^2} k^{-\alpha}, \quad k \to \infty .$$

Notice that expression (9) is just an asymptotic approximation as the lags tend to infinity, determining only the slow rate of decaying characterizing the strong dependence processes. It does not specify neither the correlations for any fixed finite lag nor the fact that each individual correlation can be arbitrarily small making difficult the detection of long memory properties in the time series of interest.

In our case, expression (9) allows us to distinguish between the persistence or the memory of the process, given by the value of α from what we could call the *size* of the process, given by the value of $\vartheta(\lambda, \xi) = \lambda\left(\gamma_y(0) + \sigma_\varepsilon^2\right)^{-1}$. The greater the value of α, more difficult it becomes to tell whether the autocorrelations follow the hyperbolic decay which characterizes the long memory processes or an exponential decay as for a short memory processes. It is clear also from the fact that

(10) $$\frac{\partial \vartheta(\lambda, \xi)}{\partial \xi} < 0 ,$$

and the continuity of the size function that, given λ, there exists a sufficiently large signal-to-ratio ξ^* for which the autocovariance function $\rho_x^*(k) \sim \vartheta(\lambda, \xi^*) k^{-\alpha}$ will be arbitrarily small. Moreover, since it is well-known that the autocovariances of a long memory process are positive for $\alpha \in (0,1)$, then, under (1)-(4), we have obtained a class of long memory processes, say x_t^*, with positive but arbitrary small autocorrelations with near-observational problems in the sense of distinguishing between autocorrelations with hyperbolic decay from autocorrelations with exponential decay on the basis of the shape of the autocorrelation function.

3. Estimated Autocorrelations

Consider now the sample counterpart of the theoretical moments reported in the previous section. For this, define the sample autocovariances and autocorrelations based on T observations of an arbitrary stationary process z_t as follows:

$$(11) \qquad \hat{\rho}_z(k) = \frac{\hat{\gamma}_z(k)}{\hat{\gamma}_z(0)}, \qquad \hat{\gamma}_z(k) = T^{-1}\sum_{t=1}^{T-k}(z_t - \bar{z})(z_{t+k} - \bar{z}), \qquad k = 0,1,\ldots,T-1,$$

and where $\bar{z}$ denotes the sample mean.

One of the standard methods in time series analysis is to construct the plot of the estimated autocorrelations (and partial autocorrelations) of the time series of interest against the lag k and, as a simple rule, to consider as significant at the 5% level all those correlations outside the band $\pm 2/\sqrt{T}$. The justification for this approach relies on the fact that, if the true autocorrelations are zero, then under some regularity conditions, it can be proved that $\sqrt{T}\hat{\rho}_z(k)$ are asymptotically independent standard normal random variables.

With long memory processes, indeed, the correlogram is not a useful diagnostic tool for detecting long range dependence. As we have seen in the previous section and is widely known, long memory processes have autocorrelations decaying at a slow rate proportional to $k^{-\alpha}$. Hence, a plot of the sample autocorrelations should exhibit this slow decay. In this sense, however, Beran (1994, Chapter 4, Figures 4.7 and 4.8) provides clear graphical evidence that for values of α close to one, it is very difficult to distinguish between short and long memory processes. Another difficulty with the use of the correlogram in the presence of long range dependence reported also by Beran is that, since long memory is an asymptotic notion, then we should analyzed the correlogram at high lags, which, in turn, cannot be estimated in a reliably way.

Moreover, as we already noted, even in the case where the value of α of a particular long memory process is small, its size can be small as well, implying that the bound $\pm 2/\sqrt{T}$ will not detect the presence of long range dependence in the data. In this section we shall ilustrate this possibility by deriving the asymptotic behaviour of the estimated autocorrelations of the perturbed long memory process x_t. For this, one needs to introduce more structure to the underlying processes.

Assumption 1:

(*a*) $y_t = \mu + \sum_{j=0}^{\infty} \phi_j \eta_{t-j}, \quad \eta_t \sim iid(0, \sigma_\eta^2), \ \sigma_\eta^2 < \infty.$

(*b*) $\phi_j \sim \delta j^{-\beta}$ *as* $j \to \infty$, *where* $\delta > 0$ *and* $\beta = \frac{1}{2}(1+\alpha)$.

(*c*) η_t and ε_τ *are uncorrelated Gaussian process for all* t, τ.

Two of the more popular models of long memory processes, namely, the fractional Gaussian noise (Mandelbrot and Van Ness, 1968) and the fractional *ARIMA* processes (Granger and Joyeux, 1980; Hosking, 1981) have representations of the form (*a*) and (*b*). Notice, however, that in general, property (1) does not imply (*a*) and (*b*). On the other hand, Assumption 1.*c* is a convenient assumption for our purposes but can be relaxed in certain circumstances and replaced, for instance, by the requirement of uniformly bounded fourth moments.

Under Assumption 1, the asymptotic bias and covariance of the $\hat{\rho}_x(k)$, $k \geq 1$, can be characterized using the following proposition whose proof is given in the Appendix at the end of the paper.

Proposition 1: *Under Assumption* 1, *as* $T \to \infty$,

(12) $$E(\hat{\rho}_x(k) - \rho_x(k)) \sim \frac{-2}{(1-\alpha)(2-\alpha)} \vartheta(\lambda, \xi)(1 - \rho_x(k))T^{-\alpha}.$$

(13) *If* $0 < \alpha < \frac{1}{2}$,

$$\mathrm{cov}(\hat{\rho}_x(k), \hat{\rho}_x(l)) \sim 2\vartheta^2(\lambda, \xi)(1 - \rho_x(k))(1 - \rho_x(l))\Re_2 T^{-2\alpha},$$

where $\Re_2$ *is a modified Rosenblatt distribution defined by Hosking* (1996, *expressions* (12)-(15)).

(14) *If* $\alpha = \frac{1}{2}$,

$$\mathrm{cov}(\hat{\rho}_x(k), \hat{\rho}_x(l)) \sim 4\vartheta^2(\lambda, \xi)(1 - \rho_x(k))(1 - \rho_x(l))T^{-1} \log T.$$

(15) *If* $\frac{1}{2} < \alpha < 1$,

$$\mathrm{cov}(\hat{\rho}_x(k), \hat{\rho}_x(l)) \sim T^{-1} \sum_{j=-\infty}^{\infty} \big(\rho_x(j)\rho_x(j+k-l) + \rho_x(j)\rho_x(j+k+l)$$

$$+2\rho_x(k)\rho_x(l)\rho_x^2(j) - 2\rho_x(k)\rho_x(j)\rho_x(j+l) - 2\rho_x(l)\rho_x(j)\rho_x(j+k)\big).$$

From expression (12), we learn that the sample autocorrelations of the perturbed long memory process x_t actually underestimate their theoretical counterparts, with the negative bias decaying at a slowly rate as the sample size increases. However, as we already noted in Section 2, there always exists a sufficiently large signal-to-ratio ξ^* for which $\vartheta(\lambda, \xi^*)$, and consequently $\rho_x^*(k)$ and the asymptotic bias will be arbitrarily small, uniformly on $\alpha \in (0,1)$. For such values of ξ we shall expect sample autocorrelations to be arbitrarily close from below of their theoretical counterparts and hence, a correlogram of x_t with small and positive estimated autocorrelations.

Such a long memory series, x_t^*, will have a correlogram unidentifiable from a white noise process on the basis of the standard bounds $\pm 2/\sqrt{T}$.

Moreover, from expressions (13)-(15), it follows that for large enough ξ, the asymptotic covariance of the $\hat{\rho}_x(k)$ will tend to zero uniformly on $\alpha \in (0,1)$ and the sample correlations at different lags of x_t^* will appear uncorrelated with each other.

4. Perturbed Fractional White Noise Process

In this section we shall give explicit expressions of the theoretical moments of the perturbed long memory process x_t through the study of a particular member of the family of *ARFIMA* processes, namely, the so-called *fractional white noise* process, a discrete time version of the fractional Brownian motion process.

This process is defined as

(16) $\quad \Delta^d y_t = \eta_t,$

with $\eta_t \sim iid\left(0, \sigma_\eta^2\right)$, d is a noninteger number and

$$\Delta^d = (1-L)^d = 1 - dL + d(d-1)L^2/2! - d(d-1)(d-2)L^3/3! + \ldots,$$

and where by simplicity we assumed that $\mu = 0$.

The probability properties of this process were developed by Granger and Joyeux (1980) and Hosking (1981). They proved that the process is weakly stationary with long memory if and only if $0 < d < 1/2$, with Wold decomposition

(17) $\quad y_t = \sum_{j=0}^{\infty} \phi_j \eta_{t-j},$

(18) $\quad \phi_j = \dfrac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \sim \dfrac{1}{\Gamma(d)} j^{d-1}$ as $j \to \infty$.

Moreover, its autocovariance and autocorrelation functions are

(19) $$\gamma_y(0) = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma^2(1-d)},$$

(20) $$\gamma_y(k) = \sigma_\eta^2 \frac{\Gamma(1-2d)\Gamma(k+d)}{\Gamma(1-d)\Gamma(k+1-d)\Gamma(d)} \sim \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma(1-d)\Gamma(d)} k^{2d-1} \text{ as } k \to \infty,$$

(21) $$\rho_y(1) = \frac{d}{1-d},$$

and

(22) $$\rho_y(k) = \frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(k+1-d)\Gamma(d)} \sim \frac{\Gamma(1-d)}{\Gamma(d)} k^{2d-1} \text{ as } k \to \infty,$$

respectively, with $\Gamma(\circ)$ denoting the gamma or generalized factorial function and where the asymptotic approximations in (18), (20) and (22) follow from Shephard's Lemma.

Consequently, using (19)-(22) and the notation in Section 2, it follows that

(21) $$\lambda = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma(1-d)\Gamma(d)},$$

with $\alpha = 1-2d$,

(22) $$\gamma_x(0) = \sigma_\eta^2 \frac{\Gamma(1-2d)}{\Gamma^2(1-d)} + \sigma_\varepsilon^2,$$

(23) $$\gamma_x(k) = \sigma_\eta^2 \frac{\Gamma(1-2d)\Gamma(k+d)}{\Gamma(1-d)\Gamma(k+1-d)\Gamma(d)} \sim \lambda k^{2d-1} \text{ as } k \to \infty,$$

(24) $$\xi = \frac{\sigma_\varepsilon^2}{\sigma_\eta^2} \frac{\Gamma^2(1-d)}{\Gamma(1-2d)},$$

(25) $$\rho_x(1) = \frac{\sigma_\eta^2 \Gamma(1-2d)}{\sigma_\eta^2 \Gamma(1-2d) + \sigma_\varepsilon^2 \Gamma^2(1-d)} \frac{d}{(1-d)},$$

$$(26)\qquad \rho_x(k)=\frac{\sigma_\eta^2\Gamma(1-2d)}{\sigma_\eta^2\Gamma(1-2d)+\sigma_\varepsilon^2\Gamma^2(1-d)}\frac{\Gamma(1-d)\Gamma(k+d)}{\Gamma(k+1-d)\Gamma(d)}$$

$$\sim\frac{\sigma_\eta^2\Gamma(1-2d)}{\sigma_\eta^2\Gamma(1-2d)+\sigma_\varepsilon^2\Gamma^2(1-d)}\frac{\Gamma(1-d)}{\Gamma(d)}k^{2d-1}\text{ as }k\to\infty,$$

so that in this case the size of the perturbed long memory process x_t is given by

$$(27)\qquad \vartheta(\lambda,\xi)=\frac{\sigma_\eta^2\Gamma(1-2d)}{\sigma_\eta^2\Gamma(1-2d)+\sigma_\varepsilon^2\Gamma^2(1-d)}\frac{\Gamma(1-d)}{\Gamma(d)}.$$

5. Other Heuristic Approaches to Detecting Long Memory

In order to detect long memory from an heuristic approach, a more suitable plot than the correlogram suggested by the literature would be the so-called *log-log correlogram*, obtained by plotting $\log|\rho(k)|$ against $\log k$, the idea being that, if the asymptotic decay of the autocorrelations of the underlying series is hyperbolic, then the points of this plot should be scattered, for large lags, around a straight line with negative slope given by $-\alpha$, whereas that, if the series of interest has short memory, then the log-log correlogram should show divergence to minus infinity at an exponential rate.

Indeed, Beran (1994, pages 90-91) provides evidence on the fact that the log-log correlogram is mainly useful in cases of high long range dependence, with α near zero or for very long time series. Otherwise, it is very difficult to decide whether there is long memory in the data by looking at this device only. Essentially the same criticism applies to other heuristic statistics, such as the *R / S Plot*, the *Variance Plot* or the *Variogram Plot* (cf. Beran, 1994, chapter 4 for further details).

In our case, it follows from expression (9) that

$$(28)\qquad \log|\rho_x(k)|\sim\log|\vartheta(\lambda,\xi)|-\alpha\log k$$

for large lags, and then the size of the process does not affect the slope of the log-log correlogram which in this sense is a robust device against the near-observational problems found with the standard correlogram with perturberd long memory series.

On the other hand, knowing the autocovariance function of a process is equivalent to knowing its spectral density. Therefore, perturbed long memory dependence can also be analyzed in the *frequency domain*. In this sense, if we denote the spectral density of an arbitrary stationary time series z_t by

$$f_z(\omega) = \frac{\gamma_z(0)}{2\pi} \sum_{k=-\infty}^{\infty} \rho_z(k) \exp(ik\omega), \quad \omega \in [0, \pi]$$

then, under our set-up, it is straightforward to show that

$$(29) \qquad f_x(\omega) = f_y(\omega) + f_\varepsilon(\omega) = f_y(\omega) + \frac{\sigma_\varepsilon^2}{2\pi}, \quad \omega \in [0, \pi],$$

so that the spectrum of the perturbed long memory process x_t is just the sum of the spectrum of the long memory process y_t plus a constant, which is the spectrum of the white noise component of expression (2).

Consequently, it follows from expression (9) and Beran (1994, Theorem 2.1) that

$$(30) \qquad f_x(\omega) = f_y(\omega) \sim \frac{\lambda}{\pi} \Gamma(1-\alpha) \sin\left(\alpha \frac{\pi}{2}\right) \omega^{\alpha-1}, \quad \omega \to 0,$$

and hence, both processes have exactly the same spectral shape at low frequencies, tending for $\alpha < 1$ to infinity at the origin, in spite of the fact that $\rho_x(k) < \rho_y(k), k = 1,2,\ldots$. In the case of a perturbed fractional white noise process, the corresponding expression (29) for the exact population spectrum is given by

$$f_x(\omega) = \frac{\gamma_y(0)}{2\pi} \left(2 \sin\left(\frac{\omega}{2}\right)\right)^{-2d} + \frac{\sigma_\varepsilon^2}{2\pi}$$

(31) $$= \frac{\left(\gamma_y(0) + \sigma_\varepsilon^2\left(2\sin\left(\frac{\omega}{2}\right)\right)^{2d}\right)}{2\pi}\left(2\sin\left(\frac{\omega}{2}\right)\right)^{-2d} = \frac{\gamma_x^*(0)}{2\pi}\left(2\sin\left(\frac{\omega}{2}\right)\right)^{-2d}$$

In this manner, it follows from (22) and (31) that, for the consistency of model (2) in this particular case, the adding white noise process must have a variance given by

(32) $$\tilde{\sigma}_\varepsilon^2 = \sigma_\varepsilon^2\left(2\sin\frac{\omega}{2}\right)^{2d}.$$

On the other hand, from expressions (21) and (30) we obtain the asymptotic approximation to the population spectrum of the perturbed fractional white noise process at low frequencies

(33) $$f_x(\omega) = f_y(\omega) \sim \frac{\sigma_\eta^2\Gamma(1-2d)\Gamma(2d)}{\pi\Gamma(1-d)\Gamma(d)}\sin\left((1-2d)\frac{\pi}{2}\right)\omega^{-2d}, \quad \omega \to 0.$$

This asymptotic approximations, in turn, directly connect with the observation made by Granger (1966) that the typical shape of the spectral density for economic time series would be well approximated as a function with a pole at the origin. It seems, therefore, that in order to differentiate between a stationary series with long range dependence and one with short range dependence the spectral domain is also a more robust diagnostic tool than the plot of autocorrelations.

6. Portmanteau Tests

Jointly with the correlogram, most standard econometric packages frequently include by default the so-called Box-Pierce statistic

(34) $$Q_K = T\sum_{k=1}^{K}\hat{\rho}_x^2(k),$$

in order to test if x_t is well approximated by a white noise process. Under this null hypothesis, Q_K asymptotically is chi-square distributed. If x_t, however, is a short memory or a long memory process, from the Ergodic Theorem we know that $\hat{\rho}_x(k) \xrightarrow{p} \rho_x(k)$, $k \geq 1$, and hence, from Slutsky's Theorem, it follows that $\sum_{k=1}^{K} \hat{\rho}_x^2(k) \xrightarrow{p} \sum_{k=1}^{K} \rho_x^2(k)$, K fixed, so that

(35) $$Q_K \xrightarrow{p} \infty,$$

and the Box-Pierce test is consistent against short and long memory alternatives.

Notwithstanding, in our framework, we know that there may exists a sufficiently large signal-to-noise ratio, ξ^*, for which the long memory series x_t is undistinguible from a white noise on the basis of the correlogram. For this class of series, say x_t^*, the estimated autocorrelations, by assumption, satisfy the inequality

(36) $$\hat{\rho}_x^*(k) < \frac{2}{\sqrt{T}}, \ k \geq 1,$$

for given T, and hence

(37) $$Q_K < 4K$$

for any K. Consequently, it is enterely possible for the class of long memory processes x_t^* not to reject the null hypothesis of independence using the Box-Pierce test

In this sense, the so-called Ljung-Box test,

(38) $$Q_K' = T(T+2)\sum_{k=1}^{K} \frac{1}{T-k} \hat{\rho}_x^2(k),$$

by providing a greater bound,

(39) $$Q_K < 4\sum_{k=1}^{K} \frac{T+2}{T-k} = K', \qquad K' > K,$$

can improve the power of the portmanteau tests against the class of perturbed long memory processes x_t^*.

7. Common Factors

As a final section, let us now be concerned with the possibility of the existence of common factors among a set of perturbed long memory series. For simplicity, we shall only consider the bivariate case

(40) $x_{1t} = y_{1t} + \varepsilon_{1t}$,

(41) $x_{2t} = y_{2t} + \varepsilon_{2t}$,

with y_{1t}, y_{2t} two long memory processes of order α_1, α_2, respectively, and $\varepsilon_{1t}, \varepsilon_{2t}$ two orthogonal white noise processes with variances σ_1^2, σ_2^2 , respectively, and independent of their corresponding long memory processes y_{1t}, y_{2t}, giving rise to signal-to-noise ratios ξ_1, ξ_2, such that x_{1t}, x_{2t} are two perturbed long memory processes with sizes $\vartheta(\lambda_1, \xi_1)$ and $\vartheta(\lambda_2, \xi_2)$, respectively.

Under this framework, we shall be interesting on the following situation

(42) $y_{2t} = \wp y_{1t}, \qquad \alpha_1 = \alpha_2 = \alpha$,

for all t, so that x_{1t} and x_{2t} share a common long memory factor. For example, this situation could correspond with financial series of two related stock markets with different new information $\varepsilon_{1t}, \varepsilon_{2t}$ in each session.

Substituting (42) into (40) and (41) and rearranging, yields

(43) $x_{2t} - \wp x_{1t} = z_t$,

where $z_t = \varepsilon_{2t} - \wp \varepsilon_{1t}$, so that the error term z_t is a iid white noise process with zero mean and variance $\sigma_z^2 = \sigma_2^2 + \wp^2 \sigma_1^2$. Therefore, it follows that x_{1t} and x_{2t} are cointegrated long memory processes $CI(\alpha, \alpha)$.

Assume now, however, that the signal-to-noise ratio of x_{1t} is so large that x_{1t} has the identification problems that we are trying to illustrate along the different sections of this paper. Once more, in such case, the series x^*_{1t} will appear white noise on the basis of the estimated correlogram, concluding that the two series, x^*_{1t} and x_{2t} are unbalanced.

In this event, however, the log-log correlogram appears to be a particularly useful device in order to correct the above misspecification. Indeed, since z_t is white noise, it follows that

(44) $\log \rho_z(k) = 1, \; k \geq 1,$

and hence, if we plot $\log \rho_{x_1}(k)$, $\log \rho_{x_2}(k)$ and $\log \rho_z(k)$ against $\log k$, we shall obtain two paralell lines of slope $-\alpha$ corresponding to x^*_{1t} and x_{2t} plus a constant line at the level 1, corresponding to z_t.

Since in general $\wp$ is unknown, so is the z_t series. However, a consistent estimator of z_t can be constructed trough the least squares estimation of $\wp$ in expression (43), $\hat{z}_t = x_{2t} - \hat{\wp} x_{1t}$, since

$$\hat{\wp} = \frac{T^{-1}\sum_{t=1}^{T} x_{1t}x_{2t}}{T^{-1}\sum_{t=1}^{T} x_{1t}^2} \xrightarrow{p} \frac{E(x_{2t}x_{xt})}{E(x_{1t}^2)} = \wp .$$

7. Conclusions And Example

It is suggested that a correlogram shape that occurs fairly frequently with financial data can easily be explained. If the explanation is accepted, it follows that important properties of some financial series may be missed if the effects of a strong white noise are not allowed for.

An extreme form of the property being considered is to be found in Steigerwald (1997). Using the squares of the daily prices changes of the New York Stock Exchange Composite Index for a period January 1, 1990 to November 29, 1996, giving a sample of size 1748, Table 1 shows the autocorrelations and corresponding Box-Pierce (Q) statistics for lags 1 to 36. Although these autocorrelations are very small, all but one are positive. Thus, it would seem that the process contains long memory but the autocorrelations are all greatly discounted, and so the theory proposed above applies.

8. Appendix: Proof of Proposition 1

Along the proof of this result, we shall assume, without loss of generality and in the sake of simplicity that $E(x_t) = E(y_t) = 0$. Alternatively, suppose that we are working with the centered time series.

Under expressions (1)-(3) and Assumption 1, it follows from the Wold Representation Theorem and the Gaussianity assumption that the perturbed series x_t is a Gaussian long memory series

(A1) $\gamma_x(k) \sim \lambda k^{-\alpha}$, $\lambda > 0$, $\alpha \in (0,1)$,

as $k \to \infty$ with $\sum_{k=-\infty}^{\infty} \gamma_x(k)^2 < +\infty$ and $\sum_{k=-\infty}^{\infty} |\gamma_x(k)| = +\infty$, with a purely linearly decomposition given by

(A2) $$x_t = \sum_{h=0}^{\infty} \psi_h \upsilon_{t-h},$$

with $\upsilon_t \sim iid(0, \sigma_\upsilon^2)$, $\psi_0 = 1$ and $\sum_{p=0}^{\infty} \psi_p^2 < \infty$.

Consequently, using expression (2) and Assumption 1a we have that

$$\upsilon_t + \psi_1 \upsilon_{t-1} + \psi_2 \upsilon_{t-2} + \cdots = \eta_t + \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + \cdots + \varepsilon_t,$$

and collecting terms of the same power, yields

(A3) $\psi_j = \phi_j \sim \delta j^{-\beta}$,

as $j \to \infty$, with $\beta = (1+\alpha)/2$ under Assumption 1b.

Finally, since Assumption 1c implies that the disturbance term υ_t is Normally distrubuted with finite second moment, Theorem 6 of Hosking (1996) applies to the perturbed long memory process x_t and Proposition 1 follows. *Q.E.D.*

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Correlogram of SDCOMPOSITE

Date: 04/01/97 Time: 09:30 Sample 1: 1748 Included Observation: 1747		
	AC	*Q*-Stat
1	0.061	6.5672
2	0.091	21.196
3	0.045	24.757
4	0.102	42.866
5.	0.063	49.722
6	0.143	85.520
7	0.012	85.755
8	0.018	86.301
9	0.064	93.603
10	0.036	95.840
11	0.085	108.41
12	0.020	109.11
13	0.083	121.20
14	0.095	137.14
15	0.056	142.68
16	0.052	147.49
17	0.001	147.49
18	0.040	150.30
19	0.046	154.02
20	0.109	174.91
21	-0.002	174.92
22	0.045	178.50
23	0.014	178.83
24	0.052	183.70
25	0.066	191.33
26	0.024	192.37
27	0.036	194.68
28	0.020	195.35
29	0.058	201.26
30	0.038	203.82
31	0.030	205.48
32	0.046	209.30
33	0.050	213.83
34	0.026	215.04
35	0.014	215.38
36	0.023	216.30