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AN INTERIOR POINT ALGORITHM FOR COMPUTING EQUILIBRIA IN ECONOMIES WITH INCOMPLETE ASSET MARKETS*

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Abstract

Computing equilibria in general equilibria models with incomplete asset (GEI) markets is technically difficult. The standard numerical methods for computing these equilibria are based on homotopy methods. Despite recent advances in computational economics, much more can be done to enlarge the catalogue of techniques for computing GEI equilibria. This paper presents an interior-point algorithm that exploits the special structure of GEI markets. We prove that the algorithm converges globally at a quadratic rate, rendering it particularly effective in solving large-scale GEI economies. To illustrate its performance, we solve relevant examples of GEI markets.

Keywords: General equilibrium, incomplete markets, computation of equilibria, interior point methods.

JEL classification: C68; C63.

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1 Introduction

Since the time of Radner's (1972) paper, the general equilibrium analysis of sequential economies with incomplete markets has been the subject of extensive research in economic theory. These models, known as *General Equilibrium with Incomplete markets* (GEI), represent the most general and versatile tool in competitive economic theory, as they deal simultaneously with all real and financial markets and their interactions. The size and complexity of these models often demands the use of numerical methods to compute their equilibria. Surveys in this area are due to Magill and Shafer (1991), Magill and Quinzii (1996) and Hens (1998).

The literature on the computation of GEI equilibria is based on *path following* or *homotopy methods*. There are three main approaches to compute equilibria in the GEI model. The first method was given by DeMarzo and Eaves (1996), who considered the excess demand function defined on prices and elements of the Grassmannian manifold (see e.g. Duffie and Shafer (1985)). By applying the work of Brown et al. (1996a), they computed equilibria via a homotopy algorithm. The second algorithm for computing fixed points was developed by Brown et al. (1996b), who considered the excess demand function to be a function of prices only. Because this excess demand function is discontinuous, Brown et al. introduced an auxiliary asset to define a family of homotopies. The third approach based on the work of Schmedders (1998, 1999), who computed equilibria with homotopy techniques using the first-order conditions of the no-arbitrage agents' problems. To avoid discontinuities in the excess demand correspondence, he considered one agent with penalties for transactions on the asset markets instead of assuming lower bounds on short sales. By making these penalties smaller and smaller, the solutions of the homotopy function move closer and closer to the GEI equilibrium. Other contributions of note are those of Kubler and Schmedders (2000), Kubler (2001) and Herings and Kubler (2002).

Homotopy methods possess powerful theoretical properties, but these methods may be inappropriate from a practical point of view, due to the difficulty of setting up the homotopy function (see Kubler, 2001) and dealing with inequalities (see Watson, 2000). Furthermore, homotopy methods may fail to produce a solution even for relatively simple systems of nonlinear equations (e.g. see Nocedal and Wright (1999), Example 11.2). The focus of this paper is a proposal for an alternative algorithm for problems of practical relevance that cannot be solved by the existing approaches.

This paper introduces a tailor-made interior-point algorithm to compute equilibria in economies with incomplete asset markets. We consider a set of optimality conditions which are necessary and sufficient conditions for the existence of a GEI equilibrium, assuming the standard convexity assumptions for the agents' problem. These optimality conditions are the Kuhn-Tucker first-order conditions of the agents' utility maximization problems, the market clearing and no-arbitrage conditions. A distinctive characteristic in solving these optimality conditions is that they often include inequalities (e.g. equilibria can be restricted-domain). Homotopy or continuation methods can provide a solution, but sometimes a difficult one. We propose an alternative interior-point approach which is tailored to deal effectively with inequality-constrained nonlinear systems of equations.

Although interior-point methods are closely related to central path continuation methods (see Gill et al., 1986), the solution procedures are completely different. A relevant property of interior-point methods is their typical requirement for significantly less function and derivative evaluations and linear algebra operations than are required by homotopy methods, (see Garcia and Zangwill, 1981). This feature makes interior-point algorithms an attractive alternative to homotopy when considering large-scale GEI models such as the pricing of financial assets. In a recent survey, Esteban-Bravo (2004) suggests the application of interior-point methods to compute equilibria in complete markets. In this paper we fully explore this approach for solving GEI models, the complexity and scale of which demand efficient algorithms to compute equilibria.

We propose an interior-point algorithm that enjoys some computational advantages over the standard algorithms. In particular, we introduce two devices for increasing the speed of computations. Following the Gauss-Newton arguments, the second-order information of the nonlinear elements of the problem is neglected, a strategy that reduces the number of function evaluations needed. Second, we exploit the sparsity properties of GEI models. As a consequence, our algorithm finds accurate solutions with less computational costs than does a standard interior-point method. To be rigorous, we prove that the algorithm globally converges at a quadratic rate (i.e., – that the algorithm will eventually find an equilibrium if the economy has any).

Computational examples are presented, documenting the fact that the implementation is capable of solv-

ing relevant examples of GEI markets robustly and efficiently. Even though numerical comparisons lie beyond the scope of this paper, we compute equilibria for GEI models considered by Schmedders (1998), showing competitive running times. Larger models are also solved to provide evidence of its strong performance. Furthermore, we illustrate that the algorithm diverges when we consider economies in which equilibria do not exist and prove that this algorithm is particularly suited to solving problems in which some or all of the variables of interest are bounded. Such problems are commonly found in the literature of incomplete markets, for example, when considering short-selling constraints.

The remainder of the paper is organized as follows. Section 1 presents a two-period general equilibrium model with incomplete asset markets and the characterization of the equilibrium as the first-order conditions of the agents' utility maximization problems, the market clearing, and no-arbitrage conditions. In Section 2 we present an interior-point algorithm to compute such an equilibrium and study its convergence properties. Finally, the algorithm has been implemented for relevant examples of GEI markets, illustrating its strong performance, as shown in Section 3.

2 The GEI model

The basic GEI model describes an exchange economy over two time periods ($t = 0, 1$), with uncertainty over the state of nature in Period 1. At time $t = 0$ the economy is in state $s = 0$ which is known by each of the I consumers participating in the economy (i.e. all relevant information is symmetric across the economy). However, it is not known which of the S possible states at time $t = 1$ will occur. Trade occurs sequentially over time.

We assume that the markets on which commodities and financial assets are traded are competitive. In each state there are D goods, and for each good d in state s there exists a spot market with spot price p_{sd}. Hence, the commodity space is $\mathbb{R}_+^{D(S+1)}$. For any $x \in \mathbb{R}_+^{D(S+1)}$, x^T denotes the transpose of x, a $D(S+1)$-dimensional row vector. For any x, $y \in \mathbb{R}_+^{D(S+1)}$, $x \cdot y = x^T y$ denotes the inner product of vectors x and y. There is a finite number C of assets traded on financial markets. Let $\theta_i \in \mathbb{R}^C$ denote the portfolio of traded assets by the i-th consumer. An asset c can be purchased for a price q_c at time $t = 0$, and delivers a return across the states at time $t = 1$. The return of an asset c is described by its asset matrix $A^c = (A_1^c, ..., A_S^c)^T$, defined across all states at time $t = 1$, where A_s^c is the commodity bundle which asset c delivers for state s. The matrix A^c can be specified exogenously or be given as a function of some variables observed at $t = 1$. The asset c delivers a nominal return $V_s^c = p_s \cdot A_s^c$ in state s. Therefore, the asset structure is summarized by the asset matrix A (given in units of commodities) and the nominal return matrix $V(p)$:

$$\underset{DS\times C}{A} = \begin{pmatrix} A_{11}^1 & \dots & A_{11}^C \\ \vdots & & \vdots \\ A_{SD}^1 & \dots & A_{SD}^C \end{pmatrix}, \; \underset{S\times C}{V(p)} = \begin{pmatrix} V_1^1(p_1) & \dots & V_1^C(p_1) \\ \vdots & & \vdots \\ V_S^1(p_S) & \dots & V_S^C(p_S) \end{pmatrix}.$$

We assume that $S \geq C$. The completeness condition is important in the context of GEI markets. The financial markets are said to be *complete* if $\operatorname{rank}(V(p)) = S$ for any p. Under this condition, agents can insure themselves against any type of contingency in Period $t = 1$. When $\operatorname{rank}(V(p)) < S$, the financial markets are said to be *incomplete* (see Magill and Shafer, 1991).

Each consumer is described by a consumption set $\mathbb{R}_+^{D(S+1)}$, a set of traded assets $\mathbb{R}^C$, initial endowments for the $D(S+1)$ goods in each state $w_i = (w_{i0}, w_{i1}, ..., w_{iS}) \in \mathbb{R}_+^{D(S+1)}$, and a preference relation. The utility function $u_i : \mathbb{R}_+^{D(S+1)} \longrightarrow \mathbb{R}$ that represents the i-th consumer's preferences is assumed to be continuously differentiable, concave, strictly monotonous and increasing. Given $p \in \mathbb{R}_+^{D(S+1)}$ and $q \in \mathbb{R}^C$, each consumer faces the following problem:

$$\begin{array}{ll} \max_{x,\theta} & u_i(x_{i0}, ..., x_{iS}) \\ \text{s.t.} & p_0 \cdot x_{i0} \leq p_0 \cdot w_{i0} - q \cdot \theta_i, \\ & p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s \; \theta_i, \; \forall s. \end{array} \tag{1}$$

An equilibrium for the GEI economy is defined to be the prices (p^*, q^*) and the allocation (x^*, θ^*) satisfying: i) (x_i^*, θ_i^*) is an optimal solution to Problem (1), $\forall i = 1, ..., I$; ii) $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} w_i$, and $\sum_{i=1}^{I} \theta_i^* = 0$.

The next theorem provides a characterization of GEI equilibria.

Theorem 1 ***Characterization of GEI equilibria.*** *Assume* $rank\,(A_s) = D, \forall s$. *The allocation* $(x^*, \theta^*, p^*, q^*)$ *is an equilibrium for the economy if and only if there exist* $\delta^* \in \mathbb{R}^{I(S+1)}_{++}$ *and* $(x^*, \theta^*, p^*, q^*) \in \mathbb{R}^{ID(S+1)}_{+} \times \mathbb{R}^{C} \times \mathbb{R}^{ID(S+1)}_{++} \times \mathbb{R}^{C}$ *that satisfies the following optimality conditions:*

$$\begin{cases} \delta^*_{is} \nabla_{x_{is}} u_i\left(x^*_i\right) - p^{*T}_s = 0,\ \forall i,\ \forall s, \\ p^{*T}_0\ \left(x^*_{i0} - w_{i0}\right) + q^{*T}\theta^*_i = 0,\ \forall i, \\ p^{*T}_s\ \left(x^*_{is} - w_{is} - A_s\theta^*_i\right) = 0,\ \forall i, \forall s, \end{cases} \tag{2}$$

$$\sum_{s=1}^{S} \frac{\delta^*_{i0}}{\delta^*_{is}} p^* A_s - \phi^* = 0,\ \forall i, \tag{3}$$

$$\begin{cases} \sum\limits_{i=1}^{I} x^*_i = \sum\limits_{i=1}^{I} w_i, \\ \sum\limits_{i=1}^{I} \theta^*_i = 0. \end{cases} \tag{4}$$

Equation (3) implies that in equilibrium the asset prices q^* do not allow arbitrage – namely that there exists a $\beta \in \mathbb{R}^{S}_{++}$ so that $\sum\limits_{s=1}^{S} \beta_s p^* A_s - \phi^* = 0$.

Note that the constraint qualification under differentiability, $\text{rank}\,(A_s) = D$, $\forall s$, does not imply that the financial markets are complete. For example, if $A_s = A_{s'}$, $\forall s \neq s'$, and $\text{rank}\,(A_s) = D$, we have that $rank\,(p^* A) = 1 \neq S$.

3 An interior-point algorithm

In this section we present an interior-point algorithm to compute GEI equilibria using the characterization given in Theorem 1. To facilitate the exposition, we denote this system of nonlinear equations by $H\left(z^*\right) = 0$, where all the variables are assumed to be lower-bounded by zero (i.e. $z^* = (\delta^*, x^*, \theta^*, p^*, \phi^*) \in \mathbb{R}^{N}_{+}$). Reformulating as a least-squares problem and introducing logarithmic barrier terms in order to remove the non-negativity bounds, we introduce the unconstrained problem:

$$\min \frac{1}{2} \left\| H\left(z\right) \right\|^2_2 - \mu \log z, \tag{5}$$

where $\mu > 0$ is a scalar called the *barrier parameter*. By letting μ converge to zero, the sequence of solutions $\left\{z^*_\mu\right\}$ to (5) converges to a solution z^* of $\min \left\| H\left(z\right) \right\|^2_2$. We also consider a tolerance parameter ϵ_{SIZE}, which forces the solutions $\left\{z^*_\mu\right\}$ to satisfy $\left\| H\left(z^*_\mu\right) \right\|^2_2 \leq \epsilon_{SIZE}$. To ensure that the local minimizer z^* is a solution of the original nonlinear system $H\left(z^*\right) = 0$, the tolerance parameter is decreased from one barrier problem to the next and must converge to zero. Therefore, we aim to compute the sequence of solutions to (5) with $\left\| H\left(z^*_\mu\right) \right\|^2_2 \leq \epsilon_{SIZE}$.

A minimizer for problem (5) must satisfy the perturbed Karush-Kuhn-Tucker (KKT) conditions:

$$J(z)^T H(z) - \mu Z^{-1} = 0,$$

where $J(z)$ denotes the Jacobian matrix of $H(z)$; $Z = diag\,(z)$ defines a diagonal matrix, the diagonal entries of which are the components of vector z; and Z^{-1} is the inverse matrix of Z. Let $e = (1, ..., 1)^T$ denote the vector of ones. Notice that $Ze = z$ and $Z^{-1}e = \left(1/\,z_1, ..., 1/\,z_n\right)$.

Introducing an auxiliary variable w, so that $W = \mu Z^{-1}$ where $W = diag\,(w)$, the KKT conditions can be rewritten as follows:

$$\begin{array}{c} J(z)^T H(z) - w = 0, \\ Z\ W - \mu e = 0. \end{array} \tag{6}$$

Note that $w \in \mathbb{R}^{N}_{+}$ because the vector z has N components lower bounded by zero.

In essence, an interior-point method consists of the application of Newton's method to find a solution to the nonlinear system (6). Newton's method provides search directions $(\Delta z, \Delta w)$ from the first-order Taylor series expansion for (6) about the values (z, w). Let (z_k, w_k) be the current iterate. Then the search direction $(\Delta z, \Delta w)$ is the solution of the following system of linear equations:

$$\begin{pmatrix} J_k^T J_k + L_k H_k & -I \\ W_k & Z_k \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta w \end{pmatrix} = - \begin{pmatrix} J_k^T H_k - w_k \\ Z_k \, W_k - \mu_k \, e \end{pmatrix}, \tag{7}$$

where $H_k = H(z_k)$ denotes the system H evaluated at (z_k, w_k), $J_k = J(z_k)$ denotes Jacobian of H_k evaluated at (z_k, w_k), and L_k denotes the Hessian of H evaluated at (z_k, w_k); and the next iterate (z_{k+1}, w_{k+1}) is defined as $(z_k, w_k) + (\Delta z, \Delta w)$. Rather than solving each system (7) as the standard interior-point method would do, we are content with an approximate solution (z_k, w_k) satisfying

$$\begin{pmatrix} J_k^T J_k & -I \\ W_k & Z_k \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta w \end{pmatrix} = - \begin{pmatrix} J_k^T H_k - w_k \\ Z_k \, W_k - \mu_k \, e \end{pmatrix}. \tag{8}$$

In other words, we omit the second-order information of the system of equations $H(z) = 0$. The second-order term $L_k H_k$ can be neglected, as H_k is small near the solution z_k of the (6). Most of the computational cost of an interior-point method is associated with the computation of the search direction. By exploiting the special features of our problem, we reduce the computational cost within the algorithm in terms of function evaluations and number of iterations.

In addition, the next iterate (z_{k+1}, w_{k+1}) should be forced to remain strictly positive, a requirement that is achieved by rescaling $(\Delta z, \Delta w)$. To ensure this requirement, scalars α_{z_k} and α_{w_k} must be chosen such that $z_k + \alpha_{z_k} \, \Delta z > 0$ and $w_k + \alpha_{w_k} \Delta w > 0$. These parameters are called *steplength parameters*. Therefore, at iteration k,

$$\begin{aligned} \alpha_{z_k} &= \min \left\{ 1, \min \left\{ -\frac{z_{ki}}{\Delta z_i} \text{ s.t. } \Delta z_i < 0 \right\} \right\}, \text{ and} \\ \alpha_{w_k} &= \min \left\{ 1, \min \left\{ -\frac{w_{ki}}{\Delta w_i} \text{ s.t. } \Delta w_i < 0 \right\} \right\}, \end{aligned}$$

where (z_{ki}, w_{ki}) and $(\Delta z_i, \Delta w_i)$ are the i-th component of vectors (z_k, w_k) and $(\Delta z, \Delta w)$. An additional condition on α_{z_k} is required to ensure global convergence of $\{z_k\}$; namely, the scalar α_{z_k} should be chosen such that the objective function $\frac{1}{2} \|H(z)\|_2^2 - \mu \log z$ decreases sufficiently in each iteration z_k (Armijo's rule) and the choice α_{z_k} is not too far from a minimizer of the objective function (Goldstein's rule). If these requirements are not satisfied, α_{z_k} should be modified. Because these criteria help us to find an appropriate step length α_{z_k}, they are called line-search methods. This procedure relies on a univariate function called merit function $m(\alpha)$, to measure the progress of the algorithm. A suitable merit function for our algorithm is

$$m(\alpha) = \frac{1}{2} \|H(z + \alpha \Delta z)\|_2^2 - \mu \log(\, z + \alpha \Delta z).$$

(Other examples of merit functions can be found in Nocedal and Wright, 1999). Then α_{z_k} results in a sufficient decrease if

$$m(\alpha_{z_k}) \leq m(0) + \rho \, \alpha_{z_k} \, \bigtriangledown m(0)^T \, \Delta z \tag{9}$$

$$\left| \nabla m(\alpha_{z_k})^T \, \Delta z \right| \leq \eta \left| \nabla m(0)^T \, \Delta z \right| \tag{10}$$

where $0 < \rho < \eta < 1$. Otherwise α_{z_k} should be reduced until conditions (9) and (10) are satisfied. In particular, when α_{z_k} does not satisfy (9) and (10), then we consider an update of the steplength as $\alpha_{z_k}/2$.

The complete iteration of the algorithm requires an update of the barrier parameter μ. This update should be carefully defined to obtain a rapidly convergent algorithm. The choice of μ is based on the satisfaction of the complementarity conditions $ZW = \mu$. Then the new value of μ at the k -th iteration is:

$$\mu_{k+1} = \gamma \frac{z_k^T w_k}{n}, \tag{11}$$

where $0 \leq \gamma < 1$ and n is the dimension of vector z. This definition ensures that $\mu \to 0$ if Problem (5) has a solution. When we choose γ close to 0, we are requiring a rapid convergence of μ to 0. The choice of the updating parameter γ may significantly affect the efficiency of the overall method.

The algorithm terminates when the following stopping criteria are satisfied:

$$\left\| \begin{pmatrix} J^T H - w \\ ZW - \mu\, e \end{pmatrix} \right\|_2^2 \leq \epsilon_{SIZE}, \tag{12}$$

$$\|H(z)\|_2^2 \leq \epsilon_{TOL}. \tag{13}$$

The stopping criterion (12), hereafter called *the first stopping criterion,* is related to the fulfilment of the first-order KKT conditions for problem (5) and guarantees the boundedness of the variables. The stopping criterion (13) , which we call *the second stopping criterion,* guarantees satisfaction of the optimality conditions for the existence of equilibria under convexity assumptions. Therefore, if the algorithm converges, it converges to an equilibrium of the economy.

3.1 Practical implementation issues

The algorithm described in the preceding section includes certain parameters and conditions that have not yet been completely specified. In the following paragraphs we indicate how these implementation issues were treated.

3.1.1 Choosing the initial point

The algorithm performs better if the starting point (z_0, w_0) , where $z_0 = (\delta_0, x_0, \theta_0, p_0, \phi_0)$ meets the bound constraints. For simplicity, the algorithm sets the auxiliary variables w_0 and $\delta_0, \theta_0, p_0, \phi_0$ equal to a vector of ones and x_0 equal to the initial endowment.

3.1.2 Choosing the parameters

In our implementation, a current iterate is considered optimal when $\epsilon_{TOL} = 10^{-14}$ and $\epsilon_{SIZE} = 10^{-10}$. The Armijo and Goldstein parameters are $\rho = 0.0001$ and $\eta = 0.9$. The choice of the parameter μ is based on the satisfaction of the complementarity conditions and depends on the parameter γ. Typically, $\gamma = 0.1$.

3.1.3 Computing the search direction

The computationally most expensive part of an interior-point algorithm is the computation of the Newton search direction, because this calculation involves the solution of a potentially large system of linear equations. It is important to note that the matrix

$$\begin{pmatrix} J_k^T J_k & -I \\ W_k & Z_k \end{pmatrix} \tag{14}$$

is sparse – that it contains a significant number of zero-valued elements. Note that matrices I, W_k, and Z_k are diagonal and, furthermore, that J_k is of the form:

$$J_k = \begin{pmatrix} \nabla_x u & \nabla_x^2 u & 0 & -I & 0 \\ 0 & P & V & X & U \\ 0 & 0 & 0 & \Phi A & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{pmatrix},$$

where $\nabla_x u$ and $\nabla_x^2 u$ are diagonal matrices because the vector has the form $(u_1(x_{10}, ..., x_{1S}), ..., u_I(x_{I0}, ..., x_{IS}))$; P is the diagonal matrix of the prices vector $(p_0, p_1, ..., p_S)$; X is the diagonal matrix of the consumption allocations for all consumers $(x_{10}, ..., x_{1S}, ..., x_{i0}, ..., x_{iS}, ..., x_{I0}, ..., x_{IS})$; Φ is the diagonal matrix of the auxiliary variables $(\delta_{10}, ..., \delta_{1S}, ..., \delta_{i0}, ..., \delta_{iS}, ..., \delta_{I0}, ..., \delta_{IS})$; V is the full matrix of returns; and U is an auxiliary matrix, respectively defined as

$$V = \begin{pmatrix} q \\ -A \end{pmatrix}; \quad U = \begin{pmatrix} e_{1\times C} \\ 0_{S\times C} \end{pmatrix}.$$

This sparsity can and should be exploited to improve the computational efficiency. In the computational results reported in the next section, we exploit the sparsity properties of the full matrix (14), which reduces computation time by eliminating operations on zero elements.

3.2 The algorithm

In the preceding description of the algorithm, we have considered simple bounds of the form $z^* \geq 0$ in order to simplify the exposition. The generalization of this algorithm to problems such that $H(z^*) = 0$ where $l \leq z^* \leq u$ is straightforward. Some of the entries in l could be equal to $-\infty$, and some of those in u could be equal to ∞. If we rewrite the problem as $H(z^*) = 0$ with $z^* - l \geq 0$ and $u - z^* \geq 0$, then the finite bounds will be included in the objective function via logarithmic barrier terms, and therefore we consider two auxiliary variables $w^1 = \mu (z - l)^{-1}$ and $w^2 = \mu (u - z)^{-1}$.

A summary of the proposed interior-point algorithm is:

Step 1. Select the parameters $\rho \in (0,1)$, $\theta \in (0,1)$, $\gamma \in [0,1)$, the tolerance ϵ_{SIZE}, and the final stop tolerance ϵ_{TOL}. Initialize variables $\left(z, w^1, w^2\right)$ and set the initial value of the barrier parameter μ as

$$\mu = \gamma \frac{(z-l)^T w^1 + (u-z)^T w^2}{2N}. \tag{15}$$

Step 2. Evaluate the objective function and its derivatives at $\left(z, w^1, w^2\right)$. Repeat until

$$\left\| \begin{pmatrix} J^T H - w^1 + w^2 \\ (Z-L) W^1 - \mu\, e \\ (U-Z) W^2 - \mu\, e \end{pmatrix} \right\| \leq \epsilon_{TOL}$$

and (13) are satisfied:

Step 2.1. Compute a Newton search direction:

$$\begin{pmatrix} J^T J & -I & I \\ W^1 & (Z-L) & 0 \\ -W^2 & 0 & (U-Z) \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta w^1 \\ \Delta w^2 \end{pmatrix} = - \begin{pmatrix} J^T H - w^1 + w^2 \\ (Z-L) W^1 - \mu\, e \\ (U-Z) W^2 - \mu\, e \end{pmatrix}.$$

Step 2.2. Compute scalars α_z, α_{w^1} and α_{w^2} such that:

$$\begin{aligned} \alpha_z &= \min\left\{1, \min\left\{-\frac{z_i}{\Delta z_i} \text{ s.t. } \Delta z_i < 0\right\}\right\}, \\ \alpha_{w^1} &= \min\left\{1, \min\left\{-\frac{w_i^1}{\Delta w_i^1} \text{ s.t. } \Delta w_i^1 < 0\right\}\right\}, \\ \alpha_{w^2} &= \min\left\{1, \min\left\{-\frac{w_i^2}{\Delta w_i^2} \text{ s.t. } \Delta w_i^2 < 0\right\}\right\}, \end{aligned}$$

and update α_z until (9) and (10) are satisfied.

Step 2.3. Update variables as $z + \alpha_z\, \Delta z$ and $w + \alpha_w \Delta w$.

Step 2.4. Update parameter μ as described in (15).

There are many different types of interior-point algorithms with certain common mathematical themes having to do with the logarithmic barrier function. The distinguishing feature of our algorithm is the use of a Gauss-Newton approximation of the search direction. In Appendix A we analyze the convergence properties of this variant and prove that our algorithm will find an equilibrium if the economy has any. However, when there are no equilibria, our algorithm diverges (or converges to infinity), because the matrix J is singular (due to the collinearity of the returns matrix). As a consequence, the search direction Δz is infinite. Example 3 in the next section illustrates the behavior of the algorithm when there are no equilibria. In Appendix A we also prove that our algorithm converges quadratically, which broadly means that the number of correct figures in z_k eventually doubles at each step.

4 Numerical examples

The algorithm has been implemented in MATLAB 6.5 on an Intel Centrino Pentium M 1.6 GHz with machine precision 10^{-16}. The first example is intended to show how the algorithm is set up and to compute a GEI equilibrium.

Example 2 ***Two-period exchange economy. DeMarzo and Eaves (1996).***

Consider a two-period exchange economy with three consumers, three states in the second period, two assets, and two goods. Each consumer i has a utility function of the form $u_i(x) = \sum_{s=1}^{3} \pi_s \left(B - x_{s1}^{\alpha_i} x_{s2}^{1-\alpha_i}\right)$, with parameters $B = 57$, $\pi = \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\alpha_1 = \alpha_2 = \frac{3}{4}$ and $\alpha_3 = \frac{1}{4}$. The agent's endowments are $w_1 = w_2 = (10, 10; 25, 20; 20, 20; 15, 20)^T$ and $w_3 = (20, 20; 5, 10; 10, 10; 15, 20)^T$. The asset matrix A is given by

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 2 & -1 \end{bmatrix}.$$

For this example, $H(\cdot) = 0$ is a system of 52 equations with 52 variables (the auxiliary variables $\delta^* \in \mathbb{R}^{12}_{++}$ and the equilibrium $(x^*, \theta^*, p^*, \phi^*) \in \mathbb{R}^{40}$). Taking as an initial point $\delta_0 = e$, $x_{10} = w_1$, $x_{20} = w_2$, $x_{20} = w_3$ and $\theta_0, p_0, \phi_0 = e$, where e is a vector of ones, the algorithm converges in 0.56 seconds (20 iterations) with an error of $\epsilon_{SIZE} = 10^{-14}$. Table 1 shows the iterates of the portfolio allocations for each iteration until convergence. $\theta(i, c)$ denotes the portfolio decision of the asset c by agent i.

Iterations	θ(1,1)	θ(1,2)	θ(2,1)	θ(2,2)	θ(3,1)	θ(3,2)
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	7.5882	-7.4157	13.6431	-13.4706	-20.3205	21.7970
3	5.3402	-3.1748	11.1594	-7.7269	-16.4392	10.9132
4	1.6257	-4.5227	10.2507	-12.8887	-11.8786	17.4080
5	4.5801	-6.4406	8.3884	-9.3628	-12.9683	15.8037
6	1.6009	-5.3987	2.5987	-6.2855	-4.1996	11.6842
7	-1.2995	-3.9109	-1.0984	-4.0964	2.3979	8.0074
8	-0.7307	-4.3251	-0.7121	-4.3375	1.4428	8.6626
9	-0.6793	-4.4071	-0.6604	-4.4248	1.3397	8.8319
10	-0.6463	-4.4258	-0.6464	-4.4253	1.2927	8.8512
11	-0.6359	-4.4387	-0.6342	-4.4404	1.2700	8.8791
12	-0.6355	-4.4379	-0.6356	-4.4378	1.2711	8.8757
13	-0.6340	-4.4397	-0.6338	-4.4398	1.2678	8.8795
14	-0.6342	-4.4393	-0.6342	-4.4393	1.2685	8.8786
15	-0.6340	-4.4395	-0.6340	-4.4396	1.2680	8.8791
16	-0.6341	-4.4395	-0.6341	-4.4395	1.2681	8.8790
17	-0.6340	-4.4395	-0.6340	-4.4395	1.2681	8.8790
18	-0.6340	-4.4395	-0.6340	-4.4395	1.2681	8.8790
19	-0.6340	-4.4395	-0.6340	-4.4395	1.2681	8.8790
20	-0.6340	-4.4395	-0.6340	-4.4395	1.2681	8.8790

Table 1: Iterates of the portfolio allocations approaching convergence in Example 2.

Figure 1 shows the values of the first and second stopping criteria and the computation time for each iteration until convergence, which reveal the strong performance of the algorithm, even though it started from a poor initial point.

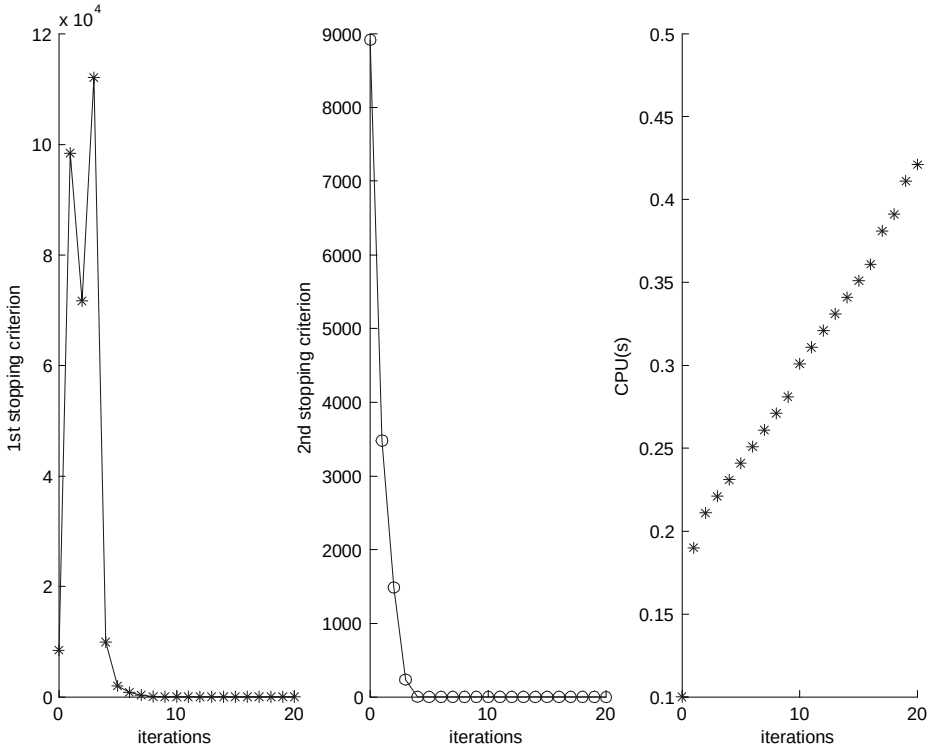

Figure 1. The 1st and 2nd stopping criteria and the cumulative CPU time (s) until convergence.

The following computations are intended to demonstrate how the algorithm behaves in large-scale markets.

Example 3 ***Large-scale computations.***

We consider four variations of the two-period exchange economy described in DeMarzo and Eaves (1996). All the models consider a two-period exchange economy with two goods, two assets with the asset matrix A given in Example 1, and three states in the second period. Endowments and constants of the utility function were randomly generated using: $w_i \sim U[0.75, 1.25]$ and $\alpha_i \sim U[0, 1]$ for each consumer i. However, Model 1 considers 3 agents, Model 2 has 15 agents, Model 3 consists of 30 agents, and Model 4 has 60 agents. Table 2 shows the number of variables and equations for each model and the number of iterations and running times until convergence with an error of $\epsilon_{SIZE} = 10^{-14}$. For all models, the algorithm converges to an equilibrium in a moderate number of iterations, which illustrates its rapid convergence. The scale of the problem only affects the cost of computation, mainly because of the cost of function evaluations.

	Number variables/equations	Number of iterations	CPU (seconds)
Model 1: 3 agents	52	22	.51
Model 2: 15 agents	260	26	4.09
Model 3: 30 agents	520	29	20.71
Model 4: 60 agents	1040	32	216.89

Table 2: Number of iterations and running times for four different models.

The third example illustrates the behavior of the algorithm if the economy has no equilibrium.

Example 4 ***One-period exchange economy with no equilibria.***

Suppose there is no first period. Consider a one-period exchange economy with two consumers, two states, two assets, and two goods. Each consumer i has a utility function of the form $U_i(x) = u_i(x_1) + u_i(x_2)$, with

$$u_i(x_1) = \frac{1}{3}\log(x_{s1}) + \frac{2}{3}\log(x_{s2}),\ u_i(x_2) = \frac{2}{3}\log(x_{s1}) + \frac{1}{3}\log(x_{s2}).$$

The agent's endowments are $w_1 = (1-\varpi, 1-\varpi, \varpi, \varpi)^T$ and $w_2 = (\varpi, \varpi, 1-\varpi, 1-\varpi)^T$ for some positive $\varpi < 1/2$. For this example, $H(\delta^*, x^*, \theta^*, p^*, \phi^*) = 0$ is a system of 20 equations with 20 variables. We assume that there are bounds only on the auxiliary variables $\delta^* \in \mathbb{R}^4_{++}$, i.e. $l \leq \delta^*$ with $l = (0,0,0,0)$.

This economy has no equilibrium (see Hart, 1975) and our algorithm does not converge. The collinearity of the return matrix makes system (7) incompatible, and consequently the algorithm indicates that an unbounded search direction in the portfolios iterates should be taken from the initial point (the computed search direction is infinite). In other words, the algorithm fails to converge.

The following example is intended to illustrate that this algorithm can effectively deal with inequality-constrained nonlinear systems of equations.

Example 5 ***Two-period exchange economy with short-selling constraints.***

The easiest way to guarantee the existence of equilibria in GEI models is to impose a short-sales constraint (see Hart, 1975). We impose exogenous bounds on the portfolio variables of Example 4 to illustrate the practicability of our algorithm in solving inequality-constrained nonlinear system of equations.

Consider Example 4 with lower bounds on portfolios, i.e. $\theta \geq -E$ for some $E \in \mathbb{R}^4_+$. Again, we must solve the same system of 20 equations and 20 variables as before, $H(\delta^*, x^*, \theta^*, p^*, \phi^*) = 0$, but now there is a lower bound l on the auxiliary variables δ^* and portfolios θ^*, i.e. $l \leq (\delta^*, \theta^*)$ with $l = (0,0,0,0,-E,-E,-E,-E)$. Assuming that $\varpi = .3$ and $E = 1$, the algorithm converges to the equilibrium in 0.39 seconds (41 iterations) with an error of $\epsilon_{SIZE} = 10^{-14}$. Table 3 shows the last iterates of the portfolio allocations for each iteration until convergence.

Iterations	θ(1,1)	θ(1,2)	θ(2,1)	θ(2,2)
30	0.119	0.094	-0.119	-0.094
31	0.1198	0.0955	-0.1198	-0.0955
32	0.1373	0.0747	-0.1373	-0.0747
33	0.137	0.0734	-0.137	-0.0734
34	0.1366	0.075	-0.1366	-0.075
35	0.1358	0.0752	-0.1358	-0.0752
36	0.126	0.0847	-0.126	-0.0847
37	0.1258	0.0845	-0.1258	-0.0845
38	0.126	0.0842	-0.126	-0.0842
39	0.126	0.0842	-0.126	-0.0842
40	0.126	0.0842	-0.126	-0.0842
41	0.126	0.0842	-0.126	-0.0842
42	0.126	0.0842	-0.126	-0.0842

Table 3: Last iterates of the portfolio allocations approaching convergence in Example 5.

Note that the lower bounds on the portfolios are not binding, but they prevent the problem of rank-deficiency faced in Example 4. Essentially, when bounds on the portfolio allocations are considered, it becomes necessary to search numerically for new auxiliary variables w. Given (8), the search direction of (z, w) is computed from:

$$\begin{aligned} \left(J^T J + Z^{-1}W\right)\Delta z &= -\left(J^T F - \mu\, Z^{-1}e\right), \\ \Delta w &= Z^{-1}W\Delta z - W + \mu\, Z^{-1}e. \end{aligned}$$

In Example 4, the search direction Δz of z_k was defined as:

$$\left(J^T J + diag\left(w_1/\delta, 0, ..., 0\right)\right)\Delta z = -\left(J^T F - \mu\ Z^{-1} e\right),$$

where $diag\left(w_1/\delta, 0, ..., 0\right)$ is the diagonal matrix, the diagonal entries of which are the components of vector $\left(w_1/\delta, 0, ..., 0\right)$. However, in the case in which there are bounds on portfolios, the search direction Δz of z_k is given as:

$$\left(J^T J + diag\left(w_1/\delta, 0, ..., 0,\ w_2/\theta, 0, ., 0\right)\right)\Delta z = -\left(J^T F - \mu\ Z^{-1} e\right).$$

The term w_2/θ is non-null by definition and its summation corrects the singularity of $J^T J$; therefore, the economy has an equilibrium.

As the interior-point approach treats lower bounds on portfolio allocations as a barrier function $-\mu \log\left(\theta + E\right)$ and the degree of influence of the barrier term is determined by the size of μ which goes to zero as $k \to \infty$, in certain instances these bounds can be considered as a computational artifact to help the computation of equilibria.

5 Summary and conclusions

In this paper we describe an efficient algorithm for the computation of equilibria in general equilibrium models with incomplete asset markets. The procedure is based on an interior-point method to define the search direction for the new iterates. Particular care has been taken to reduce the computational cost, avoiding the use of second-order information for the more complicated elements of the problem. The algorithm has proven to be globally convergent and the local convergence rate is quadratic.

Given its practicability and efficiency, this algorithm seems to be a promising alternative for computing equilibria when the existing homotopy continuation approaches are difficult to apply.

6 Appendix A. Convergence properties of the algorithm

To prove convergence properties of the proposed algorithm, we should have sufficient descent on the merit function in every iteration, and this function should be bounded below. The global convergence of the algorithm is obtained under the following assumptions:

A.1. The second derivatives of H are Lipschitz continuous on the region defined by the bounds.

A.2. Strict complementarity holds at all first-order KKT points.

A.3. The matrix J has full rank at all first-order KKT points and for a positive constant M,

$$\frac{1}{M}\left\|v\right\|^2 \leq v^T J^T\ Jv \leq M\left\|v\right\|^2.$$

A.4. The iterates $\{z_k\}$ generated by the algorithm remain in a compact set.

For the sake of a clearer presentation, denote $w_k = \left(w_k^1, w_k^2\right)^T$. We start by stating the boundedness of dual variables.

Lemma 6 *For a fixed μ, the lower bounds and the upper bounds of the box constraints in the dual step-size rule are bounded away from zero and bounded from above, if the corresponding components of z_k are also bounded above and away from zero.*

Proof. See Yamashita (1998). ■

As a consequence, the elements of $Z_k^{-1} W_k$ are bounded above and non-zero and, by Assumption A.3, there exists a positive constant N such that $\frac{1}{N}\left\|v\right\|^2 \leq v^T\left(J_k^T J_k + Z_k^{-1} W_k\right) v \leq N\left\|v\right\|^2$.

We now show that Δz is a descent direction for the merit function $m\left(\alpha\right)$.

Theorem 7 *For a fixed μ, Δz is a descent direction for the merit function $m(\alpha)$, i.e. $\bigtriangledown m(0)^T \Delta z \leq 0$.*

Proof. Note that the direction Δz is given as:

$$\left(J_k^T J_k + Z_k^{-1} W_k\right) \Delta z = -\left(J_k^T H_k - \mu\ Z_k^{-1} e\right).$$

Then

$$\Delta z^T \bigtriangledown m(0) = \Delta z^T \left(J_k^T H_k - \mu X^{-1} e\right) = -\Delta z^T \left(J_k^T J_k + Z_k^{-1} W_k\right) \Delta z - N \left\|\Delta z\right\|^2 \leq 0.$$

■

By definition, the step size α is always bounded. We next prove that α is also non-zero.

Lemma 8 *The step size α is always bounded and non-zero.*

Proof. Assume that α is reduced infinitely. This means that the descent condition does not hold; therefore

$$m(0) - m(\alpha) \leq -\rho\ \alpha\ \bigtriangledown m(0)^T\ \Delta z^T, \text{ for all } \alpha. \tag{16}$$

From the Taylor expansion, we have $m(\alpha) - m(0) = \alpha \bigtriangledown m(0)^T \Delta z + o(\alpha)$. Then for a small enough α, using Theorem 7,

$$m(\alpha) - m(0) - \rho\ \alpha\ \bigtriangledown m(0)^T\ \Delta z^T = (1-\rho)\ \alpha\ \bigtriangledown m(0)^T\ \Delta z^T \leq 0$$

which contradicts (16). ■

The next two results prove the global convergence of the algorithm for a fixed μ.

Lemma 9 *For a fixed μ, assume also that at any initial point z_0, the set $\{z : m(z_k;\mu) \leq m(z_0;\mu)\}$ is bounded. Then $\lim_{k\to\infty} \|\Delta z\| = 0$.*

Proof. Assume $\lim_{k\to\infty} \|\Delta z\| \geq \varepsilon > 0$. Using the descent lemma (see Bertsekas p. 553)

$$m(z_{k+1};\mu) - m(z_k;\mu) \leq \alpha \bigtriangledown m(z_k)^T \Delta z + \frac{1}{2} K \alpha^2 \left\|\Delta z\right\|^2 \leq \frac{1}{2} K \alpha^2 \left\|\Delta z\right\|^2$$

for some positive constant K, as m is a Lipschitz continuous function and Δz is a descent direction of the merit function m, using Theorem 7.

By hypothesis, the sequence $\{m(z_k;\mu)\}$ converges and $\lim_{k\to\infty} (m(z_k;\mu) - m(z_{k+1};\mu)) = 0$, thus

$$m(z_k;\mu) - m(z_{k+1};\mu) \geq \frac{1}{2} L \alpha^2 \left\|\Delta z\right\|^2 \to 0,$$

which implies $\|\Delta z\| \to 0$, using Lemma 8. ■

Theorem 10 *For a fixed μ, the algorithm terminates at a point, satisfying the perturbed optimality conditions (6).*

Proof. As $\Delta w = -\left(W_k - \mu\ Z_k^{-1} e + Z_k^{-1} W_k \Delta z\right)$ and $\lim_{k\to\infty} \|\Delta z\| = 0$, by Lemma 9, the following holds:

$$\lim_{k\to\infty} \left\|w_k + \Delta w - \mu\ Z_k^{-1} e\right\| = 0.$$

Therefore, there exists a vector w^* such that $\lim_{k\to\infty} \|w_k + \Delta w\| = \mu\ Z_*^{-1} e$. Furthermore, the first equation of system (6) satisfies

$$\left(J_k^T J_k + Z_k^{-1} W_k\right) \Delta z = -\left(J_k^T H_k - \mu\ Z_k^{-1} e\right),$$

and letting $k \to \infty$,

$$\lim_{k\to\infty} \left\|J_k^T H_k - \mu\ Z_k^{-1} e\right\| = 0,$$

i.e. there exists a z^* such that $J_*^T H_* - \mu\ Z_*^{-1} e = 0$. Then (z^*, w^*) is a solution of the perturbed optimality conditions (6). ■

The convergence of the algorithm can be proven using the next theorem.

Theorem 11 *Suppose that Assumptions A.1 to A.4 hold. The limit of sequence* $\{(z_k, w_k)\}$ *exists and satisfies the optimality conditions of Problem* (5).

Proof. Note that the definition of μ_k ensures that $\{\mu_k\}$ is positive and monotonically decreasing with $\mu \to 0$. Thus, it follows from theorems 10 and 11 that the algorithm terminates at a point (z_k, w_k), satisfying the optimality conditions of Problem (5).

Using analogous arguments to those discussed in Akrotirianakis and Rustem (2000), it is satisfied $\lim_{k\to\infty} \|\mathcal{F}_\mu(z_k, w_k)\|_2 = 0$. ■

Next we prove the Q-quadratic convergence of the algorithm. First we present an auxiliary result. Denote

$$\mathcal{F}_\mu(z_k, w_k) = \begin{pmatrix} J_k^T H_k - w_k \\ Z_k W_k - \mu_k e \end{pmatrix}, \mathcal{H}_\mu(z_k, w_k) = \begin{pmatrix} J_k^T J_k & -I \\ W_k & Z_k \end{pmatrix}.$$

Lemma 12 *Suppose that Assumptions A.1 to A.4 hold. Then*

$$\|\Lambda_k - I\| \leq O(\|\mathcal{F}_\mu(z_k, w_k)\|) + O(\mu_k),$$

where $\Lambda_k = diag(\alpha_k)$.

Proof. See Yamashita and Yabe (1996). ■

Using this lemma, we establish the quadratic convergence of the algorithm.

Theorem 13 *Suppose that Assumptions A.1 to A.4 hold and that the sequence* $\{(z_k, w_k)\}$ *generated by the proposed algorithm converges to the solution* (z^*, w^*). *Assume that* $\mu_k = O(\|\mathcal{F}_\mu(z_k, w_k)\|)$. *Then there exists* $\varepsilon > 0$ *such that for all* $(z_0, w_0) \in B((z^*, w^*), \varepsilon)$, *the sequence* $\{(z_k, w_k)\}$ *is well defined and converges to* (z^*, w^*) *Q-quadratically.*

Proof. Denote $h_k = (z_k, w_k)$ and $h^* = (z^*, w^*)$. Since $h_0 \in B(h^*, \varepsilon)$, $\|h_0 - h^*\| < \varepsilon$. Assume $\|h_k - h^*\| < \varepsilon$, then

$$\begin{aligned} h_{k+1} - h^* &= h_k + \Lambda_k(\Delta z, \Delta w) - h^* = h_k - \Lambda_k \mathcal{H}_\mu(h_k)^{-1} \mathcal{F}_\mu(h_k) - h^* = \\ &= \mathcal{H}_\mu(z_k, w_k)^{-1} [\Lambda_k \mathcal{F}_\mu(h^*) - \Lambda_k \mathcal{F}_\mu(h_k) + \mathcal{H}_\mu(z_k, w_k)(h_k - h^*)] = \\ &= \Lambda_k \mathcal{H}_\mu(z_k, w_k)^{-1} [\mathcal{F}_\mu(h^*) - \mathcal{F}_\mu(h_k) - \mathcal{H}_\mu(z_k, w_k)(h_k - h^*)] + (\Lambda_k - 1)(h_k - h^*). \end{aligned}$$

Taking Euclidean norms, we have

$$\begin{aligned} &\|h_{k+1} - h^*\| \leq \left\|\mathcal{H}_\mu(z_k, w_k)^{-1}\right\| O\left(\|h_k - h^*\|^2\right) + \|\Lambda_k - 1\| \|h_k - h^*\| \leq \\ \leq\ & O\left(\|h_k - h^*\|^2\right) + [O(\|\mathcal{F}_\mu(h_k)\|) + O(\mu_k)] \|h_k - h^*\|. \end{aligned}$$

From Assumption A.1, H is Lipschitz continuous, and therefore there exists a constant $N > 0$ such that, for all $(z_k, w_k) \in B((z^*, w^*), \varepsilon)$,

$$\|\mathcal{F}_\mu(h_k)\| = \|\mathcal{F}_\mu(h_k) - \mathcal{F}_\mu(h^*)\| \leq N \|h_k - h^*\|.$$

Then there exists a constant $\xi > 0$, such that $\|h_{k+1} - h^*\| \leq \xi \|h_k - h^*\|^2$. Hence the sequence $\{(z_k, w_k)\}$ converges Q-quadratically to (z^*, w^*). ■

7 References

Akrotirianakis, I., Rustem, B., 2000. A primal-dual interior point algorithm with an exact and differentiable merit function for nonlinear programming. Optimization Meth. & Soft. 14, 1-36.

Bertsekas, D. P., 1995. Nonlinear Programming. Athena Scientific, Belmont, Massachussets.

Brown, D. J., DeMarzo, P.M., Eaves, B.C., 1996a. Computing Zeros of Sections of Vector Bundles using Homotopies and Relocalization. Mathematics of Operations Research 21, 26-43.

Brown, D. J., DeMarzo, P.M., Eaves, B.C., 1996b. Computing Equilibria when Asset Markets are Incomplete. Econometrica 64, 1-27.

DeMarzo, P.M., Eaves, B.C., 1996. Computing Equilibria of GEI by relocalization on a Grassmann Manifold. Journal of Mathematical Economics 26, 479-497.

Duffie, D., 1992. The Nature of Incomplete Markets.In: J. J. Laffont, ed., Advances in Economic Theory; Sixth World Congress, Vol. II. Cambridge University Press, Cambridge, U.K., pp. 214-262.

Duffie, D., Shafer, W., 1985. Equilibrium in Incomplete Markets I. Journal of Mathematical Economics 14, 285-300.

Esteban-Bravo, M., 2004. Computing Equilibria in General Equilibrium Models via Interior-point Methods. Computational Economics, 23, 147-171.

Fiacco, A.V., McCormick, G. P., 1968. Nonlinear Programming: Sequential Unconstrained Minimization Techniques. John Wiley & Sons, Inc., New York.

Garcia, C. B., Zangwill, W. I., 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Gill, P. E., Murray, W., Sauders, M. A., Tomlin, J. A. and Wright, M. H., 1986. On projected Newton barrier methods for linear programming and an equivalence to Karmarkar's projective method. Mathematical Programming 36, 183-209.

Hart, O. D., 1975. On the Optimality of Equilibrium when the Market Structure is Incomplete. Journal of Economic Theory 11, 418-443.

Hens, T., 1998, Incomplete Markets. In: Kirman, A., ed., Elements of General Equilibrium Theory, Festschrift in Honor of Gérard Debreu (Oxford University Press, Oxford, pp. 139-210.

Herings, P. J.-J., Kubler, F., 2002. Computing Equilibria in Finance Economies. Forthcoming in Mathematics of Operations Research.

Kubler, F., Schmedders, K., 2000. Computing Equilibria in Stochastic Finance Economies. Computational Economics 15, 145-172.

Kubler, F., 2001. Computable general equilibrium with financial markets, Economic Theory 18, 73-96.

Magill, M. J. P., Shafer W., 1991. Incomplete Markets. In: W. Hildenbrand and H. Sonnenschein, eds., Handbook of Mathematical Economics, Volume IV. North-Holland, New York, pp. 1523-1614.

Magill, M. J. P., Quinzii, M., 1996. Theory of Incomplete Markets, Vol. 1. The MIT Press, Massachusetts.

Nocedal, J., Wright, S. J., 1999. Numerical Optimization. Springer Series in Operations Research. Springer-Verlag, New York.

Radner, R., 1972. Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets. Econometrica 40, 289-303.

Schmedders, K., 1998. Computing Equilibria in the General Equilibrium Model with Incomplete Asset Markets. Journal of Economic Dynamics and Control 22, 1375-1401.

Schmedders, K., 1999. A homotopy algorithm and an index theorem for the general equilibrium model with incomplete asset markets. Journal of Mathematical Economics 32, 225-241.

Yamashita, H., Yabe, H., 1996. Superlinear and quadratic convergence of some primal-dual interior point methods for constrained optimization. Mathematical Programming 75, 377-397.

Yamashita, H., 1998. A globally convergent primal-dual interior point methods for constrained optimization. Optimization Methods & Software 10, 443-469.

Watson, L. T., Billups, S. C., Morgan, A. P., 1987. Algorithm 652: HOMPACK: A suite of codes for globally convergent homotopy algorithms. ACM Trans. Math. Software 13, 281-310.

Watson, L. T., Sosonkina, M., Melville, R. C., Morgan, A. P., Walker, H. F., 1997. HOMPACK90: A suite of Fortran 90 codes for globally convergent homotopy algorithms. ACM Trans. Math. Softw. 23, 4, 514-549.

Watson, L. T., 2000. Theory of globally convergent probability-one homotopies for nonlinear programming. SIAM Journal of Optimization 11(3), 761-780.

Wright, M. H., 1992. Interior methods for constrained optimization. In: A. Iserles (ed.), Acta Numerica 1992. Cambridge University Press, N.Y., pp. 341-402.

Wright, M. H., 1998. Ill-conditioning and computational error in primal-dual methods for nonlinear programming. SIAM Journal of Optimization 10, 443-469.