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MAXMIN PORTFOLIOS IN FINANCIAL IMMUNIZATION

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Abstract

The existence of maxmim bond portfolios is proved in very general contexts, and so for instance, this existence holds if an immunized portfolio does not exist but all the considered portfolios have duration equal to the investor planning period. To characterize the maxmin portfolio, saddle point conditions are found, and from them, an algorithm is given. This algorithm permits to find the maxmin portfolio in practical situations. Relations between maxmin portfolios and the ones minimizing the dispersion measures (for instance, the M-squared or the Ñ measure) are also studied. In particular, it will be proved that minimizing the dispersion measure and looking for maxmin portfolio are equivalent strategies only when we are working with pure discount bonds. Finally, as a consecuence of the obtained results, two new strategies to invest are proposed.

Key Words and Phrases

Maxmin Portfolio, Immunized Portfolio, Saddle Point Condition, Dispersion Measure.

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Abstract

Literature on immunization has shown that an immunized portfolio is a maxmin portfolio, but the opposite is not necessarily true.. In models where immunization is not feasible, in addition to matching duration, many strategies has been proposed, *i.e.*, minimizing dispersion measures M^2 or Ñ, to include a maturity matching bond, *etc.* However, in these models the maxmin portfolios have never been computed, and it seems that the proposed strategies are halfway between a matching duration and a maxmin portfolio.

In this paper we shall show that maxmin portfolios are characterized by saddle point conditions and from them an algorithm is given to compute the maxmin portfolios. Our model is specialized on the very general set of shocks from which the dispersion measures M^2 and Ñ have been developed. We shall show that by minimizing the dispersion measure, subject to matching duration, and by computing the maxmin portfolio both are only equivalent strategies if we work with zero coupon bonds. We shall compute the maxmin portfolios with examples using coupon bonds, and from them, two new strategies will be proposed.

I. Introduction

Several authors have studied maxmin portfolios in financial immunization theory. The concept was introduced initially by Bierwag and Khang (1979) revealing that maxmin portfolios guarantee the largest amount of money after an additive shock on the interest rates. Bierwag and Khang (1979), Khang (1983) and Prisman (1986) have proved in different models, and under different assumptions on the shocks on interest rates, that immunized portfolios are always maxmin ones, and are also matching duration portfolios.

In a recent paper, Balbás and Ibáñez (1995) show that the opposite fails in models for which total immunization is not possible. Furthermore, in these models, besides a matching duration, many strategies have been proposed. For instance, Fong and Vasicek (1984) (see also Montrucchio and Peccati (1991)) show that the M^2 measure gives us a bound on the possible capital losses after a shock, and therefore, this dispersion measure should be minimized. Another dispersion measure (which should also be minimized) is given in Balbás and Ibáñez (1995). Bierwag *et al.* (1993) and others show that the strategy that works best empirically is including a maturity matching bond. Prisman and Shores (1988) propose to minimize other dispersion measures without matching duration.

However, in these models the maxmin portfolio has been never computed, and it seems that these proposed strategies (defined for example as "Risk Minimizing Strategies for Portfolio Immunization", Fong and Vasicek (1984)) are halfway between a matching duration and a maxmin portfolio. Moreover, in these models, Balbás and Ibáñez (1995) prove that a maxmin portfolio always exist and that both concepts, maxmin and immunized are equivalent only if the latter can be found. Therefore we have that the concept of maxmin portfolio clearly extends and generalizes the concept of the immunized one beyond more general models. All these precedents show that studing and computing the maxmin portfolio is not only a new work and a important task by themselves, but is closely related to some puzzles in this literature, and therefore form the object of the present paper.

In this paper we follow the model of Balbás and Ibáñez (1995) where, amongst other things, they prove the existence of maxmin portfolios amongst bonds under three very general assumptions. We begin the paper by extending the existence results amongst bonds up to a convex subset of feasible portfolios, with a finite number of extreme points, *e.g.*, matching duration portfolios. Then, following an very common approach in game theory, we show that maxmin portfolios are characterized by saddle point conditions, and therefore, by means of an equations system. This system is non-linear and more difficult to solve than the one that usually appears in game theory. Furthermore, the system cannot be solved with a linear program and consequently, an algorithm is developed, which leads to the maxmin bond portfolio.

The model is specialized on the set of shocks from which the dispersion measures M^2 (Fong and Vasicek (1984)) and $\tilde{N}$ (Balbás and Ibáñez (1995)) are developed. The sets of shocks have bounded derivative and have bounded variations between two arbitrary instants, respectively. Both set of shocks are very general and they allow almost any change on the instantaneous forward interest rates. We show that the four strategies by minimizing both dispersion measures or computing the maxmin strategies are equivalent, only if we work with zero coupon bonds.

Finally, we compute the maxmin portfolio in two examples for both sets of shocks with coupon bonds, amongst bonds and also amongst matching duration portfolios because this is the classical immunization result, see Fisher and Weil (1971). By computing the maxmin portfolio we also obtain the worst shock and the guaranteed value of this portfolio. These two values can be very interesting to the investor. We compute the maxmin portfolio for many values for parameter λ, to see the path of the maxmin portfolio. As a consequence of the results obtained and from the theoretical advantages of the bounded shocks, two new strategies are proposed for the shocks of Balbás and Ibáñez (1995). First, estimating the parameter λ and computing the maxmin portfolio amongst bonds, which is theoretically the best strategy. Second, estimating the parameter λ and computing the maxmin portfolio amongst matching duration portfolios, because these portfolios work well empirically and do not depend very much on parameter λ.

The paper's outline is the following. The second section establishes the set of hypotheses, and from them, the existence of maxmin portfolios is proved in a general context. The third section is devoted to characterizing the maxmin portfolios by means of saddle point conditions. The fourth one compares the maxmin portfolio with the one obtained if we apply other proposed strategies, and in particular, if we minimize some dispersion measures. In the Fifth section we solve the maxmin portfolio under two examples with coupon bonds, by applying a previously developed algorithm. Finally, the last section points out the most important conclusions.

II. Existence of Maxmin Portfolios

In this section we will follow the notation introduced in Balbás and Ibáñez (1995). Let $[0,T]$ be a time interval being $t=0$ the present moment. Let us consider n default free and option free bonds with maturity less or equal than T, and with prices $P_1, P_2, \cdots, P_n$ respectively. Let K be the set of admissible shocks on the interest rate, K being a subset of the vector space of real valued functions defined on $[0,T]$.

Let m, $(0 < m < T)$ represent the investor planning period, and the real valued functionals

$$V_i : K \rightarrow R \quad i = 1, 2, \cdots, n$$

be such that $V_i(k)$ (where $k \in K$ is any admissible shock) is the *i-th* bond value at time m if shock k takes place.

In Balbás and Ibáñez (1995) were assumed the following three hypotheses:

H1: K is a convex set.

H2: V_i is a convex functional for $i = 1, 2, \cdots, n$.

H3: $V_i(k) > 0$ for $i = 1, ..., n$ and for any $k \in K$.

These assumptions are quite simple and clear.

Let $C > 0$ be the total amount to invest, and let $q = (q_1, q_2, \cdots, q_n)$ be a vector such that q_i, $i = 1, 2, \cdots, n$, represents the number of units of the *i-th* bond that the investor is going to buy. The constraints

$$\sum_{i=1}^{n} q_i P_i = C \ , \qquad q_i \geq 0 \quad i = 1, ..., n. \tag{1}$$

are clear, and we will represent by Q the set of portfolios q such that expression (1) holds.

The functional

$$V(q,k) = \sum_{i=1}^{n} q_i V_i(k) \tag{2}$$

gives us the value for time m of portfolio q if the k shock takes place, and it is linear in the q variable and convex in the k one.

Let us define the guaranteed amount by portfolio q as follows

$$\overline{V}(q) = Inf\{V(q,k);\ k \in K\}$$

We will say that q^* is a maxmin portfolio in Q if it solves the program

$$\left.\begin{array}{c} Max\ \overline{V}(q) \\ q \in Q \end{array}\right\} (PQ)$$

Now we will introduce the concept of maxmin portfolio in any convex closed subset of Q.

If Q^* is a convex closed subset of Q then q^* is a maxmin portfolio in Q^* if it solves

$$\left.\begin{array}{c} Max\ \overline{V}(q) \\ q \in Q^* \end{array}\right\} (PQ^*)$$

Let us point out that if q' is maxmin in Q and q^* is maxmin in Q^* then the inequality

$$\overline{V}(q^*) < \overline{V}(q')$$

could hold, that is, the guaranteed amount by portfolios in Q could be bigger than the guaranteed amount in Q^*. Balbás and Ibáñez (1995) show that program (PQ) always has a solution, *i.e.*, there always exists a maxmin portfolio. Now we are interested in generalizing the latter result to convex subset Q^* with a finite number of extreme points.

Theorem 2.1. If Q^* has a finite number of extreme points, then program (PQ^*) has a solution, *i.e.*, there always exists a maxmin portfolio $q^* \in Q^*$.

Proof. See the Appendix. □

The interest of the latter result would be clearer if we consider the set Q^* as the set of feasible portfolios with a duration equal to the investor planning period. This is the classical strategy for immunizing a bond portfolio against additive shocks. If the shocks are continuously differentiable (as in Fong and Vasicek (1984)) then an immunized portfolio does not exist, but there are maxmin portfolios in Q and also in Q^*. We have an analogous situation if we consider integrable and bounded shocks (as in Balbás and Ibáñez (1995)).

III. The Saddle Point Conditions

Once we know that maxmin portfolios do exist, we will study the general conditions for characterizing them in practical situations. If we carefully analyze the proof of theorem 2.1, we will obtain that for a portfolio q^* maxmin in Q^*

$$\overline{V}(q^*) = Inf\{U(k);\ k \in K\}C \tag{3}$$

where U is the real valued functional given in (25). Therefore, if we consider the minimization program

$$\left.\begin{matrix} Min\ U(k) \\ k \in K \end{matrix}\right\} (PK)$$

and $k^* \in K$ is its solution, then

$$\overline{V}(q^*) = U(k^*)C \tag{4}$$

The functional U may be also given by

$$U(k) = Max\{\frac{V(q,k)}{C};\ q \in Q^*\} \tag{5}$$

since for a fixed shock k, V is linear in the q variable and then its maximun must be attained in an extreme point of Q^*. Therefore, (4) may be written as

$$\underset{\{q \in Q^*\}}{Max}\ \underset{\{k \in K\}}{Inf}\ V(q,k) = \underset{\{k \in K\}}{Inf}\ \underset{\{q \in Q^*\}}{Max}\ V(q,k) \tag{6}$$

The latter equality is well known in game theory, characterizes the existence of saddle points for two persons zero sum games. This fact may be applied in immunization theory to obtain the maxmin portfolios by means of saddle point conditions.

Definiton 3.1. We will say that a pair $(q^*, k^*) \in Q^* \times K$ is a saddle point of functional V in $Q^* \times K$ if for any portfolio $q \in Q^*$ and for any admissible shock $k \in K$ we have

$$V(q,k^*) \leq V(q^*,k^*) \leq V(q^*,k)$$

Prisman (1986) shows that a portfolio q is immunized if and only if $(q, 0)$ is a saddle point of V. The following result may be considered as an extension of Prisman's (1986), and may be applied in models for which total immunization is not possible.

Theorem 3.2. Given a portfolio $q^* \in Q$ and a shock $k^* \in K$, then q^* is maxmin in Q^* and k^* solves (PK)[1] if and only if (q^*, k^*) is a saddle point of V in $Q^* \times K$.

Proof. See the Appendix[2]. □

Let us introduce a system of equations to characterize the saddle points of $V(q, k)$ in $Q^* \times K$. To do this, we are going to consider that the set $\{q^1, q^2, \cdots, q^l\}$ of extreme points of Q^* is known, and therefore, portfolios in Q^* are given by their linear convex combinations. We are also going to assume, that set K is included in a normed space, that all its points are interior, and that functionals $V_1, V_2, \cdots, V_n$ are Gateaux differentiable (see Luenberger (1968)).

It may be easily proved that (q^*, k^*) is a saddle point of V if and only if

$$q^* = \sum_{i=1}^{l} \alpha_i^* q^i$$

and

$$\sum_{i=1}^{l} \alpha_i^* = 1 \qquad (7)$$

$$\alpha_i^*[V(q^i, k^*) - Max\{V(q^j, k^*); \ j = 1, \cdots, l\}] = 0 \quad i = \{1, \cdots, l\} \qquad (8)$$

$$\alpha_i^* \geq 0 \quad i = \{1, \cdots, l\} \qquad (9)$$

$$\sum_{i=1}^{l} \alpha_i^* \left. \frac{\partial V(q^i, k)}{\partial k} \right|_{k=k^*} = 0 \qquad (10)$$

where the derivative in equation (10) is the Gateaux differential of functional V with respect to its k variable evaluated in $k = k^*$ (see Balbás and Ibáñez (1995)). As it will be shown, this are only the partial derivatives with respect to the shock parameters when working with reasonable kind of shocks.

To prove equations (7) to (10) let us point out that if (q^*, k^*) is a saddle point then (10) clearly follows from $V(q^*, k) \geq V(q^*, k^*)$, (7) and (9) are obvious, and (8) is due to $V(q^*, k^*) \geq V(q, k^*)$ and for a linear program, any maximum is a linear convex combination of points which are extremes and maximums.

Conversely, if the system of equations holds, then in order to prove that (q^*, k^*) is a saddle point we only have to bear in mind that the necessary optimality conditions for convex minimization programs (or linear maximization programs) are also sufficient.

IV. Is Minimizing the Dispersion Measures equivalent to looking for Maxmin Portfolios?

Following the usual assumptions in immunization, let us consider that the q portfolio pays a continuous coupon $c(t)$ $(0 \leq t \leq T)$. If $g(s)$ $(0 \leq s \leq T)$ represents the instantaneous forward interest rates and $k(s)$ is a shock on $g(s)$, then the q portfolio value at time m is given by

$$V(q,k) = \int_0^T c(t) exp \left[\int_t^m (g(s) + k(s)) ds \right] dt \tag{11}$$

Denoting the capitalization rate between 0 and m by

$$R = exp \left[\int_0^m g(s) ds \right] \tag{12}$$

and the coupon present value by

$$c(t,0) = c(t) exp \left[- \int_0^t g(s) ds \right] \tag{13}$$

we have[3]

$$V(q,k) = R \int_0^T c(t,0) exp \left[\int_t^m k(s) ds \right] dt \tag{14}$$

Many dispersion measures have been introduced by the literature (see for instance Alexander and Resnick (1985)) but perhaps only two allow us to bound the possible capital losses after a shock on the interest rates. These measures are given by

$$M^2 = \int_0^T (m-t)^2 \frac{c(t,0)}{C} dt$$

and

$$\tilde{N} = \int_0^T \mid m - t \mid \frac{c(t,0)}{C} dt$$

and were defined in FV and BI respectively[4].

In this section we will consider the set of shocks for FV and BI since they are very general (FV only assumes shocks with a bounded derivative and BI works with bounded shocks) and, as already mentioned, possible capital losses may be measured. By working with these two kinds of shocks, our purpose is to compare the final result for the investor if he or she follows one of the four following strategies in choosing the portfolio.

a) Look for the maxmin portfolio in the FV case.
b) Look for the maxmin portfolio in the BI case.
b) Minimize the dispersion measure M^2.
c) Minimize the dispersion measure $\tilde{N}$.

Let us analyze two different situations. First, it will be proved that if we work with pure discount bonds, then options *a*), *b*), *c*) or *d*) lead to the same solution (the same final portfolio). However, in the general case (that is, if we consider coupon bonds) then *a*), *b*), *c*) or *d*) are far from being equivalent, as will be shown in some examples.

Let us consider that we are in the FV case. Then, any shock $k(s)$ is continuously differentiable and there exists a constant λ such that

$$\frac{dk(s)}{ds} \leq \lambda \text{ for } 0 \leq s \leq T.$$

Then, if we define the worst shocks $k^*(s)$ by

$$k^*(s) = \lambda_0 + \lambda(s - m) \tag{15}$$

being $\lambda_0 = k(m)$, it clearly follows from the mean value theorem (see also BI) that

$$\left.\begin{array}{l} k(s) \geq k^*(s) \text{ if } s \leq m \\ k(s) \leq k^*(s) \text{ if } s > m \end{array}\right\} \tag{16}$$

From (14) and (16) we obtain

$$V(q,k) \geq V(q,k^*) \tag{17}$$

for any portfolio q, and therefore

$$U(k) \geq U(k^*) \tag{18}$$

being U the functional introduced in (25). Since U is a convex functional (it is the maximun of a finite number of convex functionals as stated by expression (3)), in order to prove that program (PK) has an interior solution, it is sufficient to show that $U(k^*) \to +\infty$ if $\lambda_0 \to +\infty$ or $\lambda_0 \to -\infty$. This may be easily proved if we take into account (14) and (15) to evaluate $V(q, k^*)$, and also amongst the n feasible bonds there is at least one coupon paid before m and another paid after m.

By analogy, in the BI case, there exists a constant λ such that

$$| k(s_1) - k(s_2) | \leq \lambda \quad \text{for} \quad 0 \leq s_1 \leq s_2 \leq T$$

and therefore, given a shock $k(s)$, one can find a real number λ_0 (see BI) such that

$$\left. \begin{array}{l} k(s) \geq \lambda_0 - \frac{\lambda}{2} \ \text{ if } \ s \leq m \\ k(s) \leq \lambda_0 + \frac{\lambda}{2} \ \text{ if } \ s > m \end{array} \right\} \tag{19}$$

The worst shocks now are given by the step function

$$k^*(s) = \left\{ \begin{array}{ll} \lambda_0 - \frac{\lambda}{2} & \text{if } \ s \leq m \\ \lambda_0 + \frac{\lambda}{2} & \text{if } \ s > m \end{array} \right. \tag{20}$$

and following the ideas in the latter case, it may be easily proved that program (PK) also has an interior solution in this case. Hence, the results of the second section can be applied to both cases, either FV shocks or BI shocks.

A. The case of pure discount bonds

Proposition 4.1. Let us assume that the n considered bonds are pure discount bonds, and let $q^* \in Q$ be a feasible portfolio. Then, the four following conditions are equivalent.

i) q^* is maximin in Q if we consider the FV shocks.

ii) q^* is maximin in Q if we consider the BI shocks.

iii) q^* solves the program

$$\left. \begin{array}{c} Min \ M^2 \\ q \in Q^* \end{array} \right\}$$

where M^2 represents the FV dispersion measure and Q^* is the set of matching duration portfolios in Q.

iv) q^* solves the program

$$\left.\begin{array}{l} Min\ \tilde{N} \\ q \in Q^* \end{array}\right\}$$

where $\tilde{N}$ represents the BI dispersion measure and Q^* is the same as iii).

Proof. See the Appendix. □

If $T_1 < T_2 \cdots < T_n$ represents the maturity of the n considered bonds and if $T_i \neq m$ for $i = 1, \cdots, n$, we show in the proof that the q^* portfolio satisfying any of these four conditions has a duration equal to m, there are only two bonds in q^*, and these two bonds have a couple (T_i, T_j) of maturities such that $T_i < m < T_j$, T_i is the largest maturity amongst the ones smaller than m, and T_j is the smallest maturity amongst the ones greater than m.

The surprising result derived from Proposition 4.1 is that amongst zero coupon bonds the maxmin portfolio always has a duration equal to the investor planning period. From this, it is clear that some strategies proposed in Prisman and Shores (1988) as alternative strategies to the one proposed by FV (1984) are not very reasonable, since the FV strategy is the maxmin strategy.

B. The case of coupon bonds

If the bonds pay coupons the situation is quite different, and maxmin portfolios must be determined by equations (7) to (10). It may be difficult in practice to solve this system of equations, since we have to simultaneously determine shock k^* and the extreme portfolios weights. In any case the system becomes far easier if we know the shock k^*.

We will now present an algorithm to find the maxmin portfolios for both BI shocks and FV shocks. As has already been stated, $U(k)$ is a convex functional which can be analyzed by means of real function $U(\lambda_0)$[5] for which we always have an interior global minimum λ_0^*. Once we know the set $\{q^1, q^2, \cdots, q^l\}$ of extreme points of Q^* (or Q) we can evaluate $U(\lambda_0)$ by

$$U(\lambda_0) = Max\{\frac{V(q^1, \lambda_0)}{C}, \cdots, \frac{V(q^l, \lambda_0)}{C}\} \tag{21}$$

To calculate λ_0^* we start at an initial value $\lambda_0(1)$ and consider the sequence

$$\lambda_0(1), \lambda_0(2), \lambda_0(3), \cdots,$$

where $\lambda_0(i+1) = \lambda(i) + s$ being $s > 0$ the step. This setp may be taken as small as necessary.

Then we consider the sequence

$$U(\lambda_0(1)), U(\lambda_0(2)), U(\lambda_0(3)), \cdots \tag{22}$$

where $U(\lambda_0(i))$ is given by (21).

Since U is a convex function, λ_0^* will be determined when the secuence (22) begins to increase.

Once we have λ_0^*, the maxmin portfolio may be easily determined from the system of equations (7) to (10).

V. Solving maxmin portfolio in some examples

We are going to apply the latter algorithm to solve a simulation model which tests how maxmin portfolios and the ones minimizing the dispersion measures work out in practice. The algorithm has been applied by taking the step $s = 10^{-8}$.

We will take an investor planning period of five years, $m = 5$, along the line of empirical studies on immunization, see Bierwag *et al.* (1993). We will assume a plane term structure on the interest rates, $r = 10\%$, to make it easier.

Let us consider the set of coupon bonds presented in Table 1 and denote by Q_1 (respectively Q_2) the set of feasible portfolios (see (1)) (respectively, the set of feasible portfolios which do not contain bond number thirteen). The first column in Table 1 is the bond number, the second one is its maturity, the third is the coupon (as a percentage), the fourth is the coupon periodicity (in months), the fifth is the bond duration, in years, the sixth is its M^2 measure, and the last one is its $\tilde{N}$ measure.

In Table 2 (respectively 3) we give the extreme points of Q_1^* (respectively Q_2^*), which are the set of portfolios in Q_1 (Q_2) matching duration. The first column is the portfolio number, the second is the first bond in the portfolio, the third is the second bond in the portfolio, the fourth is the first bond percentage and the last columns are their M^2 and Ñ dispersion measures. The portfolios are arranged according to their Ñ measure.

By applying the algorithm, we have solved the five following questions, and the results are given in Tables (4) to (11).
i) The maxmin portfolios in Q_1, Q_2, Q_1^*, Q_2^*.
ii) The weights of the different bonds (or portfolios) in the maxmin strategy.
iii) The worst shock, *i.e.*, number λ_0^*.
iv) The value at m guaranteed by the maxmin portfolio as a percentage with respect to the promised amount.
v) The maxmin portfolio duration.

Let us remark that in order to choose the set of bonds we have taken into account the empirical result revealed by in Bierwag *et al.* (1993). They empirically show that the best immunization strategy is matching duration but including a maturity matching bond. This strategy is better than a bullet or a barbell matching duration strategies, a maturity matching strategy and the FV strategy. At the time the work was carried out the Ñ measure had not been developed.

The reason for studying two different situations, (*i.e.* working with or without bond number thirteen) is that this bond must play an important role in immunization strategies. In fact, its maturity is exactly five years, which means (according to Bierwag *et al.* (1993) result) that this bond will probably be in the "best strategy". On the other hand, this bond pays the lowest coupon (only 9% per year) which will be useful to minimize the dispersion measures. The remaining bonds can be considered very normal bonds found in the market. The differences amongst them arise from their maturities, and from their periodicity in paying the coupons (one or two coupons per year).

A. The FV and the BI shocks.

Now we point out three arguments given in BI about the advantages of the BI shocks over the FV shocks.

First, the bounded shocks have a theoretical argument in their favor with respect to the FV shocks. In the bounded shocks the parameter λ can be understood as a volatility measure, as how much the shocks on the forward instantaneous interest rates can differ between two instants, and this parameter can be estimated. On the other hand, the FV shocks parameter which is a derivative, has a more complex economic meaning and it is more difficult to estimate.

Second, shocks with a bounded derivative are also bounded but the opposite is false. Shocks with small variations could have a very big derivative. Then, we have that bounded shocks include the Fong and Vasicek shocks but the opposite is false.

Finally, the worst shocks on the term structure of interest rates, in the FV situation are very unreal, because they imply very big values when t is far from m, see BI.

B. The Risk Immunization Measures M^2 and Ñ.

Before starting the maxmin portfolio analysis, we can spend some time discussing measures M^2 and Ñ.

An initial result that we can observe in table 2 is that the portfolio minimizing the Ñ measure includes a maturity matching bond (bond number 13). If we work without the bond number 13 (see table 3) then we obtain six portfolios with almost the same Ñ measure and portfolios number 3,5 and 6 (which include a maturity matching bond) have a Ñ measure very close to the minimum value of this dispersion measure. On the other hand, the portfolio with the minimum M^2 measure, is far from including a maturity matching bond.

The reason why portfolio 1 in table 2 has the mininum Ñ but a very high M^2 could be the following. Bond thirteen has a lower duration and therefore, the portfolio must invest more money in another bond to match the duration with m. Since M^2 is a quadratic dispersion measure, the second bond makes it increase a lot, while the Ñ measure does not grow as fast in this second bond. Portfolio number 4 has the minimum M^2 because it is composed by 60% of bond number nine (which has a maturity of six years, a duration close five years, and not large M^2).

Likewise, in Table 1 regarding the first twelve bonds, we observe that the second (or eigth) bonds, which are maturity matching bonds, have an Ñ measure which is approximately half of the adjacent bonds, the first and third (or seventh and neinth). These bonds mature upon four or six years respectively. There is a drastic reduction in the dispersion of the second (or eigth) bond when using Ñ measure. This reduction is the greatest between each pair of consecutive bonds. However, if we use M^2 measure, we can see that the reduction is not as large. Moreover, the said reduction is the smallest between each pair of consecutive bonds.

It is seems that there exists a closer relation between a maturity matching bond and the BI strategy than between a maturity matching bond and the FV strategy.

Let us now analyze results obtained on the maxmin portfolio.

C. Maxmin Portfolio amongst bonds.

In all the situations considered (FV shocks, BI shocks, working with or without bond number thirteen) we can see in Tables 4,5,8 and 9 the maxmin portfolio paths. We think these paths are very robust since they do not seem to depend on the considered bonds nor on the plane term structure (r=10%) initially taken. When parameter λ is big enough, the maxmin portfolio is almost composed only by a maturity matching bond. However, there is always little percentage invested in a bond with a maturity bigger than five years in order to avoid a shock ($\lambda_0 \rightarrow -\infty$) which implies that the value of the porfolio would be nothing at m. If λ decreases, the percentage invested in the maturity matching bond also decreases and the portfolio duration increases. This duration is always smaller than five years, because the bonds pay their coupon before they pay their principal. When λ closes to zero the maxmin portfolio converges to the minimum M^2 (for FV shocks) or to nearly the minimum $\tilde{N}$ (for BI shocks).

Regarding BI shocks, we can see in Table 4 that the maxmin portfolio is always composed by the same bonds (12 and 13) and only the percentage of bonds in the portfolio changes. In Table 5, when parameter λ is lower than 0.11 then the maxmin portfolio is not unique. Bonds 8 or 9 combined with bonds 10,11 or 12 also make a maxmin portfolio, although only one solution appears in Table 5. For FV shocks, in Tables 8 and 9, we can see that the maxmin portfolio does not follow such a robust path. There is a value for parameter λ from which a maturity matching bond appears in the maxmin portfolio. Then, it seems there is more coherence in the results obtained with BI shocks than with FV shocks. The maturity matching bond that appears in all the tables is the bond with the lowest annual coupon.

The presence of a maturity matching bond is very clear because if λ takes big values, then the worst shocks are big and the principal paid by this bond is completely riskfree.

We can see that the six month coupon bonds do not appear in the maxmin porfolio, *ceteris paribus*, these bonds have more dispersion than the annual coupon bonds.

The path for the worst shock, independent of the type of shocks, is also very easy to understand. If λ grows, then the rate invested in the maturity matching bond or in bond number 9 also grows. Therefore, the duration of the maxmin portfolio decreases, which implies that the worst shock will be a lower interest rate.

So, from the results of Table 4 and 5 and the theoretical advantages of BI shocks we propose the following strategy. The investor must estimate parameter λ for BI shocks and compute the maxmin portfolio for the estimated parameter and for the set of feasible bonds. This strategy could include a maturity matching bond.

D. Maxmin portfolio among matching duration portfolios

The reasons for looking for the maxmin portfolio among matching duration portfolios are clear if we remember that a matching duration portfolio is the classical result for immunization (Fisher and Weil (1971)). Also, parallel changes on the interest rates are a strong proportion of the total change as shown empirically by Litterman and Scheinkman (1991). Finally, we consider the more recent empirical study on immunization by Bierwag *et al.* (1993). Five of the six strategies that they empirically test to see which is the best strategy for immunization have a duration equal to m, and the sixth one, a maturity matching strategy, has a lower duration and has the worst empirical behaviour. Bierwag *et al.* (1993) show that the best immunization strategy consists of a matching duration and including a maturity matching bond. This strategy is better than a barbell, bullet or FV strategy[6].

Regarding BI shocks we can observe in tables 6 and 7 that the maxmin portfolio is independent of parameter λ (which is a very interesting property). This portfolio includes a maturity matching bond with the lowest annual coupon at the greatest percentage. So, an alternative strategy to the above mentioned one may be to estimate paramater λ for BI shocks and to compute the maxmin portfolio amongst the duration matching portfolios. When there are several portfolios minimizing the $\tilde{N}$ measure (see table 3, portfolios 1 to 6), this strategy would allow to select one of these portfolios for a small parameter λ. This strategy could be consistent with the empirical result of Bierwag *et al.* (1993).

When we consider FV shocks, the maxmin portfolio minimizes the M^2 for small values for parameter λ. When parameter λ is large enough, we can see in tables 10 and 11 that the maxmin portfolio contains the solution for the maxmin portfolio for BI shocks (tables 6 and 7) which increases proportionally to λ.

It is worth emphasizing that the value guaranteed by the maxmin portfolio amongst matching duration portfolios (for reasonable values for parameter λ) is very close to the value guaranteed by the maxmin portfolio amongst bonds. However, those portfolios do not seem to depend too much on parameter λ for both FV and BI shocks.

Finally, some maxmin portfolios have associated worst shocks which lead to negative interest rates. This is a very undesirable property of the model. But we think that it is not that important because negative interest rates only appear when λ is very large, which is not very reasonable for both FV and BI shocks. If we prevented the worst shock in zero from avoiding negative interest rates, the maxmin portfolio and its guaranteed value would change only slightly. Then, from a qualitative point of view, things would not change. We also assume that the interest rates can go to $+\infty$ or $-\infty$ to easily prove that there is an interior global minimum, but the theory developed in the first and the second section is general enough to analyze more complicated situations.

VI. Conclusions

We have shown that when it is not possible to immunize a bond portfolio, we can still find a maxmin portfolio. Moreover, if we want to match the portfolio duration, to immunize against additive shocks, we could still find a maxmin portfolio among these duration matching portfolios.

Once the existence of maxmin portfolios has been proved we characterize them by means of a saddle point condition and also by means of an equation system.

We have analyzed the strategy of minimizing the risk immunization measures (M^2 and $\tilde{N}$) and the maxmin strategy for both bounded shocks and shocks with a bounded derivative. We have proved that these strategies are only equivalent when we consider pure discount bonds. Minimizing dispersion measures is equivalent to minimizing the worst shock effect, for each set of shocks considered. If we consider coupon bonds, then the strategies for minimizing dispersion measures are not maxmin strategies since they solve different programs. We have also given an algorithm For FV shocks and BI shocks in order to find the maxmin portfolios. The worst shock may also be obtained from these programs.

With two examples (two sets of bonds) we have computed the maxmin portfolio amongst bonds, amongst matching duration portfolios and for both FV shocks and BI shocks.

For BI shocks the parameter λ has a sound economic meaning. It can be understood as the volatility of interest rates. From looking at the maxmin portfolio amongst bonds (tables 4 and 5) we have suggested to estimate parameter λ and to compute the maxmin portfolio. This is theoreticaly, in our point of view, the best strategy for immunization. This strategy has a duration slightly lower than m and could be empirically tested.

If we wishs to immunize against additive shocks, then an alternative strategy is to estimate parameter λ for BI shocks, and to compute the maxmin portfolio amongst the duration matching portfolios. This strategy could be an alternative strategy for minimizing the $\tilde{N}$ measure and it could be consistent with the empirical results of Bierwag *et al.* (1993).

Once we have computed the maxmin portfolio and the worst shock, it is easy to compute how much money may be lost. This is another interesting property of this model.

The FV shocks have a disadvantage, which is the meaning of the parameter λ. Is a large or a small number for λ reasonable? With FV shocks, and for a small λ the maxmin portfolio, amongst bonds or amongst matching duration portfolios and the portfolio with the minimum M^2, may be obtained by buying the same bonds. If λ increases, a maturity matching bond appears in the maxmin portfolio, and its percentage increases with λ.

Appendix

Proof of Theorem 2.1. Let $\{q^1, q^2, \cdots, q^l\}$ be the set of extreme points in Q^*. Then the functionals $V(q^i, k)$ $i = 1 \cdots, l$ are convex and positive in the k variable. Therefore the hypotheses H2 and H3 still hold and we are under the assumptions of Lemma 1.1 in Balbás and Ibáñez (1995) which proves the existence of a riskless shadow asset that guarantees a return μ_0.

Since Q^* is the convex hull of $\{q^1, q^2, \cdots, q^l\}$, then given any $q \in Q^*$ one can find $\lambda_1, \lambda_2, \cdots, \lambda_l$ non negative real numbers such that

$$\begin{aligned} q &= \sum_{i=1}^{l} \alpha_i q^i \\ 1 &= \sum_{i=1}^{l} \alpha_i \end{aligned} \tag{24}$$

Let us consider the functional

$$U(k) = Max\{\frac{V(q^i, k)}{C};\ i = 1, \cdots, l\} \tag{25}$$

and let

$$\mu_0^* = Inf\{U(k);\ k \in K\} \tag{26}$$

Clearly $\mu_0^* \geq 0$ and we are going to prove that

$$\overline{V}(q) \leq \mu_0^* C \tag{27}$$

holds for any $q \in Q^*$.

For any $k \in K$ we have that

$$V(q, k) = \sum_{i=1}^{l} \alpha_i V(q^i, k) \leq C \sum_{i=1}^{l} \alpha_i U(k) = CU(k)$$

and therefore

$$\overline{V}(q) = Inf\{V(q, k);\ k \in K\} \leq Inf\{U(k);\ k \in K\}C = \mu_0^* C.$$

It follows from (27), that if we can find a portfolio q^* in Q^* such that $\overline{V}(q^*) = \mu_0^* C$ then q^* will be maxmin in Q^* and the theorem will be proved. To find this q^*, let us remark that for any $k \in K$ we have

$$U(k) \geq \mu_0^*$$

and therefore, since $U(k)$ is given by (25) there exist $i \in \{1, \cdots, l\}$ (which depends of k) such that

$$\frac{V(q^i, k)}{C} \geq \mu_0^*$$

Now, the existence of q^* trivially follows from Lemma 1.1 of Balbás and Ibáñez (1995). □

Proof of Theorem 3.2. Let us assume that q^* is a maxmin portfolio in Q^* and that k^* solves (PK). Then we have from (8) that $\overline{V}(q^*) = U(k^*)C$.

For any q' portfolio in Q^* we have

$$V(q', k^*) \leq Max\{V(q, k^*);\ q \in Q^*\} = U(k^*)C =$$
$$= \overline{V}(q^*) = Inf\{V(q^*, k)\ k \in K\} \leq V(q^*, k^*)$$

Furthermore, for any k' admissible shock

$$V(q^*, k') \geq Inf\{V(q^*, k);\ k \in K\} = \overline{V}(q^*) =$$
$$= U(k^*)C = Max\{V(q, k^*);\ q \in Q^*\} \geq V(q^*, k^*)$$

Conversely, let us assume that (q^*, k^*) is a saddle point and let us prove that q^* is maxmin. Since $V(q^*, k^*) \leq V(q^*, k)$ for any $k \in K$ we have that

$$\overline{V}(q^*) = V(q^*, k^*)$$

and q^* will be maxmin if we show that $\overline{V}(q) \leq V(q^*, k^*)$ for any $q \in Q^*$. This is true since

$$\overline{V}(q) = Inf\{V(q, k);\ k \in K\} \leq V(q, k^*) \leq V(q^*, k^*)$$

Let us finally prove that k^* solves (PK). Since $V(q, k^*) \leq V(q^*, k^*)$ for any $q \in Q^*$ we have that

$$U(k^*) = \frac{V(q^*, k^*)}{C}$$

and k^* will solve (PK) if we show that $U(k) \geq \dfrac{V(q^*, k^*)}{C}$ for any $k \in K$. But

$$U(k) = Max\{\frac{V(q, k)}{C};\ q \in Q\} \geq \frac{V(q^*, k)}{C} \geq \frac{V(q^*, k^*)}{C}. \quad \square$$

Proof of Proposition 4.1. Let us consider that the n bonds have their maturity at instants $T_1 < T_2 \cdots < T_n$ respectively and that the n bonds pay one monetary unit at maturity. The proposition is obvious if there exists a bond i such that $T_i = m$ since the solution for the four programs is to invest capital C in this bond.

Let us assume that $T_i \neq m$ for $i = 1, \cdots, n$.

If we assume that condition ii) holds, then, as has been stated, the minimums of functional U are attained in shocks $k^*(s)$ with the form given in (20). There, $\lambda > 0$ is constant and therefore the shock is given by λ_0. In order to achieve an easier notation we indentify the shock with λ_0, and since all the bonds are pure discount bonds, (14) becomes in this case

$$V(q^i, \lambda_0) = CR\ exp\left[\lambda_0(m - T_i) - \frac{\lambda}{2} \mid m - T_i \mid\right] \tag{28}$$

and therefore

$$\frac{\partial V(q^i, \lambda_0)}{\partial \lambda_0} = CR\ exp\left[\lambda_0(m - T_i) - \frac{\lambda}{2} \mid m - T_i \mid\right](m - T_i) \tag{29}$$

where q^i, $(i = 1, \cdots, n)$ is any extreme point of set Q, and clearly, it consists of a portfolio which has invested capital C in the i-*th* bond. If

$$q^* = \sum_{i=1}^{l} \alpha_i^* q^i$$

is the maximin portfolio, conditions (7) to (10) must hold, and then, from (28) and (29) we obtain

$$\sum_{i=1}^{h} \alpha_i^* = 1 \tag{30}$$

$$\lambda_0(m - T_i) - \frac{\lambda}{2} \mid m - T_i \mid \text{ has its maximum in } j \text{ for } j = 1, \cdots, h \tag{31}$$

$$\alpha_i^* \geq 0 \quad i = \{1, \cdots, h\}$$

$$\sum_{i=1}^{h} \alpha_i^*(m - T_i) = 0 \tag{32}$$

where we have assumed that the portfolios $q^i, \cdots, q^h$ are the only ones in the maxmin portfolio, again in order to achieve easier notation.

Expressions (30) and (32) imply that the maxmin portfolio q^* is a matching duration portfolio, *i.e.*, $q^* \in Q^*$.

Since q^* has a duration equal to m and none of the bonds has m maturity, at least two bonds (i and j) must be in the maxmin portfolio, and the following inequality must hold $T_i < m < T_j$.

We have from (31)

$$\lambda_0(m - T_i) - \frac{\lambda}{2} \mid m - T_i \mid = \lambda_0(m - T_j) - \frac{\lambda}{2} \mid m - T_j \mid$$

from where the worst shock will be

$$\lambda_0^* = \frac{\lambda(2m - T_j - T_i)}{2(T_j - T_i)}$$

and therefore

$$\lambda_0^*(m - T_i) - \frac{\lambda}{2} \mid m - T_i \mid = \frac{-\lambda}{2(T_j - t_i)}(m - T_i)(T_j - m)$$

since this expression has to be maxima (see (31)), it will be proved that a maxmin portfolio only has two bonds, if we prove that

$$F(T_i, T_j) = \frac{(m - T_i)(T_j - m)}{T_j - T_i} \quad i, j = 1, \cdots, h \ , \ i \neq j \tag{33}$$

only has one minimum.

Clearly

$$\frac{\partial F}{\partial T_i} = -\left[\frac{T_j - m}{T_j - T_i}\right]^2 < 0$$

$$\frac{\partial F}{\partial T_j} = \left[\frac{m - T_i}{T_j - T_i}\right]^2 > 0$$

and F increases with T_j and decreases with T_i. Given that $T_i < m < T_j$ the minimum is attained at the point (T_i, T_j) closest to m.

By analogy, if we assume that i) holds, then following the ideas in the latter case, the worst shock will be given by

$$\lambda_0^* = \frac{\lambda(2m - T_j - T_i)}{2}$$

and the only two bonds in the maxmim portfolio will be obtained by minimizing the expression

$$\frac{\lambda}{2}(m - T_i)(T_j - m) \text{ subject to } T_i < m < T_j.$$

Therefore, the maxmin portfolio will be the one computed in the latter case.

Let us assume that iv) holds.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the percentages invested in bonds $1, 2, \cdots, n$ respectively and clearly q^* solves the following program

$$\begin{gathered} \text{Min } \sum_{i=1}^{n} \alpha_i \mid m - T_i \mid \\ \left.\begin{aligned} \text{subject to } \textstyle\sum_{i=1}^{n} \alpha_i T_i &= m \\ \textstyle\sum_{i=1}^{n} \alpha_i &= 1 \\ \alpha_i \geq 0 \quad i &= 1, \cdots, n \end{aligned}\right\} \end{gathered}$$

Since the program is linear, the minimum must be attained at an extreme point. On the other hand the basic feasible solutions (extreme points) have only two non-zero variables (there are two constraints), the first constraint shows that the two bonds in the solution must have maturities smaller and greater than m respectively. If α_i and α_j ($T_i < m < T_j$) are non zero in the solution, then

$$\left.\begin{aligned} \alpha_i T_i + \alpha_j T_j &= m \\ \alpha_i + \alpha_j &= 1 \end{aligned}\right\}$$

from where

$$\alpha_i \mid m - T_i \mid + \alpha_j \mid m - T_j \mid = \frac{2(m - T_i)(T_j - m)}{T_j - T_i}$$

and it has already been proved that the latter expression becomes minima when the couple (T_i, T_j) is as near as possible to m.

If we assume that iii) holds we obtain the same solution q^* in an analogous way. □

References

Alexander G.J. and B.G. Resnick. "Using Linear and Goal Programming to Immunize Bond Portfolios," *Journal of Financial and Quantitative Analysis*, (1985).

Balbás A. and A. Ibáñez (1995). "When can you Immunize a Bond Portfolio?." II Jornadas de Economía Financiera, (June 1995), Bilbao (Spain).

Bierwag, G.O., Fooladi I. and G. S. Roberts. "Designing an Immunized Portfolio is M-square the key?." *Journal of Banking and Finance*,17 (1993), 1147-1170.

Bierwag, G.O. and C. Khang. "An Immunization Strategy is a Maximin Strategy." *The Journal of Finance*, XXXVII, (May 1979), 379-389.

Fisher, L. and R. Weil. "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies." *Journal of Business*, 52 (1971), 51-61.

Fong, H.G. and O. A. Vasicek. "A Risk Minimizing Strategy for Portfolio Immunization." *The Journal of Finance*, XXXIX, 5 (December 1984), 1541-1546.

Khang, C. "A Dynamic Global Portfolio Immunization Strategy in the World of Multiple Interest Rate Changes: A Dynamic Immunization and Minimax Theorem." *Journal of Financial and Quantitative Analysis*, 18 (September 1983), 355-363.

Litterman, R. and J.A. Scheinkman "Commong Factors affecting Bond Returns." *The Journal of Fixed Income*, 1 (June 1991), 54-61.

Luenberger, D.G. *Optimization by Vector Space Methods.* New York: Wiley. (1969)

Montrucchio, L. and L. Peccati. "A note On Shiu-Fisher-Weil Immunization Theorem." *Insurance Mathematics and Economics*, 10 (1991), 125-131.

Prisman, E.Z. "Immunization as a Maxmin Strategy a New Look." *Journal of Banking and Finance*, 10 (1986), 491-509.

Prisman, E.Z. and M.R. Shores. "Duration Measures for specific term structure estimations and applications to bond portfolio immunization." *Journal of Banking and Finance*, 12 (1988), 493-504.

Prisman, E.Z. and Y. Tian. "Immunization in Markets with Tax-Clientele Effects: Evidence from the Canadian Markets." *Journal of Financial and Quantitative Analysis*, 29 (June 1994), 301-321.

Footnotes

[1]From now on, if k^* solves (PK) it is called the "worst shock".

[2]We are going to show the proof although it is very similar to the very well known proof in game theory.

[3]We observe that having defined $V(q,k)$ as in (11) the Hyphotesis **H4** of Balbás and Ibáñez (1995) holds, which says that $V_i(0) = RP_i$, $i = 1, \cdots, n$.

[4]From now on, FV and BI will mean Fong and Vasicek (1984) and Balbás and Ibáñez (1995) respectively.

[5]$U(\lambda_0)$ represents the number denoted by $U(k)$ in the second section.

[6]Prisman and Tian (1994) in an empirical work on the effects of taxes on immunization point out that there is no theory underyliyng the strategy proposed by Bierwag *et al.* (1993), and consequently, there are no guidelines regarding the weight of such bond in the portfolio.

Table 1: Set of Bonds

Bond number	Maturity (years)	coupon in %	coupon (monthly)	Duration (years)	M^2	$\tilde{N}$
1	4	5	6	3.39029	3.78821	1.60970
2	5	5	6	4.04855	3.12539	0.95144
3	6	5	6	4.64453	3.71997	1.55018
4	7	5	6	5.18397	5.34059	2.09212
5	8	5	6	5.67211	7.78769	2.58253
6	9	5	6	6.11372	10.8897	3.02620
7	4	10	12	3.48232	3.27130	1.51767
8	5	10	12	4.16101	2.59822	0.83898
9	6	10	12	4.77597	3.22315	1.45880
10	7	10	12	5.33285	4.90932	2.02011
11	8	10	12	5.83689	7.45157	2.52818
12	9	10	12	6.29287	10.6728	2.98782
13	5	9	12	4.21496	2.43114	0.78503

Table 2: Matching-duration Portfolios, Q_1^*.

number	1th bond	2th bond	% (1th)	M^2	$\tilde{N}$
1	12	13	37.77	5.544	1.617
2	11	13	48.40	4.861	1.628
3	10	13	70.22	4.171	1.652
4	9	10	59.77	3.901	1.684
5	9	11	78.88	4.116	1.684
6	8	10	28.40	4.252	1.684
7	9	12	85.23	4.323	1.684
8	8	11	49.93	5.027	1.684
9	8	12	60.64	5.775	1.684
10	6	13	41.34	5.928	1.711
11	6	9	16.74	4.507	1.721
12	5	9	24.99	4.364	1.739
13	2	10	25.91	4.446	1.743
14	5	13	53.87	5.316	1.753
15	6	8	42.96	6.160	1.778
16	2	11	46.79	5.427	1.790
17	3	10	48.35	4.334	1.792
18	4	9	54.90	4.385	1.806
19	5	8	55.52	5.479	1.807
20	2	12	57.60	6.325	1.814
21	3	11	70.18	4.832	1.841
22	4	13	81.01	4.788	1.843
23	3	12	78.43	5.219	1.860
24	4	8	82.01	4.847	1.866
25	3	4	34.10	4.787	1.907
26	2	4	16.20	4.981	1.907
27	3	5	65.40	5.127	1.907
28	3	6	75.80	5.454	1.907
29	2	5	41.39	5.857	1.907
30	2	6	53.92	6.702	1.907
31	7	10	17.98	4.614	1.929
32	1	10	17.13	4.717	1.949
33	4	7	89.18	5.116	2.030
34	1	4	10.25	5.181	2.042
35	7	11	35.54	5.965	2.169
36	1	11	34.20	6.198	2.214
37	5	7	69.30	6.401	2.255
38	1	5	29.45	6.609	2.295
39	7	12	46.00	7.268	2.311
40	1	12	44.54	7.606	2.373
41	6	7	57.67	7.665	2.387
42	1	6	40.89	7.985	2.446

Table 3: Matching-duration Portfolios, Q_2^*.

number	1th bond	2th bond	% (1th)	M^2	Ñ
1	9	10	59.77	3.901	1.6846132
2	9	11	78.88	4.116	1.6846172
3	8	10	28.40	4.252	1.6846198
4	9	12	85.23	4.323	1.6846211
5	8	11	49.93	5.027	1.6846342
6	8	12	60.64	5.775	1.6846482
7	6	9	16.74	4.507	1.7212876
8	5	9	24.99	4.364	1.7397250
9	2	10	25.91	4.446	1.7431487
10	6	8	42.96	6.160	1.7787185
11	2	11	46.79	5.427	1.7903170
12	3	10	48.35	4.334	1.7928698
13	4	9	54.90	4.385	1.8065498
14	5	8	55.52	5.479	1.8070234
15	2	12	57.60	6.325	1.8147429
16	3	11	70.18	4.832	1.8417463
17	3	12	78.43	5.219	1.8602143
18	4	8	82.01	4.847	1.8667515
19	3	4	34.10	4.787	1.9072982
20	2	4	16.20	4.981	1.9072993
21	3	5	65.40	5.127	1.9073001
22	3	6	75.80	5.454	1.9073020
23	2	5	41.39	5.857	1.9073042
24	2	6	53.92	6.702	1.9073090
25	7	10	17.98	4.614	1.9297429
26	1	10	17.13	4.717	1.9497946
27	4	7	89.18	5.116	2.0300177
28	1	4	10.25	5.181	2.0426442
29	7	11	35.54	5.965	2.1690124
30	1	11	34.20	6.198	2.2140047
31	5	7	69.30	6.401	2.2556923
32	1	5	29.45	6.609	2.2959836
33	7	12	46.00	7.268	2.3115433
34	1	12	44.54	7.606	2.3739805
35	6	7	57.67	7.665	2.3877217
36	1	6	40.89	7.985	2.4469345

Table 4: Maxmin Portfolios among bonds
BI shocks. Set of bonds Q_1.

λ	1th bond	2th bond	% (1th)	λ_0^*	% value	duration
0.001	12	13	36.9774	-0.0005	99.9192	4.9833
0.002	12	13	36.8506	-0.0010	99.8388	4.9806
0.004	12	13	36.5985	-0.0021	99.6786	4.9754
0.006	12	13	36.3484	-0.0031	99.5195	4.9702
0.008	12	13	36.1004	-0.0042	99.3615	4.9651
0.010	12	13	35.8544	-0.0052	99.2045	4.9599
0.012	12	13	35.6103	-0.0063	99.0485	4.9549
0.014	12	13	35.3683	-0.0074	98.8935	4.9498
0.016	12	13	35.1281	-0.0084	98.7395	4.9448
0.018	12	13	34.8900	-0.0095	98.5866	4.9399
0.020	12	13	34.6537	-0.0105	98.4346	4.9350
0.022	12	13	34.4193	-0.0116	98.2836	4.9301
0.024	12	13	34.1868	-0.0126	98.1336	4.9253
0.026	12	13	33.9561	-0.0137	97.9845	4.9205
0.028	12	13	33.7273	-0.0148	97.8365	4.9157
0.030	12	13	33.5004	-0.0158	97.6894	4.9110
0.032	12	13	33.2752	-0.0169	97.5432	4.9063
0.034	12	13	33.0518	-0.0179	97.3980	4.9017
0.036	12	13	32.8302	-0.0190	97.2537	4.8971
0.038	12	13	32.6103	-0.0200	97.1103	4.8925
0.040	12	13	32.3922	-0.0211	96.9679	4.8880
0.042	12	13	32.1758	-0.0221	96.8264	4.8835
0.044	12	13	31.9611	-0.0232	96.6858	4.8790
0.046	12	13	31.7481	-0.0242	96.5461	4.8746
0.048	12	13	31.5368	-0.0253	96.4072	4.8702
0.050	12	13	31.3272	-0.0263	96.2693	4.8659
0.060	12	13	30.3033	-0.0316	95.5928	4.8446
0.070	12	13	29.3186	-0.0368	94.9375	4.8241
0.080	12	13	28.3716	-0.0421	94.3027	4.8045
0.090	12	13	27.4603	-0.0473	93.6877	4.7855
0.100	12	13	26.5834	-0.0525	93.0917	4.7673
0.110	12	13	25.7392	-0.0577	92.5140	4.7498
0.120	12	13	24.9264	-0.0629	91.9541	4.7329
0.130	12	13	24.1437	-0.0681	91.4113	4.7166
0.140	12	13	23.3897	-0.0733	90.8849	4.7009
0.150	12	13	22.6632	-0.0785	90.3745	4.6858
0.160	12	13	21.9631	-0.0837	89.8795	4.6713
0.170	12	13	21.2883	-0.0889	89.3992	4.6573
0.180	12	13	20.6377	-0.0940	88.9334	4.6438
0.190	12	13	20.0103	-0.0992	88.4813	4.6307
0.200	12	13	19.4052	-0.1044	88.0426	4.6181
0.250	12	13	16.6839	-0.1301	86.0341	4.5616
0.300	12	13	14.4005	-0.1557	84.2983	4.5141
0.350	12	13	12.4765	-0.1813	82.7930	4.4742
0.400	12	13	10.8490	-0.2067	81.4833	4.4404
0.450	12	13	9.46736	-0.2322	80.3400	4.4116
0.500	12	13	8.29019	-0.2575	79.3388	4.3872
0.550	12	13	7.28392	-0.2828	78.4593	4.3663
0.600	12	13	6.42096	-0.3081	77.6842	4.3483
0.650	12	13	5.67854	-0.3334	76.9990	4.3329
0.700	12	13	5.03784	-0.3586	76.3914	4.3196
0.750	12	13	4.48324	-0.3838	75.8510	4.3081
0.800	12	13	4.00172	-0.4089	75.3690	4.2981
0.850	12	13	3.58240	-0.4341	74.9379	4.2894
0.900	12	13	3.21617	-0.4592	74.5511	4.2817
0.950	12	13	2.89539	-0.4844	74.2031	4.2751
1.0	12	13	2.61358	-0.5095	73.8893	4.2692

Table 5: Maxmin Portfolios among bonds

BI shocks. Set of bonds Q_2.

λ	1th bond	2th bond	% (1th)	λ_0^*	% value	duration
0.001	10	9	39.8212	-0.0005	99.9159	4.9977
0.002	10	9	39.3521	-0.0010	99.8320	4.9951
0.004	10	9	38.4190	-0.0020	99.6651	4.9899
0.006	10	9	37.4922	-0.0030	99.4993	4.9847
0.008	10	9	36.5718	-0.0040	99.3344	4.9796
0.010	10	9	35.6577	-0.0050	99.1706	4.9745
0.012	10	9	34.7498	-0.0060	99.0079	4.9694
0.014	10	9	33.8482	-0.0070	98.8461	4.9644
0.016	10	9	32.9528	-0.0080	98.6853	4.9594
0.018	10	9	32.0635	-0.0090	98.5255	4.9545
0.020	10	9	31.1803	-0.0100	98.3667	4.9496
0.022	10	9	30.3031	-0.0110	98.2089	4.9447
0.024	10	9	29.4319	-0.0120	98.0521	4.9398
0.026	10	9	28.5667	-0.0130	97.8962	4.9350
0.028	10	9	27.7074	-0.0141	97.7413	4.9302
0.030	10	9	26.8540	-0.0151	97.5874	4.9255
0.032	10	9	26.0064	-0.0161	97.4344	4.9207
0.034	10	9	25.1646	-0.0171	97.2823	4.9161
0.036	10	9	24.3285	-0.0181	97.1312	4.9114
0.038	10	9	23.4981	-0.0191	96.9810	4.9068
0.040	10	9	22.6734	-0.0201	96.8317	4.9022
0.042	10	9	21.8542	-0.0211	96.6834	4.8976
0.044	10	9	21.0407	-0.0221	96.5359	4.8931
0.046	10	9	20.2327	-0.0231	96.3894	4.8886
0.048	10	9	19.4302	-0.0241	96.2438	4.8841
0.050	10	9	18.6331	-0.0251	96.0990	4.8797
0.060	10	9	14.7284	-0.0302	95.3885	4.8579
0.070	10	9	10.9541	-0.0352	94.6993	4.8369
0.080	10	9	73.0585	-0.0402	94.0308	4.8166
0.090	10	9	37.7896	-0.0453	93.3824	4.7970
0.100	10	9	36.9186	-0.0503	92.7533	4.7780
0.110	9	8	97.3620	-0.0553	92.1429	4.7597
0.120	9	8	94.4887	-0.0603	91.5505	4.7420
0.130	9	8	91.7097	-0.0654	90.9757	4.7249
0.140	9	8	89.0219	-0.0704	90.4177	4.7084
0.150	9	8	86.4220	-0.0754	89.8761	4.6924
0.160	9	8	83.9070	-0.0805	89.3503	4.6770
0.170	9	8	81.4739	-0.0855	88.8398	4.6620
0.180	9	8	79.1198	-0.0905	88.3442	4.6475
0.190	9	8	76.8420	-0.0955	87.8628	4.6335
0.200	9	8	74.6378	-0.1005	87.3953	4.6200
0.250	9	8	64.6355	-0.1256	85.2503	4.5584
0.300	9	8	56.1283	-0.1507	83.3907	4.5061
0.350	9	8	48.8776	-0.1758	81.7738	4.4615
0.400	9	8	42.6846	-0.2009	80.3640	4.4235
0.450	9	8	37.3831	-0.2259	79.1312	4.3909
0.500	9	8	32.8347	-0.2510	78.0502	4.3629
0.550	9	8	28.9233	-0.2760	77.0994	4.3388
0.600	9	8	25.5517	-0.3011	76.2607	4.3181
0.650	9	8	22.6384	-0.3261	75.5188	4.3002
0.700	9	8	20.1148	-0.3511	74.8605	4.2847
0.750	9	8	17.9233	-0.3762	74.2749	4.2712
0.800	9	8	16.0153	-0.4012	73.7524	4.2595
0.850	9	8	14.3499	-0.4262	73.2848	4.2492
0.900	9	8	12.8925	-0.4512	72.8654	4.2403
0.950	9	8	11.6136	-0.4762	72.4880	4.2324
1.0	9	8	10.4886	-0.5013	72.1477	4.2255

Table 6: Maxmin Portfolios among duration-matching portfolios. BI shocks. Set of portfolios Q_1^*.

λ	1th portf.	2th portf.	% (1th)	λ_0^*	% value
0.001	1	1	100.	-.00004	99.9192
0.002	1	1	100.	-.00009	99.8385
0.004	1	1	100.	-.00019	99.6776
0.006	1	1	100.	-.00029	99.5172
0.008	1	1	100.	-.00038	99.3574
0.010	1	1	100.	-.00048	99.1982
0.012	1	1	100.	-.00058	99.0394
0.014	1	1	100.	-.00068	98.8812
0.016	1	1	100.	-.00077	98.7236
0.018	1	1	100.	-.00087	98.5664
0.020	1	1	100.	-.00097	98.4098
0.022	1	1	100.	-.00107	98.2538
0.024	1	1	100.	-.00116	98.0982
0.026	1	1	100.	-.00126	97.9432
0.028	1	1	100.	-.00136	97.7887
0.030	1	1	100.	-.00146	97.6348
0.032	1	1	100.	-.00156	97.4813
0.034	1	1	100.	-.00166	97.3284
0.036	1	1	100.	-.00175	97.1760
0.038	1	1	100.	-.00185	97.0241
0.040	1	1	100.	-.00195	96.8727
0.042	1	1	100.	-.00205	96.7218
0.044	1	1	100.	-.00215	96.5715
0.046	1	1	100.	-.00225	96.4216
0.048	1	1	100.	-.00235	96.2723
0.050	1	1	100.	-.00245	96.1234
0.060	1	1	100.	-.00295	95.3867
0.070	1	1	100.	-.00345	94.6623
0.080	1	1	100.	-.00396	93.9499
0.090	1	1	100.	-.00446	93.2495
0.100	1	1	100.	-.00498	92.5607
0.110	1	1	100.	-.00549	91.8834
0.120	1	1	100.	-.00601	91.2174
0.130	1	1	100.	-.00653	90.5624
0.140	1	1	100.	-.00705	89.9182
0.150	1	1	100.	-.00757	89.2848
0.160	1	1	100.	-.00810	88.6618
0.170	1	1	100.	-.00864	88.0491
0.180	1	1	100.	-.00917	87.4465
0.190	1	1	100.	-.00971	86.8539
0.200	1	1	100.	-.01025	86.2709
0.250	1	1	100.	-.01300	83.4968
0.300	1	1	100.	-.01583	80.9418
0.350	1	1	100.	-.01874	78.5873
0.400	1	1	100.	-.02173	76.4164
0.450	1	1	100.	-.02481	74.4136
0.500	1	1	100.	-.02797	72.5647
0.550	1	1	100.	-.03122	70.8568
0.600	1	1	100.	-.03455	69.2782
0.650	1	1	100.	-.03797	67.8182
0.700	1	1	100.	-.04148	66.4669
0.750	1	1	100.	-.04508	65.2154
0.800	1	1	100.	-.04878	64.0556
0.850	1	1	100.	-.05256	62.9800
0.900	1	1	100.	-.05644	61.9818
0.950	1	1	100.	-.06042	61.0547
1.0	1	1	100.	-.06448	60.1930

Table 7: Maxmin Portfolios among duration-matching portfolios. BI shocks. Set of portfolios Q_2^*.

λ	1th portf.	2th portf.	% (1th)	λ_0^*	% value
0.001	6	6	100.	-.00004	99.9158
0.002	6	6	100.	-.00009	99.8318
0.004	6	6	100.	-.00019	99.6642
0.006	6	6	100.	-.00029	99.4971
0.008	6	6	100.	-.00038	99.3306
0.010	6	6	100.	-.00048	99.1647
0.012	6	6	100.	-.00058	98.9994
0.014	6	6	100.	-.00068	98.8346
0.016	6	6	100.	-.00077	98.6704
0.018	6	6	100.	-.00087	98.5067
0.020	6	6	100.	-.00097	98.3436
0.022	6	6	100.	-.00107	98.1810
0.024	6	6	100.	-.00116	98.0190
0.026	6	6	100.	-.00126	97.8575
0.028	6	6	100.	-.00136	97.6966
0.030	6	6	100.	-.00146	97.5362
0.032	6	6	100.	-.00156	97.3764
0.034	6	6	100.	-.00166	97.2171
0.036	6	6	100.	-.00175	97.0583
0.038	6	6	100.	-.00185	96.9001
0.040	6	6	100.	-.00195	96.7424
0.042	6	6	100.	-.00205	96.5852
0.044	6	6	100.	-.00215	96.4286
0.046	6	6	100.	-.00225	96.2725
0.048	6	6	100.	-.00235	96.1169
0.050	6	6	100.	-.00245	95.9619
0.060	6	6	100.	-.00295	95.1944
0.070	6	6	100.	-.00345	94.4398
0.080	6	6	100.	-.00396	93.6978
0.090	6	6	100.	-.00446	92.9682
0.100	6	6	100.	-.00498	92.2507
0.110	6	6	100.	-.00549	91.5451
0.120	6	6	100.	-.00601	90.8513
0.130	6	6	100.	-.00653	90.1690
0.140	6	6	100.	-.00705	89.4981
0.150	6	6	100.	-.00757	88.8382
0.160	6	6	100.	-.00810	88.1893
0.170	6	6	100.	-.00864	87.5510
0.180	6	6	100.	-.00917	86.9233
0.190	6	6	100.	-.00971	86.3060
0.200	6	6	100.	-.01025	85.6988
0.250	6	6	100.	-.01300	82.8090
0.300	6	6	100.	-.01583	80.1475
0.350	6	6	100.	-.01874	77.6949
0.400	6	6	100.	-.02173	75.4335
0.450	6	6	100.	-.02481	73.3472
0.500	6	6	100.	-.02797	71.4213
0.550	6	6	100.	-.03122	69.6422
0.600	6	6	100.	-.03455	67.9978
0.650	6	6	100.	-.03797	66.4769
0.700	6	6	100.	-.04148	65.0693
0.750	6	6	100.	-.04508	63.7657
0.800	6	6	100.	-.04878	62.5576
0.850	6	6	100.	-.05256	61.4371
0.900	6	6	100.	-.05644	60.3973
0.950	6	6	100.	-.06042	59.4316
1.0	6	6	100.	-.06448	58.5340

Table 8: Maxmin Portfolios among bonds
FV shocks. Set of bonds Q_1.

λ	1th bond	2th bond	% (1th)	λ_0^*	% value	duration
0.001	10	9	38.6416	-.00151	99.8063	4.9911
0.002	10	9	37.0122	-.00302	99.6154	4.9820
0.004	10	9	33.8142	-.00605	99.2417	4.9642
0.006	10	9	30.6958	-.00908	98.8787	4.9469
0.008	10	9	27.6548	-.01210	98.5262	4.9299
0.010	10	9	24.6890	-.01513	98.1838	4.9134
0.012	10	9	21.7965	-.01815	97.8514	4.8973
0.014	10	9	18.9753	-.02118	97.5287	4.8816
0.016	10	9	16.2235	-.02420	97.2155	4.8663
0.018	10	9	13.5392	-.02722	96.9115	4.8513
0.020	10	9	10.9206	-.03025	96.6166	4.8367
0.022	10	9	8.36603	-.03327	96.3306	4.8225
0.024	10	9	5.87367	-.03629	96.0532	4.8086
0.026	10	9	3.44193	-.03931	95.7843	4.7951
0.028	10	9	1.06918	-.04233	95.5237	4.7819
0.030	9	9	100.	-.04251	95.2701	4.7759
0.032	9	9	100.	-.04028	95.0182	4.7759
0.034	9	9	100.	-.03806	94.7674	4.7759
0.036	9	9	100.	-.03584	94.5175	4.7759
0.038	9	9	100.	-.03362	94.2687	4.7759
0.040	9	9	100.	-.03140	94.0209	4.7759
0.042	9	9	100.	-.02919	93.7740	4.7759
0.044	9	13	98.9017	-.02949	93.5290	4.7698
0.046	9	13	97.4048	-.03077	93.2877	4.7614
0.048	9	13	95.9394	-.03205	93.0503	4.7531
0.050	9	13	94.5048	-.03332	92.8167	4.7451
0.060	9	13	87.7642	-.03960	91.7026	4.7073
0.070	9	13	81.6788	-.04579	90.6722	4.6731
0.080	9	13	76.1695	-.05189	89.7176	4.6422
0.090	9	13	71.1687	-.05790	88.8319	4.6142
0.100	9	13	66.6184	-.06385	88.0087	4.5887
0.110	9	13	62.4687	-.06973	87.2425	4.5654
0.120	9	13	58.6763	-.07555	86.5284	4.5441
0.130	9	13	55.2033	-.08131	85.8618	4.5246
0.140	9	13	52.0169	-.08702	85.2387	4.5067
0.150	9	13	49.0880	-.09269	84.6555	4.4903
0.160	9	13	46.3911	-.09831	84.1090	4.4752
0.170	9	13	43.9038	-.10390	83.5961	4.4612
0.180	9	13	41.6060	-.10945	83.1141	4.4483
0.190	9	13	39.4801	-.11497	82.6608	4.4364
0.200	9	13	37.5103	-.12045	82.2337	4.4254
0.250	9	13	29.5569	-.14751	80.4291	4.3807
0.300	9	13	23.9067	-.17408	79.0458	4.3490
0.350	9	13	19.7715	-.20031	77.9581	4.3258
0.400	9	13	16.6626	-.22631	77.0830	4.3084
0.450	9	13	14.2682	-.25212	76.3648	4.2950
0.500	9	13	12.3832	-.27780	75.7647	4.2844
0.550	9	13	10.8700	-.30337	75.2555	4.2759
0.600	9	13	9.63413	-.32887	74.8179	4.2690
0.650	9	13	8.60899	-.35430	74.4373	4.2632
0.700	9	13	7.74710	-.37968	74.1030	4.2584
0.750	9	13	7.01377	-.40501	73.8070	4.2543
0.800	9	13	6.38321	-.43031	73.5428	4.2507
0.850	9	13	5.83594	-.45558	73.3055	4.2477
0.900	9	13	5.35702	-.48082	73.0912	4.2450
0.950	9	13	4.93482	-.50604	72.8966	4.2426
1.0	9	13	4.56015	-.53124	72.7191	4.2405

Table 9: Maxmin Portfolios among bonds
FV shocks. Set of bonds Q_2.

λ	1th bond	2th bond	% (1th)	λ_0^*	% value	duration
0.001	10	9	38.6416	-.00151	99.8063	4.9911
0.002	10	9	37.0122	-.00302	99.6154	4.9820
0.004	10	9	33.8142	-.00605	99.2417	4.9642
0.006	10	9	30.6958	-.00908	98.8787	4.9469
0.008	10	9	27.6548	-.01210	98.5262	4.9299
0.010	10	9	24.6890	-.01513	98.1838	4.9134
0.012	10	9	21.7965	-.01815	97.8514	4.8973
0.014	10	9	18.9753	-.02118	97.5287	4.8816
0.016	10	9	16.2235	-.02420	97.2155	4.8663
0.018	10	9	13.5392	-.02722	96.9115	4.8513
0.020	10	9	10.9206	-.03025	96.6166	4.8367
0.022	10	9	8.36603	-.03327	96.3306	4.8225
0.024	10	9	5.87367	-.03629	96.0532	4.8086
0.026	10	9	3.44193	-.03931	95.7843	4.7951
0.028	10	9	1.06918	-.04233	95.5237	4.7819
0.030	9	9	100.	-.04251	95.2701	4.7759
0.032	9	9	100.	-.04028	95.0182	4.7759
0.034	9	9	100.	-.03806	94.7674	4.7759
0.036	9	9	100.	-.03584	94.5175	4.7759
0.038	9	9	100.	-.03362	94.2687	4.7759
0.040	9	9	100.	-.03140	94.0209	4.7759
0.042	9	9	100.	-.02919	93.7740	4.7759
0.044	9	9	100.	-.02698	93.5282	4.7759
0.046	9	9	100.	-.02477	93.2833	4.7759
0.048	9	8	99.3094	-.02432	93.0398	4.7717
0.050	9	8	98.0602	-.02533	92.7995	4.7640
0.060	9	8	92.1045	-.03039	91.6481	4.7274
0.070	9	8	86.6030	-.03544	90.5751	4.6935
0.080	9	8	81.5172	-.04048	89.5742	4.6623
0.090	9	8	76.8121	-.04553	88.6398	4.6333
0.100	9	8	72.4560	-.05057	87.7665	4.6065
0.110	9	8	68.4197	-.05561	86.9495	4.5817
0.120	9	8	64.6767	-.06064	86.1845	4.5587
0.130	9	8	61.2031	-.06568	85.4674	4.5373
0.140	9	8	57.9766	-.07071	84.7945	4.5175
0.150	9	8	54.9774	-.07574	84.1626	4.4991
0.160	9	8	52.1870	-.08077	83.5685	4.4819
0.170	9	8	49.5887	-.08579	83.0094	4.4659
0.180	9	8	47.1673	-.09082	82.4827	4.4510
0.190	9	8	44.9086	-.09584	81.9860	4.4371
0.200	9	8	42.8001	-.10086	81.5172	4.4242
0.250	9	8	34.1318	-.12596	79.5264	4.3709
0.300	9	8	27.8234	-.15103	77.9913	4.3321
0.350	9	8	23.1291	-.17609	76.7803	4.3032
0.400	9	8	19.5591	-.20113	75.8044	4.2812
0.450	9	8	16.7871	-.22617	75.0028	4.2642
0.500	9	8	14.5924	-.25120	74.3330	4.2507
0.550	9	8	12.8233	-.27623	73.7649	4.2398
0.600	9	8	11.3740	-.30125	73.2768	4.2309
0.650	9	8	10.1692	-.32627	72.8526	4.2235
0.700	9	8	9.15476	-.35129	72.4803	4.2173
0.750	9	8	8.29052	-.37630	72.1507	4.2120
0.800	9	8	7.54673	-.40132	71.8568	4.2074
0.850	9	8	6.90076	-.42633	71.5930	4.2034
0.900	9	8	6.33517	-.45134	71.3549	4.1999
0.950	9	8	5.83637	-.47635	71.1388	4.1969
1.0	9	8	5.39360	-.50136	70.9418	4.1941

Table 10: Maxmin Portfolios among duration-matching portfolios.
FV shocks. Set of portfolios Q_1^*.

λ	1th portf.	2th portf.	% (1th)	λ_0^*	% value
0.001	4	4	100.	.00085	99.8052
0.002	4	4	100.	.00170	99.6110
0.004	4	4	100.	.00341	99.2245
0.006	4	4	100.	.00512	98.8404
0.008	4	4	100.	.00682	98.4587
0.010	4	4	100.	.00852	98.0793
0.012	4	4	100.	.01022	97.7023
0.014	4	4	100.	.01192	97.3276
0.016	4	4	100.	.01362	96.9553
0.018	4	4	100.	.01531	96.5852
0.020	4	4	100.	.01700	96.2174
0.022	4	4	100.	.01869	95.8519
0.024	4	4	100.	.02038	95.4887
0.026	4	4	100.	.02207	95.1277
0.028	4	4	100.	.02375	94.7689
0.030	4	4	100.	.02544	94.4123
0.032	4	4	100.	.02712	94.0579
0.034	4	4	100.	.02880	93.7056
0.036	4	4	100.	.03047	93.3556
0.038	4	4	100.	.03215	93.0076
0.040	4	4	100.	.03382	92.6618
0.042	4	4	100.	.03549	92.3181
0.044	4	4	100.	.03716	91.9765
0.046	4	4	100.	.03883	91.6370
0.048	4	4	100.	.04049	91.2995
0.050	4	4	100.	.04216	90.9641
0.060	4	4	100.	.05044	89.3173
0.070	4	4	100.	.05867	87.7197
0.080	4	4	100.	.06685	86.1694
0.090	4	4	100.	.07497	84.6649
0.100	4	4	100.	.08305	83.2046
0.110	4	7	74.9576	.08372	81.7944
0.120	4	7	27.3614	.07632	80.4794
0.130	7	7	100.	.07403	79.2619
0.140	7	7	100.	.08100	78.0996
0.150	7	7	100.	.08812	76.9793
0.160	7	7	100.	.09537	75.8983
0.170	7	7	100.	.10273	74.8540
0.180	7	7	100.	.11020	73.8442
0.190	7	7	100.	.11776	72.8666
0.200	7	1	97.3069	.12100	71.9211
0.250	7	1	71.4077	.10388	67.8691
0.300	7	1	54.0713	.08447	64.7538
0.350	7	1	42.0529	.06355	62.3297
0.400	7	1	33.4502	.04158	60.4157
0.450	7	1	27.1071	.01886	58.8805
0.500	7	1	22.3003	-.00439	57.6298
0.550	7	1	18.5660	-.02805	56.5955
0.600	7	1	15.5995	-.05200	55.7283
0.650	7	1	13.1966	-.07617	54.9917
0.700	7	1	11.2170	-.10050	54.3588
0.750	7	1	9.56230	-.12496	53.8094
0.800	7	1	8.16191	-.14952	53.3278
0.850	7	1	6.96415	-.17414	52.9023
0.900	7	1	5.93036	-.19882	52.5234
0.950	7	1	5.03104	-.22355	52.1839
1.0	7	1	4.24330	-.24830	51.8778

Table 11: **Maxmin Portfolios among duration-matching portfolios. FV shocks. Set of portfolios Q_2^*.**

λ	1th bond	2th bond	% (1th)	λ_0^*	% value
0.001	1	1	100.	.00085	99.8052
0.002	1	1	100.	.00170	99.6110
0.004	1	1	100.	.00341	99.2245
0.006	1	1	100.	.00512	98.8404
0.008	1	1	100.	.00682	98.4587
0.010	1	1	100.	.00852	98.0793
0.012	1	1	100.	.01022	97.7023
0.014	1	1	100.	.01192	97.3276
0.016	1	1	100.	.01362	96.9553
0.018	1	1	100.	.01531	96.5852
0.020	1	1	100.	.01700	96.2174
0.022	1	1	100.	.01869	95.8519
0.024	1	1	100.	.02038	95.4887
0.026	1	1	100.	.02207	95.1277
0.028	1	1	100.	.02375	94.7689
0.030	1	1	100.	.02544	94.4123
0.032	1	1	100.	.02712	94.0579
0.034	1	1	100.	.02880	93.7056
0.036	1	1	100.	.03047	93.3556
0.038	1	1	100.	.03215	93.0076
0.040	1	1	100.	.03382	92.6618
0.042	1	1	100.	.03549	92.3181
0.044	1	1	100.	.03716	91.9765
0.046	1	1	100.	.03883	91.6370
0.048	1	1	100.	.04049	91.2995
0.050	1	1	100.	.04216	90.9641
0.060	1	1	100.	.05044	89.3173
0.070	1	1	100.	.05867	87.7197
0.080	1	1	100.	.06685	86.1694
0.090	1	1	100.	.07497	84.6649
0.100	1	1	100.	.08305	83.2046
0.110	1	4	74.5761	.08372	81.7944
0.120	1	4	27.6143	.07632	80.4794
0.130	4	4	100.	.07403	79.2619
0.140	4	4	100.	.08100	78.0996
0.150	4	4	100.	.08812	76.9793
0.160	4	4	100.	.09537	75.8983
0.170	4	4	100.	.10273	74.8540
0.180	4	4	100.	.11020	73.8442
0.190	4	4	100.	.11776	72.8666
0.200	4	4	100.	.12541	71.9193
0.250	4	6	89.8112	.14296	67.6174
0.300	4	6	69.5939	.12617	64.2043
0.350	4	6	54.9612	.10725	61.5334
0.400	4	6	44.1790	.08684	59.4181
0.450	4	6	36.0764	.06538	57.7192
0.500	4	6	29.8635	.04315	56.3346
0.550	4	6	25.0046	.02035	55.1898
0.600	4	6	21.1332	-.00286	54.2303
0.650	4	6	17.9957	-.02640	53.4158
0.700	4	6	15.4137	-.05018	52.7165
0.750	4	6	13.2599	-.07416	52.1098
0.800	4	6	11.4416	-.09828	51.5785
0.850	4	6	9.89074	-.12252	51.1094
0.900	4	6	8.55564	-.14686	50.6921
0.950	4	6	7.39708	-.17127	50.3185
1.0	4	6	6.38452	-.19575	49.9819