Recent trends in orthogonal polynomials and their applications*

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Abstract

In this contribution we summarize some new directions in the theory of orthogonal polynomials. In particular, we emphasize three kinds of orthogonality conditions which have attracted the interest of researchers from the last decade to the present time. The connection with operator theory, potential theory and numerical analysis will be shown.

1 Introduction

Orthogonal polynomials are associated in numerical analysis with quadrature formulas and spectral methods for boundary value problems among others. For the first question, a modern presentation with a summary of recent results are [24]. For the second one, see [10, 22].

In spite of the popularity of orthogonal polynomials as an useful tool in the above domains, there is a very rich development in other areas as harmonic analysis, stochastic processes, operator theory, matrix analysis, number theory, etc., both from theoretical and applied purposes. We remember the fact that the usual concept of orthogonality is related to a positive Borel measure supported on the real line.

In fact, in the linear space $\mathbb{P}$ of polynomials with real coefficients we can associate with a positive Borel measure μ supported on the real line an inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(t)q(t)d\mu(t). \tag{1}$$

In such a situation, we can define a sequence (p_n) of orthonormal polynomials taking into account the very well known Gram-Schmidt method for the canonical sequence (t^n).

The key relation that such a sequence (p_n) satisfies is the three-term recurrence relation

$$\begin{gathered} tp_n(t) = a_{n+1}p_{n+1}(t) + b_np_n(t) + a_np_{n-1}(t), \quad n = 0, 1, \ldots, \\ p_{-1}(t) = 0, \quad p_0(t) = 1, \end{gathered} \tag{2}$$

where $a_n > 0$ and $b_n \in \mathbb{R}$.

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The modern theory of orthogonal polynomials is based on the analysis of the behaviour of the sequences of parameters (a_n) and (b_n) because the information about the analytic properties of polynomials, the measure of orthogonality, and the distribution of the zeros of polynomials can be deduced from them.

The aim of our contribution is to analyze some kind of inner products (which we will call nonstandard inner products) which yield sequences of orthogonal polynomials whose properties are quite different of those associated with (1):

1. Orthogonal polynomials on the unit circle, i.e., polynomials orthogonal with respect to measures supported on the unit circle in such a way that the inner product is defined by
$$\langle p, q\rangle = \int_0^{2\pi} p(e^{i\theta})\overline{q(e^{i\theta})}d\mu(\theta).$$
This will be the aim of Section 2.

2. Orthogonal polynomials with respect to Sobolev inner products
$$\langle p, q\rangle = \sum_{k=0}^{N} \int_{\Delta_k} f^{(k)}(z)\overline{g^{(k)}(z)}d\mu_k(z),$$
where $(\mu_k)_{k=0}^N$ are Borel regular measures with $\operatorname{supp}\mu_k = \Delta_k \subset \mathbb{C}$.

   This kind of orthogonality appears when we consider least square approximation with smoothness conditions. We will present some recent trends about this subject in Section 3.

3. Multiple orthogonal polynomials (MOP) are a family of polynomials generated by several orthogonality conditions, i.e., when the orthogonality conditions are distributed over r ($r \in \mathbb{N}$) intervals of $\mathbb{R}$ (or r arc/curve of the complex plane) with r measures. In Section 4 we will define such a kind of mathematical objects, and explain how the MOP are closely related with the simultaneous rational approximation (Hermite-Padé approximants) of a system of Markov (or Stieltjes) functions. Instead of only one function now a vector $(m_1(z), \ldots, m_r(z))$ of r functions is simultaneously approximated by a vector $\left(\frac{p_1(z)}{q_{\vec{n}}(z)}, \ldots, \frac{p_r(z)}{q_{\vec{n}}(z)}\right)$ of rational functions with the same denominator $q_{\vec{n}}(z)$.

   Special emphasis will paid to several examples of Hermite-Padé polynomials (multiple orthogonal polynomials) when the simultaneous orthogonality conditions are considered with respect to continuous and discrete "classical" measures. Discrete multiple orthogonal polynomials constitute a natural generalization of the continuous multiple orthogonal polynomials and also of the very classical orthogonal polynomials: Hahn, Meixner, Kravchuk and Charlier (for discrete case) and Jacobi, Laguerre and Hermite (for continuous case). These new families of orthogonal polynomials satisfy a Rodrigues type formula as well as an $r+2$ recurrence relation, where r is the dimension of the vector of Markov (or Stieltjes) functions or, equivalently, the number of orthogonality conditions.

2 Orthogonal polynomials on the unit circle

The role of orthogonal polynomials on the unit circle in circuit and system theory is very well recognized. Orthogonal polynomials with respect to measures supported on the unit circle are directly related to problems of Fourier expansions, characterization of positive functions and stable polynomials, least square polynomial approximations, and spectral operator analysis.

A set of orthogonal polynomials on the unit circle is classically associated with a linear functional $\mathcal{U}$ on the linear space of the Laurent polynomials $\Lambda = \text{span}\,\{z^k,\, k \in \mathbb{Z}\}$ such that if $\mathcal{U}_n = \langle \mathcal{U}, z^n \rangle$, then $\mathcal{U}_{-n} = \overline{\mathcal{U}_n}$, $n \in \mathbb{Z}$. Using this linear functional, we can introduce a bilinear form on $\mathbb{P}$, the linear space of polynomials with complex coefficients in the following way

$$\langle p, q \rangle = \left\langle \mathcal{U}, p(z)\overline{q}(z^{-1}) \right\rangle, \quad p, q \in \mathbb{P}.$$

Notice that in such a case, the shift operator is unitary with respect to the above bilinear form, which we will assume to be quasi-definite, i.e., the principal submatrices T_n, $n \in \mathbb{N}$, of the Gram matrix $T = [\mathcal{U}_{i-j}]_{i,j=0}^{\infty}$ are nonsingular for every $n \in \mathbb{N}$.

The infinite Gram matrix T is a Toeplitz matrix because of the shift operator is a unitary operator with respect to the above inner product. If the principal submatrices $(T_n)_{n\in\mathbb{N}}$ are positive definite then the linear functional $\mathcal{U}$ is said to be positive definite. In such a case, there exists a unique positive Borel measure μ supported on the unit circle $\mathbb{T}$ such that

$$\langle \mathcal{U}, p \rangle = \int_{\mathbb{T}} p(z) d\mu(z).$$

The Toeplitz matrix has the structure of the covariance matrix of a discrete stationary stochastic process [1].

Definition 2.1 *A sequence of monic polynomials (ϕ_n) is said to be a sequence of monic orthogonal polynomials (SMOP) with respect to the quasi-definite linear functional $\mathcal{U}$ if*

i) $\deg \phi_n = n$.

ii) $\langle \phi_n, \phi_m \rangle = M_n \delta_{n,m}$, $M_n \neq 0$.

From elementary properties of the Toeplitz matrices we get $|\phi_n(0)| \neq 1$. The values $(\phi_n(0))$ are called the reflection (or Schur) parameters of the linear functional $\mathcal{U}$.

The SMOP (ϕ_n) satisfies a forward recurrence relation

$$\phi_n(z) = z\phi_{n-1}(z) + \phi_n(0)\phi_{n-1}^*(z), \quad \phi_0(z) = 1, \quad n = 1, 2, \ldots,$$

which is the polynomial version of the Levinson algorithm widely used by statisticians in least square estimation problems.

On the other hand, the SMOP satisfies a backward recurrence relation

$$\phi_n(z) = (1 - |\phi_n(0)|^2) z\phi_{n-1}(z) + \phi_n(0)\phi_n^*(z), \quad \phi_0(z) = 1, \quad n = 1, 2, \ldots$$

In both recurrence relations $\phi_n^*(z) = z^n \overline{\phi_n}(z^{-1})$ is called the reversed polynomial associated with ϕ_n.

Theorem 2.2

i) If $\mathcal{U}$ is a positive definite linear functional, then the zeros of (ϕ_n) lie inside the unit circle.

ii) If $\mathcal{U}$ is a quasi-definite linear functional, then zeros of (ϕ_n) lie on $\mathbb{C} \setminus \mathbb{T}$.

Taking into account the first statement of the above theorem, the backward recurrence relation reads the Schur-Cohen algorithm in the stability theory of discrete linear systems.

In the theory of orthogonal polynomials on the real line there is a very important set of polynomials related to the spectral theory of linear differential (and difference) operators of second order with polynomial coefficients. The symmetrization factors for such operators are very well known distribution functions (beta, gamma and normal distributions of the probability theory) which lead to Jacobi, Laguerre and Hermite polynomials when we consider differential operators.

In the case of the unit circle, there are very few explicit examples of sequences of monic orthogonal polynomials. One of the reasons is the fact that they enter into a different class of problems than the classical orthogonal polynomials on the real line do.

In the last decade a hard work has been achieved in such a direction taking into account some different approaches from the theory of perturbation of measures [25, 26], the reflection parameters [53] or the polynomials themselves [14, 28].

We can not guess easily the asymptotic behaviour of the reflection parameters from the entries $(\mathcal{U}_n)_{n\in\mathbb{Z}}$ of the Toeplitz matrix.

It is very useful to select classes of orthogonal polynomials whose reflection parameters satisfy a kind of one-dimensional recurrence relation. This will happen where such recurrence relations will be associated with differential relations that the SMOP satisfies.

Speaking of differential relations, the famous Sonin-Hahn characterization of "classical" orthogonal polynomials as having derivatives which are also orthogonal polynomials fails short on the unit circle, as it only works for $\phi_n(z) = z^n$, the SMOP associated with the Lebesgue measure supported on $\mathbb{T}$ [39].

In the language of linear functionals defined on Λ, one defines the product of a function $f \in \Lambda$ and a linear functional $\mathcal{U}$ as the linear functional $\mathcal{U}$ such that

$$\langle f(z)\mathcal{U}, p(z)\rangle = \langle \mathcal{U}, f(z)p(z)\rangle, \quad p \in \Lambda.$$

The derivative $D\mathcal{U}$ of $\mathcal{U}$ is the linear functional such that

$$\langle D\mathcal{U}, p(z)\rangle = -i \langle \mathcal{U}, zp'(z)\rangle.$$

This latter strange definition corresponds actually to a simple relation for positive definite linear functionals, i.e., for positive Borel measures such that their Lebesgue decomposition has at most a finite number of Dirac masses [1, 37]. Assume P is a Laurent polynomial vanishing at these mass points. Then if $\langle \mathcal{U}, q\rangle = \frac{1}{\pi}\int_0^{2\pi} q(e^{i\theta})d\mu(e^{i\theta})$

$$\begin{aligned}\langle D(P\mathcal{U}), q\rangle &= -i \langle (P\mathcal{U}), zq'(z)\rangle \\ &= \frac{1}{2i\pi}\int_0^{2\pi} P(e^{i\theta})e^{i\theta}\frac{dq(e^{i\theta})}{de^{i\theta}}w(\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} q(e^{i\theta})\frac{d\left[P(e^{i\theta})w(\theta)\right]}{d\theta}d\theta,\end{aligned}$$

i.e. $D(P_n)$ has the integral representation involving $\frac{d\left[P(e^{i\theta})w(\theta)\right]}{d\theta}$. Here w is the Radon-Nikodym derivative of μ.

Remark that, in order to get rid of annoying boundary terms in the integration by parts, P must vanish at all Dirac mass points and other discontinuities of w.

Definition 2.3 *A linear functional $\mathcal{U}$ is said to semiclassical if it satisfies $D(A(z)\mathcal{U}) = B(z)\mathcal{U}$ where A, B are Laurent polynomials.*

In particular, if $\mathcal{U}$ has an integrable representation involving a positive Borel measure $d\mu = w(\theta)d\theta$, then

$$\frac{w'(\theta)}{w(\theta)} = \frac{B(e^{i\theta}) - ie^{i\theta}\frac{dA(e^{i\theta})}{de^{i\theta}}}{A(e^{i\theta})}, \tag{3}$$

with $A(e^{i\theta}) = 0$ at the singular points of w [37].

Examples

1. Let $d\mu(\theta) = \left(\cos\frac{\theta}{2}\right)^{2\beta}\left|\sin\frac{\theta}{2}\right|^{2\alpha} d\theta$ be the Jacobi measure on the unit circle. Then the corresponding linear functional is semiclassical and

$$\begin{aligned} A(z) &= z^2 - 1, \\ B(z) &= i\left[(\alpha+\beta+2)z^2 + 2(\alpha-\beta)z + \alpha+\beta\right]. \end{aligned}$$

2. The square wave with Fourier coefficients $\mathcal{U}_0 = 1$, $\mathcal{U}_{2n} = 0$, $n \neq 0$, and $\mathcal{U}_{2n+1} = (-1)^n/(2n+1)\pi$ is semiclassical.
3. The weight function $w(\theta) = |\theta|$ is not semiclassical.

With the positive weight function w we can associate a function v defined by $w(z) = \exp(-v(z))$.

The function v is said to be an external field for our inner product.

Theorem 2.4 *[29] Let $w(z)$ be differentiable in a neighborhood of the unit circle and assume that the integrals $\int_{\mathbb{T}} \frac{v'(z)-v'(y)}{z-y} y^n w(y)\frac{dy}{iy}$ exists for every $n \in \mathbb{Z}$. Then the corresponding orthonormal polynomials $\psi_n(z) = \frac{\phi_n(z)}{\|\phi_n\|}$ satisfy the differential relation*

$$\psi_n'(z) = n\frac{\|\phi_n\|}{\|\phi_{n-1}\|}\psi_{n-1}(z) + M_n(z)\psi_n(z) + N_n(z)\psi_n^*(z), \tag{4}$$

where

$$\begin{aligned} M_n(z) &= i\int_{\mathbb{T}} \frac{v'(z)-v'(y)}{z-y}\psi_n(y)\overline{\psi_n(y)}w(y)dy \\ N_n(z) &= -i\int_{\mathbb{T}} \frac{v'(z)-v'(y)}{z-y}\psi_n(y)\overline{\psi_n^*(y)}w(y)dy. \end{aligned}$$

We can rewrite (3) in the form

$$\psi_n'(z) = -A_n(z)\psi_n(z) + B_n(z)\psi_{n-1}(z), \tag{5}$$

taking into account the forward recurrence relation, and assuming $\psi_n(0) \neq 0$. Next we can define differential operators $\mathcal{L}_{n,1}$ and $\mathcal{L}_{n,2}$ by

$$\begin{aligned} \mathcal{L}_{n,1} &= \frac{d}{dz} + A_n(z) \\ \mathcal{L}_{n,2} &= -\frac{d}{dz} + A_{n-1}(z) + \frac{B_{n-1}(z)}{z}\frac{\|\phi_{n-2}\|}{\|\phi_{n-1}\|} + B_{n-1}(z)\frac{\|\phi_{n-2}\|}{\|\phi_n\|}\frac{\psi_{n-1}(0)}{\psi_n(0)}. \end{aligned}$$

Thus, the operators $\mathcal{L}_{n,1}$ and $\mathcal{L}_{n,2}$ are annihilation and creation operators in the sense that they satisfy

$$\begin{aligned}\mathcal{L}_{n,1}\psi_n(z) &= B_n(z)\psi_{n-1}(z)\\ \mathcal{L}_{n,2}\psi_{n-1}(z) &= \frac{B_{n-1}(z)}{z}\frac{\psi_{n-1}(0)}{\psi_n(0)}\frac{\|\phi_{n-2}\|}{\|\phi_{n-1}\|}\psi_n(z).\end{aligned}$$

Hence we deduce the second order differential equation [15, 29]

$$\mathcal{L}_{n,2}\left[\frac{1}{B_n(z)}\mathcal{L}_{n,1}\right]\psi_n(z) = \frac{B_{n-1}(z)}{z}\frac{\psi_{n-1}(0)}{\psi_n(0)}\frac{\|\phi_{n-2}\|}{\|\phi_{n-1}\|}\psi_n(z).$$

But if we assume (3) holds, the above relation can also be written in the following way

$$C(z;n)\psi_n''(z) + D(z;n)\psi_n'(z) + E(z;n)\psi_n(z) = 0, \tag{6}$$

where $C(z;n)$, $D(z;n)$, and $E(z;n)$ are polynomials in z of degree independent of n.

Example

Consider the weight function

$$w\left(e^{i\theta}\right) = \frac{1}{2\pi I_0(t)}\exp(t\cos\theta),$$

where I_ν is the modified Bessel function. The corresponding system of orthogonal polynomials arise from studies of the lenght of longest increasing subsequences of random permutations and unitary matrix models.

Proposition 2.5 *[29, 63] The reflection coefficients $\phi_n(0;t) = a_n(t)$ for the above system of orthogonal polynomial satisfy a discrete Painlevé II equation*

$$-2\frac{n}{t}\frac{a_n}{1-a_n^2} = a_{n+1} + a_{n-1}, \ \textit{for } n \geq 1, \ a_0(t) = 1, \ a_1(t) = -\frac{I_1(t)}{I_0(t)}.$$

There is also a differential relation which such reflection parameters satisfy or, equivalently, a differential relation in t for the orthogonal polynomials themselves.

Proposition 2.6

i)

$$\begin{aligned}2\frac{d\psi_n}{dt} &= \left[\frac{I_1(t)}{I_0(t)} + \frac{\psi_{n+1}(0)}{\psi_n(0)}\frac{\|\phi_{n+1}\|}{\|\phi_n\|}\right]\psi_n(z)\\ &-\frac{\|\phi_n\|}{\|\phi_{n-1}\|}\left[1 + \frac{\psi_{n+1}(0)}{\psi_n(0)}\frac{\|\phi_{n+1}\|}{\|\phi_n\|}z\right]\psi_{n-1}(z), \ \textit{for } n \geq 1.\end{aligned}$$

ii)

$$\begin{aligned}a_n'' &= \frac{1}{2}\left[\frac{1}{a_n+1} - \frac{1}{a_n-1}\right](r_n')^2 - \frac{1}{t}a_n'\\ &-a_n(1-a_n^2) + \frac{n^2}{t^2}\frac{a_n}{1-a_n^2},\end{aligned}$$

with the boundary conditions determined by the expansion

$$a_n(t) \sim \frac{(-\frac{1}{2}t)^n}{n!}\left[1 + \left(\frac{n}{n+1} - \delta_{n,1}\right)\frac{1}{4}t^2 + \mathcal{O}(t^4)\right], \ t \to 0,$$

for $n \geq 1$.

The parameter a_n *is related by*

$$a_n(t) = \frac{z_n(t)+1}{z_n(t)-1},$$

to $z_n(t)$ *which satisfies the Painlevé transcendent P-V equation with the parameters* $\alpha = \beta = \frac{n^2}{8}$, $\gamma = 0$, $\delta = -2$.

iii) Finally, the coefficients in (5) become

$$\begin{aligned} A_n(z) &= \frac{t}{2z}\frac{\|\phi_n\|}{\|\phi_{n-1}\|}\frac{\psi_{n-1}(0)}{\psi_n(0)} \\ B_n(z) &= \frac{\|\phi_n\|}{\|\phi_{n-1}\|}\left[n + \frac{t}{2z} + \frac{t}{2}\frac{\|\phi_n\|}{\|\phi_{n-1}\|}\frac{\psi_{n-1}(0)}{\psi_n(0)}\right. \\ &\left. -\frac{t}{2}\overline{\psi_{n+1}(0)}\psi_n(0)\,\|\phi_{n+1}\|\,\|\phi_n\|\right] \end{aligned}$$

In the general case of standard orthogonal polynomials on the real line, it is very well known [43] that a second order differential equation as (6) characterizes semiclassical linear functionals. Our open problem is to verify if such a differential equation characterizes semiclassical linear functionals on the unit circle when the polynomial solutions satisfy a backward (forward) recurrence relation.

The interest of this second order differential equation is related to the electrostatic interpretation of the zero distribution of orthogonal polynomials on the unit circle.

Let $\{z_{n,i}\}_{i=1}^n$ be the set of zeros of the polynomial $\phi_n(z)$. We can introduce the real function $|T(y_1, \ldots, y_n)|$ where

$$T(y_1, y_2, \ldots, y_n) = \prod_{j=1}^{n} y_j^{-n+1}\frac{e^{-v(y_j)}}{B_n(y_j)}\prod_{1\leq j<k\leq n}(y_j - y_k)^2,$$

such that the zeros are the stationary points of this function. It represents the total energy function for n unit charges in the unit disk interacting with a one-body confining potential $v(z) + \ln B_n(z)$, and attractive logarithmic potential with a charge $n-1$ at the origin, $(n-1)\ln z$, and repulsive logarithmic two-body potentials $\ln \frac{1}{|y_i - y_j|}$ between pairs of charges. However, all the stationary points are saddle points, a natural consequence of the analyticity on the unit disk.

3 Sobolev orthogonal polynomials

Orthogonal polynomials with respect to inner products involving derivatives (we will call them Sobolev orthogonal polynomials) have been intensively studied in the last decade. The core of the research activity was focused in the comparative analysis of the basic differences in terms of the standard ones. A very important contribution is done by several Spanish teams. Some survey papers about the state of the art appeared [20, 38, 45, 46].

An inner product is said to be a Sobolev inner product in the linear space $\mathbb{P}$ of polynomials with complex coefficients if

$$\langle f, g\rangle = \sum_{k=0}^{N}\int_{\Delta_k} f^{(k)}(z)\overline{g^{(k)}(z)}d\mu_k(z), \tag{7}$$

where $(\mu_k)_{k=0}^N$ is a vector of regular Borel measures with $\Delta_k = \operatorname{supp} \mu_k \subset \mathbb{C}$, $k = 0, 1, \ldots, N$.

Definition 3.1 *A sequence of polynomials (q_n) is said to be orthonormal with respect to the inner product (7) if*

i) $\deg q_n = n$.

ii) $\langle q_n, q_m \rangle = \delta_{n,m}$, $n, m \in \mathbb{N}$.

This family is unique if we assume the leading coefficient of q_n is positive for every $n \in \mathbb{N}$. We will denote (Q_n) the corresponding sequence of monic polynomials.

For a Sobolev inner product and a sequence of orthonormal polynomials (q_n) we can associate an infinite Hessenberg matrix $H = (h_{i,j})_{i,j=0}^{\infty}$ such that their entries are the Fourier coefficients of the images of the elements of the orthonormal polynomials (q_n) by the shift operator, i.e.,

$$zq_n(z) = \sum_{i=0}^{n+1} h_{i,n} q_i(z).$$

Notice that when we consider the case $N = 0$, two canonical situations appears.

If $\operatorname{supp} \mu_0 \subset \mathbb{R}$, then the Hessenberg matrix becomes a symmetric and tridiagonal (Jacobi) matrix. It represents a self-adjoint operator if the moment problem is a determinate problem, or, in particular, if $\mathbb{P}$ is a dense subset of $L^2(\mu_0)$.

If $\operatorname{supp} \mu_0 \subset \mathbb{T}$, the Hessenberg matrix is the representation of a unitary operator if and only if the measure μ_0 does not belong to the Szegő class, i.e. $\ln \mu_0' \notin L^1$, or, equivalently, $\mathbb{P}$ is a dense subset of $L^2(\mu_0)$.

The structural properties of H constitute the starting point for the understanding of the analytic properties of the orthonormal polynomials (q_n).

Indeed, the zeros of such polynomials are the eigenvalues of the principal submatrices of H. In the case $N = 0$ when $\operatorname{supp} \mu_0 \subset \mathbb{R}$, the zeros are real, interlace and live in the convex hull of the $\operatorname{supp} \mu_0$. In the case $N = 0$ when $\operatorname{supp} \mu_0 \subset \mathbb{T}$ we shown in Section 2 the zeros are inside the unit circle, in other words, the principal submatrices are no unitary matrices but we can obtain some bound for their norms in terms of the norm of the operator H.

In both cases, and taking into account the special structure of the Hessenberg matrix, it seems to be a natural question to obtain the explicit relation between the Hessenberg matrix associated with (7) and the Hessenberg matrices associated with the vector regular Borel measures $(\mu_k)_{k=0}^N$. In other words, the Hessenberg matrix for the shift operator can be characterized in terms of the Sobolev inner product (7)?.

Notice that if the measures $(\mu_k)_{k=1}^N$ are discrete measures, i.e., the sum of a finite number of Dirac masses, instead of the shift operator we can use a polynomial operator of the shift operator. Thus, when $\operatorname{supp} \mu_0 \subset \mathbb{R}$, this polynomial operator is symmetric with respect to (7) and, in fact, the corresponding infinite matrix is a symmetric band matrix which is strongly related with the Jacobi matrix associated with μ_0. This kind of questions appear in the study of higher order recurrence relations and the identification of inner products such that the corresponding sequences of orthonormal polynomials verify such recurrence relations [17, 18, 19].

On the other hand, there is a matrix which gives a relevant information about an inner product. It is the so called moment matrix $M = (c_{i,j})_{i,j=0}^{\infty}$ where $c_{i,j} = \langle z^i, z^j \rangle$, $i, j \in \mathbb{N}$ are said to be the moments of the inner product.

Definition 3.2 *Given an infinite and hermitian matrix M, the moment problem is said to be a definite Sobolev moment problem if there exists an inner product (7) such that the entries of M are the moments of (7). The moment problem is said to be determinate if the Sobolev inner product is unique.*

In [9] it is proved that the definiteness and determination of the moment problem when the support of the measures is real can be reduced to the same problem for the corresponding canonical decomposition in terms of the moment matrices associated with the vector measure $(\mu_k)_{k=0}^N$. In fact, we get

Proposition 3.3 *Given a symmetric matrix M, the Sobolev moment problem is definite (determinate) if and only if $M = \sum_{k=0}^N S^k M_k (S^k)^T$ where M_k $(k = 0, 1, \ldots, N)$ are Hankel matrices, $S = (s_{i,j})$ with $s_{i,j} = i$, $i - j = 1$, 0 otherwise and the moment problems for M_k are definite (determinate) for each k.*

Later on, in [41] and [42] we have considered a general situation and we obtain an efficient algorithm to deduce the above canonical decomposition.

Finally, taking into account the classical moment theory for the standard case, it seems to be natural to give sufficient conditions in terms of the entries of the Hessenberg matrix H in order to have a determinate moment problem. Notice that this question can be reformulated in terms of the density of $\mathbb{P}$ in weighted Sobolev space. Some results about this density problem have been obtained, independently of the moment theory, in [55].

Concerning the asymptotic properties of the polynomials (q_n) a remarkable advance is done in the last years. The first general results are obtained in [34] when $\operatorname{supp}\mu_0 = [a, b] \subset \mathbb{R}$, $\mu_0' > 0$ a.e. and $(\mu_k)_{k=1}^N$ are discrete measures. In fact, the polynomials (q_n) behave as the polynomials (p_n) orthonormal with respect to the measure μ_0. If ϕ denotes the conformal mapping of $\mathbb{C} \setminus [a, b]$ on the exterior of the unit disk, then

Proposition 3.4

$$\frac{q_n(z)}{p_n(z;\mu_0)} \rightrightarrows \prod_{j=1}^{m} \left[\frac{(\phi(z) - \phi(c_j))^2}{2\phi(z)(z - c_j)} \right]^{n_j+1},$$

locally uniformly in compact sets contained in $\overline{\mathbb{C}} \setminus \mathbb{S}$ where $\mathbb{S}$ is union of $[a, b]$ and the set of mass points $\{c_j\}$ of the measure $(\mu_i)_{i=1}^N$, and n_j is the higher order of the derivative at c_j.

When the measures $(\mu_i)_{i=1}^N$ have an infinite set as support, the complexity of the asymptotic problems increases very much. As a first approach the study was focused for $N = 1$. Here the sequence of orthonormal polynomials with respect to the measure μ_1 plays a central role. If we denote them by $p_n(z; \mu_1)$ then we get

Proposition 3.5 *If $\Delta = \operatorname{supp}\mu_0 = \operatorname{supp}\mu_1$ is a compact subset of the complex plane and Ω is the unbounded connected component of $\overline{\mathbb{C}} \setminus \Delta$, then*

$$\frac{Q_n(z)}{P_n(z;\mu_1)} \rightrightarrows \frac{1}{\phi'(z)}, \quad z \in \Omega, \tag{8}$$

locally uniformly. Here ϕ is the conformal mapping of Ω into $\mathbb{C} \setminus \mathbb{D}$ ($\mathbb{D}$ is the unit disk) such that $\lim_{z\to\infty} \frac{\phi(z)}{z} = 1$.

From the above result we deduce that the set of accumulation points of zeros of polynomials (q_n) is contained in Δ.

In [44] this result is proved under general hypothesis when Δ is a smooth Jordan arc/curve and the measures $(\mu_i)_{i=0}^{1}$ belong to the Szegő class.

If $\Delta_0 \not\subset \Delta_1$, the study of the relative asymptotics (8) remains open. In particular, and as a first step, the analysis of the case $\Delta_0 \cap \Delta_1 = \emptyset$ would be welcome.

Another asymptotic behavior is the so-called nth root asymptotics, i.e., the study of $\lim_{n\to\infty} |q_n(z)|^{\frac{1}{n}}$.

In [36] a first approach is done using logarithmic potential tools. From a heuristic point of view, if the nth root asymptotics holds, i.e.

$$\lim_{n\to\infty} |q_n(z)|^{\frac{1}{n}} = |\alpha(z)|, \tag{9}$$

then taking logarithmic derivatives we get

$$\lim_{n\to\infty} \frac{1}{n}\frac{q_n'(z)}{q_n(z)} = \frac{\alpha'(z)}{\alpha(z)},$$
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{z - z_{n,i}} = \frac{\alpha'(z)}{\alpha(z)}.$$

Thus, we obtain the weak convergence for the counting unit measure $\nu_n = \frac{1}{n}\sum \delta(z - z_{n,i})$ associated with the zeros $(z_{n,i})_{i=1}^{n}$ of the Sobolev orthonormal polynomial q_n. Notice that (9) must hold in a zero free region and thus we need to have information about the location of them.

If the shift operator is a bounded operator then its norm is an upper bound for the set of the zeros of q_n. Thus, we need necessary and sufficient conditions for the boundedness of such an operator in terms of the vector of measures $(\mu_k)_{k=0}^{N}$. In [35], the concept of sequentially dominated Sobolev inner product is introduced. This means that $\frac{d\mu_i}{d\mu_{i-1}} \in L^\infty(\Delta_{i-1})$, $i = 1, \ldots, N$ and thus, assuming that $\Delta_i \subset \Delta_{i-1}$ for $i = 1, \ldots, N$ we get a sufficient condition for the boundedness of the shift operator which, in some sense, is also a necessary condition (see [56]).

From such a condition, if the measures $(\mu_i)_{i=0}^{N}$ belong to the class of the regular measures, i.e.,

$$\lim_{n\to\infty} \left[\frac{\|r_n\|_{\Delta_k}}{\|r_n\|_{L^2(\mu_k)}}\right]^{\frac{1}{n}},$$

for every sequence of polynomials (r_n), $\deg r_n \leq n$, we get

Proposition 3.6 *Assume the unbounded connected component of $\mathbb{C} \setminus \Delta_0$ is regular with respect to the Dirichlet problem and the Sobolev inner problem is sequentially dominated. Then, for all $j \in \mathbb{N}$*

$$\overline{\lim_{n\to\infty}} \left|Q_n^{(j)}(z)\right|^{\frac{1}{n}} = C(\Delta_0)\exp(g_{\Delta_0}(z,\infty)),$$

for every $z \in \mathbb{C}$ up to a set of capacity zero and

$$\lim_{n\to\infty} \left|Q_n^{(j)}(z)\right|^{\frac{1}{n}} = C(\Delta_0)e^{g_{\Delta_0}(z,\infty)},$$

uniformly on compact subsets of $\mathbb{C} \setminus \{z \ : \ |z| \leq 2\|H\|\}$.

$C(\Delta_0)$ is the logarithmic capacity of Δ_0 and $g_{\Delta_0}(z,\infty)$ is the Green function with singularity at infinity.

Furthermore, $\nu_n \xrightarrow{*} w_{\Delta_0}$, the equilibrium measure of Δ_0.

The analysis of more general conditions for the existence of nth root asymptotics is an open problem.

Finally, if the supporting sets of the measures are unbounded sets, then scaling variable techniques are needed. We have obtained some results for the Hermite and Laguerre Sobolev polynomials when $N = 1$ (see [40]) but a general theory for unbounded supports remains open.

4 Multiple orthogonal polynomials

In this section we present the basic notions, definitions, and notations related with Multiple Orthogonal Polynomials (MOP). Special attention we will paid to the type II multiple discrete orthogonal polynomials. Furthermore, we will sketch the recent progress of the subject for the past few years, and discuss the results presented in this contribution.

During the past fifteen years there has been increased interest in multiple orthogonal polynomials, particularly promoted by the eastern European mathematical school. There are several survey papers about the topic (see, e.g., [3, 4, 13, 60] and Chapter 4 in the Nikishin and Sorokin book [52]), where the connection with the Hermite-Padé approximants is shown.

Recently, in [5, 6, 7] multiple orthogonal polynomials with respect to discrete measures have been considered. Other efforts in this direction have been considered (see [30, 31, 32, 52, 57, 58, 59]). A quite complete collection of them are surveyed in [61].

For multiple orthogonal polynomials we will need multi-indices consisting in a r-upple of positive integers, for which we use the notation $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$.

When the orthogonality conditions are distributed over r real intervals $\mathbb{D}_1, \dots, \mathbb{D}_r$ with r different measures $\mu_1, \mu_2, \dots, \mu_r$, to extend the classical orthogonal polynomials two different ways appear: The so-called type I and type II multiple orthogonal polynomials. Let us start with the last ones:

Type II multiple orthogonal polynomials

Definition 4.1 *A polynomial $q_{\vec{n}}(x)$ is said to be a multiple orthogonal polynomial of a multi-index $\vec{n}$ with respect to positive Borel measures*

$$\mu_1, \mu_2, \dots, \mu_r \quad \text{such that} \quad \text{supp}\, \mu_i \in \mathbb{R}, \quad i = 1, 2, \dots, r,$$

if it satisfies the following conditions:

$$\begin{gathered} \deg q_{\vec{n}} \leq |\vec{n}| := n_1 + n_2 + \cdots + n_r, \\ \int_{\mathbb{D}_i} q_{\vec{n}}(x) x^k d\mu_i(x) = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r. \end{gathered} \tag{10}$$

For $r = 1$ the multiple orthogonal polynomial becomes the standard orthogonal polynomial.

The *existence* of $q_{\vec{n}}(x) = \sum_{k=0}^{|\vec{n}|} a_{k,\vec{n}} x^k$ is always guaranteed, because for its $|\vec{n}| + 1$ unknown coefficients the orthogonality conditions (10) give a system of $|\vec{n}|$ linear algebraic homogeneous equations, which always has a nontrivial solution. However, the matrix of coefficients for such a system can be singular. Therefore the *uniqueness* is in general not guaranteed. A simple *counterexample* is when the measures $\mu_1, \mu_2, \dots, \mu_r$ are all identical over the same interval. Hence, we need some extra conditions on the r vector of measures in order that the above multiple orthogonal polynomial is unique.

Remark 4.2 *If one considers the non-Hermitian complex orthogonality with respect to a set of complex valued functions*

$$m_1(z) = \sum_{k=0}^{\infty} \frac{m_{1,k}}{z^{k+1}}, \ldots, m_r(z) = \sum_{k=0}^{\infty} \frac{m_{r,k}}{z^{k+1}}, \tag{11}$$

over the contours $\Gamma_i \subset \mathbb{C}$ ($i = 1, 2, \ldots, r$) then, the previous Definition is generalized, i.e., the notion of multiple orthogonal polynomials.

Hence, a multiple orthogonal polynomial $q_{\vec{n}}(z)$ with respect to complex weights (11) verifies

$$\deg q_{\vec{n}} \le |\vec{n}|,$$
$$\oint_{\Gamma_i} q_{\vec{n}}(z) z^k m_i(z) dz = 0, \quad k = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, r.$$

Two different sequences of indices are usually considered. The so-called diagonal and step-line sequences (see [3, 31, 60]).

Type I multiple orthogonal polynomials

Definition 4.3 *A vector of polynomials $(v_{\vec{n},1} v_{\vec{n},2}, \ldots, v_{\vec{n},r})$ is said to be a multiple orthogonal polynomial vector of type I if each polynomial $v_{\vec{n},i}$, where $i = 1, 2, \ldots, r$, satisfies the conditions*

$$\deg v_{\vec{n},i} \le n_i - 1,$$
$$\sum_{i=1}^{r} \int_{\mathbb{D}_i} v_{\vec{n},i}(x) x^k d\mu_i(x) = 0, \quad k = 0, 1, \ldots, |\vec{n}| - 2, \quad i = 1, 2, \ldots, r. \tag{12}$$

When $r = 1$ one recovers the standard orthogonal polynomials.

Again the *existence* of all the polynomials $v_{\vec{n},i}$ ($i = 1, 2, \ldots, r$) determined by Definition 4.3 is guaranteed, because there are $|\vec{n}| - 1$ orthogonality conditions which give $|\vec{n}| - 1$ linear algebraic homogeneous equations for the $|\vec{n}|$ unknown coefficients. Therefore the type I multiple orthogonal polynomial vector is determined up to a constant factor.

Connection with Hermite-Padé simultaneous rational approximants

Multiple orthogonal polynomials are intimately related to simultaneous Padé approximation, which is often known as Hermite-Padé approximation.

Let $\mathbb{D}_i = (a_i, b_i)$, $i = 1, 2, \ldots, r$, be intervals on the real line, and $\mu_1, \mu_2, \ldots, \mu_r$ are Borel measures on $\mathbb{R}$ with infinitely many points of increase such that $\operatorname{supp} \mu_i \subset \mathbb{D}_i$, $i = 1, 2, \ldots, r$. The Markov functions (or Stieltjes functions)

$$m_i(z) = \int_{\mathbb{D}_i} \frac{d\mu_i}{z - x}, \quad z \notin \mathbb{D}_i \quad i = 1, 2, \ldots, r,$$

can be approximated simultaneously by rational functions with prescribed order near infinity. Two different ways are considered to study such a kind of problems.

The first one: For the multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ of nonnegative integers is well known [52] how to find a polynomial $q_{\vec{n}}(z) \not\equiv 0$ of degree at most $|\vec{n}|$, in such a way that the expressions

$$q_{\vec{n}}(z)m_i(z) = p_{\vec{n},i}(z) + \frac{\zeta_i}{z^{n_i+1}} + \cdots, \quad i = 1, 2, \ldots, r, \tag{13}$$

are verified, being $p_{\vec{n},i}(z)$ certain polynomials. Notice that $q_{\vec{n}}(z)$ always exists, because the relations (13) lead us to a system of $|\vec{n}| + 1$ homogeneous linear equations. This approximation procedure, where one needs to find the polynomials $q_{\vec{n}}(z)$ and $p_{\vec{n},i}(z)$, is called type II Hermite-Padé approximation.

Through the rational function

$$\pi_i(\vec{n}, z) = \frac{p_{\vec{n},i}(z)}{q_{\vec{n}}(z)}, \quad i = 1, 2, \ldots, r. \tag{14}$$

we denote the simultaneous Hermite-Padé approximants of the r Markov (or Stieltjes) functions $m_1(z), m_2(z), \ldots, m_r(z)$, being $q_{\vec{n}}(z)$ the common denominator of the simultaneous approximants. This polynomial is precisely the multiple orthogonal polynomial due to the connection between the relations (13) and (10).

Thus, type II Hermite-Padé approximation is a rational approximation of the functions $m_i(z)$ $(i = 1, 2, \ldots, r)$ with the same denominator.

In [51], the author considered the simultaneous Hermite-Padé approximants of several Markov (or Stieltjes) functions, as well as the connection of this kind of approximation with the construction of linear forms on Markov (or Stieltjes) functions with polynomial coefficients.

For a fixed index-vector $\vec{n}$, although the existence of $q_{\vec{n}}(z)$ is guaranteed, the uniqueness is not determined by equalities (10) up to a normalizing constant. Even the rational functions $\pi_1(\vec{n}, z), \pi_2(\vec{n}, z), \ldots, \pi_r(\vec{n}, z)$ are not constructed in a unique way, using the vector-index $\vec{n}$ and the measures $\mu_1, \mu_2, \ldots, \mu_r$. There are two cases where $q_{\vec{n}}(z)$ is unique determined up to a constant factor: The so-called Angelesco and Nikishin cases [52].

Chebyshev system of functions.

Definition 4.4 *A set of continuous real functions $\{f_k(x)\}_0^n$ on the interval $\mathbb{D}$ is called a Chebyshev system (or T-system) of order n, if the determinant*

$$V(x_0, \ldots, x_n) = \det \begin{pmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(x_n) & f_1(x_n) & \cdots & f_n(x_n) \end{pmatrix}, \tag{15}$$

does not vanish for arbitrary different values $x_0, \ldots, x_n$ of $\mathbb{D}$.

The determinant (15) can be interpreted as the determinant of the system of homogeneous linear equations

$$\sum_{k=0}^{n} \lambda_k f_k(x_i) = 0, \quad i = 0, 1, \ldots, n,$$

where

$$\sum_{k=0}^{n} \lambda_k^2 > 0, \quad \lambda_k \in \mathbb{R}. \tag{16}$$

Then, the definition 4.4 is equivalent to the following statement [52]: The above set of functions $\{f_k(x)\}_0^n$ forms a T-system of order n, if every linear combination

$$F(x) = \sum_{k=0}^{n} \lambda_k f_k(x),$$

satisfying the condition (16), has at most n zeros on $\mathbb{D}$.

For type II multiple orthogonal polynomials there is an useful system of functions.

Definition 4.5 *An AT-system consists of r weights $\{\rho_k(x)\}_1^r$ supported on the same interval $\mathbb{D}$ such that*

$$\rho_1(x), x\rho_1(x), \ldots, x^{n_1-1}\rho_1(x), \ldots, \rho_r(x), x\rho_r(x), \ldots, x^{n_r-1}\rho_r(x), \tag{17}$$

is a T-system on $\mathbb{D}$ for every multi-index $\vec{n} = (n_1, \ldots, n_r)$.

In this contribution we will focus our attention on the set of weights (the extension to measures is straightforward) which forms an AT-system. See [5] for more information about the normality of the system of weights).

Brief description on the classical discrete polynomials.

The classical discrete orthogonal polynomials are those named after Hahn, Meixner, Kravchuk and Charlier. There exist several approaches to the study (or characterization) of these polynomials. The more standard one is based on the fact that these discrete orthogonal polynomials are special cases of the basic hypergeometric series [23]. Other usual approaches are the group-theoretical approach [62] and the difference-equation approach on a lattice with a constant mesh [47, 49]. In the present paper all the classical discrete orthogonal polynomials are considered as special cases of the type-II Hermite-Padé polynomials.

The Hahn polynomials $h_n^{(\alpha_0,\alpha_1)}(x,N)$ are polynomials of degree n which are orthogonal to all lower degree polynomials with respect to the weight function $\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_1+1)/(\Gamma(x+1)\Gamma(N-x))$ over the set of points $x \in [0, N-1]$ (mass points), where $\alpha_0, \alpha_1 > -1$; consequently, they satisfy the orthogonality conditions

$$\sum_{x=0}^{N-1} h_n^{(\alpha_0,\alpha_1)}(x,N) \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_1+1)}{\Gamma(x+1)\Gamma(N-x)} x^k = 0, \quad k = 0, \ldots, n-1. \tag{18}$$

The Meixner polynomials $m_n^{(\gamma,\upsilon)}(x)$ (being $\gamma > 0$ and $0 < \upsilon < 1$) are orthogonal over the set of points $x \in [0, \infty)$ to all lower degree polynomials with respect to the *Pascal distribution* $\upsilon^x (\gamma)_x / \Gamma(x+1)$, where $(\gamma)_x := \Gamma(\gamma+x)/\Gamma(\gamma)$, so that,

$$\sum_{x=0}^{\infty} m_n^{(\gamma,\upsilon)}(x) \frac{\upsilon^x \Gamma(\gamma+x)}{\Gamma(x+1)} x^k = 0, \quad k = 0, \ldots, n-1. \tag{19}$$

The Kravchuk polynomials $k_n^{(p)}(x,N)$ are orthogonal to all polynomials of degree less than n over the set of points $x \in [0, N]$ with respect to the *binomial distribution* $N!p^x(1-p)^{N-x}/(\Gamma(x+1)\Gamma(N+1-x))$, where $p, (1-p) > 0$. Therefore, the orthogonality conditions are

$$\sum_{x=0}^{N} k_n^{(p)}(x,N) \frac{N!p^x(1-p)^{N-x}}{\Gamma(x+1)\Gamma(N+1-x)} x^k = 0, \quad k = 0, \ldots, n-1. \tag{20}$$

Finally, the Charlier polynomials $c_n^{(a)}(x)$ are polynomials of degree n which are orthogonal to all lower degree polynomials with respect to the *Poisson distribution* $a^x/\Gamma(x+1)$ $(a>0)$ on the mass points $x \in [0,\infty)$. They have the orthogonality conditions

$$\sum_{x=0}^{\infty} c_n^{(a)}(x) \frac{a^x}{\Gamma(x+1)} x^k = 0, \quad k = 0, \dots, n-1. \tag{21}$$

The above four families of discrete orthogonal polynomials satisfy several properties which also allow to characterize them in a number of ways. Before proceed to comment these properties of classical discrete orthogonal polynomials, let define the forward and backward difference operators

$$\begin{aligned} \Delta y(x(s)) &:= \frac{\triangle}{\triangle x(s)} y(x(s)) = \frac{y(s+h)-y(s)}{x(s+h)-x(s)}, \\ \nabla y(x(s)) &:= \frac{\nabla}{\nabla x(s)} y(x(s)) = \Delta y(s-h), \end{aligned} \tag{22}$$

respectively, on a function given in arbitrary partition $x(s)$ with mesh h. For more simplicity, let us choose $x(s) = s$ and $\triangle x(s) = h = 1$. Such a situation we will name canonical one, nevertheless by canonical variable we will use x instead of s. Thus, the formula

$$\nabla^n y(x) = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!(n-i)!} y(x-i) = \sum_{i=0}^{n} \frac{(-n)_i}{i!} y(x-i), \tag{23}$$

holds. It can be proved easily by induction.

Other important property of the operator ∇ (and equivalently of Δ) is the formula of summation by parts

$$\sum_{x=a}^{b} y(x) \nabla z(x) = y(x)z(x)|_{a-1}^{b} - \sum_{x=a}^{b} z(x-1) \nabla y(x), \tag{24}$$

whose proof is straightforward by virtue of the relation

$$\nabla[y(x)z(x)] = y(x) \nabla z(x) + z(x-1) \nabla y(x). \tag{25}$$

One of the most useful properties of the classical discrete orthogonal polynomials are that these verify the hypergeometric-type difference equation

$$\sigma(x) \triangle \nabla y(x) + \tau(x) \triangle y(x) + \lambda_n y(x) = 0, \tag{26}$$

where σ and τ are polynomials independent of the degree n, such that, $\deg \sigma \le 2$, $\deg \tau = 1$, and λ_n is a constant depending on n.

This equation can be written in the self-adjoint form

$$\triangle[\sigma(x)\rho(x) \nabla y(x)] + \lambda_n \rho(x) y(x) = 0,$$

if the function $\rho(x)$ satisfies the *Pearson-type* equation

$$\triangle[\sigma(x)\rho(x)] = \tau(x)\rho(x). \tag{27}$$

A simple consequence of the second order linear difference equation (26) is that their polynomial solutions satisfy a finite-difference analog of the Rodrigues formula, i.e.,

$$p_n(x) = \frac{c_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right], \quad \nabla^n := \underbrace{\nabla \cdots \nabla}_{n\text{-times}}, \tag{28}$$

where c_n is a constant normalizing factor depending on n.

Based on (28) we can find the relation between $\triangle p_n(x)$ and the polynomials themselves. Hence, the first finite difference of the discrete orthogonal polynomials are again orthogonal polynomials of the same family, but with different parameters, i.e.,

$$\triangle p_n(x) = -\lambda_n \frac{c_n}{\tilde{c}_{n-1}} p_{n-1}(x),$$

where λ_n and c_n are the corresponding eigenvalue of (26) and normalizing factor of (28). The coefficient $\tilde{c}_{n-1}$ is the normalizing constant in the Rodrigues formula (28) for the polynomial $p_{n-1}(x)$ obtained by replacing $\rho(x)$ by $\sigma(x+1)\rho(x+1)$. In fact

$$\begin{cases} \triangle h_n^{(\alpha_0,\alpha_1)}(x,N) = \dfrac{h_{n-1}^{(\alpha_0+1,\alpha_1+1)}(x,N-1)}{(n+\alpha_0+\alpha_1+1)^{-1}}, \quad c_n = \dfrac{(-1)^n}{n!}, \\ \triangle m^{(\gamma,\upsilon)}(x) = \dfrac{n(\upsilon-1)}{\upsilon} m_{n-1}^{(\gamma+1,\upsilon)}(x), \quad c_n = \upsilon^{-n}, \\ \triangle k_n^{(p)}(x,N) = k_{n-1}^{(p)}(x,N-1), \quad c_n = \dfrac{(p-1)^n}{n!}, \\ \triangle c_n^{(a)}(x) = -\dfrac{n}{a} c_{n-1}^{(a)}(x) \quad c_n = a^{-n}. \end{cases} \tag{29}$$

All these families verify the three-term recurrence relation

$$x p_n(x) = \frac{a_n}{a_{n+1}} p_{n+1}(x) + \left[\frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \right] p_n(x) + \frac{a_{n-1}||p_n||^2}{a_n||p_{n-1}||^2} p_{n-1}(x), \tag{30}$$

as well as the structure relation

$$\sigma(x) \nabla p_n(x) = \frac{\lambda_n}{n\tau_n'} \left[\tau_n(x) p_n(x) - \frac{c_n}{c_{n+1}} p_{n+1}(x) \right], \tag{31}$$

being $p_n(x) = a_n x^n + b_n x^{n-1} +$ *lower terms*, and $\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x)$.

For the classical discrete orthogonal polynomials starting from any of the properties (26)-(31), or from one of the orthogonality conditions (18)-(21), can be deduced any of the other properties. So all the characterization ways coincide in the classical case, which is not true in general for other kind of polynomial families.

Discrete polynomials with simultaneous orthogonality

Now we will present five families of multiple discrete orthogonal polynomials which constitute an AT system (see [5]). Here we give their Rodrigues-type formulas [7].

Examples

Multiple Hahn polynomials. The multiple Hahn polynomials are orthogonal polynomials associated with an AT system consisting of Hahn weights on $[0, N-1]$. These polynomials verify simultaneous orthogonality conditions with respect to r measures over the same mass points belonging to the interval $[0, N-1]$. This system has different singularities at 0 and the same singularity at 1, when the set of mass points tends to infinity and one changes the variable $x \in [0, N-1]$ by $(N-1)s$. Let us discuss this affirmation in more detail. It is natural to expect that the Hahn polynomials $h_n^{(\alpha_0,\alpha_1)}(x, N)$, after the linear change of variable $x = (N-1)s$, which transforms the interval $[0, N-1]$ into $[0, 1]$, will tend to the Jacobi polynomials $P_n^{(\alpha_0,\alpha_1)}(s)$ when N tends to infinity (i.e., when the step $h = \triangle s = 1/(N-1)$ in the new variable s tends to 0), and that the weight function $\rho(x)$ will tend, up to a constant factor, to the weight function $x^{\alpha_0}(1-x)^{\alpha_1}$, where α_0, α_1¿-1 for the Jacobi polynomials, orthogonal on $[0, 1]$.

More precisely, replacing x by $(N-1)s$ in the Hahn weight one has

$$\rho(x) = \frac{\Gamma(N - Ns + \alpha_1)\Gamma(Ns + 1 + \alpha_0)}{\Gamma(N - Ns)\Gamma(Ns + 1)}.$$

Using the well known relation [49]

$$\frac{\Gamma(z+a)}{\Gamma(z)} = z^a \left[1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right], \quad |\arg z| \le \pi - \delta, \quad \delta > 0, \tag{32}$$

or, equivalently, the limit

$$\lim_{z\to\infty} \frac{\Gamma(z+a)}{z^a \Gamma(z)} = 1, \tag{33}$$

one gets that $\rho(x)$ behaves as $N^{\alpha_0}N^{\alpha_1}s^{\alpha_0}(1-s)^{\alpha_1}$ when $N \to \infty$.

Let $\alpha_0 > -1$ and $\alpha_1, \ldots, \alpha_r$ be such that each $\alpha_i > -1$, $i = 1, 2, \ldots, r$, and $\alpha_i - \alpha_j \in \mathbb{Z}$ whenever $i \neq j$. The function $\hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x, N)$ denotes the monic multiple Hahn polynomial of degree $|\vec{n}| < N-1$ for the multi-index $\vec{n} \in \mathbb{N}^r$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$ that satisfies the orthogonality conditions

$$\sum_{x=0}^{N-1} \hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x, N) \frac{\Gamma(N + \alpha_0 - x)\Gamma(x + \alpha_i + 1)}{\Gamma(x+1)\Gamma(N-x)} x^k = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r. \tag{34}$$

In [7] is proved the following

Proposition 4.6 *The following finite-difference analog of the Rodrigues formula*

$$\hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x, N) = \frac{(-1)^{|\vec{n}|}}{\prod_{i=1}^{r} (|\vec{n} + n_i \vec{e}_i| + \alpha_0 + \alpha_i)_{n_i}} \times \frac{\Gamma(N-x)}{\Gamma(N + \alpha_0 - x)} \mathcal{D} \frac{\Gamma(N + \alpha_0 - x)}{\Gamma(N - |\vec{n}| - x)}, \tag{35}$$

where

$$\mathcal{D} := \prod_{i=1}^{r} \mathcal{D}_{i,n_i}, \quad \mathcal{D}_{i,n_i} = \frac{\Gamma(x+1)}{\Gamma(x + \alpha_i + 1)} \nabla^{n_i} \frac{\Gamma(x + \alpha_i + n_i + 1)}{\Gamma(x+1)},$$

Remark 4.7 *The product of the r difference operators $\mathcal{D}_{i,n_i}$ can be taken in any order since these operators are commuting.*

From the orthogonality relations (34) we can write

$$\sum_{x=0}^{N-1} \hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x,N) \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i+1)}{\Gamma(x+1)\Gamma(N-x)} \bigtriangledown \pi_{k+1}(x+1) = 0,$$
$$k = 0,1,\ldots,n_i, \quad i = 1,2,\ldots,r,$$

where

$$\pi_k(x) = x(x-1)\cdots(x-k+1) = \frac{x!}{(x-k)!} = \frac{\bigtriangledown \pi_{k+1}(x+1)}{k+1}. \tag{36}$$

Hence, using the summation by parts (24), and the fact that $\pi_{k+1}(0) = 0$ and $\Gamma^{-1}(0) = 0$, one obtains

$$\sum_{x=0}^{N} \bigtriangledown \left[\hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x,N) \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i+1)}{\Gamma(x+1)\Gamma(N-x)} \right] \pi_{k+1}(x) = 0,$$
$$k = 0,1,\ldots,n_i-1, \quad i = 1,2,\ldots,r,$$

from which it is easily deduced the raising operators

$$\begin{gathered} \bigtriangledown \left[\hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x,N) \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i+1)}{\Gamma(x+1)\Gamma(N-x)} \right] \\ = c_{n,i} \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i)}{\Gamma(x+1)\Gamma(N-x+1)} \hat{h}_{\vec{n}+\vec{e}_i}^{(\alpha_0-1,\vec{\alpha}-\vec{e}_i)}(x,N+1), \quad i = 1,2,\ldots,r, \end{gathered} \tag{37}$$

where the multiplier constants $c_{n,i}$ are found comparing the coefficients of the power $|\vec{n}|+1$ of x on the two sides of (37), i.e.,

$$c_{i,n} = -(|\vec{n}| + \alpha_0 + \alpha_i).$$

If we apply the above operator $(l_i - 1)$ times, where $l_i \in \mathbb{N}$, $(l_i > 1)$ on the expression (37), the resulting operation leads us to the formulas

$$\begin{gathered} \bigtriangledown^{l_i} \left[\hat{h}_{\vec{n}}^{(\alpha_0,\vec{\alpha})}(x,N) \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i+1)}{\Gamma(x+1)\Gamma(N-x)} \right] = (-1)^{l_i} \\ (|\vec{n}| + \alpha_0 + \alpha_i)_{l_i} \frac{\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i)}{\Gamma(x+1)\Gamma(N-x+1)} \hat{h}_{\vec{n}+\vec{e}_i}^{(\alpha_0-l_i,\vec{\alpha}-\vec{e}_{l_i})}(x,N+l_i), \\ i = 1,2,\ldots,r. \end{gathered} \tag{38}$$

The convenient multiplication by the ratios $\Gamma(x+1)\Gamma(N-x+1)/\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_i)$ and $\Gamma(N+\alpha_0-x)\Gamma(x+\alpha_j)/\Gamma(x+1)\Gamma(N-x+1)$ on both sides of expressions (38), being $(j = 1,\ldots,i-1,i+1\ldots,r)$, and the successive application of $\bigtriangledown^{l_j}$ on both sides of the equalities leads to (35).

Similarly to the multiple Hahn polynomials we can consider the discrete orthogonal polynomials of simultaneous orthogonality over the mass points $x = 0,1,2,\ldots$, with respect to r different *Pascal distributions*. Here can be distinguished two different cases that we will show in more detail below.

Multiple Meixner polynomials (first kind)

The multiple Meixner polynomials of the first kind $\hat{m}_{\vec{n}}^{(\vec{\gamma},\upsilon)}(x)$ are given by the weights $\upsilon^x\Gamma(x+\gamma_i)/\Gamma(x+1)\Gamma(\gamma_i)$, where $\upsilon\in(0,1)$ and $\gamma_i>0$, for $i=1,2,\ldots,r$. The assumption $\gamma_i-\gamma_j\notin\mathbb{Z}$ guarantees the AT property of the system of Meixner weights (or Pascal distributions).

So, the orthogonality conditions which determine the polynomial $\hat{m}_{\vec{n}}^{(\vec{\gamma},\upsilon)}(x)$ are the following

$$\sum_{x=0}^{\infty}\hat{m}_{\vec{n}}^{(\vec{\gamma},\upsilon)}(x)x^k\frac{\upsilon^x\Gamma(\gamma_i+x)}{\Gamma(x+1)}=0,\quad k=0,\ldots,n_i-1,\quad i=1,\ldots,r. \tag{39}$$

From (39) is rather easy to find the raising operators

$$\mathcal{D}_i\hat{m}_{\vec{n}}^{(\vec{\gamma},\upsilon)}(x)=\frac{\upsilon-1}{\upsilon(\gamma_i-1)}m_{\vec{n}+\vec{e}_i}^{(\vec{\gamma}-\vec{e}_i,\upsilon)}(x),\quad \mathcal{D}_i:=\frac{\Gamma(x+1)}{\upsilon^x(\gamma_i-1)_x}\bigtriangledown\frac{\upsilon^x(\gamma_i)_x}{\Gamma(x+1)}, \tag{40}$$

where $(a)_x:=\Gamma(a+x)/\Gamma(a)$ is the Pochhammer symbol.

Repeated use of the raising operator (40) gives the Rodrigues-type formula

$$\hat{m}_{\vec{n}}^{(\vec{\gamma},\upsilon)}(x)=\left(\frac{\upsilon}{\upsilon-1}\right)^{|\vec{n}|}\left[\prod_{i=1}^{r}\frac{(\gamma_i+n_i-1)!}{(\gamma_i-1)!}\right]\Gamma(x+1)\mathcal{D}\,\Gamma^{-1}(x+1), \tag{41}$$

where $\mathcal{D}:=\prod_{i=1}^{r}\mathcal{D}_{i,n_i}$, and $\mathcal{D}_{i,n_i}:=\upsilon^{-x}\left(\gamma_i\right)_x^{-1}\bigtriangledown^{n_i}\upsilon^x\left(\gamma_i+n_i\right)_x$.

Multiple Meixner polynomials (second kind)

The other family of multiple Meixner polynomials appears when the simultaneous orthogonality conditions are distributed over the same set of discrete points $\{x\}=\mathbb{N}_0$ with respect to the weights $\upsilon_i^x(\gamma)_x/x!$, $(i=1,\ldots,r)$. To avoid the system of measures will be identical system we assume $\upsilon_i\neq\upsilon_j$ whenever $i\neq j$. The restrictions $\gamma>0$, and $\upsilon_i\in(0,1)$, $(i=1,\ldots,r)$ are essentially inherited from the classical case (19). Thus, the AT property is again guaranteed. Hence, the polynomial $\hat{m}_{\vec{n}}^{(\gamma,\vec{\upsilon})}(x)$ determined by the following orthogonality conditions

$$\sum_{x=0}^{\infty}\hat{m}_{\vec{n}}^{(\gamma,\vec{\upsilon})}(x)x^k\frac{\upsilon_i^x\Gamma(\gamma+x)}{\Gamma(x+1)}=0,\quad k=0,\ldots,n_i-1,\quad i=1,\ldots,r, \tag{42}$$

is called multiple Meixner polynomial of the second kind.

Analogous procedure as that carried out for the multiple Meixner polynomial of the first kind allows to obtain a finite-difference analog of the Rodrigues formula

$$\begin{aligned}\hat{m}_{\vec{n}}^{(\gamma,\vec{\upsilon})}(x)&=(\gamma+|\vec{n}|-1)!^2\left[\prod_{i=1}^{r}\left(\frac{\upsilon_i}{\upsilon_i-1}\right)^{n_i}\frac{1}{(\gamma+n_i-1)!}\right]\\&\times\Gamma(x+1)\mathcal{D}\,\Gamma^{-1}(x+1),\end{aligned} \tag{43}$$

where $\mathcal{D}:=\prod_{i=1}^{r}\mathcal{D}_{i,n_i}$, and $\mathcal{D}_{i,n_i}=\upsilon_i^{-x}\left(\gamma+|\vec{n}|-n_i\right)_x^{-1}\bigtriangledown^{n_i}\upsilon_i^x\left(\gamma+|\vec{n}|\right)_x$.

Multiple Kravchuk polynomials

The multiple Kravchuk polynomials are orthogonal polynomials of degree $|\vec{n}| < N$, associated to an AT system of Kravchuk weights (*binomial distributions).* The function $\hat{k}_{\vec{n}}^{(\vec{p})}(x,N)$ denotes the monic multiple Kravchuk polynomials for which the r orthogonality conditions are given on the same set of finite number of points $x = 1, 2 \ldots, N$ with respect to distinct binomial distribution. Thus, the orthogonality conditions are

$$\begin{gathered}\sum_{x=0}^{N} \hat{k}_{\vec{n}}^{(\vec{p})}(x,N) x^k \frac{N! p_i^x (1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N+1-x)} = 0, \quad p_i, 1-p_i > 0, \\ k = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, r.\end{gathered} \tag{44}$$

From (44), using (36) and (24) one obtains the raising operators

$$\begin{aligned}\mathcal{D}_i \hat{k}_{\vec{n}}^{(\vec{p})}(x,N) &= -\frac{1}{p_i(1-p_i)} \hat{k}_{\vec{n}+\vec{e}_i}^{(\vec{p})}(x, N+1), \quad i = 1, 2, \ldots, r \\ \mathcal{D}_i &:= \frac{N! p_i^x (1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N+1-x)} \bigtriangledown \frac{(N+1)! p_i^x (1-p_i)^{N+1-x}}{\Gamma(x+1)\Gamma(N+2-x)}.\end{aligned} \tag{45}$$

An appropriate combination of the raising operators (45) leads to the Rodrigues-type formula

$$\hat{k}_{\vec{n}}^{(\vec{p})}(x,N) = (-1)^{|\vec{n}|} \left[\prod_{i=1}^{r} p_i^{n_i}\right] \Gamma(x+1) \mathcal{D}\, \Gamma^{-1}(x+1), \tag{46}$$

where $\mathcal{D} := \prod_{i=1}^{r} \mathcal{D}_{i,n_i}$, and $\mathcal{D}_{i,n_i} = \frac{\Gamma^{-1}(N+1) p_i^{-x} (1-p_i)^x}{\Gamma^{-1}(N+1-x)} \bigtriangledown^{n_i} \frac{\Gamma(N-n_i+1) p_i^x (1-p_i)^{-x}}{\Gamma(N-n_i+1-x)}$.

Multiple Charlier polynomials

Finally, let the Poisson discrete measures

$$\mu_i = \sum_{x=0}^{\infty} \frac{a_i^x}{\Gamma(x+1)}, \quad a_i > 0, \quad i = 1, 2, \ldots, r,$$

determines the associated system of simultaneous orthogonal polynomials $\hat{c}_{\vec{n}}^{(\vec{a})}(x)$, with $\vec{a} = (a_1, a_2, \ldots, a_r)$ such that $a_i \neq a_j$, under the orthogonality conditions

$$\sum_{x=0}^{\infty} \hat{c}_{\vec{n}}^{(\vec{a})}(x) \frac{a_i}{\Gamma(x+1)} x^k = 0, \quad k = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, r. \tag{47}$$

Thus, $\hat{c}_{\vec{n}}^{(\vec{a})}(x)$ is the multiple Charlier polynomial (see also [8]).

In the same sense as we have proceeded in the above cases, the raising operators

$$\bigtriangledown \left[\hat{c}_{\vec{n}}^{(\vec{a})}(x) \frac{a_i^x}{\Gamma(x+1)}\right] = -\frac{1}{a_i} \frac{a_i^x}{\Gamma(x+1)} \hat{c}_{\vec{n}+\vec{e}_i}^{(\vec{a})}(x), \quad i = 1, 2, \ldots, r, \tag{48}$$

can be obtained from (47).

Repeated use of the raising operators (48) gives the Rodrigues-type formula

$$\hat{c}_{\vec{n}}^{(\vec{a})}(x) = (-1)^{|\vec{n}|} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r} \Gamma(x+1) \underbrace{\left[\prod_{i=1}^{r} a_i^{-x} \bigtriangledown^{n_i} a_i^x\right]}_{\mathcal{D}} \Gamma^{-1}(x+1). \tag{49}$$

Recurrence relation for multiple discrete orthogonal polynomials

Let $\mathcal{D}_{n_i}$, where n_i is the i-th coordinate of the vector-index $\vec{n}$, be a difference operator defined as bellow

$$\mathcal{D}_{n_i} := g_{i,1}(x;\alpha_1,\ldots,\alpha_n)\bigtriangledown^{n_i} g_{i,2}(x;\beta_1,\ldots,\beta_n), \quad \alpha_k,\beta_k\in\mathbb{R}, \quad k=1,\ldots,n,$$

being $g_{i,1}$ and $g_{i,2}$ certain functions depending, in general, on the i-th orthogonality measure.

The Rodrigues-type formulas allow us to introduce the following notation for the MDOP

$$q_{\vec{n}}(x) = c_{\vec{n},r} f_1(x)\mathcal{D}_{\vec{n}} f_2(x), \quad \mathcal{D}_{\vec{n}} := \prod_{i=1}^{r}\mathcal{D}_{n_i}, \tag{50}$$

where the coefficient $c_{\vec{n},r}$ depends on the parameters of the measures $\mu_1,\ldots,\mu_r$, and $f_1(x)$, $f_2(x)$ can also depend, in general, on $|\vec{n}|$ and on $\mu_1,\ldots,\mu_r$.

Lemma 4.8 *Let $n_k\in\mathbb{N}$. Then, the product of n_k backward difference operators over the function $xf(x)$ can be written in the following manner*

$$\bigtriangledown^{n_k} xf(x) = n_k\bigtriangledown^{n_k-1} f(x) + (x-n_k)\bigtriangledown^{n_k} f(x), \tag{51}$$

where $\bigtriangledown^{n_k} := \underbrace{\bigtriangledown\cdots\bigtriangledown}_{n_k\text{times}}$.

Using the relations (23) and (25), the proof is straightforward by induction, i.e.,

$$\begin{aligned}\bigtriangledown^{n_k} xf(x) &= \bigtriangledown^{n_k-1}[\bigtriangledown xf(x)] = \bigtriangledown^{n_k-1} f(x) + \bigtriangledown^{n_k-1}[(x-1)\bigtriangledown f(x)]\\ &= 2\bigtriangledown^{n_k-1} f(x) + \bigtriangledown^{n_k-2}\left[(x-2)\bigtriangledown^2 f(x)\right] = \cdots = \\ &= n_k\bigtriangledown^{n_k-1} f(x) + (x-n_k)\bigtriangledown^{n_k} f(x).\end{aligned}$$

Corollary 4.9 *The relation*

$$\mathcal{D}_{n_i} xf(x) = n_i\mathcal{D}_{n_i-1} f(x) + (x-n_i)\mathcal{D}_{n_i} f(x),$$

holds.

Lemma 4.10 *Let $\mathcal{D}_{n_i}$, where n_i is the i-th coordinate of the vector-index $\vec{n}$, be a difference operator defined in (50). Then,*

$$\mathcal{D}_{\vec{n}-\vec{e}} xf(x) = \left[\sum_{i=1}^{r}(n_i-1)\prod_{j=1}^{r}\mathcal{D}_{n_j-\delta_{j,i}-1} + (x-|\vec{n}|+r)\,\mathcal{D}_{\vec{n}-\vec{e}}\right] f(x), \tag{52}$$

where $\mathcal{D}_{\vec{n}-\vec{e}} := \mathcal{D}_{n_1-1}\cdots\mathcal{D}_{n_r-1} = \prod_{j=1}^{r}\mathcal{D}_{n_j-1}$ and $\delta_{i,j}$ is the delta Kronecker.

Applying r times the Corollary 4.9 the result (52) is obtained.

If the conditions

$$\bigtriangledown[g_{n_i,2}(x) f_2(x)] = g_{n_i,2}(x)[ax+b] f_2(x),$$

is verified by the set of functions $g_{n_i,2}(x)$ $(i=1,\ldots,r)$. Then, the MDOP $q_{\vec{n}}(x)$ satisfies a $r+2$ recurrence relation.

Theorem 4.11 *The multiple discrete orthogonal polynomial satisfies a recurrence relation of $r+2$ order (for every row and column in the Padé Table), where r is the number of orthogonality conditions.*

4.1 Application of MOP and open problems

This section deals with the application and open problems in which are involved the MOP and various fields of mathematics. Among them the number theory, the special functions and the spectral analysis of non-symmetric banded Hessenberg operators will be commented.

Number theory

The number theory is perhaps the more natural field of application of MOP. Indeed, the roots of these mathematical objects go back to the nineteenth century. More precisely, in 1878 C. Hermite used the Hermite-Padé approximants to prove the transcendence of "e" [27]. Traditionally, the Hermite method has been considered the main tool to investigate the arithmetic properties of real numbers. Related with the transcendence of π we remind the solution of the famous old problem about the quadrature of the circle given by Lendemann in 1882.

The multiple orthogonal polynomials seem to be an useful tool to prove the irrationality and transcendence of certain numbers.

Let us comment the problem about the arithmetic nature of the values of Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

at the odd points, because the first result in this direction has been obtained just in 1979 by Apéry.

The proof of Apéry is based on the following elementary lemma:

Lemma 4.12 *Let x be a real number, and p_n, q_n two sequences of integers ($n \in \mathbb{N}$). If p_n and q_n are such that*

i) $q_n x - p_n \neq 0$, *for all* $n \in \mathbb{N}$,

ii) $\lim_{n\to\infty} (q_n x - p_n) = 0$,

then x is irrational.

Theorem 4.13 (Apéry [2]) $\zeta(3)$ *is irrational.*

Apéry found the sequence of numbers

$$\begin{aligned} p_n &= \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2 \left[\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{j} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right] \\ q_n &= \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n}^2, \end{aligned} \qquad (53)$$

which after some normalization give a sequence of integers $\bar{p}_n$ and $\bar{q}_n$ such that

$$0 \neq r_n = \bar{p}_n - \bar{q}_n \zeta(3) \to 0, \quad n \to \infty.$$

Thus, using the previous Lemma 4.12 the statement holds.

The Beukers' contributions [11, 12] helped to understand where the sequences of integers (53) come from. On the other hand, the proofs given by Sorokin [] and Van Assche [60] based on multiple orthogonal polynomials about the irrationaly of $\zeta(3)$ are very constructive and interesting by themselves.

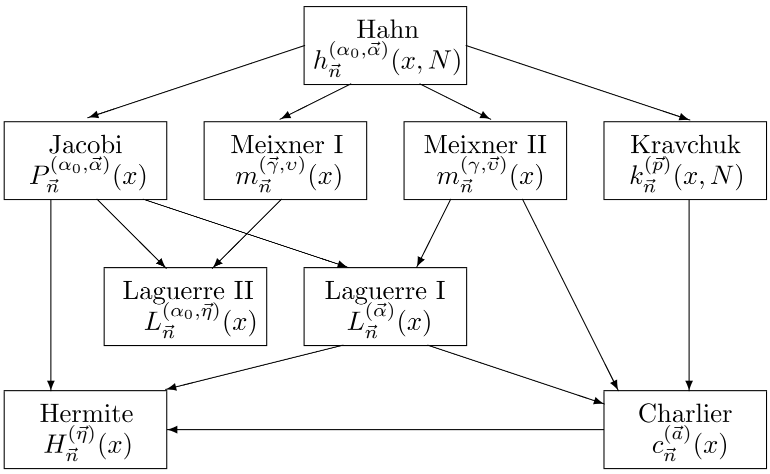
Concerning to the arithmetic nature of the values of Riemann zeta-function there is a challenging open problem which consists in proving the irrationality of $\zeta(5), \zeta(7), \ldots$, as well as the transcendency of $\zeta(3), \ldots$, etc.

Special functions and limit relations

The Rodrigues-type formula for the MOP suggests that the MOP could be expressed in terms of hypergeometric series. So, will be a very good contribution from the point of view of special functions to clarify this question.

Other interesting problem is to clasify all the classical multiple orthogonal polynomials and establish the limit relations between them. It would be a good idea to do this starting from the multiple Askey-Wilson polynomials because all other cases like multiple q-Hanh, q-Meixner, q-Kravchuk and q-Charlier can be obtained by limit transitions (see [6] for the q examples of MOP on the linear q lattice).

In [61] the authors show how some families of classical MOP are connected by limit passages. In [7] is shown the limit relations between discrete MOP and continous MOP for AT system (see the table below).



Non-symmetric band operators

Here we will show the relationship between MOP and the spectral theory of non-symmetric operators. Let us start mentioning few classical results. Assuming that the hight order difference operator are represented by a band matrix, i.e.,

$$H = (a_{i,j})_{i,j=0}^{\infty}, \quad a_{i,j} = 0, \quad (i > j + k,\ j > i + m), \quad \text{and} \quad a_{n,n-k} \neq 0,\ a_{n,n+m} \neq 0, \quad (54)$$

in the standard basis of the Hilbert space $l_2(\mathbb{N}_0)$

$$e_k = (\underbrace{0, \ldots, 0}_{k-\text{times}}, 1, 0, \ldots), \quad k \in \mathbb{N}_0,$$

one concludes that if H is a Jacobi matrix (i.e., symmetric tridiagonal matrix with real coefficient and positive extreme diagonals) then, the moment problem associated with H is determined. Hence by the Stone theorem, the class of closed operators

$$He_n = a_n e_{n-1} + b_n e_n + a_{n+1} e_{n+1}, \quad \text{where} \quad \left\{ \begin{array}{l} a_n = a_{n,n-1} \\ b_n = a_{n,n} \end{array} \right. ,$$

coincides with the class of simple spectrum self-adjoint operators (also known as Lebesgue operators). Therefore, by the spectral theorem of Von Neumann there exists a unique operator valued measure $\mathcal{F}_t$, for which H admits the representation

$$H = \int_{\mathbb{R}} t\mathcal{F}_t .$$

The spectral measure for the operator H is the positive Borel measure

$$\mu(t) = \langle \mathcal{F}_t e_0, e_0 \rangle .$$

Thus, the Weyl function is

$$\mathcal{S}(z) = \langle \mathcal{R}_z e_0, e_0 \rangle , \tag{55}$$

where $\mathcal{R}_z$ is the resolvent operator defined as

$$\mathcal{R}_t = (zI - H)^{-1} = \int_{\mathbb{R}} \frac{d\mathcal{F}_t}{z-t}.$$

For the self-adjoint operator the Weyl function becomes the Markov (or Stieltjes) function

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}. \tag{56}$$

The theory of orthogonal polynomial enjoys a very important result, known as Favard's theorem, which connects the spectral theory of self-adjoint operators and the theory of orthogonal polynomials. This theorem says that a sequence of polynomials $q_n(t)$ which verifies the recurrence relation (2) is always the orthogonal polynomial sequence with respect to the spectral measure

$$\int_{\mathbb{R}} q_n(t) t^k d\mu(t) = 0, \quad k = 0, \ldots, n-1. \tag{57}$$

Consecuently, the orthogonality (57) and the recurrence relation (2) are two equivalents ways to describe orthogonal polynomials.

Remark 4.14 *The three-term recurrence relation (2) can be written as*

$$\underbrace{\begin{pmatrix} b_0 & a_1 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} \end{pmatrix}}_{H_n} \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ \vdots \\ q_{n-1}(t) \end{pmatrix} = t \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \\ \vdots \\ q_{n-1}(t) \end{pmatrix} - a_n q_n(t) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Since the zeros of orthogonal polynomials $t_{n,i}$ $(i = 1, \ldots, n)$ are simple, one concludes that the eigenvalues of the Jacobi matrix H_n are the zeros of $q_n(t)$. Hence, the connection with the spectral theory of self-adjoint operators is clearly established when one considers the infinite matrix H (instead of H_n) acting as an operator $H : l_2 \to l_2$ on an appropiate domains.

On the other hand, the rational function $\pi_n(z) = \frac{p_n(z)}{q_n(z)}$ (see (14) for the multiple case) being $p_n(z)$ the other linearly independent solution of the difference equation (2), i.e.,

$$
\begin{aligned}
ty_n(t) = a_{n+1}y_{n+1}(t) + b_n y_n(t) + a_n y_{n-1}(t), \quad n = 0, 1, \ldots, \\
p_1(t) = \frac{!}{a_1}, \quad p_{-1}(t) = 0,
\end{aligned} \tag{58}
$$

is the diagonal Padé approximant for the Markov (or Stieltjes) function

$$
m(z) - \pi_n(z) = \frac{\zeta_n}{z^{2n+1}} + \ldots .
$$

Despite non-symmetric property of certain operators H, a properly chosen of the rational approximants for the Weyl functions (55) (see also (56)), of the operator H, guarantees the connection with the entries of the matrix H.

In [31] the author considers an example of an operator H associated to non-symmetric $p+2$ diagonal matrix (54). This example shows how the three-term recurrence relation and the Jacobi matrix have a natural extension for multiple orthogonal polynomials.

Let $veck(n)$ ($n \in \mathbb{N}$) be a sequence of multi-indices such that $n = kr+j$, where $0 \leq j < r$, and then set

$$
\vec{k}(n) = (\underbrace{k+1, \ldots, k+1}_{j-times}, k, \ldots, k). \tag{59}
$$

If all these indices are normal, then we have a weakly complete system. This condition is guaranted if we consider the spectral problem for H i.e., if $q_n(z)$ and $p_n^{(j)}(z)$ ($j = 1, \ldots, r$) are the $p+1$ linearly independent solutions of $(p+1)$-order difference equation

$$
\begin{aligned}
zy_n = a_{n,n-r}y_{n-r} + \cdots + a_{n,n}y_n + a_{n,n+1}y_{n+1}, \\
p_j^{(j)} = \frac{1}{a_{j-1,j}}, \quad p_n^{(j)} = 0, \quad n < j, \quad j = 1, \ldots, r.
\end{aligned}
$$

Then, the connection between the Hermite-Padé approximants for the system of functions

$$
m_j(z) = \langle \mathcal{R}_z e_{j-1}, e_0 \rangle, \quad j = 1, \ldots, r, \tag{60}
$$

and the spectral problem is given in the following

Theorem 4.15 (Kalyagin [31]) *For $n = kr + j$ the vector of rational functions*

$$
(\pi_1(\vec{n}, z), \ldots, (\pi_r(\vec{n}, z)),
$$

(see (14)) is the Hermite-Padé approximant of index (59) for the system (60)

Notice that in general for non-symmetric operators the notion of spectral positive measure loses sense. However, for non-symmetric operators the multiple orthogonal polynomials (Hermite-Padé polynomials) can be used, instead of the notion of standard orthogonal polynomials with respect to the positive spectral measure supported on the real line.

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